Effect of compact term for maximization problem on Trudinger-Moser inequality

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Abstract

We study the maximization problem on the Trudinger-Moser inequality involving compact term. This study is generalization of results in [5]. We prove that decaying speed of compact term plays a crucial role on existence and nonexistence of maximizer.

Keywords: maximization problem, Trudinger-Moser inequality, variational problem

1. Introduction

Assume that $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain. The classical Trudinger-Moser inequality asserts that

$$\sup_{\substack{u \in W_0^{1,N}(\Omega) \\ \|\nabla u\|_N \leq 1}} \int_\Omega e^{\alpha |u|^{\frac{N}{N-1}}} dx \begin{cases} < +\infty & (\alpha \leq \alpha_N), \\ = +\infty & (\alpha > \alpha_N), \end{cases}$$

where ω_{N-1} is the surface area of $(N-1)$-dimensional unit sphere and $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$. There are many results concerning this inequality so far. The origin of this inequality is the embeddings of $W_0^{1,N}(\Omega)$ by [17]. It was shown that $W_0^{1,N}(\Omega)$ is embedded continuously to the Orlicz space $L^{\phi_*}(\Omega)$ where $\phi_*(t) = e^{|t|^{\frac{N}{N-1}}} - 1$ and this embedding is sharp. After that the classical Trudinger-Moser inequality was shown by [14]. On the variational problem, the existence of a maximizer is known for any $\alpha \in (0, \alpha_N]$. When

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$\alpha \in (0, \alpha_N)$, we can find existence of maximizer since the Trudinger-Moser functional is continuous with respect to weak convergence sequence in the set $\left\{ u \in W_0^{1,N}(\Omega) \middle| \|\nabla u\|_N \leq 1 \right\}$. On the other hand, when $\alpha = \alpha_N$, the functional is not continuous. Thus existence and non-existence of maximizer is nontrivial in this case. The first result on the existence of a maximizer is [2] in the case of a unit ball. In general bounded domain case, the existence result was shown in 2-dimensional case by [4]. In the N-dimensional general bounded domain case, existence of the maximizer was shown by [10]. Besides these, some results related to existence of maximizer was obtained in [3, 13].

In $\mathbb{R}^N$ case, the situation is different. There are many studies in this case also, for instance [1], [6], and so on. Here, we introduce the paper only related to the variational problems. In [9], they proved the following inequality:

$$\sup_{\substack{u \in W^{1,N}(\mathbb{R}^N) \\ \|\nabla u\|_N + \|u\|_N \leq 1}} \int_{\mathbb{R}^N} \left(e^{\alpha |u|^{\frac{N}{N-1}}} - \sum_{j=0}^{N-2} \frac{\alpha^j |u|^{\frac{N}{N-1}j}}{j!} \right) dx \begin{cases} < +\infty & (\alpha \leq \alpha_N), \\ = +\infty & (\alpha > \alpha_N). \end{cases}$$

In addition, when $N \geq 3$, for $\alpha \in (0, \alpha_N]$ existence of maximizer was proved by [9, 8]. However, when $N = 2$, not only existence results by [15, 8] but also non-existence result was shown depending on α by [8]. Specifically, it was shown that a maximizer exists when $\alpha \in (\alpha_*, \alpha_2)$ for some constant α_*, and maximizer does not exist when α is sufficiently small. The cause is lack of compactness by vanishing phenomenon of bounded sequences.

In bounded domain case also, non-existence results for the Trudinger-Moser functional with perturbations exist. In two dimensional case, [16, 5, 7] investigated the maximization problem on the following

$$\sup_{\substack{u \in H_0^1(\Omega) \\ \|\nabla u\|_2^2 \leq \alpha_2}} \int_\Omega (1 + g(u)) e^{u^2} dx$$

and they clarified the form of g on the borderline of existence and nonexistence of a maximizer. In [16], the author studied two speeds on a blowing up sequence. One is the speed of remainder which comes from the concentration around the origin, and another one is that of vanishing on the annular region. Then the optimal nonlinearity on the behavior of g is shown taking these two speeds into account. In [7], they studied the unit ball case. They focus only on the concentration phenomena by using a cut off function and showed the optimal growth of g more strictly. In addition, they also studied the variational problem on the inequality in [6]. The variational problem on Adimurthi-Druet type was also studied by [12] and non-existence result was obtained. In [5], the author studied the following variational problem

$$C(\lambda, f) := \sup_{\substack{u \in H_0^1(B) \\ \|\nabla u\|_2 \le 1}} \int_B \left(e^{\alpha_2 u^2} - \lambda |u|^p \right) dx,$$

where λ is a positive constant and $p \ge 1$. This is the case of $g(s) = -\lambda |s|^p / (\alpha_2^{p/2} e^{s^2})$. However, since this perturbation decays rapidly as $s \to +\infty$ this situation is different from that in [16, 7] essentially. Before the studies by [5], existence result obtained by [3] for $p = 2$ and $\lambda < \alpha_2$. As the extended results due to [5], a maximizer exists for $p > 2, \lambda > 0$ or for $p \in [1, 2]$, sufficiently small λ, and maximizer does not exist for $p \in [1.2]$ and large λ. On this results, the crucial property is the speed of decaying of Lebesgue term $|\cdot|^p$. Thus, in this paper, we focus on this decaying speed in more detail.

We study the maximization problem

$$C(\lambda, f) := \sup_{\substack{u \in H_0^1(B) \\ \|\nabla u\|_2 \le 1}} \int_B \left(e^{\alpha_2 u^2} - \lambda f(|u|) \right) dx,$$

where $f \in C([0, +\infty), [0, +\infty))$ satisfies

$$f(0) = 0, \quad \text{and} \quad f(s) \le K e^{\alpha s^2} \ (s > S) \text{ for some } K, S > 0 \text{ and } \alpha \in (0, \alpha_2). \tag{1}$$

Throughout this paper, X denotes the set of all functions in $C([0, +\infty), [0, +\infty))$ satisfying (1). We set (I), (II), X_I, X_{II} as follows

- (I) There is a maximizer for any $\lambda > 0$.
- (II) There exists a threshold $\lambda_* = \lambda_*(f)$ such that if $\lambda \in (0, \lambda_*)$ a maximizer exists, and if $\lambda > \lambda_*$ maximizer does not exist.

$$X_I := \{ f \in X | \text{(I) holds.} \}, \quad X_{II} := \{ f \in X | \text{(II) holds.} \}$$

We will show that $X_I \cap X_{II} = \emptyset$ and $X_I \cup X_{II} = X$ in the section 2. Our purpose of this paper is to clarify conditions of $f \in X_I$ and $f \in X_{II}$ by using only decaying speed of f as $s \to 0$.

Remark 1.1. The second condition in (1) guarantees the compactness of f, that is if u_n satisfies $\|\nabla u_n\|_2 \le 1$ and $u_n \rightharpoonup u_0$ weakly in $H_0^1(B)$, then $\int_B f(u_n) dx \to \int_B f(u_0) dx$.

The main theorem is as follows.

Theorem 1.1. *(i) $f \in X_I$ if there exists $g \in X \cap C^1$ such that*

$$f(s) \leq g(s) \text{ for any } s \in [0, +\infty), \quad \text{and} \quad \lim_{s \to 0} \frac{g'(s)}{s} = 0. \tag{2}$$

(ii) $f \in X_{II}$ if $f \in X$ satisfies as follows:
There exist positive constants c_1 such that

$$\lim_{s \to 0} \frac{f(s)}{s^2} \geq c_1,$$

and for any sufficiently small $\varepsilon > 0$,

$$\inf_{s \geq \varepsilon} f(s) = f(\varepsilon) > 0.$$

(iii) Assume that $f \in X_{II}$ and f satisfies

$$f(s) = s^2 \quad \text{for } s \in [0, s_1), \;\; s_1 > 0,$$

or

$$f \in C^1 \quad \textit{and} \quad \lim_{s \to 0} \frac{f'(s)}{s} = 1,$$

then $\lambda_*(f) \geq \alpha_2 + 2e|B|$.

(iv) Assume that $f \in X_{II} \cap C^1$, and

$$\lim_{s \to 0} \frac{f'(s)}{s} = +\infty.$$

Then $C(\lambda_*, f)$ is attained.

This theorem is extended results in [5]. Indeed, $|s|^p$ satisfies the condition of the part (i) for $p > 2$, and the part (ii) for $p \in [1, 2]$. We note that if we consider the elliptic equation corresponding to the variational problem $C(\lambda, f)$, f should be C^1. Since this theorem is the argument on only the maximization problem, the function space X needs not to be differentiable. As in the section 3 and 4, we need the differentiability of f only in the typical case. The proof of this theorem is based on the techniques in [5]. However, in order to complete the proof, we need some preparations which will be introduced in the section 2.

Remark 1.2. The part (i) of Theorem 1.1 does not need the positivity of f. Indeed, we can prove the same result for any $f \in C([0,+\infty),\mathbb{R})$ such that

$$f(0)=0, \quad |f(s)| \le Ke^{\alpha s^2} \ (s>S) \text{ for some } K,S>0 \text{ and } \alpha \in (0,\alpha_2), \quad (3)$$

and (2). However, in this case, there is the possibility of $\tilde{X}_I \cup \tilde{X}_{II} \neq \tilde{X}$, where $\tilde{X} = \{f \in C([0,+\infty),\mathbb{R}) | f \text{ satisfies (3).}\}$, $\tilde{X}_I := \left\{ f \in \tilde{X} \middle| \text{(I) holds.} \right\}$, and $\tilde{X}_{II} := \left\{ f \in \tilde{X} \middle| \text{(II) holds.} \right\}$.

This paper is organized as follows. In Section 2, we prepare some lemmas and propositions to prove the main theorem. In Section 3 we prove the part (i). We will use the blow up analysis. In Section 4, we prove the part (ii). Also in this section, we will use the blow up analysis, but this techniques are a little bit different from Section 3 since we consider the case of $\lambda \to +\infty$. In Section 5, we prove the part (iii) and (iv). The strategies are based on Section 3 and 4.

2. Preliminaries

First, we fix some notations. The $L^q(B)$-norm is written as $\|\cdot\|_q$. For simplicity, sometimes we write function $v(r)$ as the radially symmetric function $v(x)$ by supposing that $r=|x|$. For a function v, we define v_+ and v_- as $v_+ := \max\{v,0\}$ and $v_- := \min\{v,0\}$. Unless otherwise stated, we assume that $f \in X$.

We prepare some lemmas and propositions to prove Theorem 1.1. We set

$$C_{rad}(\lambda,f) = \sup_{\substack{u \in H^1_{0,rad}(B) \\ \|\nabla u\|_2 \le 1}} \int_B \left(e^{\alpha_2 u^2} - \lambda f(|u|) \right) dx,$$

where $H^1_{0,rad}(B)$ is the set of radially symmetric functions in $H^1_0(B)$. By the symmetrization of function in $H^1_0(B)$, we can see that $C(\lambda,f) = C_{rad}(\lambda,f)$ and existence of maximizer of $C(\lambda,f)$ is equivalent to existence of maximizer of $C_{rad}(\lambda,f)$.

We take a sequence $\{u_n\}$ satisfying

$$\{u_n\} \subset H^1_{0,rad}(B), \quad \|\nabla u_n\|_2 \le 1, \quad u_n \rightharpoonup 0 \quad \text{weakly in } H^1_0(B)$$
$$\lim_{n\to\infty} \|\nabla u_n\|_2 \to 1, \quad \lim_{n\to\infty} \|\nabla u_n\|_{L^2(B\setminus B_\varepsilon)} = 0 \quad \text{for any } \varepsilon > 0.$$

We call $\{u_n\}$ satisfying the above conditions a normalized concentrating sequence. Then we have the following upper bound:

Proposition 2.1 ([2]). *For any normalized concentrating sequence* $\{u_n\}$*, we have*

$$\limsup_{n\to\infty} \int_B e^{\alpha_2 u_n^2} dx \le (1+e)|B|.$$

Proposition 2.2 ([3]). *There exists a normalized concentrating sequence* $\{y_n\}$ *such that*

$$\lim_{n\to\infty} \int_B e^{\alpha_2 y_n^2} dx = (1+e)|B|.$$

More precisely, for sufficiently large n*,* y_n *satisfies*

$$\int_B e^{\alpha_2 y_n^2} dx = (1+e)|B| + \varepsilon_n,$$

where ε_n *is a positive constant such that* $\varepsilon_n \to 0$ *as* $n \to \infty$*.*

The following lemma follows from the definition of X, $C(\lambda, f)$ and Proposition 2.2.

Lemma 2.3. *(i)* $C(\lambda, f)$ *is continuous and non-increasing with respect to* λ*.*

(ii) It follows that $C(\lambda, f) \ge (1+e)|B|$ *for any* λ *and* $f \in X$*.*

Proposition 2.4. *Assume that* f *is* C^1*. For any* $t \in [0,1)$*, we have*

$$\sup_{\substack{u\in H_0^1(B)\\ \|\nabla u\|_2\le t}} \int_B \left(e^{\alpha_2 u^2} - \lambda f(|u|)\right) dx < C(\lambda, f).$$

Proof. Set

$$C_t(\lambda, f) = \sup_{\substack{u\in H_0^1(B)\\ \|\nabla u\|_2\le t}} \int_B \left(e^{\alpha_2 u^2} - \lambda f(|u|)\right) dx$$

and assume that $C_t(\lambda, f) = C(\lambda, f)$. By the part (ii) of Lemma 2.3, we can see that 0 is not maximizer. We take a maximizing sequence $\{u_n\} \subset H_0^1(B)$, that is,

$$\|\nabla u_n\|_2 \le t, \quad \lim_{n\to\infty} \int_B \left(e^{\alpha_2 u_n^2} - \lambda f(|u_n|)\right) dx = C(\lambda, f).$$

Then we have $u_n \rightharpoonup u_0$ weakly in $H_0^1(B)$ and $\|\nabla u_0\|_2 = \tilde{t} \le t$. Moreover, by the compactness of the Trudinger-Moser functional and the functional $\int_B f(|\cdot|)dx$, it follows that

$$\int_B \left(e^{\alpha_2 u_0^2} - \lambda f(|u_0|)\right) dx = \lim_{n\to\infty} \int_B \left(e^{\alpha_2 u_n^2} - \lambda f(|u_n|)\right) dx = C(\lambda, f).$$

In addition, we may assume that $u_0 > 0$ and that $u_0 \in H^1_{0,rad}(B)$ by the symmetrization. Since u_0 is also a maximizer of

$$\sup_{\substack{u\in H_0^1(B)\\ \|\nabla u\|_2=\tilde{t}}} \int_B \left(e^{\alpha_2 u^2} - \lambda f(|u|)\right) dx,$$

there exists the Lagrange multiplier M such that

$$M\int_B \nabla u_0 \nabla \phi dx - \int_B \left(2\alpha_2 u_0 e^{\alpha_2 u_0^2} - \lambda f'(u_0)\right) \phi dx = 0 \tag{4}$$

for any $\phi \in H_0^1(B)$. On the other hand, for $s \in [0, 1/\tilde{t}]$ we set

$$H(s) := \int_B \left[e^{\alpha_2 (su_0)^2} - \lambda f(s|u_0|)\right] dx.$$

Then since $H'(s)|_{s=1} = 0$ we have

$$\int_B \left(2\alpha_2 u_0^2 e^{\alpha_2 u_0^2} - |u_0| f'(|u_0|)\right) dx = 0,$$

and hence $M = 0$. From this and (4), it follows that

$$\int_B \left(2\alpha_2 u_0 e^{\alpha_2 u_0^2} - \lambda f'(|u_0|)\right) \phi dx = 0$$

for any $\phi \in H_0^1(B)$. Hence

$$2\alpha_2 u_0 e^{\alpha_2 u_0^2} - \lambda f'(|u_0|) = 0$$

for any $x \in B \setminus \{0\}$ since u_0 is continuous in any annular domain due to $u_0 \in H^1_{0,rad}(B)$. Thus from $f(0) = 0$ it follows that $\lambda f(|s|) = e^{\alpha_2 s^2} - 1$ for $s \in [0, \|u_0\|_\infty]$. However, if $\lambda f(|s|) = e^{\alpha_2 s^2} - 1$ for $s \in [0, \|u_0\|_\infty]$, it follows that

$$(1+e)|B| \le C(\lambda, f) = C_t(\lambda, f) = \int_B \left(e^{\alpha_2 u_0^2} - \lambda f(u_0)\right) dx = |B|,$$

which is a contradiction by the part (ii) of Lemma 2.3 again. □

Lemma 2.5. *(i) If $C(\lambda, f) > (1+e)|B|$, then maximizer of $C(\lambda, f)$ exists.*

(ii) If there exists λ_* such that $C(\lambda_*, f) = (1+e)|B|$, then for $\lambda > \lambda_*$ maximizer does not exist.

Proof. We prove (i). Assume that $\{u_n\}$ is a maximizing sequence of $C(\lambda, f)$, namely, $\{u_n\}$ satisfies

$$\{u_n\} \subset H^1_{0,rad}(B), \quad \|\nabla u_n\|_2 \le 1, \quad \lim_{n\to\infty} \int_B \left(e^{\alpha_2 u_n^2} - \lambda f(|u_n|)\right) dx = C(\lambda, f).$$

Since $\{u_n\}$ is bounded sequence, there exists u_0 such that up to a subsequence $u_n \rightharpoonup u_0$ weakly in $H^1_0(B)$, and $\|\nabla u_0\|_2 \le 1$. By the assumption and Proposition 2.1, we can see that $\{u_n\}$ is not normalized concentrating sequence. Therefore by the theorem in [11] we have

$$\lim_{n\to\infty} \int_B \left(e^{\alpha_2 u_n^2} - \lambda f(|u_n|)\right) dx = \int_B \left(e^{\alpha_2 u_0^2} - \lambda f(|u_n|)\right) dx.$$

Consequently u_0 is the maximizer.

We prove (ii). Assume that $\lambda > \lambda_*$ and $u_\lambda \in H^1_{0,rad}(B)$ is a maximizer of $C(\lambda, f)$. Then we have

$$\begin{aligned}(1+e)|B| \;&\le\; C(\lambda, f) = \int_B \left(e^{\alpha_2 u_\lambda^2} - \lambda f(|u_\lambda|)\right) dx \\ &<\; \int_B \left(e^{\alpha_2 u_\lambda^2} - \lambda_* f(|u_\lambda|)\right) dx \le C(\lambda_*, f) = (1+e)|B|.\end{aligned}$$

This is a contradiction. □

The next lemma follows from the monotonicity of $C(\lambda, f)$ on f.

Lemma 2.6. *Assume that $f_1 \in X_I$. Then for any $f \in X$ satisfying $f(s) \le f_1(s)$ for all $s \in [0, +\infty)$, $f \in X_I$. On the other hand, assume that $f_2 \in X_{II}$. Then for any $f \in X$ satisfying $f(s) \ge f_2(s)$ for all $s \in [0, +\infty)$, $f \in X_{II}$.*

Proposition 2.7. *It follows that $X_I \cap X_{II} = \emptyset$ and $X_I \cup X_{II} = X$.*

Proof. The first assertion follows from the definitions of X_I and X_{II}. Assume that $f \notin X_I$. This implies the existence of Λ such that $C(\Lambda, f)$ is not attained. By the part (ii) of Lemma 2.3 and the part (i) of Lemma 2.5 it follows that

$C(\Lambda, f) = (1+e)|B|$. Thus by the part (ii) of Lemma 2.5 $C(\lambda, f)$ is not attained for any $\lambda > \Lambda$. We set

$$\lambda_* := \inf\{\lambda > 0 | C(\lambda, f) = (1+e)|B|\}.$$

By this definition, for $\lambda < \lambda_*$, $C(\lambda, f) > (1+e)|B|$ and $C(\lambda, f)$ is attained. On the other hand, as we confirmed that, for $\lambda > \lambda_*$, $C(\lambda, f) = (1+e)|B|$ and $C(\lambda, f)$ is not attained. Therefore $f \in X_{II}$. □

3. Proof of Theorem 1 (i)

In this section, we prove the part (i) of Theorem 1.1. The strategies is based on [5].

We assume that for $f \in X$ there is $g \in X \cap C^1$ satisfying (2). By Lemma 2.6, we only have to prove that $g \in X_I$. We may assume that

$$g(s) = K_1 e^{\alpha s^2} \quad (s > S), \qquad \sup_{s\in[0,S]} |g'(s)| \le K_2. \tag{5}$$

for some $K_1, K_2, S > 0$ and $\alpha \in (\alpha, \alpha_2)$. Fix $\lambda > 0$ and assume that $u_n \in H_0^1(B)$ is a maximizer of

$$C_n(\lambda, g) := \sup_{\substack{u\in H_0^1(B)\\ \|\nabla u\|_2 \le 1}} \int_B \left(e^{\alpha_n u^2} - \lambda g(|u|)\right) dx,$$

where α_n is a sequence of real numbers such that $\alpha_n \nearrow \alpha_2$ as $n \to \infty$. Since $C_n(\lambda, g) \to C(\lambda, g)$ as $n \to \infty$, we have

$$C(\lambda, g) = \lim_{n\to\infty} \int_B \left(e^{\alpha_n u_n^2} - \lambda g(|u_n|)\right) dx.$$

In addition, since u_n is a bounded sequence, we have $u_n \rightharpoonup u_0$ weakly in $H_0^1(B)$ up to a subsequence. We will show the following proposition.

Proposition 3.1. *If $u_n \rightharpoonup 0$ weakly in $H_0^1(B)$ as $n \to \infty$, then we have*

$$\|\nabla u_n\|_2^2 \ge \frac{\alpha_2}{\alpha_n}\left(1 + \frac{1 + \alpha_2(2e|B|)^{-1}}{\alpha_2^2}\frac{1}{\|u_n\|_\infty^4}\right) + o(\|u_n\|_\infty^{-4}).$$

By this proposition, if $u_n \rightharpoonup 0$ weakly in $H_0^1(B)$ holds, this is in contradiction to the constraint of $C_n(\lambda, g)$. Thus there is $u_0 \in H_0^1(B)$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(B)$. Consequently

$$\lim_{n\to\infty} \int_B \left(e^{\alpha_n u_n^2} - \lambda g(|u_n|)\right) dx = \int_B \left(e^{\alpha_2 u_0^2} - \lambda g(|u_0|)\right) dx,$$

and u_0 is a maximizer.

3.1. Preliminaries of the proof of Proposition 3.1

We prepare to prove Proposition 3.1. We note that α_n is a sequence with $\alpha_n \nearrow \alpha_2$ and that $u_n \in H_0^1(B)$ is a maximizer of $C_n(\lambda, g)$ again. By the symmetrization and similar result to Proposition 2.4, we have

$$\|\nabla u_n\|_2 = 1, \quad u_n \in H^1_{0,rad}(B), \quad u_n > 0, \quad \text{and} \quad \frac{\partial u_n}{\partial r} \leq 0.$$

Concerning g, we recall Remark 1.2.

Assume that $u_n \rightharpoonup 0$ weakly in $H_0^1(B)$. By the embedding theorem, we have

$$u_n(x) \to 0 \ \text{ in } B \setminus \{0\}.$$

Moreover, by the part (ii) of Lemma 2.3 we have

$$(1+e)|B| \leq \lim_{n\to\infty} \int_B \left(e^{\alpha_n u_n^2} - \lambda g(u_n) \right) dx,$$

and this implies that

$$\sup_{x\in B} u_n(x) = u_n(0) \to +\infty.$$

By the Lagrange multiplier theorem, u_n is a solution of

$$\begin{cases} -\Delta u = \frac{\alpha_n}{M_n} \left(u e^{\alpha_n u^2} - \frac{\lambda}{2\alpha_2} g'(u) \right), & u > 0, \quad \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where

$$M_n := \alpha_n \int_B \left(u_n^2 e^{\alpha_n u_n^2} - \frac{\lambda}{2\alpha_2} u_n g'(u_n) \right) dx.$$

By setting $v_n := \alpha_n^{1/2} u_n$, v_n satisfies

$$\begin{cases} -\Delta v_n = \frac{\alpha_n}{M_n} \left(v_n e^{v_n^2} - \frac{\lambda}{2\alpha_2^{1/2}} g'(\alpha_n^{-1/2} v_n) \right), & v_n > 0, \quad \text{in } B, \\ v_n = 0 & \text{on } \partial B, \end{cases} \tag{6}$$

and

$$\|\nabla v_n\|_2^2 = \alpha_n, \quad M_n = \int_B \left(v_n^2 e^{v_n^2} - \frac{\lambda}{2\alpha_n^{1/2}} v_n g'(\alpha_n^{-1/2} v_n) \right) dx.$$

By the elliptic regularity theory if follows that $v_n \in C^2(B)$. In addition, we note that $\lim_{n\to\infty} v_n = 0$ in $B \setminus \{0\}$ and $\lim_{n\to\infty} v_n(0) = +\infty$. We can also find that

$$\|\nabla v_n\|_2^2 = \frac{\alpha_n}{M_n} \int_B \left(v_n^2 e^{v_n^2} - \frac{\lambda}{2\alpha_2^{1/2}} v_n g'(\alpha_n^{-1/2} v_n) \right) dx.$$

We will study these two terms in the right hand side. By obtaining Proposition 3.2 in the subsection 3.2 and Proposition 3.12 in the subsection 3.3 we complete the proof.

For simplicity, we set

$$c_n := v_n(0) = \sup_{x\in B} v_n(x).$$

3.2. Estimate of the compact term

We focus on proving the following proposition:

Proposition 3.2. *It follows that*

$$\frac{\alpha_n^{1/2}}{2M_n} \lambda \int_B v_n g'(\alpha_n^{-1/2} v_n) dx = o(c_n^{-4})$$

as $n \to \infty$.

Proof of Proposition 3.2.

Lemma 3.3. *We have*

$$\lim_{n\to\infty} \int_B e^{v_n^2} dx = (1+e)|B|, \tag{7}$$

and

$$\lim_{n\to\infty} \int_B v_n g'(\alpha_n^{-1/2} v_n) dx = 0. \tag{8}$$

Proof. We show that

$$\lim_{n\to\infty} \int_B e^{\alpha_n u_n^2} dx = (1+e)|B|,$$

and

$$\lim_{n\to\infty} \int_B \alpha_n^{1/2} u_n g'(u_n) dx = 0.$$

As we confirmed that in Subsection 3,1, u_n is a normalized concentrating sequence. Thus by Proposition 2.1, we have

$$\begin{aligned}(1+e)|B| &\leq \liminf_{n\to\infty}\int_B \left(e^{\alpha_n u_n^2}-\lambda g(u_n)\right)dx\\ &\leq \liminf_{n\to\infty}\int_B e^{\alpha_n u_n^2}dx\\ &\leq \limsup_{n\to\infty}\int_B e^{\alpha_n u_n^2}dx\\ &\leq (1+e)|B|.\end{aligned}$$

The following estimate comes from (5) and the compactness of the embedding $H_0^1(B)$ into $L^q(B)$ for any $q\geq 1$. This yields the second claim.

$$\begin{aligned}\left|\int_B u_n g'(u_n)dx\right| &\leq K_2\int_B u_n dx + K_1\int_B 2\alpha u_n^2 e^{\alpha u_n^2}dx\\ &\leq K_2\int_B u_n dx + 2K_1\alpha\left(\int_B u_n^{2\frac{\alpha_2}{\alpha_2-\alpha}}dx\right)^{1-\frac{\alpha}{\alpha_2}}\left(\int_B e^{\alpha_2 u_n^2}dx\right)^{\frac{\alpha}{\alpha_2}}\\ &\leq K_2\int_B u_n dx + 2K_1\alpha[C(0,g)]^{\frac{\alpha}{\alpha_2}}\left(\int_B u_n^{2\frac{\alpha_2}{\alpha_2-\alpha}}dx\right)^{1-\frac{\alpha}{\alpha_2}}.\end{aligned}$$

□

Lemma 3.4. *It follows that*

$$\liminf_{n\to\infty} M_n > 0.$$

Proof. By the part (ii) of Lemma 2.3 and (8) we have

$$\begin{aligned}(1+e)|B| &\leq \int_B \left(e^{\alpha_n u_n^2}-\lambda g(u_n)\right)dx\\ &\leq \int_{[u_n\leq 1]} e^{\alpha_n u_n^2}dx + \int_{[u_n>1]} e^{\alpha_2 u_n^2}dx + o(1)\\ &\leq |B| + \int_B u_n^2 e^{\alpha_n u_n^2}dx + o(1)\\ &= |B| + \frac{1}{\alpha_n}\int_B\left(v_n^2 e^{v_n^2} - \frac{\lambda}{2\alpha_n^{1/2}}v_n g'(\alpha_n^{-1/2}v_n)\right)dx + o(1)\\ &\leq |B| + \frac{M_n}{\alpha_n} + o(1).\end{aligned}$$

Hence for sufficiently large n we have

$$M_n \geq e|B|.$$

□

Lemma 3.5. *For sufficiently large n, we have*

$$M_n \leq c_n^2 \left((1+e)|B| + o(1)\right).$$

Proof. By Lemma 3.3, we have

$$\begin{aligned} M_n &= \int_B \left(v_n^2 e^{v_n^2} - \frac{\lambda}{2\alpha_n^{1/2}} v_n g'(\alpha_n^{-1/2} v_n) \right) dx \\ &\leq c_n^2 \int_B e^{v_n^2} dx + o(1) \\ &= c_n^2 \left((1+e)|B| + o(1)\right). \end{aligned}$$

□

We set

$$r_n := \frac{\sqrt{M_n}}{\sqrt{\pi} c_n} e^{-\frac{c_n^2}{2}},$$

and

$$\begin{cases} \phi_n(y) := c_n(v_n(r_n y) - c_n), \\ \psi_n(y) := c_n^{-1} v_n(r_n y). \end{cases}$$

Note that $r_n = O(e^{-c_n^2/2})$ by Lemma 3.5. Then, from (6) we have

$$-\Delta \phi_n = 4 \left[\psi_n e^{\phi_n(1+\psi_n)} - \frac{1}{2\alpha_2^{1/2}} c_n^{-1} e^{-c_n^2} \lambda g'(\alpha_n^{-1/2} c_n \psi_n) \right] \quad \text{in } B_{1/r_n}, \qquad (9)$$

$$-\Delta \psi_n = \frac{4}{c_n^2} \left[\psi_n e^{c_n^2(\psi_n^2 - 1)} - \frac{1}{2\alpha_2^{1/2}} c_n^{-1} e^{-c_n^2} \lambda g'(\alpha_n^{-1/2} c_n \psi_n) \right] \quad \text{in } B_{1/r_n}. \qquad (10)$$

For sufficient large n since (5) and $\psi_n \leq 1$ we have

$$|g'(\alpha_n^{-1/2} c_n \psi_n)| \leq O(c_n e^{\frac{\alpha}{\alpha_n} c_n^2}) = o(e^{c_n^2}).$$

Thus we can use the elliptic regularity theory in (10). We have

$$\psi_n \to 1 \quad \text{in } C^2_{loc}(\mathbb{R}^2).$$

Moreover, in (9), by the elliptic regularity theory we have

$$\phi_n \to \phi_\infty = -\log(1+|x|^2) \quad \text{in } C^2_{loc}(\mathbb{R}^2),$$

$$-\Delta\phi_\infty = 4e^{2\phi_\infty} \quad \text{in } \mathbb{R}^2.$$

For a constant $\rho > 1$ we set

$$v_{n,\rho} := \min\left\{\frac{c_n}{\rho}, v_n\right\}.$$

Lemma 3.6.

$$\lim_{n\to\infty}\int_B |\nabla v_{n,\rho}|^2 dx = \frac{\alpha_2}{\rho}$$

We estimate the growth rate of M_n explicitly. We refer the techniques of the proof of Lemma 3.3 in [9] and the proof of Lemma 3.6 in [18]).

Lemma 3.7. *We have*

$$\liminf_{n\to\infty}\frac{M_n}{c_n^2} \geq e|B|.$$

Proof. For any fixed $\rho > 1$, by (8) and Lemma 3.6 we have

$$\begin{aligned}
\int_B e^{v_n^2}dx &= \int_{[v_n<c_n/\rho]} e^{v_n^2}dx + \int_{[v_n\geq c_n/\rho]} e^{v_n^2}dx \\
&\leq \int_B e^{v_{k,\rho}^2} + \frac{\rho^2}{c_n^2}\int_B v_n^2 e^{v_n^2}dx \\
&= |B| + o(1) + \frac{\rho^2}{c_n^2}\left(M_n + \frac{\lambda}{2\alpha_n^{1/2}}\int_B v_n g'(\alpha_n^{-1/2}v_n)dx\right) \\
&= |B| + \frac{\rho^2}{c_n^2}M_n + o(1).
\end{aligned}$$

The left hand side is $(1+e)|B| + o(1)$ by (7). Hence we obtain the inequality of the lemma. □

Lemma 3.8. *For any $\phi \in C^\infty(B)$ we have*

$$\lim_{n\to\infty}\frac{1}{M_n}\int_B c_n v_n e^{v_n^2}\phi dx = \phi(0).$$

We can prove this lemma in the same way as the proof of similar lemma in the previous works (for example, the proof of Lemma 3.6 in [9] and the proof of Lemma 3.9 in [18]).

Proposition 3.9. *We have*

$$\lim_{n\to\infty}\frac{M_n}{c_n^2}=e|B|.$$

Proof. By Lemma 3.7 we only have to show

$$\limsup_{n\to\infty}\frac{M_n}{c_n^2}\le e|B|.$$

Since $\lim_{n\to\infty} v_n=0$ in $B\setminus\{0\}$, for any $\varepsilon>0$ we have

$$\int_{B_\varepsilon} e^{v_n^2}dx=|B_\varepsilon|+e|B|+o(1).$$

We take $\phi_\varepsilon\in C^\infty(B)$ such that

$$\phi_\varepsilon(0)=1,\quad \phi_\varepsilon\le 1 \text{ in } B_\varepsilon,\quad \text{supp}\phi_\varepsilon\subset B_\varepsilon.$$

Then it follows that

$$|B_\varepsilon|+e|B|+o(1)=\int_{B_\varepsilon} e^{v_n^2}dx\ge\frac{M_n}{c_n^2}\left(\frac{1}{M_n}\int_B c_n v_n e^{v_n^2}\phi_\varepsilon dx\right).$$

By Lemma 3.8 we have

$$|B_\varepsilon|+e|B|\ge\limsup_{n\to\infty}\frac{M_n}{c_n^2}.$$

Consequently, we finish the proof. □

The following lemma follows from (5) and Proposition 3.9.

Lemma 3.10. *For any $\phi\in C^\infty(B)$ we have*

$$\lim_{n\to\infty}\frac{1}{M_n}\int_B c_n g'(\alpha_n^{-1/2}v_n)\phi dx=0.$$

Proposition 3.11. *For any $q\in[1,\infty)$, there exists a positive constant $C(q)$ such that for sufficiently large n we have*

$$\int_B v_n^q dx=\frac{C(q)+o(1)}{c_n^q}.$$

Proof. We consider the equation:

$$\begin{cases} -\Delta(c_n v_n) = \frac{\alpha_n}{M_n}\left[(c_n v_n)e^{v_n^2} - \frac{\lambda}{2\alpha_2^{1/2}} c_n g'(\alpha_n^{-1/2} v_n)\right] & \text{in } B, \\ v_n = 0 & \text{on } \partial B. \end{cases}$$

By Lemma 3.8 and Lemma 3.10 it follows that

$$\begin{aligned} -\int_B (\Delta c_n v_n)_- dx &= \frac{\alpha_n}{M_\lambda} \int_{[\Delta c_n v_n \le 0]} \left(c_n v_n e^{v_n^2} - \frac{\lambda}{2\alpha_2^{1/2}} c_n g'(\alpha_n^{-1/2} v_n)\right) dx \\ &\le \frac{\alpha_n}{M_n} \int_B c_n v_n e^{v_n^2} dx + o(1) \\ &= \alpha_2 + o(1), \end{aligned}$$

and

$$\begin{aligned} \int_B (\Delta c_n v_n)_+ dx &= \int_B (\Delta c_n v_n) dx - \int_B (\Delta c_n v_n)_- dx \\ &\le c_n \int_{\partial B} \frac{\partial v_n}{\partial \nu} d\sigma + \alpha_2 + o(1) \\ &\le \alpha_2 + o(1), \end{aligned}$$

Thus we have $\int_B |\Delta c_\lambda v_n| dx < 2\alpha_2 + o(1)$ and hence there exists $w \in W^{2,1}(B)$ such that

$$c_n v_n \rightharpoonup w \quad \text{weakly in} \quad W_0^{2,1}(B).$$

From this,

$$c_n v_n \rightharpoonup w \quad \text{weakly in} \quad W_0^{1,\gamma}(B) \quad \text{for any} \quad \gamma \in [1,2)$$

and hence

$$\int_B (c_n v_n)^q dx \to \int_B w^q dx \quad \text{for any} \quad q \in [1,\infty). \tag{11}$$

Moreover, by Lemma 3.8 and Lemma 3.10 w satisfies

$$\begin{cases} -\Delta w = \alpha_2 \delta_0 & \text{in } B, \\ w = 0 & \text{on } \partial B. \end{cases}$$

Thus w is concretely written as follows

$$w = \alpha_2 \omega_1^{-1} \log \frac{1}{|x|}. \tag{12}$$

Thus from (11) and (12) we have

$$\int_B v_n^q dx = \frac{1}{c_n^q} \left(\int_B \alpha_2 \omega_1^{-1} \log \frac{1}{|x|} dx + o(1) \right).$$

□

In order to complete the proof of Proposition 3.2 we prove

$$\int_B v_n g'(\alpha_n^{-1/2} v_n) dx = o(c_n^{-2}). \tag{13}$$

By (2), for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that for any $\delta \leq \delta_1$ we have

$$|g'(\delta)| \leq \varepsilon \delta.$$

In addition, by the properties of v_n there exists r_ε such that

$$\alpha_n^{-1/2} v_n(x) \leq \delta_1 \quad \text{for} \quad x \in B \setminus B_{r_\varepsilon}, \quad r_\varepsilon \to 0 \quad \text{as} \quad n \to \infty.$$

Thus we have

$$|g'(\alpha_n^{-1/2} v_n(x))| \leq \varepsilon \alpha_n^{-1/2} v_n(x) \quad \text{for} \quad x \in B \setminus B_{r_\varepsilon}, \quad r_\varepsilon \to 0 \quad \text{as} \quad n \to \infty.$$

Hence

$$\left| \int_{B \setminus B_{r_\varepsilon}} v_n g'(\alpha_n^{-1/2} v_n) dx \right| \leq \alpha_n^{-1/2} \varepsilon \int_B v_n^2 dx = \frac{\varepsilon}{c_n^2} \alpha_n^{-1/2} \left(C(2) + o(1) \right), \tag{14}$$

where we used (11) and (12). On the other hand, since (5) there exists $K_3 = K_3(\varepsilon)$ such that

$$|g'(\alpha_n^{-1/2} v_n(x))| \leq K_3 v_n(x) e^{\frac{\alpha}{\alpha_n} v_n^2(x)} \quad \text{for} \quad x \in B_{r_\varepsilon}$$

Thus since $r_\varepsilon \to 0$ we have

$$\begin{aligned} \left| \int_{B_{r_\varepsilon}} v_n g'(\alpha_n^{-1/2} v_n) dx \right| &\leq K_3 \int_{B_{r_\varepsilon}} v_n^2 e^{\frac{\alpha}{\alpha_n} v_n^2} dx \\ &= K_3 \left(\int_{B_{r_\varepsilon}} v_n^{2 \frac{\alpha_n}{\alpha_n - \alpha}} \right)^{1 - \frac{\alpha}{\alpha_n}} [C(0, g)]^{\frac{\alpha}{\alpha_n}} \\ &= o(c_n^{-2}) \end{aligned}$$

By combining this and (14), we finish to prove (13). Consequently, we complete the proof of Proposition 3.2. □

3.3. Estimate of the exponential term

In this subsection, we focus on proving the following proposition:

Proposition 3.12. *It follows that*

$$\frac{\alpha_n}{M_n}\int_B v_n^2 e^{v_n^2}dx \geq 4\pi + \left(4\pi + \frac{8\pi}{e}\right)\frac{1}{c_n^4} + o(c_n^{-4}),$$

Proof of Proposition 3.12. For any $\kappa_n > 0$ such that $\kappa_n \to 0$ as $n \to \infty$, we have

$$\frac{\alpha_n}{M_n}\int_{B\setminus B_{\kappa_n}} v_n^2 e^{v_n^2}dx \geq \frac{\alpha_n}{M_n}\int_{B\setminus B_{\kappa_n}} v_n^2 dx = \frac{\alpha_2\int_B (c_n v_n)^2 dx + o(1)}{(e|B| + o(1))c_n^4} = \frac{\frac{\alpha_2^2}{2e|B|} + o(1)}{c_n^4}. \tag{15}$$

We go back to the equation (6). Recall that as follows:

The function ϕ_n is defined by $\phi_n(y) := c_n(v_n(r_n y) - c_n)$ and ϕ_n satisfies

$$-\Delta_y\phi_n = 4\left(1+\frac{\phi_n}{c_n^2}\right)e^{\phi_n\left(2+\frac{\phi_n}{c_n^2}\right)} - \frac{2}{\alpha_n^{1/2}}c_n^{-1}e^{-c_n^2}\lambda g'\left(\alpha_n^{-\frac{1}{2}}c_n\left(1+\frac{\phi_n}{c_n^2}\right)\right) \quad \text{in } B_{1/r_n}.$$

We change the notation of the variable y into x again. Then $\phi_n \to \phi_\infty := -\log(1+|x|^2)$ in $C^2_{loc}(\mathbb{R}^2)$ and ϕ_∞ satisfies

$$-\Delta\phi_\infty = 4e^{2\phi_\infty} \quad \text{in } \mathbb{R}^2.$$

For sufficiently large n, we recall that

$$\left|g'\left(\alpha_n^{-\frac{1}{2}}\left(1+\frac{\phi_n}{c_n^2}\right)\right)\right| \leq O(c_n e^{\frac{\alpha}{\alpha_n}c_n^2}),$$

and thus

$$c_n^{-1}e^{-c_n^2}g'\left(\alpha_n^{-\frac{1}{2}}\left(1+\frac{\phi_n}{c_n^2}\right)\right) = O(e^{-Ac_n^2}) \quad \text{for some } A > 0.$$

By using this estimate and the strategy in [13] (the proof of Theorem 1) we get the following

Proposition 3.13. *Given a sequence $\{R_n\}$ with $R_n \in [c_n^q, e^{c_n}]$ for some $q > 2$, we have*

$$\frac{\alpha_n}{M_n}\int_{B_{R_n r_n}} v_n^2 e^{v_n^2}dx = 4\pi + \frac{4\pi}{c_n^4} + o(c_n^{-4}). \tag{16}$$

Combining (15) and (16) we obtain the estimate of the proposition. □

3.4. Proof of Proposition 3.1 completed

Recall that $v_n = \alpha_n^{1/2} u_n$. Proposition 3.2 and Proposition 3.12 yield Proposition 3.1.

4. Proof of Theorem 1.1. (ii)

In this section, we prove the part (ii) of Theorem 1.1. We define the function g_1 and g_2 as

$$g_1(s) := \begin{cases} s & (s \in [0,1]), \\ 2 - \frac{1}{s} & (s > 1), \end{cases}$$

$$g_2(s) = g_1^2(s)$$

We can check that $g_1, g_2 \in X$ and

$$g_1 \in C^1, \quad g_1(s) < 2, \quad g_1'(s) > 0 \quad \text{for any } s \in [0, +\infty),$$
$$g_2 \in C^1, \quad g_2(s) < 4, \quad g_2'(s) > 0 \quad \text{for any } s \in [0, +\infty).$$

Thus for any $f \in X$ with the properties in the part (ii), there exists $A = A_f > 0$ such that

$$A g_2(s) \leq f(s) \quad \text{for any } s \in [0, +\infty).$$

By the Lemma 2.6, we only have to prove that $A g_2 \in X_{II}$. This is equivalent to $g_2 \in X_{II}$. Indeed, assuming that $g \in X_{II}$, we can obtain that $Cg \in X_{II}$ and $\lambda_*(g) = \lambda_*(Cg)/C$ for any positive constant C. Thus we focus on proving $g_2 \in X_{II}$. The proof is organized two steps.

Step 1. $g_1 \in X_{II}$.

Step 2. $g_2 \in X_{II}$ by using that $g_1 \in X_{II}$.

The proofs of Step 1 and Step 2 are as follows. For fixed $i = 1$ or 2, g_i denotes g_1 or g_2. Assume that $g_i \in X_I$, and that u_λ is a maximizer of $C(\lambda, g_i)$ for each λ. By Proposition 2.4 we have $\|\nabla u_\lambda\|_2 = 1$. On the other hand, we obtain the following proposition.

Proposition 4.1. *There exist positive constants C_1 and C_2 such that we have*

$$\|\nabla u_\lambda\|_2^2 \leq 1 - \lambda \frac{C_1}{\|u_\lambda\|_\infty^{2+i}} + \frac{C_2}{\|u_\lambda\|_\infty^4} + o(\|u_n\|_\infty^{-4})$$

as $\lambda \to +\infty$.

However, this proposition contradicts the constraint that $\|\nabla u_\lambda\|_2 = 1$ for large λ. Hence for large λ maximizer does not exist. Consequently $g_i \in X_{II}$.

4.1. Preliminaries of the proof of Proposition 4.1

Before the preliminaries, we note the difference of the proofs in the case $i = 1$ and the case $i = 2$. As we said before, we have to prove that $g_1 \in X_{II}$ before proving that $g_2 \in X_{II}$ since we use the existence of $\lambda_*(g_1)$ in order to prove the existence of $\lambda_*(g_2)$. The proof of Lemma 4.5 is different point. The strategy is same as the proof of Theorem 1.1 in [5].

For any sequence λ_n such that $\lambda_n \to +\infty$ as $n \to \infty$, u_n denotes a sequence of maximizer of $C(\lambda_n, g_i)$. By Proposition 2.4 we can see that $\|\nabla u_n\|_2 = 1$. Thus there is $u_0 \in H_0^1(B)$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(B)$ up to a subsequence. Moreover, since

$$(1+e)|B| \leq C(\lambda_n, g_i) \leq C(0, g_i) - \lambda_n \int_B g_i(u_n)dx$$

we have

$$\int_B g_i(u_n)dx = O\left(\frac{1}{\lambda_n}\right) \quad \text{as} \quad n \to \infty,$$

which implies that $u_0 = 0$. By the compact embedding we can see that $u_n \to 0$ in $B \setminus \{0\}$. We can also see that $\lim_{n\to\infty} \sup_{x\in B} u_n(x) = u_n(0) = +\infty$ since

$$(1+e)|B| \leq C(\lambda_n, g_i) = \int_B e^{\alpha_2 u_n^2} dx - \lambda_n \int_B g_i(u_n)dx \leq \int_B e^{\alpha_2 u_n^2} dx.$$

In the same way as in Subsection 3.1, by setting $v_n := \alpha_2^{1/2} u_n$ and the Lagrange multiplier theorem, v_n satisfies

$$\begin{cases} -\Delta v_n = \frac{\alpha_2}{M_n}\left(v_n e^{v_n^2} - \frac{\lambda_n}{2\alpha_2^{1/2}} g_i'(\alpha_2^{-1/2} v_n)\right), & v_n > 0, \quad \text{in } B, \\ v_n = 0 & \text{on } \partial B, \end{cases} \tag{17}$$

and

$$\|\nabla v_n\|_2^2 = \alpha_2, \quad M_n = \int_B \left(v_n^2 e^{v_n^2} - \frac{\lambda_n}{2\alpha_2^{1/2}} v_n g_i'(\alpha_n^{-1/2} v_n)\right) dx.$$

By the elliptic regularity theory $v_n \in C^2(B)$. In addition, we note that $\lim_{n\to\infty} v_n = 0$ in $B \setminus \{0\}$ and $\lim_{n\to\infty} v_n(0) = \lim_{n\to\infty} \sup_{x\in B} v_n(x) = +\infty$.

We set $c_n = v_n(0)$. Different from Section 3, we remark that $\lambda_n \to +\infty$. Note that

$$g_1'(s) = \begin{cases} 1 & (s \in [0,1]) \\ \frac{1}{s^2} & (s > 1) \end{cases}$$

$$g_2'(s) = \begin{cases} 2s & (s \in [0,1]) \\ 2\left(2 - \frac{1}{s}\right)\frac{1}{s^2} & (s > 1) \end{cases}$$

and thus

$$sg_1'(s) \leq g_1(s) \quad \text{and} \quad sg_2'(s) \leq 2g_2(s).$$

4.2. Estimate of the compact term

In this section, we focus on proving the following proposition:

Proposition 4.2. *There exists a positive constant C_i such that*

$$\frac{\alpha_2^{1/2}}{2M_n}\int_B v_n g'(\alpha_2^{1/2} v_n)dx = \frac{C_i + o(1)}{c_n^{2+i}}$$

as $n \to \infty$.

Proof of Proposition 4.2.

Lemma 4.3.

$$\lim_{n\to\infty}\int_B e^{v_n^2}dx = (1+e)|B|,$$

$$\lim_{n\to\infty}\lambda_n\int_B v_n g_i'(\alpha_2^{-1/2}v_n)dx = 0,$$

$$\liminf_{n\to\infty} M_n > 0,$$

$$M_n \leq c_n^2\left((1+e)+o(1)\right).$$

Proof. We only prove the second equality since the proofs of the others are same as those in Subsection 3.2. From the first equality we have

$$(1+e)|B| \leq \lim_{n\to\infty}\int_B\left(e^{v_n^2} - \lambda_n g_i(\alpha_2^{-1/2}v_n)\right)dx = (1+e)|B| - \lim_{n\to\infty}\lambda_n\int_B g_i(\alpha_2^{-1/2}v_n)dx,$$

and thus

$$\lim_{n\to\infty}\lambda_n\int_B g_i(\alpha_2^{-1/2}v_n)dx = 0. \tag{18}$$

Hence

$$\lim_{n\to\infty} \lambda_n \int_B v_n g_i'(\alpha_2^{-1/2} v_n)dx \leq \lim_{n\to\infty} 2\lambda_n \int_B g_i(\alpha_2^{-1/2} v_n)dx = 0.$$

□

We set

$$r_n := \frac{\sqrt{M_n}}{\sqrt{\pi} c_n} e^{-\frac{c_n^2}{2}},$$

and

$$\begin{cases} \phi_n(y) := c_n(v_n(r_n y) - c_n), \\ \psi_n(y) := c_n^{-1} v_n(r_n y). \end{cases}$$

Note that $r_n = O(e^{-c_n^2/2})$. Then, from (17) we have

$$-\Delta\phi_n = 4\left[\psi_n e^{\phi_n(1+\psi_n)} - \frac{1}{2\alpha_2^{1/2}} c_n^{-1} e^{-c_n^2} \lambda_n g_i'(\alpha_n^{-1/2} c_n \psi_n)\right] \quad \text{in } B_{1/r_n}, \quad (19)$$

$$-\Delta\psi_n = \frac{4}{c_n^2}\left[\psi_n e^{c_n^2(\psi_n^2-1)} - \frac{1}{2\alpha_2^{1/2}} c_n^{-1} e^{-c_n^2} \lambda_n g_i'(\alpha_n^{-1/2} c_n \psi_n)\right] \quad \text{in } B_{1/r_n}. \quad (20)$$

For (20) it follows that

$$\frac{2}{\alpha_2^{1/2}} c_n^{-3} e^{-c_n^2} \lambda_n = \frac{4\int_{B_{1/r_n}} \psi_n^2 e^{c_n^2(\psi_n^2-1)} - \alpha_2}{c_n^2 \int_{B_{1/r_n}} \psi_n g_i'(\alpha_n^{-1/2} c_n \psi_n)} = \frac{I_1}{I_2}.$$

Concerning I_1 we have

$$I_1 \leq 4\int_{B_{1/r_n}} \psi_n^2 dx.$$

On the other hand, concerning I_2 by the definition of g_i there exists a positive constant L such that

$$I_2 \geq \frac{1}{L}\int_{B_{1/r_n}} \psi_n^2 dx,$$

where we used $\psi_n \leq 1$. Thus

$$\frac{2}{\alpha_2^{1/2}} c_n^{-3} e^{-c_n^2} \lambda_n \leq L.$$

Thus we can use the elliptic regularity theory in (20). By the strong maximum principle we have

$$\psi_n \to 1 \quad \text{in } C^2_{loc}(\mathbb{R}^2), \tag{21}$$

and

$$\frac{2}{\alpha_2^{1/2}} c_n^{-3} e^{-c_n^2} \lambda_n = o(1). \tag{22}$$

Since (22) we find that $c_n^{-1} e^{-c_n^2} \lambda_n = o(c_n^2)$. Moreover, since (21) we have $g'(\alpha_2^{-1/2} c_n \psi_n) = O(c_n^{-2})$ in B_R for each $R > 0$. Hence the second term of the right hand side in (19) vanishes as $n \to \infty$. From this fact, in (19), by the elliptic regularity theory we have

$$\phi_n \to \phi_\infty = -\log(1 + |x|^2) \quad \text{in } C^2_{loc}(\mathbb{R}^2),$$

$$-\Delta \phi_\infty = 4e^{2\phi_\infty} \quad \text{in } \mathbb{R}^2.$$

For a constant $\rho > 1$ we set

$$v_{n,\rho} := \min\left\{\frac{c_n}{\rho}, v_n\right\}. \tag{23}$$

We can get the next lemma same as in the subsection 3.2.

Lemma 4.4. *We have*

$$\lim_{n\to\infty} \int_B |\nabla v_{n,\rho}|^2 dx = \frac{\alpha_2}{\rho}, \tag{24}$$

$$\lim_{n\to\infty} \frac{1}{M_n} \int_B c_n v_n e^{v_n^2} \phi dx = \phi(0),$$

$$\lim_{n\to\infty} \frac{M_n}{c_n^2} = e|B|, \tag{25}$$

and there exists w such that for any $\gamma \in (1, 2)$ we have

$$c_n v_n \rightharpoonup w \quad \text{weakly in} \quad W_0^{1,\gamma}(B). \tag{26}$$

Lemma 4.5. *We have*

$$w = \alpha_2 \omega_1^{-1} \log \frac{1}{|x|}.$$

Proof in the case $i = 1$. By (17) and (26) for any $\phi \in C_0^\infty(B)$ we have

$$\int_B \nabla w \nabla \phi dx = \alpha_2 \phi(0) - \lim_{n\to\infty} \frac{\alpha_2^{1/2}}{2} \frac{\lambda_n}{M_n} \int_B c_n \phi g_1'(\alpha_2^{-1/2} v_n) dx. \tag{27}$$

If $w \equiv 0$, since $g'(s) = 1$ for $s \in [0,1]$ we have

$$o(1) = \alpha_2 \phi(0) + o(1) - \frac{\alpha_2^{1/2}}{2} \frac{\lambda_n}{M_n} c_n \left(\int_B \phi dx + o(1) \right),$$

which is a contradiction. Thus $w \not\equiv 0$. By (18) we have

$$o(1) = \lambda_n \int_B g_1(\alpha_2^{-1/2} v_n) dx \geq \frac{\lambda_n}{c_n} \left(\frac{1}{\alpha_2^{1/2}} \int_{B \setminus B_{1/2}} w dx + o(1) \right),$$

which means that

$$\lambda_n = o(c_n). \tag{28}$$

Going back to (27), by (25) we find that w satisfies

$$\begin{cases} -\Delta w = \alpha_2 \delta_0 & \text{in } B, \\ w = 0 & \text{on } \partial B, \end{cases}$$

and thus we complete to show Lemma 4.5. □

Proof in the case $i = 2$. Assume that $\lambda_*(g_1)$ exists. By Hölder's inequality, we have

$$|B|^{-\frac{1}{2}} \left(\int_B f^2 dx \right)^{\frac{1}{2}} \int_B |f| dx \leq \int_B f^2 dx.$$

Replacing $|f|$ and f^2 by $g_1(u_n)$ and $g_2(u_n)$ with u_n which are maximizers of $C(\lambda_n, g_2)$ respectively, we have

$$\begin{aligned} (1+e)|B| &< \int_B \left(e^{\alpha_2 u_n^2} - \lambda_n g_2(u_n) \right) dx \\ &< \int_B \left(e^{\alpha_2 u_n^2} - \lambda_n |B|^{-\frac{1}{2}} \left(\int_B g_2(u_n) dx \right)^{\frac{1}{2}} \int_B g_1(u_n) dx \right) dx \\ &\leq C \left(\lambda_n |B|^{-\frac{1}{2}} \left(\int_B g_2(u_n) dx \right)^{\frac{1}{2}}, g_1 \right). \end{aligned}$$

Thus from this it follows that

$$\lambda_n |B|^{-\frac{1}{2}} \left(\int_B g_2(u_n) dx \right)^{\frac{1}{2}} < \lambda_*(g_1).$$

Using Hölder's inequality again, we have

$$\lambda_n \int_B g_1(u_n) dx < \lambda_*(g_1)|B|.$$

Moreover, since $g_2'(s) \le 2g_1(s)$ for any $s \ge 0$ we have

$$\lambda_n \int_B g_2'(u_n) dx < 2\lambda_*(g_1)|B|.$$

Hence by this estimate, (17) and (25), w in (26) satisfies

$$\begin{cases} -\Delta w = \alpha_2 \delta_0 & \text{in } B, \\ w = 0 & \text{on } \partial B. \end{cases}$$

Consequently we complete the proof and we also obtain that

$$\lambda_n = O(c_n) \tag{29}$$

□

Since (25) by obtaining the following estimate we finish the proof of Proposition 4.2.

$$\int_B v_n g_i'(\alpha_2^{-1/2} v_n) dx = \frac{L_i + o(1)}{c_n^i} \quad \text{for some} \quad L_i.$$

Indeed, since $sg_1'(s) \le s$ and $sg_2'(s) \le 4s^2$ there exists a constant L such that

$$\int_B v_n g_i'(\alpha_2^{-1/2} v_n) dx \le L \int_B v_n^i dx = \frac{L}{c_n^i} \left(\int_B w^i dx + o(1) \right).$$

4.3. Estimate of the exponential term

In this subsection we prove the following proposition.

Proposition 4.6. *It follows that*

$$\frac{\alpha_2}{M_n} \int_B v_n^2 e^{v_n^2} dx \le 4\pi + \left(6\pi + \frac{8\pi}{e} \right) \frac{1}{c_n^4} + o(c_n^{-4}). \tag{30}$$

Proof of Proposition 4.6. By (28) and (29) we have

$$\lambda_n = \begin{cases} o(c_n) & (i=1), \\ O(c_n) & (i=2). \end{cases}$$

We set $\delta_n \in (0,1)$ as the minimum point δ such that

$$v_n(\delta)e^{v_n(\delta)^2} - \frac{1}{2\alpha_2^{1/2}}\lambda_n g_i'(\alpha_2^{-1/2}v_n(\delta)) = 0.$$

Since v_n is decreasing function with respect to r it follows that

$$v_n(r)e^{v_n(r)^2} - \frac{1}{2\alpha_2^{1/2}}\lambda_n g_i'(\alpha_2^{-1/2}v_n(\delta_n)) \geq 0 \quad \text{for } r \in [0,\delta_n].$$

We observe that

$$\frac{\alpha_2}{M_n}\int_B v_n^2 e^{v_n^2}dx = \frac{\alpha_2}{M_n}\int_{B_{\delta_n}} v_n^2 e^{v_n^2}dx + \frac{\alpha_2}{M_n}\int_{B_1\setminus B_{\delta_n}} v_n^2 e^{v_n^2}dx = J_1 + J_2. \quad (31)$$

First, we show that

$$J_2 \leq \frac{\frac{\alpha_2^2}{2e|B|} + o(1)}{c_n^4}. \quad (32)$$

By the rate of λ_n and g_i' for some $L>0$ and any $\theta > 1$

$$\frac{c_n}{\theta}e^{\left(\frac{c_n}{\theta}\right)^2} - \frac{1}{2\alpha_2^{1/2}}\lambda_n g_i'\left(\alpha_2^{-1/2}\left(\frac{c_n}{\theta}\right)\right) > \frac{c_n}{\theta}e^{\left(\frac{c_n}{\theta}\right)^2} - Lc_n^{-1} \to +\infty$$

as $n \to \infty$. Thus there exists $\{\theta_n\}$ such that

$$\theta_n \to +\infty, \quad \frac{c_n}{\theta_n}e^{\left(\frac{c_n}{\theta_n}\right)^2} - \frac{1}{2\alpha_2^{1/2}}\lambda_n g_i'\left(\alpha_2^{-1/2}\left(\frac{c_n}{\theta_n}\right)\right) \to +\infty.$$

For this θ_n, we see that $v_n(\delta_n) \leq c_n/\theta_n$. Thus we have $v_n(r) \leq c_n/\theta_n$ for $r \in (\delta_n, 1)$. We define v_{n,θ_n} in the same way as (23). Then by using (24) and

(26), we have

$$
\begin{aligned}
J_2 &\le \frac{\alpha_2}{M_n}\int_B v_n^2 e^{v_{n,\theta_n}^2}dx \\
&\le \frac{\alpha_2}{M_n}\left(\int_B v_n^{\frac{2\theta_n}{\theta_n-1}}\right)^{\frac{\theta_n-1}{\theta_n}}\left(\int_B e^{\theta_n v_{n,\theta_n}^2}dx\right)^{\frac{1}{\theta_n}} \\
&\le \frac{\alpha_2}{(e|B|+o(1))c_n^4}\left(\int_B w^2dx+o(1)\right)^{1-\frac{1}{\theta_n}}((1+e)|B|+1)^{\frac{1}{\theta_n}} \\
&= \frac{\frac{\alpha_2^2}{2e|B|}+o(1)}{c_n^4},
\end{aligned}
$$

where w is as in Lemma 4.5.

For J_1, we recall the estimate of λ_n and we can prove the following estimate by applying the strategies of blow up analysis in [13] (see also Subsection 3.2 in this paper, or Subsection 4.2 in [5]).

$$
J_1 \le 4\pi + \frac{6\pi}{c_n^4} + o(c_n^{-4}). \tag{33}
$$

Combining (31), (33), and (32) we complete the proof. □

4.4. Proof of Proposition 4.1 completed

We recall that $v_n = \alpha_2^{1/2}u_n$ and $\lambda_n \to +\infty$ as $n \to \infty$. Proposition 4.1 follows from Proposition 4.2 and Proposition 4.6.

□

5. Proof of Theorem (iii), (iv)

5.1. Proof of Theorem (iii)

For $f_1, f_2 \in X_{II}$ with $f_1 \le f_2$ we can check that $\lambda_*(f_1) \ge \lambda_*(f_2)$. Thus we have to check that $\lambda_*(g) \ge \alpha_2 + 2e|B|$ for $g \in C^1$ satisfying

$$
g(s) = s^2 \quad (s \le s_1), \quad g(s) = Ke^{\alpha e^{s^2}} \quad (s \le s_2), \quad \inf_{s\in(s_1,s_2)} g(s) > 0,
$$

or

$$
\lim_{s\to 0}\frac{g'(s)}{s} = 1, \quad g(s) = Ke^{\alpha e^{s^2}} \quad (s \le s_2), \quad \inf_{s\in(s_1,s_2)} g(s) > 0,
$$

for some positive constants s_1, s_2, K and $\alpha \in (0, \alpha_2)$. We can prove $\lambda_*(g) \ge \alpha_2 + 2e|B|$ in the same way as Section 3 by showing the following proposition instead of Proposition 3.1.

Proposition 5.1. *Fix $\lambda > 0$ and assume that $u_n \in H_0^1(B)$ is a maximizer of*

$$C_n(\lambda, g) := \sup_{\substack{u\in H_0^1(B)\\ \|\nabla u\|_2\le 1}} \int_B \left(e^{\alpha_n u^2} - \lambda g(|u|)\right) dx,$$

where α_n is a sequence of real numbers such that $\alpha_n \nearrow \alpha_2$ as $n \to \infty$. If $u_n \rightharpoonup 0$ weakly in $H_0^1(B)$ as $n \to \infty$, then we have

$$\|\nabla u_n\|_2^2 \ge \frac{\alpha_2}{\alpha_n}\left(1 + \frac{1 + (\alpha_2 - \lambda)\,(2e|B|)^{-1}}{\alpha_2^2}\frac{1}{\|u_n\|_\infty^4}\right) + o(\|u_n\|_\infty^{-4}),$$

Applying the strategies in Subsection 3.1-3.4 directly, we can prove this proposition.

5.2. Proof of Theorem (iv)

Assume that $f \in X_{II} \cap C^1$ satisfies the assumption in the part (iv). Set the sequence λ_n such that $\lambda_n \to \lambda_*$ as $n \to \infty$ and u_n is a maximizer of $C(\lambda_n, f)$. In order to prove the part (iv), we assume that $\sup_{x\in B} u_n(x) \to +\infty$ and derive a contradiction. The main proposition is as follows.

Proposition 5.2. *Assume that λ_n and u_n as above. For a positive constant C_1 and any large constant L it follows that*

$$\|\nabla u_n\|_2^2 \le 1 - \lambda_* \frac{L}{\|u_n\|_\infty^4} + \frac{C_1}{\|u_n\|_\infty^4} + o(\|u_n\|_\infty^{-4})$$

as $n \to \infty$.

By this proposition, we see that u_n is bounded in $L^\infty(B)$. Consequently, by the dominated convergence theorem we have

$$C(\lambda_*, f) = \lim_{n\to\infty} C(\lambda_n, f) = \lim_{n\to\infty} \int_B \left(e^{\alpha_2 u_n^2} - \lambda_n f(u_n)\right) dx = \int_B \left(e^{\alpha_2 u_0^2} - \lambda_* f(u_0)\right) dx,$$

where u_0 is the weak limit of u_n. Consequently, maximizer exists.

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