ON THE LIFTING OF HILBERT CUSP FORMS TO HILBERT-HERMITIAN CUSP FORMS

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Abstract. We construct a lifting that associates to a Hilbert cusp form a Hilbert-Hermitian cusp form. This is a generalization of the lifting of elliptic cusp forms constructed by Ikeda to arbitrary Hilbert cusp forms.

Contents

1. Introduction

The main theme of this paper is to attach a Hilbert-Hermitian cuspidal Hecke eigenform to an arbitrary Hilbert cuspidal Hecke eigenform by means of a Fourier expansion. Hecke has treated the case of holomorphic modular

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forms on the upper half-plane. Various kinds of generalization of this theory have been attempted. Holomorphic modular forms on the product of the upper half-planes over a totally real field are nowadays called Hilbert modular forms. Siegel pioneered the generalization of the theory of Hecke to modular forms on the upper half-space now named after him. Hilbert-Siegel and Hilbert-Hermitian modular forms are the natural generalizations of Hilbert modular forms to tube domains on which symplectic or unitary groups act. These modular forms are of fundamental importance in number theory and algebraic geometry, but unfortunately, their reputation does not match their importance. In contrast to the beauty of elliptic modular forms which is derived from the ubiquity of easily accessible examples, lack of attractive examples seems to be responsible for this unfortunate state.

It has been nearly 20 years since Tamotsu Ikeda has discovered a remarkable construction of Siegel and Hermitian cusp forms in [16, 17]. Analogous liftings were constructed for other tube domains in [34, 22], but there was little room for generalization in this construction. However, Ikeda subsequently invented a new approach from a representation theoretic standpoint. Starting with a Hilbert cusp form which does not have supercuspidal components, Ikeda and the author associate to it a family of Hilbert-Siegel cusp forms in [18]. In such a special case the liftings are described in terms of a concrete realization of degenerate Whittaker models called Jacquet integrals on degenerate principal series.

In the present paper we study the Hermitian case and construct liftings of arbitrary Hilbert cusp forms. To that end, we need generalizations of degenerate principal series and the Jacquet integrals. The unramified Jacquet integral is known as the Siegel series and plays a significant role in the local and global theories of quadratic forms and theta correspondence, and, ultimately, in a number of interesting problems in arithmetic (cf. [35, 12, 26]). Its generalization is of independent interest.

To be explicit, we here let E/F be a CM extension with Galois involution τ. We write $\mathbb{A} = \mathbb{A}_\infty \cdot \mathbb{A}_{\mathbf{f}}$ and $\mathbb{E} = \mathbb{E}_\infty \cdot \mathbb{E}_{\mathbf{f}}$ for their adèle rings, where $\mathbb{A}_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathbb{E}_\infty = E \otimes_{\mathbb{Q}} \mathbb{R}$ and where $\mathbb{A}_{\mathbf{f}}$ and $\mathbb{E}_{\mathbf{f}}$ are the finite parts of the adèle rings. We denote the set of real embeddings of F by $\mathfrak{S}_\infty$ and Weil's restriction of scalars from E to F by R^E_F. Let

$$\mathcal{G}_n = \left\{ g \in \mathrm{R}^E_F \mathrm{GL}_{2n} \;\middle|\; {}^t g^\tau \begin{bmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{bmatrix} g = \lambda_n(g) \begin{bmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{bmatrix}, \lambda_n(g) \in \mathrm{GL}_1 \right\}$$

be a quasisplit unitary similitude group in $2n$ variables. Let

$$\mathrm{Her}_n = \{ z \in \mathrm{R}^E_F \mathrm{M}_n \mid {}^t z^\tau = z \}$$

be the space of Hermitian forms with respect to E/F. Define three homomorphisms $\mathbf{d} : \mathrm{GL}_1 \to \mathcal{G}_n$, $\mathbf{m} : \mathrm{R}^E_F \mathrm{GL}_n \to \mathcal{G}_n$ and $\mathbf{n} : \mathrm{Her}_n \to \mathcal{G}_n$ by

$$\mathbf{d}(\xi) = \begin{bmatrix} \mathbf{1}_n & 0 \\ 0 & \xi \cdot \mathbf{1}_n \end{bmatrix}, \quad \mathbf{m}(A) = \begin{bmatrix} A & 0 \\ 0 & {}^t(A^{-1})^\tau \end{bmatrix}, \quad \mathbf{n}(z) = \begin{bmatrix} \mathbf{1}_n & z \\ 0 & \mathbf{1}_n \end{bmatrix}.$$

Let $\mathcal{P}_n = \mathbf{d}(\mathrm{GL}_1)\mathbf{m}(\mathrm{R}_F^E\mathrm{GL}_n)\mathbf{n}(\mathrm{Her}_n)$ be a parabolic subgroup of $\mathcal{G}_n$.

The identity component $\mathcal{G}_n(\mathbb{A}_\infty)^+$ of $\mathcal{G}_n(\mathbb{A}_\infty)$ acts componentwise on the Hermitian symmetric domain

$$\mathfrak{H}_n^d = \prod_{v\in\mathfrak{S}_\infty} \mathfrak{H}_v, \qquad \mathfrak{H}_v = \{Z \in \mathrm{M}_n(\mathbb{C}) \mid \sqrt{-1}({}^t\bar{Z} - Z) > 0\}.$$

We define the origin $\mathbf{i}$ of $\mathfrak{H}_n^d$ and the subgroup $\mathcal{K}_\infty^+$ of $\mathcal{G}_n(\mathbb{A}_\infty)^+$ by

$$\mathbf{i} = (\sqrt{-1}\mathbf{1}_n, \dots, \sqrt{-1}\mathbf{1}_n) \in \mathfrak{H}_n^d, \qquad \mathcal{K}_\infty^+ = \{g \in \mathcal{G}_n(\mathbb{A}_\infty)^+ \mid g(\mathbf{i}) = \mathbf{i}\}.$$

For $\xi \in \mathbb{A}_\infty^\times$ and $l \in \mathbb{R}^d$ we put $|\xi|^l = \prod_{v\in\mathfrak{S}_\infty} |\xi_v|_{F_v}^{l_v}$. For $a \in \mathbb{E}_\infty^\times$ and $\varkappa \in \mathbb{Z}^d$ we set $\varepsilon^\varkappa(a_\infty) = \prod_{v\in\mathfrak{S}_\infty}(a_v/a_v^\tau)^{\varkappa_v/2}$. When $\varkappa, \ell \in \mathbb{Z}^d$ and $\mathcal{F}$ is a function on $\mathfrak{H}_n^d$, we define a function $\mathcal{F}|_\ell^\varkappa g : \mathfrak{H}_n^d \to \mathbb{C}$ by

$$\mathcal{F}|_\ell^\varkappa g(Z) = \mathcal{F}(gZ)\varepsilon^\varkappa(\det g) j_\ell(g,Z)^{-1}, \quad j_\ell(g,Z) = \prod_{v\in\mathfrak{S}_\infty} \frac{\det(c_v Z_v + d_v)^{\ell_v}}{|\lambda_n(g_v)|^{n\ell_v/2}}$$

for $g = (g_v)_{v\in\mathfrak{S}_\infty} \in \mathcal{G}_n(\mathbb{A}_\infty)^+$, $g_v = \begin{bmatrix} * & * \\ c_v & d_v \end{bmatrix}$.

The subset Her_n^+ of $\mathrm{Her}_n(F)$ consists of totally positive definite Hermitian matrices over E. We define a holomorphic function $\mathbf{e}_\infty$ on $\mathbb{A}_\infty \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathbf{e}_\infty(z) = \prod_{v\in\mathfrak{S}_\infty} e^{2\pi\sqrt{-1}z_v}$. Let $\boldsymbol{\psi} = \prod_v \boldsymbol{\psi}_v$ be the additive character of $\mathbb{A}/F$ whose restriction to $\mathbb{A}_\infty$ coincides with $\mathbf{e}_\infty|_{\mathbb{A}_\infty}$.

A Hilbert-Hermitian cusp form $\mathcal{F}$ on $\mathcal{G}_n$ of weight ℓ with respect to the character $\varepsilon^\varkappa$ is a smooth function on $\mathcal{G}_n(F)\backslash\mathcal{G}_n(\mathbb{A})$ which transforms on the right by the character $k \mapsto \varepsilon^\varkappa(\det k) j_\ell(k,\mathbf{i})^{-1}$ of $\mathcal{K}_\infty^+$ and such that $\mathcal{F}_\Delta$ is a holomorphic function on $\mathfrak{H}_n^d$ having a Fourier expansion of the form

$$\mathcal{F}_\Delta(Z) = \sum_{B\in\mathrm{Her}_n^+} |\det B|^{\ell/2} \mathbf{w}_B(\Delta,\mathcal{F}) \mathbf{e}_\infty(\mathrm{tr}(BZ)) \tag{1.1}$$

for each $\Delta \in \mathcal{G}_n(\mathbb{A}_\mathbf{f})$, where $\mathbf{w}_B(\mathcal{F})$ is a function on $\mathcal{G}_n(\mathbb{A}_\mathbf{f})$ and the holomorphic function $\mathcal{F}_\Delta : \mathfrak{H}_n^d \to \mathbb{C}$ can be defined by

$$\mathcal{F}_\Delta|_\ell^\varkappa g_\infty(\mathbf{i}) = \mathcal{F}(g_\infty\Delta), \qquad g_\infty \in \mathcal{G}_n(\mathbb{A}_\infty)^+.$$

The Hilbert-Hermitian cusp form $\mathcal{F}$ is a cuspidal automorphic form on $\mathcal{G}_n(\mathbb{A})$ in the sense of Langlands (see Proposition A4.5 of [33]) with scalar K-type $k \mapsto \frac{\varepsilon^\varkappa(\det k)}{j_\ell(k,\mathbf{i})}$ and killed by certain differential operators (cf. Proposition 4.2 of [5]). If $\mathcal{F}$ is right invariant under an open compact subgroup $\mathcal{D}$ of $\mathcal{G}_n(\mathbb{A}_\mathbf{f})$, then $\mathcal{F}_\Delta$ is a traditional holomorphic cusp form with respect to the arithmetic subgroup $\mathcal{G}_n(F)^+ \cap \Delta\mathcal{D}\Delta^{-1}$, where $\mathcal{G}_n(F)^+ = \mathcal{G}_n(F) \cap \mathcal{G}_n(\mathbb{A}_\infty)^+$. It is important that the group $\mathcal{G}_n(\mathbb{A}_\mathbf{f})$ acts on the space of Hilbert-Hermitian cusp forms: for $\delta \in \mathcal{G}_n(\mathbb{A}_\mathbf{f})$ we define $\rho(\delta)\mathcal{F}$ by $(\rho(\delta)\mathcal{F})(g) = \mathcal{F}(g\delta)$.

Let $\pi \simeq \otimes_v' \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by a Hilbert cusp form of weight κ and central character $\hat{\omega}$. We write $\pi_\mathbf{f} = \otimes_\mathfrak{p}' \pi_\mathfrak{p}$ for its finite part. Fix an auxiliary Hecke character $\hat{\chi}$

of E whose restriction to $\mathbb{A}^\times$ coincides with $\hat{\omega}$. Denote its restriction to $\mathbb{E}_{\mathbf{f}}^\times$ by $\hat{\chi}_{\mathbf{f}}$. Take $\ell(\hat{\chi}) \in \mathbb{Z}^d$ so that the restriction of $\hat{\chi}$ to $\mathbb{E}_\infty^\times$ is $\varepsilon^{\ell(\hat{\chi})}$. Since $\mathcal{G}_1$ is the quotient of $\mathrm{R}_F^E\mathrm{GL}_1 \times \mathrm{GL}_2$ by GL_1 embedded diagonally, we can view $\hat{\chi}_{\mathbf{f}}^{-1} \boxtimes \pi_{\mathbf{f}}$ as a representation of $\mathcal{G}_1(\mathbb{A}_{\mathbf{f}})$. We take n to be odd throughout this paper as the key simplifying feature (3.2) can apply to the similitude group $\mathcal{G}_n$ for odd n. Let $\mathfrak{P}_{\mathbf{e}}$ be a parabolic subgroup of $\mathcal{G}_n$ with Levi subgroup $(\mathrm{R}_F^E\mathrm{GL}_2)^{(n-1)/2} \times \mathcal{G}_1$. Denote the modulus character of $\mathfrak{P}_{\mathbf{e}}(\mathbb{A}_{\mathbf{f}})$ by $\delta_{\mathfrak{P}_{\mathbf{e}}}$. We define the Galois twist ${}^\tau\hat{\chi}$ by composing $\hat{\chi}$ with the conjugation map. We write $\mathit{\Pi}_{\mathbf{f}}$ for the unique irreducible subrepresentation of

$$\mathrm{Ind}_{\mathfrak{P}_{\mathbf{e}}(\mathbb{A}_{\mathbf{f}})}^{\mathcal{G}_n(\mathbb{A}_{\mathbf{f}})} \delta_{\mathfrak{P}_{\mathbf{e}}}^{-1/4} \otimes \{({}^\tau\hat{\chi}_{\mathbf{f}}^{-1} \otimes \pi_{\mathbf{f}}^E)^{\boxtimes(n-1)/2} \boxtimes (\hat{\chi}_{\mathbf{f}}^{-1} \boxtimes \pi_{\mathbf{f}})\},$$

where $\pi_{\mathbf{f}}^E = \otimes_{\mathfrak{p}}' \pi_{\mathfrak{p}}^{E_{\mathfrak{p}}}$ denotes the base change of $\pi_{\mathbf{f}}$ to $\mathrm{GL}_2(\mathbb{E}_{\mathbf{f}})$.

In Sections 3 through 7 we will explicitly construct a family $\{\mathfrak{Z}_B^{\hat{\chi}_{\mathbf{f}}}\}_{B \in \mathrm{Her}_n^+}$ of nonzero linear functionals on $\mathit{\Pi}_{\mathbf{f}}$ which satisfy

$$\mathfrak{Z}_B^{\hat{\chi}_{\mathbf{f}}} \circ \mathit{\Pi}_{\mathbf{f}}(\mathbf{n}(z)\mathbf{d}(\xi)\mathbf{m}(A)) = \varepsilon^{-\ell(\hat{\chi})}(\det A)\boldsymbol{\psi}(\mathrm{tr}(Bz))\mathfrak{Z}_{\xi^{-1}{}^t A^\tau B A}^{\hat{\chi}_{\mathbf{f}}} \tag{1.2}$$

for all $z \in \mathrm{Her}_n(\mathbb{A}_{\mathbf{f}})$, $\xi \in F_+^\times$, $A \in \mathrm{GL}_n(E)$ and $B \in \mathrm{Her}_n^+$.

Theorem 1.1. *The Fourier series*

$$J_\kappa^\varkappa(f)_\Delta(Z) = \sum_{B \in \mathrm{Her}_n^+} |\det B|^{(\kappa+n-1)/2} \mathfrak{Z}_B^{\hat{\chi}_{\mathbf{f}}}(\mathit{\Pi}_{\mathbf{f}}(\Delta)f)\mathbf{e}_\infty(\mathrm{tr}(BZ))$$

defines a Hilbert-Hermitian cusp form on $\mathcal{G}_n$ of weight $\kappa+n-1$ with respect to $\varepsilon^\varkappa$, where $\varkappa = \frac{1}{2}(\kappa+n-1+\ell(\hat{\chi}))$. The map $f \mapsto J_\kappa^\varkappa(f)$ is a $\mathcal{G}_n(\mathbb{A}_{\mathbf{f}})$-intertwining embedding $\mathit{\Pi}_{\mathbf{f}}$ into the space of Hilbert-Hermitian cusp forms.

Here $\kappa + n - 1$ means the tuple $(\kappa_v + n - 1)_{v \in \mathfrak{S}_\infty} \in \mathbb{Z}^d$. Appendix A gives an explanation of how this theorem can be viewed in the framework of Arthur's classification. We can make Theorem 1.1 more precise, if none of $\pi_{\mathfrak{p}}$ is supercuspidal, i.e., there is a character $\mu_{\mathbf{f}} = \prod_{\mathfrak{p}} \mu_{\mathfrak{p}}$ of $\mathbb{A}_{\mathbf{f}}^\times$ such that $\pi_{\mathbf{f}}$ is equivalent to the unique irreducible subrepresentation $\otimes_{\mathfrak{p}}' A(\mu_{\mathfrak{p}}, \mu_{\mathfrak{p}}^{-1}\hat{\omega}_{\mathfrak{p}})$ of the principal series $\otimes_{\mathfrak{p}}' I(\mu_{\mathfrak{p}}, \mu_{\mathfrak{p}}^{-1}\hat{\omega}_{\mathfrak{p}})$ of $\mathrm{GL}_2(\mathbb{A}_{\mathbf{f}})$. To lighten notation, we put

$$\chi_{\mathfrak{p}} = {}^\tau\hat{\chi}_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}} \circ \mathrm{N}_F^E), \qquad \nu_{\mathfrak{p}} = \hat{\omega}_{\mathfrak{p}}^{(n+1)/2}\mu_{\mathfrak{p}}^{-n}.$$

Then the local component $\mathit{\Pi}_{\mathfrak{p}}$ of $\mathit{\Pi}_{\mathbf{f}}$ at $\mathfrak{p}$ turns out to be equivalent to the unique irreducible subrepresentation $A_n(\chi_{\mathfrak{p}}, \nu_{\mathfrak{p}})$ of the degenerate principal series $I_n(\chi_{\mathfrak{p}}, \nu_{\mathfrak{p}})$ of $\mathcal{G}_n(F_{\mathfrak{p}})$ that is induced from the character of $\mathcal{P}_n(F_{\mathfrak{p}})$

$$\mathbf{d}(\xi)\mathbf{m}(A)\mathbf{n}(z) \mapsto \hat{\omega}_{\mathfrak{p}}(\xi)^{(n+1)/2}\,{}^\tau\hat{\chi}_{\mathfrak{p}}(\det A)^{-1}\mu_{\mathfrak{p}}(\xi^{-n}\mathrm{N}_F^E(\det A))$$

and degenerate Whittaker functionals are given by the Jacquet integrals

$$w_B^{\chi_{\mathfrak{p}}}(h_{\mathfrak{p}}) = |\det B|_{F_{\mathfrak{p}}}^{n/2} \prod_{j=1}^n L(j, \mu_{\mathfrak{p}}^2\hat{\omega}_{\mathfrak{p}}^{-1}\epsilon_{E_{\mathfrak{p}}/F_{\mathfrak{p}}}^{n+j}) \times \int_{\mathrm{Her}_n(F_{\mathfrak{p}})} h_{\mathfrak{p}}\left(\begin{bmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{bmatrix} \mathbf{n}(z_{\mathfrak{p}})\right) \overline{\boldsymbol{\psi}_{\mathfrak{p}}(\mathrm{tr}(Bz_{\mathfrak{p}}))}\,\mathrm{d}z_{\mathfrak{p}}$$

for $B \in \mathrm{Her}_n(F_\mathfrak{p}) \cap \mathrm{GL}_n(E_\mathfrak{p})$, where $\epsilon_{E_\mathfrak{p}/F_\mathfrak{p}}$ is the character of $F_\mathfrak{p}^\times$ whose kernel is $\mathrm{N}_F^E(E_\mathfrak{p}^\times)$. See Corollary 6.4 for a precise relation between $\mathfrak{I}_B^{\hat{\chi}_\mathfrak{p}}$ and $w_B^{\chi_\mathfrak{p}}$. We form the restricted tensor product

$$A_n(\chi_\mathbf{f}, \nu_\mathbf{f}) = \otimes'_\mathfrak{p} A_n(\chi_\mathfrak{p}, \nu_\mathfrak{p}), \qquad\qquad w_B^{\chi_\mathbf{f}} = \otimes_\mathfrak{p} w_B^{\chi_\mathfrak{p}}.$$

Corollary 1.2. Notation being as above, the Fourier series

$$I_n^\kappa(h)_\Delta(Z) = \sum_{B \in \mathrm{Her}_n^+} \frac{|\det B|^{(\kappa+n-1)/2}}{\mu_\mathbf{f}(\det B)} w_B^{\chi_\mathbf{f}}(\Pi_\mathbf{f}(\Delta)h)\mathbf{e}_\infty(\mathrm{tr}(BZ))$$

is a Hilbert-Hermitian cusp form on $\mathcal{G}_n$ of weight $\kappa + n - 1$ with respect to $\varepsilon^\varkappa$ for every $h \in A_n(\chi_\mathbf{f}, \nu_\mathbf{f})$.

The series $J_\kappa^\varkappa(f)$ is left invariant under $\mathcal{P}_n(F)$ if and only if the family $\{\mathfrak{I}_B^{\hat{\chi}_\mathbf{f}}\}_{B \in \mathrm{Her}_n^+}$ is compatible in the sense of (1.2). Put $G_n = \ker \lambda_n \simeq \mathrm{U}(n,n)$. We view $\mathcal{G}_1$ as a subgroup of $\mathcal{G}_n$ via the embedding

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mapsto \left[\begin{array}{cc|cc} \mathbf{1}_{n-1} & & & \\ & \alpha & & \beta \\ \hline & & \lambda_1(g)\mathbf{1}_{n-1} & \\ & \gamma & & \delta \end{array}\right].$$

If $J_\kappa^\varkappa(f)$ is left invariant under $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathcal{G}_1(F) \subset \mathcal{G}_n(F)$, then since J_1 and $\mathcal{P}_n(F)$ generate $\mathcal{G}_n(F)$, the series $J_\kappa^\varkappa(f)$ is automorphic. One can prove this fact directly in the special case where $\pi_\mathbf{f}$ is an irreducible principal series. For the reader's convenience we give an outline. Let

$$N_{n-1}^n = \left\{ \left[\begin{array}{cc|cc} \mathbf{1}_{n-1} & x & z - y\,{}^t x^\tau & y \\ 0 & 1 & {}^t y^\tau & 0 \\ \hline & & \mathbf{1}_{n-1} & 0 \\ \mathbf{0}_n & & -\,{}^t x^\tau & 1 \end{array}\right] \middle| \begin{array}{l} x, y \in E^{n-1} \\ z \in \mathrm{Her}_{n-1} \end{array} \right\}.$$

The Schrödinger representation associated to $S \in \mathrm{Her}_{n-1}^+$ is extended to the Weil representation ω_S of $G_1(\mathbb{A}) \ltimes N_{n-1}^n(\mathbb{A})$. For $\phi \in \omega_{S,\mathbf{f}}$ we define the (S,ϕ)th Fourier-Jacobi coefficient of a function $\mathcal{F} : \mathcal{P}_n(F)\backslash\mathcal{G}_n(\mathbb{A}) \to \mathbb{C}$ by

$$\mathcal{F}_{\phi_S}^S(g') = \int_{N_{n-1}^n(F)\backslash N_{n-1}^n(\mathbb{A})} \mathcal{F}(vg')\overline{\Theta(\omega_S(vg')\phi_S)}\,\mathrm{d}v$$

for $g' \in G_1(\mathbb{A})$, where $\phi_S \in \omega_S$ is defined by taking the Gaussian at the archimedean components. When $\mathcal{F}_\Delta$ is holomorphic for every $\Delta \in \mathcal{G}_n(\mathbb{A}_\mathbf{f})$ and $\mathbf{w}_B(\Delta, \mathcal{F}) = 0$ for $B \notin \mathrm{Her}_n^+$, Lemma 7.7 of [18] tells us that $\mathcal{F}$ is left invariant under $\mathcal{G}_n(F)$ if and only if $(\rho(\delta)\mathcal{F})_{\phi_S}^S$ is left invariant under J_1 for all $S \in \mathrm{Her}_{n-1}^+$, $\phi \in \omega_{S,\mathbf{f}}$ and $\delta \in \mathcal{G}_n(\mathbb{A}_\mathbf{f})$. This notion of Fourier-Jacobi coefficients is an adèlic version of the classical one (cf. Theorems 5.1 and 6.1 of [7]). Thus it suffices to show that $I_n^\kappa(h)_{\phi_S}^S$ is left invariant under J_1.

Denote by $I_n(\chi_{\mathbf{f}})$ the restriction of $I_n(\chi_{\mathbf{f}}, \nu_{\mathbf{f}})$ to $G_n(\mathbb{A}_{\mathbf{f}})$. Lemma 10.1 defines a $G_1(\mathbb{A}_{\mathbf{f}})$-intertwining and $N_{n-1}^n(\mathbb{A}_{\mathbf{f}})$-invariant map

$$\beta_S^1 : I_n(\chi_{\mathbf{f}}) \otimes \overline{\omega_S} \to I_1(\chi_{\mathbf{f}}). \tag{1.3}$$

The proof is now accounted for by the following relation:

$$I_n^\kappa(h)_{\phi_S}^S = I_1^\kappa(\beta_S^1(h \otimes \overline{\phi_S})).$$

Since $I_1(\chi_{\mathbf{f}})$ is the restriction of the automorphic representation $\hat{\chi}_{\mathbf{f}}^{-1} \boxtimes \pi_{\mathbf{f}}$, the right hand side is the Fourier series of a modular form.

One can construct analogous Hilbert-Hermitian cusp forms on $\mathrm{U}(m,m)_{/F}$ for even m by taking the first Fourier-Jacobi coefficient of $I_{m+1}^\kappa(h)$. Hermitian cusp forms on $\mathrm{U}(\ell,\ell)_{/\mathbb{Q}}$ constructed by Ikeda in [17] are a particular case. We do not touch on this aspect as the structure of the A-packet for $\mathrm{U}(m,m)$ is not as simple as that for $\mathcal{G}_n$ (see Section 18 of [17]). A lifting analogous to Corollary 1.2 is constructed in [18] for metaplectic groups. Since the restriction of $\Pi_{\mathbf{f}}$ to $G_n(\mathbb{A}_{\mathbf{f}})$ can be reducible, the Hermitian case is more complicated. It makes our exposition simpler to deal with the group $\mathcal{G}_n$ rather than G_n.

The proof of Theorem 1.1 consists of two steps. The first step is to construct the invariant functional $\mathfrak{I}_B^{\hat{\chi}_{\mathbf{f}}}$. It is essentially local in nature, i.e., it is built out of local functionals $\mathfrak{I}_B^{\hat{\chi}_{\mathfrak{p}}}$ on the local components $\Pi_{\mathfrak{p}}$ of $\Pi_{\mathbf{f}}$. Since we restrict ourselves to odd n, a Levi subgroup of $\mathcal{P}_n(F)$ acts on Her_n^+ transitively, so that a compatible family $\{\mathfrak{I}_B^{\hat{\chi}_{\mathbf{f}}}\}_{B \in \mathrm{Her}_n^+}$ is clear from (1.2). The main difficulty in this paper is to show that $\mathfrak{I}_B^{\hat{\chi}_{\mathfrak{p}}}$ enjoys properties similar to those of $w_B^{\chi_{\mathfrak{p}}}$ when $\pi_{\mathfrak{p}}$ is supercuspidal and $E_{\mathfrak{p}} \not\simeq F_{\mathfrak{p}} \oplus F_{\mathfrak{p}}$. Proposition 7.2 proves some invariance of $\mathfrak{I}_B^{\hat{\chi}_{\mathfrak{p}}}$, which (1.2) implies. Though we give a uniform exposition, since it is cumbersome to prove the split and non-split cases at one time, the split case is also dealt with in Appendix B. We can extend the Fourier-Jacobi coefficients $J_\kappa^{\varkappa}(f)_{\phi_S}^S$ to functions on $B_2(F)\backslash \mathrm{GL}_2(\mathbb{A})$ due to the invariance (cf. Proposition 3.3, Remark 8.2(1)). Here the assumption on the parity of n is used to extend $\omega_{S,\mathbf{f}}$ to the similitude group.

The second step is to prove an analogous inductivity stated in Lemma 10.5, which implies that $J_\kappa^{\varkappa}(f)_{\phi_S}^S \in \pi$. When $\pi_{\mathfrak{p}}$ is not supercuspidal, the invariance is proved in Lemma 5.3(3) and the inductive structure is (1.3) (see Lemma 10.1). In the nonsplit supercuspidal case both properties are proved indirectly by global methods: one can prove that the unique irreducible subrepresentation of

$$\mathrm{Ind}_{\mathfrak{P}_{\mathbf{e}}(\mathbb{A})}^{\mathcal{G}_n(\mathbb{A})} \delta_{\mathfrak{P}_{\mathbf{e}}}^{-1/4} \otimes \{({}^\tau\hat{\chi}^{-1} \otimes \pi^E)^{\boxtimes (n-1)/2} \boxtimes (\hat{\chi}^{-1} \boxtimes \pi)\}$$

is residual and directly check that $\mathfrak{I}_B^{\hat{\chi}_{\mathfrak{p}}}$ occurs in the explicit factorization of Bth Fourier coefficients of those residual automorphic forms. We remark that all the results and the proofs in this paper carry over to holomorphic cusp forms on quaternion upper half-spaces with minor changes (cf. [34, 23]).

Finally, we construct cuspidal Hecke eigenforms by making Theorem 1.1 explicit. To state the formula in a style closer to the traditional one, we let $F = \mathbb{Q}$ and require π to be generated by a primitive form $f = \sum_{m=1}^{\infty} \mathbf{a}_m(f)\mathbf{q}^m \in S_\kappa(\Gamma_0(N))$ of square-free level N in this introductory section. Denote the integer ring of the imaginary quadratic field E by $\mathfrak{r}$. Put

$$\Gamma_0^{(n)}[N] = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{G}_n(\mathbb{Q})^+ \cap \mathrm{GL}_{2n}(\mathfrak{r}) \;\middle|\; C \in N\mathrm{M}_n(\mathfrak{r}) \right\}.$$

We write

$$L(s,\pi) = \sum_{m=1}^{\infty} \frac{\mathbf{a}_m(f)}{m^{s+(\kappa-1)/2}} = \prod_{p|N} \frac{1}{1-p^{-s_p-s}} \prod_{p\nmid N} \frac{1}{(1-\alpha_p p^{-s})(1-\alpha_p^{-1}p^{-s})}.$$

Denote the set of positive definite Hermitian semi-integral matrices of size n by $\mathscr{R}_n^+$. Let $\widetilde{F}_p(B,X)$ be a certain Laurent polynomial arising from the unramified Jacquet integral with respect to B defined in (9.1) and (11.1). Define a holomorphic function on $\mathfrak{H}_n$ by the Fourier series

$$H(Z) = \sum_{B\in\mathscr{R}_n^+} \mathbf{e}_\infty(\mathrm{tr}(BZ))|\det B|^{(\kappa-1)/2} \prod_{p|N} |\det B|_p^{s_p} \prod_{p\nmid N} \tilde{F}_p(B,\alpha_p).$$

The following result is included in Corollary 11.3.

Corollary 1.3. If n is odd, then $H|_{\kappa+n-1}^{(\kappa+n-1)/2}\gamma = H$ for $\gamma \in \Gamma_0^{(n)}[N]$.

When $n = 1$, the function H reduces to the well-known new vector of π. The subgroup $\mathcal{D} = \prod_p \Gamma_n[\mathfrak{r}_p, N\mathfrak{r}_p]$ of $\mathcal{G}_n(\mathbb{A}_\mathbf{f})$ is defined in (5.2) so that $\Gamma_0^{(n)}(N) = \mathcal{D} \cap \mathcal{G}_n(\mathbb{Q})^+$. Since $\mathcal{G}_n(\mathbb{A}) \neq \mathcal{G}_n(F)\mathcal{G}_n(\mathbb{R})\mathcal{D}$ in general, one needs a tuple of holomorphic modular forms on $\mathfrak{H}_n^d$ to obtain a Hermitian modular form on $\mathcal{G}_n$ (see Section 11, [17, Section 13]).

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Notation

We add the table of notations at the end of this paper for the reader's convenience. Here we list some of general notations to be used throughout this paper. For an associative ring $\mathcal{O}$ with identity element we denote by $\mathcal{O}^\times$ the group of all its invertible elements and by $\mathrm{M}_n^m(\mathcal{O})$ the $\mathcal{O}$-module of all $m \times n$ matrices with entries in $\mathcal{O}$. Put $\mathcal{O}^m = \mathrm{M}_1^m(\mathcal{O})$, $\mathrm{M}_n(\mathcal{O}) = \mathrm{M}_n^n(\mathcal{O})$ and $\mathrm{GL}_n(\mathcal{O}) = \mathrm{M}_n(\mathcal{O})^\times$. The zero element of $\mathrm{M}_n^m(\mathcal{O})$ is denoted by 0 and the identity element of the ring $\mathrm{M}_n(\mathcal{O})$ is denoted by $\mathbf{1}_n$. If $x_1, \ldots, x_k$ are

square matrices, then $\mathrm{diag}[x_1, \dots, x_k]$ denotes the matrix with $x_1, \dots, x_k$ in the diagonal blocks and 0 in all other blocks. If $\mathcal{O}$ has an involution $a \mapsto a^\tau$, then for a matrix x over $\mathcal{O}$, let ${}^t x$ be the transpose of x and ${}^t x^\tau$ the conjugate transpose of x.

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ stand for the rings of rational integers, rationals, reals, complex numbers, respectively, and $\mathbb{N}$ denotes the set of strictly positive rational integers. We write $\mathbb{R}_+^\times$ and $\mathbb{S}$ for the subgroups of $\mathbb{C}^\times$ which consist of strictly positive real numbers and complex numbers of absolute value 1, respectively. Define characters $\mathbf{e} : \mathbb{C} \to \mathbb{C}^\times$ and $\varepsilon : \mathbb{C}^\times \to \mathbb{S}$ by

$$\mathbf{e}(z) = e^{2\pi\sqrt{-1}z}, \qquad \varepsilon(u) = u/\|u\| \qquad (z \in \mathbb{C}\ u \in \mathbb{C}^\times)$$

where $\|\ \|$ is the standard absolute value in $\mathbb{C}$, not its square. If $x \in \mathbb{R}$, then $[x]$ will denote the biggest integer inferior or equal to x.

When X is a smooth affine variety over a field F and S is an F-algebra, we use $X(S)$ to denote the S-points of X, or simply X to denote its F-points. If F is a local field, then we write $\mathcal{S}(X)$ for the space of Schwartz-Bruhat functions on X. When X is a real Lie group, we denote its connected component of the identity by X^+.

2. Groups, parabolic subgroups and Weil representations

Let F for the moment be an arbitrary field and E a quadratic étale algebra over F, i.e., E is either a separable quadratic field extension of F (the inert case) or $E = F \oplus F$ (the split case). Let $x \mapsto x^\tau$ denote the nontrivial F-automorphism of E. Thus $(a, b)^\tau = (b, a)$ for $a, b \in F$ in the split case. Define the norm map $\mathrm{N}_F^E : E^\times \to F^\times$ by $\mathrm{N}_F^E(x) = xx^\tau$ and the trace map $\mathrm{T}_F^E : E \to F$ by $\mathrm{T}_F^E(x) = x + x^\tau$. Let

$$\mathrm{Her}_n = \{B \in \mathrm{R}_F^E \mathrm{M}_n \mid {}^t B^\tau = B\}, \qquad \mathrm{Her}_n^{\mathrm{nd}} = \mathrm{Her}_n \cap \mathrm{R}_F^E \mathrm{GL}_n$$

be the spaces of Hermitian matrices in $\mathrm{M}_n(E)$ or $\mathrm{GL}_n(E)$ with the right $\mathrm{R}_F^E \mathrm{GL}_n$-action given by

$$B\langle A \rangle = {}^t A^\tau B A \qquad (B \in \mathrm{Her}_n,\ A \in \mathrm{R}_F^E \mathrm{GL}_n).$$

Given $B \in \mathrm{Her}_n$ and $\Xi \in \mathrm{Her}_m$, we sometimes write $B \oplus \Xi$ instead of $\mathrm{diag}[B, \Xi] \in \mathrm{Her}_{m+n}$. The associated similitude unitary group GU_B consists of all matrices $A \in \mathrm{R}_F^E \mathrm{GL}_n$ that satisfy $B\langle A \rangle = \lambda_B(A) B$ with $\lambda_B(A) \in F^\times$. This group admits a homomorphism $\lambda_B : \mathrm{GU}_B \to F^\times$ whose kernel is the stabilizer of B in $\mathrm{R}_F^E \mathrm{GL}_n$ and denoted by U_B. We define another homomorphism $\Lambda_B : \mathrm{GU}_B \to E^\times$ by

$$\Lambda_B(A) = \lambda_B(A)^{-[n/2]} \det A.$$

Let $\mathcal{G}_n = \mathrm{GU}_{J_n} = \mathrm{GU}(n, n)$, where

$$\mathrm{GU}(n, n) = \{g \in \mathrm{R}_F^E \mathrm{GL}_{2n} \mid {}^t g^\tau J_n g = \lambda_n(g) J_n \text{ with } \lambda_n(g) \in F^\times\} \tag{2.1}$$

is a unitary similitude group considered as an F-algebraic group, where

$$J_n = \begin{bmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{bmatrix} \in \mathrm{GL}_{2n}(F).$$

The kernel of the scale map $\lambda_n = \lambda_{J_n} : \mathcal{G}_n \to F^\times$ is denoted by $G_n = \mathrm{U}_{J_n} = \mathrm{U}(n,n)$. We formally set $\mathcal{G}_0 = F^\times$. The center $\mathcal{Z}_n$ of $\mathcal{G}_n$ consists of scalar matrices over E and is naturally identified with $\mathrm{R}_F^E\mathrm{GL}_1$.

Remark 2.1. When $n = 1$, there is an accidental isomorphism

$$\mathcal{G}_1(F) \simeq E^\times \times \mathrm{GL}_2(F)/\Delta, \qquad \Delta = \{(\xi, \xi) \mid \xi \in F^\times\}.$$

The isomorphism is given by $(a, g) \mapsto a^{-1}g$. Note that

$$G_1(F) \simeq \{(a, g) \in E^\times \times \mathrm{GL}_2(F) \mid \mathrm{N}_F^E(a) = \det g\}/\Delta.$$

Given $A \in \mathrm{R}_F^E\mathrm{GL}_n$, $z \in \mathrm{Her}_n$ and $\xi \in F^\times$, we put

$$\mathbf{m}_n(A) = \begin{bmatrix} A & 0 \\ 0 & {}^t(A^{-1})^\tau \end{bmatrix}, \quad \mathbf{n}_n(z) = \begin{bmatrix} \mathbf{1}_n & z \\ 0 & \mathbf{1}_n \end{bmatrix}, \quad \mathbf{d}_n(\xi) = \begin{bmatrix} \mathbf{1}_n & 0 \\ 0 & \xi \cdot \mathbf{1}_n \end{bmatrix}.$$

We will frequently suppress the subscript $_n$. Define the maximal parabolic subgroup $\mathcal{P}_n$ with Levi subgroup $\mathcal{M}_n$ and abelian unipotent radical N_n by

$$\mathcal{M}_n = \{\mathbf{d}(\xi)\mathbf{m}(A) \mid \xi \in F^\times,\ A \in \mathrm{R}_F^E\mathrm{GL}_n\}, \quad N_n = \{\mathbf{n}(z) \mid z \in \mathrm{Her}_n\}.$$

More generally, we use the notation

$$N_i^k = \left\{ \mathbf{v}_i^k(x;y;z) = \left[\begin{array}{cc|cc} \mathbf{1}_i & x & z - y\,{}^t x^\tau & y \\ 0 & \mathbf{1}_{k-i} & {}^t y^\tau & \mathbf{0}_{k-i} \\ \hline & \mathbf{0}_k & \mathbf{1}_i & 0 \\ & & -{}^t x^\tau & \mathbf{1}_{k-i} \end{array}\right] \middle|\ \begin{array}{c} x, y \in \mathrm{R}_F^E\mathrm{M}_{k-i}^i \\ z \in \mathrm{Her}_i \end{array} \right\}.$$

We define a homomorphism

$$\iota_i^k : \mathrm{R}_F^E\mathrm{GL}_i \times \mathcal{G}_{k-i} \to \mathcal{G}_k, \quad (A, g) \mapsto \left[\begin{array}{ccc|ccc} A & & & & & \\ & \alpha & & & & \beta \\ & & & & & \\ \hline & & & & & \\ & \lambda_{k-i}(g)\,{}^t(A^{-1})^\tau & & & & \\ & & & & & \\ \hline & \gamma & & & & \delta \end{array}\right],$$

where we write an element $g \in \mathcal{G}_{k-i}$ in the form $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ with matrices α, β, γ, δ of size $k - i$ over E. The group $\mathcal{M}_i^k = \iota_i^k(\mathrm{R}_F^E\mathrm{GL}_i \times \mathcal{G}_{k-i})$ is a Levi subgroup of the maximal parabolic subgroup $\mathcal{P}_i^k$ of $\mathcal{G}_k$ whose unipotent radical is N_i^k. These parabolic subgroups form a set of representatives of the set of all $\mathcal{G}_k$-conjugacy classes of maximal parabolic subgroups.

Take natural numbers i, j, k such that $i + j = k$. The center of N_i^k is

$$Z_i^k = \{\mathbf{v}_i^k(0;0;z) \mid z \in \mathrm{Her}_i\}.$$

We define subgroups X_i^k and Y_i^k of G_k by

$$X_i^k = \{\mathbf{v}_i^k(x;0;0) \mid x \in \mathrm{R}_F^E\mathrm{M}_j^i\}, \qquad Y_i^k = \{\mathbf{v}_i^k(0;y;0) \mid y \in \mathrm{R}_F^E\mathrm{M}_j^i\}.$$

For $S \in \mathrm{Her}_i$ we define a homomorphism $\ell^S : Z_i^k \to F$ by

$$\ell^S(\mathbf{v}_i^k(0;0;z)) = \mathrm{tr}(Sz).$$

Fix $S \in \mathrm{Her}_i^{\mathrm{nd}}$. Put

$$R_S^k = \{(A,g) \in \mathrm{GU}_S \times \mathcal{G}_j \mid \lambda_S(A) = \lambda_j(g)\}.$$

We identify R_S^k with the subgroup of $\mathcal{M}_S^k = \iota_i^k(\mathrm{GU}_S \times \mathcal{G}_j) \subset \mathcal{M}_i^k$. We regard GU_S as a subgroup of $\mathcal{M}_i^k$ via the embedding $\mathbf{m}_S^k : \mathrm{GU}_S \to \mathcal{G}_k$ defined by

$$\mathbf{m}_S^k(A) = \iota_i^k(A, \mathbf{d}_j(\lambda_S(A))) = \mathbf{m}_k(\mathrm{diag}[A, \mathbf{1}_j])\mathbf{d}_k(\lambda_S(A)).$$

Then

$$\mathcal{M}_S^k = \mathrm{GU}_S \ltimes \iota_i^k(\mathbf{1}_i, \mathcal{G}_j), \qquad R_S^k = \mathrm{GU}_S \ltimes \iota_i^k(\mathbf{1}_i, G_j).$$

Note that if $A \in \mathrm{GU}_S$ and $\xi = \lambda_S(A)$, then for $A \in \mathrm{GU}_S$ and $g \in G_j$

$$\mathbf{m}_S^k(A)\iota_i^k(\mathbf{1}_i, g)\mathbf{m}_S^k(A)^{-1} = \iota_i^k(\mathbf{1}_i, \mathbf{d}_j(\xi)g\mathbf{d}_j(\xi)^{-1}), \tag{2.2}$$

$$\mathbf{m}_S^k(A)\mathbf{v}_i^k(x;y;z)\mathbf{m}_S^k(A)^{-1} = \mathbf{v}_i^k(Ax; \xi^{-1}Ay; \xi^{-1}Az\,{}^tA^\tau), \tag{2.3}$$

$$\iota_i^k(\mathbf{1}_i, g)\mathbf{v}_i^k(x;y;z)\iota_i^k(\mathbf{1}_i, g)^{-1} = \mathbf{v}_i^k((x;y)g^{-1};z). \tag{2.4}$$

In particular, R_S^k is the stabilizer of ℓ^S in $\mathcal{M}_i^k$ under the conjugation action. The quotient group $N_i^k/\mathrm{Ker}\,\ell^S$ is a Heisenberg group with center $Z_i^k/\mathrm{Ker}\,\ell^S$ and a natural symplectic structure on N_i^k/Z_i^k. We will frequently let $k = n$ and suppress the dependence on k from the notation. We sometimes write

$$\mathbf{m}_i'(A') = \mathbf{m}\left(\begin{bmatrix} \mathbf{1}_i & 0 \\ 0 & A' \end{bmatrix}\right), \qquad \mathbf{n}_i'(z') = \mathbf{n}\left(\begin{bmatrix} \mathbf{0}_i & 0 \\ 0 & z' \end{bmatrix}\right)$$

for $A' \in \mathrm{GL}_{n-i}(E)$ and $z' \in \mathrm{Her}_{n-i}$.

The ground field F is a local field of characteristic zero with normalized absolute value $\boldsymbol{\alpha}_F = |\cdot|_F$ unless the end of Section 7. Let E be a quadratic étale algebra over F and $\epsilon_{E/F}$ the character of $C_F = F^\times$ attached to E/F by class field theory. Given a character χ of $C_E = E^\times$, we denote its restriction to C_F by $\chi^\dagger$. We set $\boldsymbol{\alpha}_E(a,b) = \boldsymbol{\alpha}_F(ab)$ for $(a,b) \in C_E$ in the split case. When $\mathcal{A}$ is a locally compact topological abelian group, we write $\Omega(\mathcal{A})$ for the topological group of all continuous homomorphisms from $\mathcal{A}$ to $\mathbb{C}^\times$. Given $\mu \in \Omega(F^\times)$, we define $\Re\mu$ as the unique real number such that $\mu\boldsymbol{\alpha}_F^{-\Re\mu}$ is unitary.

We define the basic character $\boldsymbol{\psi} : F \to \mathbb{S}$ in the following way: When F is archimedean, we let $\boldsymbol{\psi} = \mathbf{e} \circ \mathrm{T}_{\mathbb{R}}^F$. When F is an extension of $\mathbb{Q}_p$, we define $\boldsymbol{\psi}(x) = \mathbf{e}(-y)$ with $y \in \mathbb{Z}[p^{-1}]$ such that $\mathrm{T}_{\mathbb{Q}_p}^F(x) - y \in \mathbb{Z}_p$. We now associate a character $\boldsymbol{\psi}^B : \mathrm{Her}_n \to \mathbb{S}$ to $B \in \mathrm{Her}_n$ by $\boldsymbol{\psi}^B(z) = \boldsymbol{\psi}(\mathrm{tr}(Bz))$.

Following [15] and [13], we aim to review a construction of Weil representations of $R_S^k \ltimes N_i^k$. The Schrödinger representation ω_S of N_i^k with central character $\boldsymbol{\psi} \circ \ell^S$ is realized on the Schwartz space $\mathcal{S}(X_i^k)$ by

$$[\omega_S(\mathbf{v}_i^k(x;y;z))\phi](u) = \phi(u+x)\boldsymbol{\psi}^S(z)\boldsymbol{\psi}(\mathrm{T}_F^E({}^tu^\tau Sy)) \tag{2.5}$$

for $\phi \in \mathcal{S}(X_i^k)$. The representation ω_S is a unique irreducible representation of N_i^k on which Z_i^k acts by $\boldsymbol{\psi} \circ \ell^S$ by the Stone-von Neumann theorem, and ω_S extends to the Weil representation $\omega_S^{\hat{\epsilon}}$ of $G_j \ltimes N_i^k$. This extension depends on the choice of a character $\hat{\epsilon}$ of C_E whose restriction to C_F is $\epsilon_{E/F}^i$. Recall the well-known formulas:

$$\begin{aligned}
[\omega_S^{\hat{\epsilon}}(\mathbf{m}_j(A))\phi](u) &= \hat{\epsilon}(\det A)|\det A|_E^{i/2}\phi(uA),\\
[\omega_S^{\hat{\epsilon}}(\mathbf{n}_j(z))\phi](u) &= \boldsymbol{\psi}^S(uz\,{}^t u^\tau)\phi(u),\\
[\omega_S^{\hat{\epsilon}}(J_j)\phi](u) &= \gamma_S[\mathcal{F}_S\phi](u)
\end{aligned} \tag{2.6}$$

for $\phi \in \mathcal{S}(X_i^k)$, $u \in X_i^k$, $A \in \mathrm{GL}_j(E)$ and $z \in \mathrm{Her}_j$, where γ_S is a certain 8th root of unity and $\mathcal{F}_S\phi$ is the Fourier transform

$$[\mathcal{F}_S\phi](u) = \int_{X_i^k} \phi(x)\boldsymbol{\psi}(\mathrm{T}_F^E({}^t u^\tau Sx))\,\mathrm{d}x.$$

The measure $\mathrm{d}x$ on X_i^k is self-dual with respect to this Fourier transform.

We further extend the Weil representation $\omega_S^{\hat{\epsilon}}$ to an action of the semi-direct product $R_S^k \ltimes N_i^k$. This material is a slight variation of that of Section 3 of [13]. Fix $\theta \in \Omega(C_E)$ and define an action of GU_S on $\mathcal{S}(X_i^k)$ by

$$[{}^\theta L(A)\phi](u) = \theta(\Lambda_S(A))|\lambda_S(A)|_F^{-ij/2}\phi(A^{-1}u).$$

We obtain a representation ${}^\theta\omega_S^{\hat{\epsilon}}$ of the group $\mathrm{GU}_S \ltimes (G_j \ltimes N_i^k)$ on $\mathcal{S}(X_i^k)$ as

$${}^\theta L(A)\omega_S^{\hat{\epsilon}}(x){}^\theta L(A)^{-1} = \omega_S^{\hat{\epsilon}}(\mathbf{m}_S^k(A)x\mathbf{m}_S^k(A)^{-1}), \qquad x \in G_j \ltimes N_i^k$$

(cf. (2.2) and [13, (3.2)]). Since $\iota_i^k(A,g) = \mathbf{m}_S^k(A)\iota_i^k(\mathbf{1}_i, \mathbf{d}_j(\xi)^{-1}g)$, where $(A,g) \in R_S^k$ and $\xi = \lambda_S(A)$, we have

$$[{}^\theta\omega_S^{\hat{\epsilon}}(\iota_i^k(A,g)v)\phi](u) = \theta(\Lambda_S(A))|\xi|_F^{-ij/2}[\omega_S^{\hat{\epsilon}}(\mathbf{d}_j(\xi)^{-1}gv)\phi](A^{-1}u)$$

for $v \in N_i^k$, $u \in X_i^k$ and $\phi \in \mathcal{S}(X_i^k)$.

3. Degenerate Whittaker functionals and Shalika functionals

Let G be a reductive group over a local field. The space of an induced representation $\mathrm{Ind}_P^G \sigma$ of an admissible representation (σ, V) of a Levi subgroup M (or its pullback to a parabolic subgroup $P = MU$) of G consists of smooth functions f on G with values in V such that

$$f(umg) = \delta_P(m)^{1/2}\sigma(m)f(g) \qquad (u \in U,\ m \in M,\ g \in G)$$

on which G acts by right translation. The modulus character δ_P of P is built into the definition in order for the representation $\mathrm{Ind}_P^G \sigma$ to be unitary whenever σ is unitary. An irreducible representation π of G is called supercuspidal if it is not a composition factor of any representation of the form $\mathrm{Ind}_P^G \sigma$ with P a proper parabolic subgroup of G. If π is a smooth representation of finite length, we write $\pi^\vee$ for the contragredient representation, and for a character U of a unipotent subgroup U of G we write $J_U^\Psi(\pi)$ for the twisted Jacquet module of π, namely, the quotient of π by the closure of

the span of $\pi(u)v - \Psi(u)v$ $(u \in U,\ v \in \pi)$. When Ψ is trivial, we frequently suppress the superscript $^{\Psi}$.

A Ψ-Whittaker functional on π is a complex linear functional λ on π which satisfies $\lambda(\pi(u)v) = \Psi(u)\lambda(v)$ for all $v \in \pi$ and $u \in U$. The space of Ψ-Whittaker functionals on π can be identified with the space of complex linear functionals on $J_U^{\Psi}(\pi)$. The group G acts on the space

$$\mathrm{Ind}_U^G \Psi = \{W : G \to \mathbb{C} \text{ smooth } \mid W(ug) = \Psi(u)W(g) \text{ for all } u \in U,\ g \in G\}$$

by right translation. The image of a nontrivial intertwining map $\pi \to \mathrm{Ind}_U^G \Psi$ is called a Ψ-Whittaker model of π. Note that π has a nonzero Ψ-Whittaker functional λ if and only if π has a Ψ-Whittaker model $\mathscr{W}(\pi)$. To obtain a model from a functional, set $\mathscr{W}(g, v) = \lambda(\pi(g)v)$, and to obtain a functional from a model, set $\lambda(v) = \mathscr{W}(e, v)$, where e denotes the neutral element of G. When $G = G_n$ and $U = N_n$, we call $\boldsymbol{\psi} \circ \ell^B$-Whittaker functionals Bth degenerate Whittaker functionals, write $\mathrm{Wh}_B(\pi)$ for the space of Bth Whittaker functionals on π and denote by $\mathrm{Her}_n^{\mathrm{nd}}(\pi)$ the subset of $\mathrm{Her}_n^{\mathrm{nd}}$ which consists of Hermitian matrices B such that $\mathrm{Wh}_B(\pi)$ is nonzero.

We define a $\mathrm{GL}_n(E)$-invariant map $\epsilon : \mathrm{Her}_n^{\mathrm{nd}} \to \{\pm 1\}$ by

$$\epsilon(B) = \epsilon_{E/F}((-1)^{n(n-1)/2} \det B).$$

The set of $\mathrm{GL}_n(E)$-orbits in $\mathrm{Her}_n^{\mathrm{nd}}$ is indexed by this map in the $\mathfrak{p}$-adic case and by the possible signatures in the archimedean case. Given $B \in \mathrm{Her}_n^{\mathrm{nd}}$, we write $\mathcal{O}_F(B)$ for the set of Hermitian matrices of the form $\xi^{-1}B\langle A\rangle$ for some $\xi \in F^\times$ and $A \in \mathrm{GL}_n(E)$.

Definition 3.1. Let Π be an admissible representation of $\mathcal{G}_n$ and χ a unitary character of C_E. We call $w_B \in \mathrm{Wh}_B(\Pi)$ a Bth Shalika functional with respect to χ if

$$w_B \circ \Pi(\mathbf{m}_B(A)) = \chi(\Lambda_B(A))w_B$$

for all $A \in \mathrm{GU}_B$. Let $\mathrm{Sh}_B^{\chi}(\Pi)$ denote the space of Bth Shalika functionals on Π with respect to χ.

Let us make some general observations on Shalika functionals.

Definition 3.2. Fix $B_0 \in \mathrm{Her}_n^{\mathrm{nd}}$. Granted a single Shalika functional $\beth_{B_0} \in \mathrm{Sh}_{B_0}^{\chi}(\Pi)$, we obtain a family of Shalika functionals $\beth_B \in \mathrm{Sh}_B^{\chi}(\Pi)$ indexed by $B \in \mathcal{O}_F(B_0)$ by setting

$$\beth_B = \chi(\xi^{-[n/2]} \det A)^{-1} \beth_{B_0} \circ \Pi(\mathbf{d}(\xi)\mathbf{m}(A)),$$

where we choose $\xi \in F^\times$ and $A \in \mathrm{GL}_n(E)$ so that $B = \xi^{-1}B_0\langle A\rangle$. The right hand side is independent of the choice of ξ and A.

Here is a noteworthy consequence of Definition 3.2:

$$\beth_B \circ \Pi(\mathbf{d}(\xi)\mathbf{m}(A)) = \chi(\xi^{-[n/2]} \det A)\beth_{\xi^{-1}B\langle A\rangle} \tag{3.1}$$

for all $\xi \in F^\times$, $A \in \mathrm{GL}_n(E)$ and $B \in \mathcal{O}_F(B_0)$.

Suppose that n is odd and Π is a representation of $\mathcal{G}_n$. For simplicity we assume that $F = \mathbb{C}$ in the archimedean case. Fix $B_0 \in \mathrm{Her}_n^{\mathrm{nd}}$ and $S \in \mathrm{Her}_{n-1}^{\mathrm{nd}}$. It is important to note that

$$\mathcal{O}_F(B_0) = \mathrm{Her}_n^{\mathrm{nd}}, \qquad \lambda_S(\mathrm{GU}_S) = F^\times. \tag{3.2}$$

We define the subgroup of R_S^n by

$$\mathcal{R}_S = \{(A, g) \in \mathrm{GU}_S \times \mathrm{GL}_2(F) \mid \lambda_S(A) = \det g\}.$$

To lighten the burden of our notation, we will use the abbreviation

$$N' = N_{n-1}^n, \quad X' = X_{n-1}^n, \quad Y' = Y_{n-1}^n, \quad Z' = Z_{n-1}^n, \quad \iota' = \iota_{n-1}^n, \quad \mathbf{v}' = \mathbf{v}_{n-1}^n.$$

Let ${}^\chi\Omega_S$ denote the restriction of ${}^\chi\omega_S^{\hat{\epsilon}}$ to $\mathcal{R}_S \ltimes N'$, which is independent of the choice of $\hat{\epsilon}$ as the symbol suggests. Let $\mathfrak{J}_{B_0} \in \mathrm{Sh}_{B_0}^\chi(\Pi)$ and construct $\{\mathfrak{J}_B\}_{B \in \mathrm{Her}_n^{\mathrm{nd}}}$. We associate to $\xi \in F^\times$ a $\boldsymbol{\psi}^\xi$-Whittaker functional $\Gamma_\xi^S\langle \mathfrak{J}_{B_0}\rangle$ on the Jacquet module $J_{N'}(\Pi \otimes \overline{{}^\chi\Omega_S})$ by

$$\Gamma_\xi^S\langle \mathfrak{J}_{B_0}\rangle(f \otimes \bar{\phi}) = |\xi|_F^{(1-n)/2} \int_{Z'Y'\backslash N'} \mathfrak{J}_{S\oplus\xi}(\Pi(v)f)\overline{[{}^\chi\Omega_S(v)\phi](0)}\,\mathrm{d}v.$$

Since $[{}^\chi\Omega_S(x)\phi](0) = \phi(x)$ for $x \in X'$, this integral is convergent and $\mathrm{U}_S \ltimes N'$-invariant. Thus we can define a function on $\mathrm{GL}_2(F)$ by

$$\Gamma_\xi^S\langle \mathfrak{J}_{B_0}\rangle(g; f \otimes \bar{\phi}) = \Gamma_\xi^S\langle \mathfrak{J}_{B_0}\rangle(\Pi(\iota'(A, g))f \otimes \overline{{}^\chi\Omega_S(\iota'(A, g))\phi}),$$

where we take $A \in \mathrm{GU}_S$ with $\lambda_S(A) = \det g$. Recall that $\chi^\dagger$ denotes the restriction of χ to $F^\times$.

Proposition 3.3. *If n is odd, then for $\mathfrak{J}_{B_0} \in \mathrm{Sh}_{B_0}^\chi(\Pi)$; $S \in \mathrm{Her}_{n-1}^{\mathrm{nd}}$; $a, b \in F^\times$; $c \in F$; $g \in \mathrm{GL}_2(F)$; $f \in \Pi$ and $\phi \in {}^\chi\Omega_S$*

$$\Gamma_1^S\langle \mathfrak{J}_{B_0}\rangle(\mathfrak{n}(c)\mathfrak{m}(a, b)g; f \otimes \bar{\phi}) = \boldsymbol{\psi}(c)\chi^\dagger(a)\Gamma_{ab^{-1}}^S\langle \mathfrak{J}_{B_0}\rangle(g; f \otimes \bar{\phi}).$$

Proof. Take $A \in \mathrm{GU}_S$ such that $ab = \lambda_S(A)$. Then

$$[{}^\chi\Omega_S(\iota'(A, \mathfrak{m}(a, b)))\phi](u) = \chi(\Lambda_S(A))|ab^{-1}|_F^{(n-1)/2}\phi(aA^{-1}u)$$

for $u \in E^{n-1}$ and $\phi \in \mathcal{S}(E^{n-1})$. Observing that

$$\iota'(A, \mathfrak{m}(a, b))^{-1}\mathbf{v}'(x; y; z)\iota'(A, \mathfrak{m}(a, b)) = \mathbf{v}'(aA^{-1}x; bA^{-1}y; abA^{-1}z\,{}^t(A^{-1})^\tau)$$

by (2.3) and (2.4), we get

$$\begin{aligned}&\int_{X'} \mathfrak{J}_{S\oplus\xi}(\Pi(x\iota'(A, \mathfrak{m}(a, b)))f)\overline{[{}^\chi\Omega_S(x\iota'(A, \mathfrak{m}(a, b)))\phi](0)}\,\mathrm{d}x \\ =&\frac{|\det A|_E}{|a|_E^{n-1}} \int_{X'} \mathfrak{J}_{S\oplus\Xi}(\Pi(\iota'(A, \mathfrak{m}(a, b))x)f)\overline{[{}^\chi\Omega_S(\iota'(A, \mathfrak{m}(a, b))x)\phi](0)}\,\mathrm{d}x.\end{aligned}$$

Since $\iota'(A, \mathfrak{m}(a, b)) = \mathbf{d}(ab)\mathbf{m}(\mathrm{diag}[A, a])$, we see by (3.1) that

$$\mathfrak{J}_{S\oplus 1} \circ \Pi(\iota'(A, \mathfrak{m}(a, b))) = \chi((ab)^{(1-n)/2}a\det A)\mathfrak{J}_{S\oplus ab^{-1}}.$$

The formula follows upon combining these observations with the identities $\mathrm{N}_F^E(\det A) = (ab)^{n-1}$ and $\Lambda_S(A) = (ab)^{(1-n)/2}\det A$. □

4. Certain nontempered representations

We regard $\boldsymbol{\psi}$ as a character of $N = N_1$ in the usual way, i.e., $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mapsto \boldsymbol{\psi}(c)$. An irreducible admissible representation π of $\mathrm{GL}_2(F)$ is said to be generic if it has a $\boldsymbol{\psi}$-Whittaker model, which is unique and denoted by $\mathscr{W}(\pi)$. Given an irreducible admissible generic unitary representation π of $\mathrm{GL}_2(F)$, we will explicitly construct degenerate Whittaker functionals $\mathfrak{I}_B^{\hat{\chi}}$ on an irreducible admissible unitary representation $A_n^{\hat{\chi}}(\pi)$ of $\mathcal{G}_n$. Appendix B will construct $A_n^{\hat{\chi}}(\pi)$ and $\mathfrak{I}_B^{\hat{\chi}}$ in a simpler manner in the split case.

Proposition 7.2 and Lemma 10.5 say that

$$\mathfrak{I}_B^{\hat{\chi}} \in \mathrm{Sh}_B^{\hat{\chi}}(A_n^{\hat{\chi}}(\pi)), \qquad \Gamma_1^S\langle \mathfrak{I}_B^{\hat{\chi}} \rangle (f \otimes \bar{\phi}) \in \mathscr{W}(\pi). \tag{4.1}$$

These properties are the technical heart for the proof of Theorem 1.1.

We will differ slightly from our previous notation. Given a free right E-module X, we denote the group of all E-linear automorphisms of X by $\mathrm{GL}_E(X)$. The free E-module $W_n = E^{2n}$ comes equipped with the split skew Hermitian form $\langle x, y \rangle = {}^t x^\tau J_n y$ for $x, y \in W_n$. We regard $\mathcal{G}_n$ as the group of similitudes of the skew Hermitian space $(W_n, \langle\ ,\ \rangle)$. When X is a totally isotropic subspace of W_n, we denote the maximal parabolic subgroup of $\mathcal{G}_n$ stabilizing X by $\mathcal{P}_X$ and their unipotent radical by N_X. We define the canonical homomorphism

$$\mathrm{proj}_X : \mathcal{P}_X \to \mathrm{GL}_E(X) \times \mathcal{G}_{n-i}, \qquad \mathrm{proj}_X(p) = (p|_X, p|_{X^\perp/X}),$$

where $\dim_E X = i$ and the subspace $X^\perp$ consists of $v \in W_n$ such that $\langle v, x \rangle = 0$ for all $x \in X$.

Fix an E-basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ for W_n which consists of isotropic vectors such that $\langle e_i, f_j \rangle = \delta_{i,j}$. Let $\mathcal{X}_i$ (resp. $\mathcal{Y}_i$) be the totally isotropic subspace spanned by $e_1, e_2, \dots, e_i$ (resp. $f_1, f_2, \dots, f_i$). We often use matrix representation (2.1) with respect to this Witt basis interchangeably. Recall that the standard maximal parabolic subgroup $\mathcal{P}_{\mathcal{X}_i}$ is denoted by $\mathcal{P}_i$.

Let $\mathfrak{X}_{2i}$ (resp. $\mathfrak{X}_{2i-1}$) be the totally isotropic subspace of W_n spanned by

$$e_1, f_2, e_3, f_4, \dots, e_{2i-1}, f_{2i} \quad (\text{resp. } e_1, f_2, e_3, f_4, \dots, e_{2i-3}, f_{2i-2}, e_{2i-1}).$$

Let $\mathfrak{Y}_{2i}$ (resp. $\mathfrak{Y}_{2i-1}$) be the totally isotropic subspace spanned by

$$f_1, e_2, f_3, e_4, \dots, f_{2i-1}, e_{2i} \quad (\text{resp. } f_1, e_2, f_3, e_4, \dots, f_{2i-3}, e_{2i-2}, f_{2i-1})$$

For brevity we will write

$$\mathfrak{P}_i = \mathcal{P}_{\mathfrak{X}_i}, \qquad \mathfrak{N}_i = N_{\mathfrak{X}_i}, \qquad \mathfrak{M}_i = \mathcal{P}_{\mathfrak{X}_i} \cap \mathcal{P}_{\mathfrak{Y}_i}, \qquad \mathfrak{P}_{\mathbf{e}} = \cap_{i=1}^{(n-1)/2} \mathfrak{P}_{2i},$$

assuming n to be odd. We denote by $\mathfrak{N}_\varepsilon$ the unipotent radical of the Borel subgroup of $\mathcal{G}_n$ which stabilizes the complete flag of isotropic subspaces $\mathfrak{X}_1 \subset \mathfrak{X}_2 \subset \cdots \subset \mathfrak{X}_n$. As in Section 2 we realize the isomorphism with respect to $\mathfrak{X}_2$

$$\mathfrak{i}_2 : \mathrm{GL}_2(E) \times \mathcal{G}_{n-2} \simeq \mathfrak{M}_2.$$

The Galois twist ${}^{\tau}\Pi$ of a representation Π of $\mathrm{GL}_m(E)$ is a representation of $\mathrm{GL}_m(E)$ defined by ${}^{\tau}\Pi(A) = \Pi(A^{\tau})$ for $A \in \mathrm{GL}_m(E)$. Let $\pi \mapsto \pi^E$ be the functorial transfer from irreducible representations of $\mathrm{GL}_m(F)$ to those of $\mathrm{GL}_m(E)$ given by quadratic base change (cf. [4, Ch. 1, §§6,7]).

Let B_m be the subgroup of upper triangular matrices in GL_m. Given $a, b \in F^{\times}$ and $c \in F$, we put

$$\mathfrak{m}(a,b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \qquad \mathfrak{n}(c) = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}.$$

The induced representation $\mathrm{Ind}_{B_2(F)}^{\mathrm{GL}_2(F)} \mu_1 \boxtimes \mu_2$ of the character $\mathfrak{m}(a,b)\mathfrak{n}(c) \mapsto \mu_1(a)\mu_2(b)$ of the Borel subgroup is called a principal series representation and denoted by $I(\mu_1, \mu_2)$. In the nonarchimedean case $I(\mu_1, \mu_2)$ is irreducible unless $\mu_1\mu_2^{-1} \in \{\boldsymbol{\alpha}_F, \boldsymbol{\alpha}_F^{-1}\}$. For $\chi \in \Omega(C_E)$ we associate to $a \in E^{\times}$ a $(\boldsymbol{\psi} \circ \mathrm{T}_F^E)^a$-Whittaker functional w_a^{χ} on $I(\chi, {}^{\tau}\chi^{-1})$ by

$$w_a^{\chi}(h) = |a|_E^{1/2} L(1, \chi^{\tau}\chi) \int_E h(J_1 \mathbf{n}(x)) \overline{\boldsymbol{\psi}(\mathrm{T}_F^E(ax))}\, \mathrm{d}x, \quad J_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The integral converges absolutely for $\Re(\chi^{\tau}\chi) > 1$ and extends to an entire function on $\Omega(C_E)$.

Throughout this paper we fix unitary characters $\hat{\omega} \in \Omega(C_F)$ and $\hat{\chi} \in \Omega(C_E)$ such that $\hat{\chi}^{\dagger} = \hat{\omega}$. If $\Re\mu > -\frac{1}{2}$, then $I(\mu, \mu^{-1}\hat{\omega})$ has a unique irreducible subrepresentation, which we denote by $A(\mu, \mu^{-1}\hat{\omega})$. Let π be an irreducible admissible unitary generic representation of $\mathrm{GL}_2(F)$ whose central character is $\hat{\omega}$. If F is nonarchimedean and π is not supercuspidal, then there is $\mu \in \Omega(C_F)$ which satisfies either $-\frac{1}{2} < \Re\mu < \frac{1}{2}$ or $\mu^2\hat{\omega}^{-1} = \boldsymbol{\alpha}_F$ and such that π is equivalent to $A(\mu, \mu^{-1}\hat{\omega})$. In the archimedean case π is equivalent to $A(\mu, \mu^{-1}\hat{\omega})$ for some $\mu \in \Omega(C_F)$ with $\Re\mu > -\frac{1}{2}$.

Definition 4.1. For an irreducible admissible representation π of $\mathrm{GL}_m(F)$ we define an irreducible admissible representation $\pi[\hat{\chi}]$ of $\mathrm{GL}_m(E)$ as

$$\pi[\hat{\chi}] = {}^{\tau}\hat{\chi}^{-1} \otimes \pi^E.$$

The $\boldsymbol{\psi}$-Whittaker model of π is denoted by $\mathscr{W}(\pi)$. Let $\mathscr{W}(\pi[\hat{\chi}])$ denote the $\boldsymbol{\psi} \circ \mathrm{T}_F^E$-Whittaker model of $\pi[\hat{\chi}]$. The representation $A_1(\chi, \mu)$ of $\mathcal{G}_1$ will be defined in the next section. Remark 2.1 allows us to identify representations of $\mathcal{G}_1$ with those of $E^{\times} \times \mathrm{GL}_2(F)$ on which Δ acts trivially. With this identification we obtain the following results:

Lemma 4.2. *Let $\mu \in \Omega(C_F)$ and π be an irreducible admissible unitary generic representation of $\mathrm{GL}_2(F)$ whose central character is $\hat{\omega}$.*

(1) ${}^{\tau}\mu[\hat{\chi}]^{-1} = (\mu^{-1}\hat{\omega})[\hat{\chi}]$.
(2) *If $\Re\mu > -\frac{1}{2}$, then*

$$A(\mu, \mu^{-1}\hat{\omega})[\hat{\chi}] \simeq A(\mu[\hat{\chi}], {}^{\tau}\mu[\hat{\chi}]^{-1}), \quad \hat{\chi}^{-1} \boxtimes A(\mu, \mu^{-1}\hat{\omega}) \simeq A_1(\mu[\hat{\chi}], \mu^{-1}\hat{\omega}).$$

(3) ${}^{\tau}\pi[\hat{\chi}]^{\vee} \simeq \pi[\hat{\chi}]$ *and the central character of* $\pi[\hat{\chi}]$ *is* $\hat{\chi}\,{}^{\tau}\hat{\chi}^{-1}$.
(4) *The stable base change of* $\hat{\chi}^{-1} \boxtimes \pi$ *is* $\pi[\hat{\chi}]$.

Proof. The last statement is Theorem 4.12 of [24]. The other assertions can easily be checked. □

Definition 4.3. Let π be an irreducible admissible unitary generic representation of $\mathrm{GL}_2(F)$ whose central character is $\hat{\omega}$. For odd n we write $A_n^{\hat{\chi}}(\pi)$ for the unique irreducible subrepresentation of $J_n^{\hat{\chi}}(\pi)$, where

$$J_n^{\hat{\chi}}(\pi) = \mathrm{Ind}_{\mathfrak{P}_{\mathbf{e}}}^{\mathcal{G}_n} \delta_{\mathfrak{P}_{\mathbf{e}}}^{-1/4} \otimes \{\mathscr{W}(\pi[\hat{\chi}])^{\boxtimes (n-1)/2} \boxtimes (\hat{\chi}^{-1} \boxtimes \mathscr{W}(\pi))\}.$$

Remark 4.4. (1) The central character of $A_n^{\hat{\chi}}(\pi)$ is $\hat{\chi}^n(\hat{\omega}^E)^{(1-n)/2}$.
(2) Define a homomorphism $\Lambda_n : \mathcal{G}_n \to C_E^1$ by $\Lambda_n(g) = \lambda_n(g)^{-n} \det g$, where C_E^1 denotes the norm one elements in C_E. When $\chi^{\dagger}$ is trivial, we can define $\tilde{\chi} \in \Omega(C_E^1)$ by $\tilde{\chi}(a/a^{\tau}) = \chi(a)$ for $a \in C_E$. Then

$$A_n^{\hat{\chi}}(\pi) \otimes \tilde{\chi} \circ \Lambda_n \otimes \nu \circ \lambda_n \simeq A_n^{\hat{\chi}\chi^{\nu^E}}(\pi \otimes \nu).$$

(3) It can be shown that $A_n^{\hat{\chi}}(\pi)$ is unitary by Proposition 6.3, Corollary 5.5(2) and a simple globalization argument (cf. Appendix C).

We define the split Hermitian matrix $H_m \in \mathrm{Her}_m^{\mathrm{nd}}$ by

$$H_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H_{2i-1} = \underbrace{H_2 \oplus \cdots \oplus H_2}_{i-1} \oplus 1, \quad H_{2i} = \underbrace{H_2 \oplus \cdots \oplus H_2}_{i}.$$

For odd n we will explicitly construct a nonzero H_nth degenerate Whittaker functional on $A_n^{\hat{\chi}}(\pi)$, which will ultimately turn out to be an H_nth Shalika functional with respect to $\hat{\chi}$. Note that for $u \in N_n$

$$\boldsymbol{\psi}(\ell^{H_n}(u)) = \boldsymbol{\psi}\left(\langle uf_n, f_n\rangle + \sum_{i=1}^{(n-1)/2} \mathrm{T}_F^E(\langle uf_{2i}, f_{2i-1}\rangle)\right).$$

For $f \in J_n^{\hat{\chi}}(\pi)$, $u \in \mathfrak{N}_{\varepsilon}$ and $g \in \mathcal{G}_n$,

$$f(ug) = f(g)\boldsymbol{\psi}\left(\langle uf_n, f_n\rangle + \sum_{i=1}^{(n-1)/2} \mathrm{T}_F^E(\langle uf_{2i}, f_{2i-1}\rangle)\right).$$

Therefore the integral

$$\mathfrak{I}_{H_n}^{\hat{\chi}}(f) = \int_{\mathfrak{P}_{\mathbf{e}} \cap N_n \backslash N_n} f(u)\overline{\boldsymbol{\psi}(\ell^{H_n}(u))}\,\mathrm{d}u$$

makes sense at least formally. Put

$$X(x,\xi) = \mathbf{m}\left(\begin{bmatrix} 1 & \xi & {}^t x^{\tau} \\ 0 & 1 & 0 \\ 0 & 0 & \mathbf{1}_{n-2} \end{bmatrix}\right), \qquad Y(y,z) = \mathbf{n}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & z & {}^t y^{\tau} \\ 0 & y & \mathbf{0}_{n-2} \end{bmatrix}\right).$$

for $x, y \in E^{n-2}$; $\xi \in E$ and $z \in F$.

Proposition 4.5. *Let n be odd and $f \in J_n^{\hat{\chi}}(\pi)$.*

(1) *$u \mapsto f(u)\overline{\boldsymbol{\psi}(\ell^{H_n}(u))}$ is a Schwartz function on $\mathfrak{P}_{\mathbf{e}} \cap N_n \backslash N_n$.*
(2) *$\mathfrak{I}_{H_n}^{\hat{\chi}}$ is nonzero on $A_n^{\hat{\chi}}(\pi)$.*
(3) *The following identity holds:*

$$\mathfrak{I}_{H_n}^{\hat{\chi}}(f) = \int_{\mathfrak{P}_2 \cap N_n \backslash N_n} \mathfrak{I}_{H_{n-2}}^{\hat{\chi}}(f(u))\overline{\boldsymbol{\psi}(\ell^{H_n}(u))}\,\mathrm{d}u.$$

Remark 4.6. The following inductive nature should be mentioned:

$$J_n^{\hat{\chi}}(\pi) = \mathrm{Ind}_{\mathfrak{P}_2}^{\mathcal{G}_n} \delta_{\mathfrak{P}_2}^{-1/4} \otimes \mathscr{W}(\pi[\hat{\chi}]) \boxtimes J_{n-2}^{\hat{\chi}}(\pi).$$

We regard $f \in J_n^{\hat{\chi}}(\pi)$ as a $J_{n-2}^{\hat{\chi}}(\pi)$-valued function on $\mathcal{G}_n$ which satisfies

$$f(\mathfrak{i}_2(\mathfrak{n}(x), \mathbf{1}_{2n-4})ug) = \boldsymbol{\psi}(\mathrm{T}_F^E(x))f(g) \qquad (x \in E,\ u \in \mathfrak{N}_2,\ g \in \mathcal{G}_n)$$

and obtain the $\mathbb{C}$-valued function $g \mapsto \mathfrak{I}_{H_{n-2}}^{\hat{\chi}}(f(g))$.

Proof. Define the subgroups $\mathfrak{y}_1$ of N_n and $\check{\mathfrak{y}}_2$ of Her_n by

$$\mathfrak{y}_1 = \left\{ \mathbf{v}_1^n(0; y; z) \;\middle|\; \begin{matrix} y \in E^{n-1} \\ z \in F \end{matrix} \right\}, \quad \check{\mathfrak{y}}_2 = \left\{ \begin{bmatrix} z_1 & z_2^\tau & {}^t z_3^\tau \\ z_2 & 0 & 0 \\ z_3 & 0 & z_4 \end{bmatrix} \;\middle|\; \begin{matrix} z_1 \in F,\ z_2 \in E \\ z_3 \in E^{n-2} \\ z_4 \in \mathrm{Her}_{n-2} \end{matrix} \right\}.$$

Then

$$\mathfrak{P}_2 \cap N_n = \mathbf{n}(\check{\mathfrak{y}}_2) = \mathfrak{y}_1 \cdot \mathbf{n}_2'(\mathrm{Her}_{n-2}), \quad \mathfrak{P}_{\mathbf{e}} \cap N_n = \mathfrak{y}_1 \cdot (\mathfrak{P}_{\mathbf{e}} \cap \mathbf{n}_2'(\mathrm{Her}_{n-2})).$$

We may assume by induction that the restriction of $f(u)\overline{\boldsymbol{\psi}^{H_n}(u)}$ to $\mathfrak{P}_{\mathbf{e}} \cap \mathbf{n}_2'(\mathrm{Her}_{n-2}) \backslash \mathbf{n}_2'(\mathrm{Her}_{n-2})$ is a Schwartz function. To prove (1), we have only to show that $f \circ Y \in \mathcal{S}(E^{n-2} \oplus F)$. Since $X(x,\xi) \in \mathfrak{N}_1$ and $X(x,\xi)f_2 = f_2$, we see that $f(X(x,\xi)g) = f(g)$ for $x \in E^{n-2}$ and $\xi \in E$. Since

$$X(x,\xi)Y(y,z)X(x,\xi)^{-1}Y(y,z)^{-1} = \mathbf{v}_1^n(0; \xi z + {}^t x^\tau y, \xi\, {}^t y^\tau; \xi\xi^\tau z + \mathrm{T}_F^E(\xi^\tau\, {}^t x^\tau y))$$

for all $x, y \in E^{n-2}$; $\xi \in E$ and $z \in F$,

$$f(Y(y,z)X(x,\xi)^{-1}) = \boldsymbol{\psi}(z\mathrm{T}_F^E(\xi) + \mathrm{T}_F^E({}^t x^\tau y))f(Y(y,z)). \tag{4.2}$$

Given $\Phi \in \mathcal{S}(E^{n-2} \oplus E)$, we put

$$[\Phi \star f](g) = \int_E \int_{E^{n-2}} \Phi(x,\xi) f(gX(x,\xi))\,\mathrm{d}x\mathrm{d}\xi.$$

Write $f = \Phi \star h$ for some $h \in J_n^{\hat{\chi}}(\pi)$ and $\Phi \in \mathcal{S}(E^{n-2} \oplus E)$. Then

$$\begin{aligned} f(Y(y,z)) &= \int_E \int_{E^{n-2}} \Phi(x,\xi) h(Y(y,z)X(x,\xi))\,\mathrm{d}x\mathrm{d}\xi \\ &= h(Y(y,z))\hat{\Phi}(y,z) \end{aligned}$$

by (4.2), where

$$\hat{\Phi}(y,z) = \int_E \int_{E^{n-2}} \Phi(x,\xi)\overline{\boldsymbol{\psi}(\mathrm{T}_F^E(\xi z + {}^t x^\tau y))}\,\mathrm{d}x\mathrm{d}\xi$$

is the Fourier transform of Φ and hence a Schwartz function on $E^{n-2} \oplus E$.

By what we have seen, we get the inductive formula

$$\mathfrak{Z}^{\hat{\chi}}_{H_n}(f) = \int_F \int_{E^{n-2}} \mathfrak{Z}^{\hat{\chi}}_{H_{n-2}}(f(Y(y,z)))\,\mathrm{d}y\mathrm{d}z \tag{4.3}$$

as claimed. Moreover, we have showed that

$$\mathfrak{Z}^{\hat{\chi}}_{H_n}(\Phi \star f) = \int_F \int_{E^{n-2}} \hat{\Phi}(y,z)\mathfrak{Z}^{\hat{\chi}}_{H_{n-2}}(f(Y(y,z)))\,\mathrm{d}y\mathrm{d}z.$$

If $\mathfrak{Z}^{\hat{\chi}}_{H_n}(\Phi\star f) = 0$ for all $\Phi \in \mathcal{S}(E^{n-2}\oplus E)$, then $\mathfrak{Z}^{\hat{\chi}}_{H_{n-2}}(f(e)) = 0$. We conclude by induction that if $\mathfrak{Z}^{\hat{\chi}}_{H_n}$ kills $A^{\hat{\chi}}_n(\pi)$, then $f(e) = 0$ for all $f \in A^{\hat{\chi}}_n(\pi)$. This is a contradiction however. □

5. Degenerate principal series representations

We will highlight some properties of Jacquet integrals on degenerate principal series representations which are needed in this paper. When π is not supercuspidal, we describe $A^{\hat{\chi}}_n(\pi)$ as a subrepresentation of degenerate principal series and relate $\mathfrak{Z}^{\hat{\chi}}_B$ to the Jacquet integral in the next section.

Let $\chi \in \Omega(C_E)$ and $\mu \in \Omega(C_F)$. Recall that $\chi^\dagger$ is the restriction of χ to C_F and that $\Re\mu$ is the unique real number such that $\mu\boldsymbol{\alpha}_F^{-\Re\mu}$ is unitary. When $E = F \oplus F$, we set $\Re\chi = \frac{1}{2}\Re\chi^\dagger$. Recall that $\mu^E = \mu \circ \mathrm{N}^E_F$. Then $\Re\mu = \Re\mu^E$. Put

$$a^n(\chi) = \prod_{j=1}^n L(1-j, \chi^\dagger \cdot \epsilon^{n+j}_{E/F}), \qquad b^n(\chi) = \prod_{j=1}^n L(j, \chi^\dagger \cdot \epsilon^{n+j}_{E/F})$$

The modulus characters of maximal parabolic subgroups of $\mathcal{G}_n$ are given by

$$\delta_{\mathcal{P}_i}(\iota_i(A,g)) = |\lambda_{n-i}(g)^{-i}\mathrm{N}^E_F(\det A)|^{2n-i}_F \tag{5.1}$$

for $1 \le i \le n$, $A \in \mathrm{GL}_i(E)$ and $g \in \mathcal{G}_{n-i}$.

Let $J_n(\chi,\mu) = \mathrm{Ind}^{\mathcal{G}_n}_{\mathcal{P}_n}(\chi\circ\det\boxtimes\mu\circ\lambda_n)$ be the normalized induced representation of the character of $\mathcal{P}_n$ defined by

$$\mathbf{d}(\xi)\mathbf{m}(A)\mathbf{n}(z) \mapsto \mu(\xi)\chi(\det A).$$

The center $\mathcal{Z}_n$ of $\mathcal{G}_n$ acts on $J_n(\chi,\mu)$ by the character $\chi^n\mu^E$. Since $\mathcal{G}_n = \{\mathbf{d}(\xi) \mid \xi \in F^\times\} \ltimes G_n$, we can identify the space $J_n(\chi,\mu)$ with the space of smooth functions $f : G_n \to \mathbb{C}$ satisfying

$$f(\mathbf{m}(A)\mathbf{n}(z)g) = \chi(\det A)|\det A|^{n/2}_E f(g)$$

for all $A \in \mathrm{GL}_n(E)$, $z \in \mathrm{Her}_n$ and $g \in G_n$. We write $I_n(\chi)$ for the representation of G_n obtained by restricting the action of $J_n(\chi,\mu)$ to G_n.

In the nonarchimedean case the field F comes equipped with a subring $\mathfrak{o}$ whose elements are called the integers of F. We denote the integer ring of E by $\mathfrak{r}$. In the split case $\mathfrak{r} = \mathfrak{o}\oplus\mathfrak{o}$. The ring $\mathfrak{o}$ has a unique nonzero prime

ideal $\mathfrak{p}$. The index $[\mathfrak{o}:\mathfrak{p}]$ is denoted by q. For fractional ideals $\mathfrak{b},\mathfrak{c}$ of $\mathfrak{r}$ which satisfy $\mathfrak{bc}\subset\mathfrak{r}$ we put

$$\Gamma_n[\mathfrak{b},\mathfrak{c}]=\left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in\mathcal{G}_n\;\middle|\; a,d\in \mathrm{M}_n(\mathfrak{r}),\ b\in\mathfrak{b}\mathrm{M}_n(\mathfrak{r}),\ c\in\mathfrak{c}\mathrm{M}_n(\mathfrak{r})\right\}. \tag{5.2}$$

Set $\Gamma_n[\mathfrak{c}]=\Gamma_n[\mathfrak{c}^{-1},\mathfrak{c}]$. For $g\in\mathcal{G}_n$ the quantity $\varepsilon_{\mathfrak{c}}(g)$ is defined by writing $g=pk$ with $p=\mathbf{d}(\lambda)\mathbf{m}(A)\mathbf{n}(z)\in\mathcal{P}_n$, $k\in\Gamma_n[\mathfrak{c}]$, and setting

$$\varepsilon_{\mathfrak{c}}(g)=|\lambda|_F^{-n}|\det A|_E.$$

Once a Haar measure $\mathrm{d}z$ on Her_n is fixed, one can 'canonically' construct a nonzero element $w_B^{\chi}\in\mathrm{Wh}_B(I_n(\chi))$ for each $B\in\mathrm{Her}_n^{\mathrm{nd}}$.

Definition 5.1 (Jacquet integrals)**.** Given $\chi\in\Omega(C_E)$ and $h\in I_n(\chi)$, we define a holomorphic section $h^{(s)}$ of $I_n(\chi\boldsymbol{\alpha}_E^s)$ by setting $h^{(s)}(g)=h(g)\varepsilon_{\mathfrak{o}}(g)^s$ for $s\in\mathbb{C}$. In the archimedean case we define $h^{(s)}$ by replacing $\Gamma_n[\mathfrak{o}]$ by a certain standard maximal compact subgroup of G_n. For $B\in\mathrm{Her}_n^{\mathrm{nd}}$ the integral

$$\mathbf{w}_B^{\chi\boldsymbol{\alpha}_E^s}(h^{(s)})=\int_{\mathrm{Her}_n}h^{(s)}(J_n\mathbf{n}(z))\overline{\boldsymbol{\psi}^B(z)}\,\mathrm{d}z$$

is defined a priori for $\Re s>\frac{n}{2}-\Re\chi$ but admits an entire analytic continuation to the whole s-plane. We can therefore evaluate $\mathbf{w}_B^{\chi\boldsymbol{\alpha}_E^s}(h^{(s)})$ at $s=0$. From now on we assume that $\Re\chi>-\frac{1}{2}$ and set

$$w_B^{\chi}(h)=|\det B|_F^{n/2}\mathbf{w}_B^{\chi}(h)b^n(\chi).$$

We define an intertwining operator

$$M_n(\chi):J_n(\chi,\mu)\to J_n({}^{\tau}\chi^{-1},(\chi^{\dagger})^n\mu)$$

by the integral

$$[M_n(\chi)f](g)=a^n(\chi)^{-1}\int_{\mathrm{Her}_n}f(J_n\mathbf{n}(z)g)\,\mathrm{d}z$$

which is convergent for $\Re\chi>\frac{n}{2}$ and extends to an entire function on $\Omega(C_E)$ by [27, Proposition 3.2, Theorem 1.3(5)]. There is a meromorphic function $c_n(\chi)$ on $\Omega(C_E)$ such that

$$\mathbf{w}_B^{{}^{\tau}\chi^{-1}}\circ M_n(\chi)=c_n(\chi)\chi^{\dagger}(\det B)^{-1}\epsilon(B)^{n-1}\mathbf{w}_B^{\chi} \tag{5.3}$$

for all $B\in\mathrm{Her}_n^{\mathrm{nd}}$ by [27, Proposition 3.1, (3.5), (3.9), §7]. Moreover, the product $c_n(\chi)b^n({}^{\tau}\chi^{-1})$ is entire and nowhere vanishing on $\Omega(C_E)$. We let $\chi=\mu[\hat{\chi}\hat{\epsilon}]$. When $\epsilon^{\dagger}=\epsilon_{E/F}^{n-1}$, we can rewrite (5.3) as the functional equation

$$\mu(\det B)^{-1}w_B^{\mu[\hat{\chi}\hat{\epsilon}]}\circ M_n((\hat{\omega}\mu^{-1})[\hat{\chi}\hat{\epsilon}])=\frac{e_n(\mu[\hat{\chi}\hat{\epsilon}])}{(\hat{\omega}\mu^{-1})(\det B)}\mathbf{w}_B^{(\hat{\omega}\mu^{-1})[\hat{\chi}\hat{\epsilon}]} \tag{5.4}$$

involving an exponential factor of proportionality

$$e_n(\mu[\hat{\chi}\hat{\epsilon}])=c_n((\hat{\omega}\mu^{-1})[\hat{\chi}\hat{\epsilon}])b^n(\mu[\hat{\chi}\hat{\epsilon}]).$$

Definition 5.2. Given any function f on a group G, we define a function $\varrho(\Delta)f$ on G by $[\varrho(\Delta)f](g) = f(g\Delta)$ for $\Delta, g \in G$. We will sometimes write $\varrho = \varrho_G$ to indicate that it pertains to G.

Lemma 5.3 ([21])**.** *Let $B \in \mathrm{Her}_n^{\mathrm{nd}}$, $\chi \in \Omega(C_E)$ and $\mu \in \Omega(C_F)$. Assume that $\Re\chi > -\frac{1}{2}$.*

1. *w_B^χ is a nonzero vector in $\mathrm{Wh}_B(I_n(\chi))$.*
2. *The space $\mathrm{Wh}_B(I_n(\chi))$ is one-dimensional at least if F is $\mathfrak{p}$-adic.*
3. *If $A \in \mathrm{GL}_n(E)$ and $\xi \in F^\times$, then*

$$w_B^\chi \circ \varrho(\mathbf{d}(\xi)\mathbf{m}(A)) = \mu(\xi)\chi^\dagger(\xi)^n\, {}^\tau\chi(\det A)^{-1} w^\chi_{\xi^{-1}B\langle A\rangle}.$$

Proof. The first part is clear. The second part is the fact proved by Karel [21]. The third part can be proved by simple changes of variables. □

For simplicity we discuss only the $\mathfrak{p}$-adic case for the rest of this section. Let $k = i+n$ and $S \in \mathrm{Her}_i^{\mathrm{nd}}$. Recall that $\epsilon^\dagger = \epsilon_{E/F}^i$. Put $\chi = \hat\epsilon\boldsymbol{\alpha}_E^{(i-n)/2}$. We denote the image of the intertwining map

$$\mathcal{S}(X_i^k) \to I_n(\chi), \qquad \phi \mapsto f_\phi(g) = [\omega_S^{\hat\epsilon}(g)\phi](0)$$

by $R_n^\chi(S)$. In the inert case there are precisely two equivalence classes $S_i^\pm$ of nondegenerate Hermitian forms of size i over E, having opposite signs $\epsilon(S_i^\pm) = \pm 1$. We will write $A_n^\pm(\chi) = R_n^\chi(S_n^\pm)$. When $I_n(\chi)$ is irreducible, we abuse notation in writing $A_n^+(\chi) = I_n(\chi)$ to make our exposition uniform. Set $A_n^-(\chi) = \{0\}$ unless E is a field and $\chi^\dagger = \epsilon_{E/F}^n$.

Proposition 5.4 ([21, 27])**.** *Let $\chi \in \Omega(C_E)$. Suppose that F is $\mathfrak{p}$-adic.*

1. *$I_n(\chi)$ is reducible if and only if $\chi^\dagger = \epsilon_{E/F}^i\boldsymbol{\alpha}_F^{i-n}$ for some integer $0 \le i \le 2n$ such that $i \ne n$ in the split case.*
2. *If $E \not\simeq F \oplus F$ and $\chi^\dagger = \epsilon_{E/F}^n$, then $A_n^\pm(\chi)$ are irreducible and*

$$I_n(\chi) = A_n^+(\chi) \oplus A_n^-(\chi), \quad \mathrm{Her}_n^{\mathrm{nd}}(A_n^\pm(\chi)) = \{B \in \mathrm{Her}_n^{\mathrm{nd}} \mid \epsilon(B) = \pm 1\}.$$

3. *If $\chi^\dagger = \epsilon_{E/F}^{n-1}\boldsymbol{\alpha}_F$, then $I_n(\chi)$ has a unique irreducible subrepresentation $A_n^+(\chi)$ and $\mathrm{Her}_n^{\mathrm{nd}}(A_n^+(\chi)) = \mathrm{Her}_n^{\mathrm{nd}}$.*
4. *$\mathrm{Wh}_B(A_n^\pm(\chi))$ is spanned by the restriction of w_B^χ for $B \in \mathrm{Her}_n^{\mathrm{nd}}$.*

Proof. All points of reducibility of $I_n(\chi)$, its complete composition series and degenerate Whittaker models of its constituents at each such point are described by Kudla and Sweet [27]. Suppose that $\chi^\dagger = \epsilon_{E/F}^{n-1}\boldsymbol{\alpha}_F$. Then

$$A_n^+(\chi) = R_n^\chi(S_{n+1}^+) \cap R_n^\chi(S_{n+1}^-) \tag{5.5}$$

by Theorem 1.2(3) of [27]. Note that

$$I_n(\chi)/A_n^+(\chi) \simeq R_n^{\chi\boldsymbol{\alpha}_E^{-1}}(S_{n-1}^+) \oplus R_n^{\chi\boldsymbol{\alpha}_E^{-1}}(S_{n-1}^-). \tag{5.6}$$

Lemma 4.4 of [27] says that $\mathrm{Her}_n^{\mathrm{nd}}(R_n^{\chi\boldsymbol{\alpha}_E^{-1}}(S_{n-1}^{\pm})) = \emptyset$. By the exactness of the Jacquet functor

$$\mathrm{Wh}_B(A_n^+(\chi)) = \mathrm{Wh}_B(I_n(\chi)) = \mathrm{Her}_n^{\mathrm{nd}}. \tag{5.7}$$

The last statement follows from Lemma 5.3. □

Corollary 5.5. Let $\chi \in \Omega(C_E)$ and $\mu \in \Omega(C_F)$. Assume F to be $\mathfrak{p}$-adic.

(1) If n is odd and $-\frac{1}{2} < \Re\chi < \frac{1}{2}$, then $J_n(\chi,\mu)$ is irreducible.
(2) If $\chi^\dagger = \epsilon_{E/F}^{n-1}\boldsymbol{\alpha}_F$, then $J_n(\chi,\mu)$ has a unique irreducible subrepresentation $A_n(\chi,\mu)$.
(3) If $\chi^\dagger = \epsilon_{E/F}^{n-1}\boldsymbol{\alpha}_F$, then $A_n(\chi,\mu) = M_n({}^\tau\chi^{-1})J_n({}^\tau\chi^{-1},(\chi^\dagger)^n\mu)$.

Proof. Proposition 6.4 of [27] and (5.5) prove (3). For all $\xi \in F^\times$

$$\mathrm{Her}_n^{\mathrm{nd}}(\varrho(\mathbf{d}(\xi))A_n^{\pm}(\chi)) = \{B \in \mathrm{Her}_n^{\mathrm{nd}} \mid \epsilon(B) = \pm\epsilon_{E/F}(\xi)^n\}.$$

The first part is now clear from Proposition 5.4(1), (2). The second part is a consequence of Proposition 5.4(3). □

Definition 5.6. When n is odd, we write $A_n(\chi,\mu)$ for the unique irreducible subrepresentation of $J_n(\chi,\mu)$.

6. Compatibility with the Jacquet integral

We will identify $A_n^{\hat\chi}(\pi)$ with a submodule of the degenerate principal series and show that $\mathfrak{I}_B^{\hat\chi}$ equals the Jacquet integral $w_B^{\mu[\hat\chi]}$ if $\pi \simeq A(\mu,\hat\omega\mu^{-1})$. Consequently, one can deduce (4.1) from the relevant properties of $w_B^{\mu[\hat\chi]}$.

For $1 \le i \le \frac{n-1}{2}$ we define isotropic vectors by

$$\varepsilon_{2i-1}^{\pm} = e_{2i-1} \mp f_{2i}, \qquad \varepsilon_{2i}^{\pm} = e_{2i} \pm f_{2i-1}.$$

When n is odd, we define anisotropic vectors by

$$\varepsilon_n^{+} = e_n - \urcorner f_n, \qquad \varepsilon_n^{-} = e_n + \urcorner f_n,$$

where $\urcorner$ is a nonzero element of E such that $\bar\urcorner = -\urcorner$. We define E-linear injections $\iota^{\pm} : E^n \hookrightarrow W_n$ by $\iota^{\pm}(x_1,\dots,x_n) = \sum_{j=1}^n x_j\varepsilon_j^{\pm}$. The restrictions of $\langle\ ,\ \rangle$ to the images $W_n^{\pm} = \iota^{\pm}(E^n)$ are nondegenerate. Moreover, the isomorphism $\iota^+(x) \mapsto \iota^-(x)$ is an anti-isometry, $W_n = W_n^+ \oplus W_n^-$ is an orthogonal decomposition and

$$\mathcal{X}_n = \{\iota^+(x) + \iota^-(x) \mid x \in E^n\}, \qquad \mathcal{Y}_n = \{\iota^+(y) - \iota^-(y) \mid y \in E^n\}.$$

Put $\varepsilon_1 = {}^t(1,0,\dots,0) \in E^n$. We write $\mathscr{X}_1$ for the line spanned by ε_1. By (5.1) the modulus character of $\mathfrak{P}_2$ is given by

$$\delta_{\mathfrak{P}_2}(p) = |\lambda_n(p)^{-2}\mathrm{N}_F^E(\det(p|_{\mathfrak{X}_2}))|_F^{2n-2}. \tag{6.1}$$

Lemma 6.1 ([28])**.** (1) *The image of* $\mathcal{P}_n \cap \mathfrak{P}_2$ *under* $\mathrm{proj}_{\mathfrak{X}_2}$ *is the parabolic subgroup* $\mathfrak{i}_2(B_2 \times \mathcal{P}_{n-2})$ *of* $\mathfrak{M}_2$*, where* B_2 *is the stabilizer of* $\mathcal{X}_1 = \mathfrak{X}_1$ *in* $\mathrm{GL}_E(\mathfrak{X}_2)$*. Its unipotent radical is* $\mathrm{proj}_{\mathfrak{X}_2}(N_n \cap \mathfrak{P}_2)$.

(2) $\mathrm{proj}_{\mathcal{X}_n}(\mathcal{P}_n \cap \mathfrak{P}_2)$ *is the parabolic subgroup* $\mathbf{d}(F^\times)\mathbf{m}(Q_{1,n-2,1})$ *of* $\mathcal{M}_n$*, where* $Q_{1,n-2,1}$ *is the parabolic subgroup of* $\mathrm{GL}_E(\mathcal{X}_n)$ *stabilizing* $\mathcal{X}_1$ *and the subspace* $\mathcal{X}_1'$ *of* $\mathcal{X}_n$ *spanned by* e_1 *and* $e_3, e_4, \dots, e_n$*. Its unipotent radical is the image of* $\mathcal{P}_n \cap \mathfrak{N}_2$ *under* $\mathrm{proj}_{\mathcal{X}_n}$*.*

(3) *For all* $\xi \in F^\times$*;* $a_0, a_1 \in E^\times$*;* $A_0 \in \mathrm{GL}_{n-2}(E)$

$$\mathrm{proj}_{\mathcal{X}_n}\left(\mathfrak{i}_2\left(\begin{bmatrix} a_0 & 0 \\ 0 & a_1 \end{bmatrix}, \mathbf{d}_{n-2}(\xi)\mathbf{m}_{n-2}(A_0)\right)\right) = \mathbf{d}(\xi)\mathbf{m}\left(\begin{bmatrix} a_0 & & \\ & \xi(a_1^{-1})^\tau & \\ & & A_0 \end{bmatrix}\right).$$

(4) *The restriction of* $\delta_{\mathfrak{P}_2}^{-1/4}$ *to* $\mathcal{P}_n \cap \mathfrak{P}_2$ *is* $\delta_{Q_{1,n-2,1}}^{-1/2} \circ \mathrm{proj}_{\mathcal{X}_n}$

Proof. If we use the notation of [28], then

$$\mathcal{X}_n = W^\Diamond, \quad \mathfrak{X}_2 = \mathscr{X}_1 \times \mathscr{X}_1, \quad \mathcal{P}_n \cap G_n = S_{W^\Diamond}, \quad \mathfrak{P}_2 \cap G_n = S_{\mathscr{X}_1 \times \mathscr{X}_1},$$
$$N_n = R_{W^\Diamond}, \quad \mathfrak{N}_2 = R_{\mathscr{X}_1 \times \mathscr{X}_1}, \qquad \mathcal{X}_1 = \mathscr{X}_1^\Diamond, \qquad \mathcal{X}_1' = (\mathscr{X}_1^\perp)^\Diamond.$$

We can therefore apply Lemma 4 of [28] to obtain analogous results for G_n. The proof can easily be modified to deal with $\mathcal{P}_n \cap \mathfrak{P}_2$. □

Proposition 6.2. *An intertwining operator*

$$\Psi_n(\chi) : J_n(\chi, \mu) \to \mathrm{Ind}_{\mathfrak{P}_2}^{\mathcal{G}_n} \delta_{\mathfrak{P}_2}^{-1/4} \otimes \{I(\chi, {}^\tau\chi^{-1}) \boxtimes J_{n-2}(\chi, \mu\chi^\dagger)\}$$

is defined, for $\Re\chi > -\frac{1}{2}$*, by*

$$[\Psi_n(\chi)h](g) : p \mapsto \frac{b^{n-2}(\chi)L(1, \chi\,{}^\tau\chi)}{b^n(\chi)\delta_{\mathfrak{P}_2}(p)^{1/4}} \int_{\mathfrak{N}_2 \cap \mathcal{P}_n \backslash \mathfrak{N}_2} h(upg)\,\mathrm{d}u.$$

If χ *and* μ *are unramified and* h *is* $\Gamma_n[\mathfrak{o}]$*-invariant, then* $[\Psi_n(\chi)h](e) = h(e)$*.*

Proof. Proposition 1 and Lemma 3(2) of [28] prove an analog for the unitary group G_n. Since the inducing character of $J_n(\chi, \mu)$ sends

$$\mathfrak{i}_2\left(\begin{bmatrix} a_0 & * \\ 0 & a_1 \end{bmatrix}, \mathbf{d}_{n-2}(\xi)\mathbf{m}_{n-2}(A_0)\mathbf{n}_{n-2}(z)\right) \to \mu(\xi)\chi(\xi a_0(a_1^{-1})^\tau \det A_0)$$

by Lemma 6.1(3), we can readily extend this result to the similitude group $\mathcal{G}_n$. The convergence is proved in Lemmas 5.1 and 5.2 of [35]. The last statement follows from Remark 3 of [28]. □

Now iteration of the operators produces an intertwining map

$$\Upsilon_n(\chi) : J_n(\chi, \mu) \to \mathrm{Ind}_{\mathfrak{P}_\mathbf{e}}^{\mathcal{G}_n} \delta_{\mathfrak{P}_\mathbf{e}}^{-1/4} \otimes (\mathscr{W}(I(\chi, {}^\tau\chi^{-1}))^{\boxtimes(n-1)/2} \boxtimes \sigma),$$

where $\sigma = \mathscr{W}(J_1(\chi, \mu(\chi^\dagger)^{(n-1)/2}))$. Since

$$\hat{\chi}^{-1} \boxtimes I(\mu, \mu^{-1}\hat{\omega}) \simeq J_1(\mu^E{}^\tau\hat{\chi}^{-1}, \mu^{-1}\hat{\omega}),$$

we obtain an intertwining map

$$\Upsilon_n(\mu^E{}^\tau\hat{\chi}^{-1}) : A_n(\mu^E{}^\tau\hat{\chi}^{-1}, \mu^{-n}\hat{\omega}^{(n+1)/2}) \to J_n^{\hat{\chi}}(I(\mu, \mu^{-1}\hat{\omega})),$$

where $A_n(\mu^{E\tau}\hat{\chi}^{-1}, \mu^{-n}\hat{\omega}^{(n+1)/2})$ is defined in Definition 5.6. By Corollary 5.4 of [37] (cf. Remark 5.5 of [37]) it factors to yield the intertwining map

$$\Upsilon_n(\mu^{E\tau}\hat{\chi}^{-1}) : A_n(\mu^{E\tau}\hat{\chi}^{-1}, \mu^{-n}\hat{\omega}^{(n+1)/2}) \to J_n^{\hat{\chi}}(A(\mu, \mu^{-1}\hat{\omega})).$$

Proposition 6.3. *Assume either* $-\frac{1}{2} < \Re\mu < \frac{1}{2}$ *or* $\mu^2\hat{\omega}^{-1} = \boldsymbol{\alpha}_F$. *Then*

$$w_{H_n}^{\mu[\hat{\chi}]} = \hat{\omega}(-1)^{(n-1)/2}\mathfrak{I}_{H_n}^{\hat{\chi}} \circ \Upsilon_n(\mu[\hat{\chi}]).$$

In particular, $\mathfrak{I}_{H_n}^{\hat{\chi}} \in \mathrm{Sh}_{H_n}^{\hat{\chi}}(A_n^{\hat{\chi}}(A(\mu, \mu^{-1}\hat{\omega})))$. *Moreover, if* F *is nonarchimedean, then* $\Upsilon_n(\mu[\hat{\chi}])$ *induces a* $\mathcal{G}_n$*-intertwining isomorphism*

$$A_n(\mu[\hat{\chi}], \mu^{-n}\hat{\omega}^{(n+1)/2}) \simeq A_n^{\hat{\chi}}(A(\mu, \mu^{-1}\hat{\omega})).$$

Proof. We can infer from Lemma 5.3(2) that

$$\mathfrak{I}_{H_n}^{\hat{\chi}} \circ \Upsilon_n(\mu[\hat{\chi}]) \in \mathrm{Wh}_{H_n}(I_n(\mu[\hat{\chi}]))$$

is proportional to the Jacquet integral $w_{H_n}^{\mu[\hat{\chi}]}$ at least if F is nonarchimedean. However, we will argue directly to prove the stated identity. We may assume that $n \geq 3$. Put

$$N_k^- = \left\{ \mathbf{n}_k^-(z) = \begin{bmatrix} \mathbf{1}_k & 0 \\ z & \mathbf{1}_k \end{bmatrix} \,\middle|\, z \in \mathrm{Her}_k \right\}, \quad Z(x,\xi) = \mathbf{n}_n^- \left(\begin{bmatrix} \xi & 0 & {}^t x^\tau \\ 0 & 0 & 0 \\ x & 0 & \mathbf{0}_{n-2} \end{bmatrix} \right)$$

for $x \in E^{n-2}$ and $\xi \in F$. The map $(x,\xi) \mapsto Z(x,\xi)$ defines an isomorphism $E^{n-2} \oplus F \simeq \mathfrak{P}_2 \cap N_n^- \backslash N_n^-$. For $\chi \in \Omega(C_E)$, $\Xi \in \mathrm{Her}_k^{\mathrm{nd}}$ and $a \in E^\times$ we set

$$\bar{w}_\Xi^\chi = w_\Xi^\chi \circ \varrho_{\mathcal{G}_k}(J_k), \qquad \bar{w}_a^\chi = w_a^\chi \circ \varrho_{\mathrm{GL}_2(E)}(J_1)$$

(see Definition 5.2 for the definition of ϱ_G). Observe that for $f \in I_k(\chi)$

$$\bar{w}_\Xi^\chi(f) = \int_{\mathrm{Her}_k} f(\mathbf{n}_k^-(z))\boldsymbol{\psi}^\Xi(z)\,\mathrm{d}z.$$

Lemma 8 of [28] now implies that

$$\bar{w}_{H_n}^{\mu[\hat{\chi}]}(h) = \int_{E^{n-2}} \int_F (\bar{w}_1^{\mu[\hat{\chi}]} \boxtimes \bar{w}_{H_{n-2}}^{\mu[\hat{\chi}]})([\Psi_n(\mu[\hat{\chi}])h](Z(x,\xi)))\,\mathrm{d}\xi\mathrm{d}x$$

for all $h \in J_n(\mu[\hat{\chi}], \mu^{-n}\hat{\omega}^{(n+1)/2})$. Put

$$I_n' = \mathbf{m}\left(\mathrm{diag}\left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{1}_{n-2} \right] \right) \in \mathcal{M}_2, \qquad J_n' = \mathfrak{i}_2(J_1, J_{n-2}) \in \mathfrak{P}_2.$$

Since $J_n = I_n' J_n'$, we arrive at

$$\bar{w}_{H_n}^\chi = \chi(-1) w_{H_n}^\chi \circ \varrho(J_n'), \qquad J_n' Z(x,\xi)(J_n')^{-1} = Y(-x,-\xi)$$

by Lemma 5.3(3). These considerations give

$$\begin{aligned} w_{H_n}^{\mu[\hat{\chi}]}(h) &= \hat{\chi}(-1) \int_{E^{n-2}} \int_F (w_1^{\mu[\hat{\chi}]} \boxtimes w_{H_{n-2}}^{\mu[\hat{\chi}]})([\Psi_n(\mu[\hat{\chi}])h](Y(x,\xi)))\,\mathrm{d}\xi\mathrm{d}x \\ &= \hat{\omega}(-1) \int_{\mathfrak{P}_2 \cap N_n \backslash N_n} (w_1^{\mu[\hat{\chi}]} \boxtimes w_{H_{n-2}}^{\mu[\hat{\chi}]})([\Psi_n(\mu[\hat{\chi}])h](u))\overline{\boldsymbol{\psi}(\ell^{H_n}(u))}\,\mathrm{d}u. \end{aligned}$$

We see by induction that

$$\begin{aligned}&(w_1^{\mu[\hat\chi]}\boxtimes w_{H_{n-2}}^{\mu[\hat\chi]})([\Psi_n(\mu[\hat\chi])h](u))\\ =&\hat\omega(-1)^{(n-3)/2}(w_1^{\mu[\hat\chi]}\boxtimes \mathfrak{J}_{H_{n-2}}^{\hat\chi}\circ\Upsilon_{n-2}(\mu[\hat\chi]))([\Psi_n(\mu[\hat\chi])h](u))\\ =&\hat\omega(-1)^{(n-3)/2}\mathfrak{J}_{H_{n-2}}^{\hat\chi}([\Upsilon_n(\mu[\hat\chi])h](u)).\end{aligned}$$

Lemma 4.5(3) now prove the first identity.

Since $w_{H_n}^{\mu[\hat\chi]}$ is nonzero on $J_n(\mu[\hat\chi],\mu^{-n}\hat\omega^{(n+1)/2})$, the image of $\Upsilon_n(\mu[\hat\chi])$ contains the unique irreducible submodule $A_n^{\hat\chi}(A(\mu,\mu^{-1}\hat\omega))$. The invariance of $\mathfrak{J}_{H_n}^{\hat\chi}$ amounts to the relevant property of $w_{H_n}^{\mu[\hat\chi]}$ stated in Lemma 5.3(3). Assume F to be $\mathfrak{p}$-adic. Since $H_n\in \mathrm{Her}_n^{\mathrm{nd}}(A_n^+(\mu[\hat\chi]))$ by Proposition 5.4, the Jacquet integral $w_{H_n}^{\mu[\hat\chi]}$ is nonzero on $A_n(\mu[\hat\chi],\mu^{-n}\hat\omega^{(n+1)/2})$ on account of Proposition 5.4(4), and so is $\Upsilon_n(\mu[\hat\chi])$. The operator $\Upsilon_n(\mu[\hat\chi])$ therefore sends the unique irreducible subrepresentation of $J_n(\mu[\hat\chi],\mu^{-n}\hat\omega^{(n+1)/2})$ bijectively onto that of $J_n^{\hat\chi}(A(\mu,\mu^{-1}\hat\omega))$. □

It is worth reminding the key simplifying feature (3.2), which explains the reason why we consider the similitude group for odd n. We construct $\mathfrak{J}_B^{\hat\chi}\in \mathrm{Sh}_B^{\hat\chi}(A_n^{\hat\chi}(A(\mu,\mu^{-1}\hat\omega)))$ for all $B\in\mathcal{O}_F(H_n)$, following Section 3.

Corollary 6.4. Assume either $-\frac12<\Re\mu<\frac12$ or $\mu^2\hat\omega^{-1}=\boldsymbol{\alpha}_F$. Then for $B\in\mathcal{O}_F(H_n)$

$$\mathfrak{J}_B^{\hat\chi}\circ\Upsilon_n(\mu[\hat\chi])=\hat\omega(-1)^{(n-1)/2}\mu((-1)^{(n-1)/2}\det B)^{-1}w_B^{\mu[\hat\chi]}.$$

Proof. Take $\xi\in F^\times$ and $A\in\mathrm{GL}_n(E)$ such that $\xi^{-1}H_n\langle A\rangle=B$. By Lemma 5.3(3) and Proposition 6.3

$$\begin{aligned}w_B^{\mu[\hat\chi]}=&\mu(\xi)^n\hat\omega(\xi)^{-(n+1)/2}(\mu^2\hat\omega^{-1})(\xi)^{-n}\,\tau\mu[\hat\chi](\det A)w_{H_n}^{\mu[\hat\chi]}\circ\varrho(\mathbf{d}(\xi)\mathbf{m}(A))\\ =&\mu(\xi)^{-n}\hat\omega(-\xi)^{(n-1)/2}\,\tau\mu[\hat\chi](\det A)\mathfrak{J}_{H_n}^{\hat\chi}\circ\Upsilon_n(\mu[\hat\chi])\circ\varrho(\mathbf{d}(\xi)\mathbf{m}(A))\\ =&\mu(\xi)^{-n}\hat\omega(-\xi)^{(n-1)/2}\,\tau\mu[\hat\chi](\det A)\hat\chi(\xi^{(1-n)/2}\det A)\mathfrak{J}_B^{\hat\chi}\circ\Upsilon_n(\mu[\hat\chi]).\end{aligned}$$

Since $\det B=(-1)^{(n-1)/2}\xi^{-n}\mathrm{N}_F^E(\det A)$, we obtain the stated identity. □

7. Shalika functionals on $A_n^{\hat\chi}(\pi)$

Let π be an irreducible admissible unitary generic representation of $\mathrm{GL}_2(F)$ whose central character is $\hat\omega$. This section verifies that the degenerate Whittaker functional $\mathfrak{J}_{H_n}^{\hat\chi}$ is a Shalika functional on $A_n^{\hat\chi}(\pi)$ with respect to $\hat\chi$. When π is not supercuspidal, this result follows from Corollary 6.4 and Lemma 5.3(3). When $E\simeq F\oplus F$, Proposition 7.2 can be proved directly from Proposition B.2(2).

Lemma 7.1. *Let $\mathcal{P}_{H_n}$ be the parabolic subgroup of GU_{H_n} stabilizing the line $\mathscr{X}_1$. If n is odd and $\mathfrak{J}_{H_{n-2}}^{\hat\chi}\in\mathrm{Sh}_{H_{n-2}}^{\hat\chi}(A_{n-2}^{\hat\chi}(\pi))$, then*

$$\mathfrak{J}_{H_n}^{\hat\chi}\circ\varrho(\mathbf{m}_{H_n}(A))=\hat\chi(\Lambda_{H_n}(A))\mathfrak{J}_{H_n}^{\hat\chi},\qquad A\in\mathcal{P}_{H_n}.$$

Proof. Denote the unipotent radical of $\mathcal{P}_{H_n}$ by N_{H_n}. By Lemma 6.1(2) $\mathbf{m}_{H_n}(N_{H_n})$ is contained in $\mathfrak{N}_2$. We may assume that $A = \mathrm{diag}[a, \xi(a^{-1})^\tau, A_0]$, where $a \in E^\times$, $A_0 \in \mathrm{GU}_{H_{n-2}}$ and $\xi = \lambda_{H_{n-2}}(A_0)$. Since

$$(\mathbf{d}(\xi)\mathbf{m}(A))^{-1} Y(y,z)\mathbf{d}(\xi)\mathbf{m}(A) = Y(aA_0^{-1}y, \xi^{-1}\mathrm{N}_F^E(a)z),$$

we can see that $\mathfrak{J}^{\hat{\chi}}_{H_n}(\varrho(\mathbf{d}(\xi)\mathbf{m}(A))f)$ is equal to

$$|\xi|_F |a|_E^{1-n} |\det A_0|_E \int_F \int_{E^{n-2}} \mathfrak{J}^{\hat{\chi}}_{H_{n-2}}(f(\mathbf{d}(\xi)\mathbf{m}(A)Y(y,z)))\,\mathrm{d}y\mathrm{d}z$$

for all $f \in A_n^{\hat{\chi}}(\pi)$ by (4.3). The integral is equal to

$$\begin{aligned}&\int_F \int_{E^{n-2}} \mathfrak{J}^{\hat{\chi}}_{H_{n-2}}(f(\mathfrak{i}_2(a\mathbf{1}_2, \mathbf{d}_{n-2}(\xi)\mathbf{m}'_2(A_0))Y(y,z)))\,\mathrm{d}y\mathrm{d}z\\ =&\hat{\chi}(a)\,{}^\tau\hat{\chi}(a)^{-1}\hat{\chi}(\Lambda_{n-2}(A_0))|\xi^{-1}\mathrm{N}_F^E(a)|_F^{n-1}\mathfrak{J}^{\hat{\chi}}_{H_n}(f)\end{aligned}$$

by Lemma 6.1(3), (6.1) and the assumption on $\mathfrak{J}^{\hat{\chi}}_{H_{n-2}}$. The proof is complete in view of $|\det A_0|_E = |\xi|_F^{n-2}$. □

If we knew that $\dim \mathrm{Wh}_{H_n}(A_n^{\hat{\chi}}(\pi)) \le 1$, then we could trivially see that degenerate Whittaker functionals on $A_n^{\hat{\chi}}(\pi)$ are necessarily Shalika functionals. However, due to the lack of the knowledge of the uniqueness, it is far from formal to show that $\mathfrak{J}^{\hat{\chi}}_{H_n}$ is a Shalika functional. We resort to global means. Let $\mathbf{E}$ be a quadratic extension of a number field $\mathbf{F}$ with adèle ring $\mathbf{A}$. When $\mathcal{F}$ is a smooth function on $\mathcal{P}_n(\mathbf{F})\backslash\mathcal{G}_n(\mathbf{A})$ and $B \in \mathrm{Her}_n(\mathbf{F})$, let

$$W_B(g,\mathcal{F}) = \int_{\mathrm{Her}_n(\mathbf{F})\backslash\mathrm{Her}_n(\mathbf{A})} \mathcal{F}(\mathbf{n}(z)g)\overline{\boldsymbol{\psi}^B(z)}\,\mathrm{d}z$$

be the Bth Fourier coefficient of $\mathcal{F}$. Note that

$$W_B(\mathbf{d}(\xi)\mathbf{m}(A)g,\mathcal{F}) = W_{\xi^{-1}B\langle A\rangle}(g,\mathcal{F}) \tag{7.1}$$

for all $B \in \mathrm{Her}_n(\mathbf{F})$, $\xi \in \mathbf{F}^\times$, $A \in \mathrm{GL}_n(\mathbf{E})$ and $g \in \mathcal{G}_n(\mathbf{A})$. Appendix C says that $\mathfrak{J}^{\hat{\chi}}_B$ appears in the local factor of the Bth Fourier coefficient of a certain residual automorphic form on $\mathcal{G}_n(\mathbb{A})$.

Proposition 7.2. *If n is odd, then $\mathfrak{J}^{\hat{\chi}}_{H_n} \in \mathrm{Sh}^{\hat{\chi}}_{H_n}(A_n^{\hat{\chi}}(\pi))$.*

Proof. We may suppose that $n \ge 3$ and π is supercuspidal in view of Proposition 6.3. By using a Poincare series we can now embed π as a local component of an irreducible cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbf{A})$ at a prime $\mathfrak{p}$ of $\mathbf{F}$ such that σ_v is not supercuspidal for all primes $v \ne \mathfrak{p}$ (cf. [14, Appendice 1]). That is, $\mathbf{F}_\mathfrak{p} \simeq F$ and $\sigma_\mathfrak{p} \simeq \pi$. Take a quadratic extension $\mathbf{E}$ of $\mathbf{F}$ so that the global base change $\sigma^{\mathbf{E}}$ remains cuspidal. We extend the central character ω of σ to a Hecke character χ of $\mathbf{E}$. Appendix C constructs a residual automorphic representation $A_n^\chi(\sigma)$ which is equivalent to $\otimes'_v A_n^{\chi_v}(\sigma_v)$.

Fix a factorizable vector $f^0 = \otimes_v f_v^0 \in A_n^\chi(\sigma)$ such that $\mathfrak{Z}_{H_n}^{\chi_v}(f_v^0) \neq 0$ for all v. The identities (7.1) and (C.1) give

$$\mathfrak{Z}_{H_n}^{\chi_\mathfrak{p}}(\varrho(\mathbf{m}_{H_n}(A))f_\mathfrak{p}) \prod_{v\neq\mathfrak{p}} \mathfrak{Z}_{H_n}^{\chi_v}(\varrho(\mathbf{m}_{H_n}(A))f_v^0) = \mathfrak{Z}_{H_n}^{\chi_\mathfrak{p}}(f_\mathfrak{p}) \prod_{v\neq\mathfrak{p}} \mathfrak{Z}_{H_n}^{\chi_v}(f_v^0)$$

for all $A \in \mathrm{GU}_{H_n}(\mathbf{F})$ and $f_\mathfrak{p} \in A_n^{\chi_\mathfrak{p}}(\pi)$. Since $\mathfrak{Z}_{H_n}^{\chi_v} \in \mathrm{Sh}_{H_n}^{\chi_v}(A_n^{\chi_v}(\sigma_v))$ for all $v \neq \mathfrak{p}$ by Proposition 6.3, the equality

$$\mathfrak{Z}_{H_n}^{\chi_\mathfrak{p}}(\varrho(\mathbf{m}_{H_n}(A))f_\mathfrak{p}) = \chi_\mathfrak{p}(\Lambda_{H_n}(A))\mathfrak{Z}_{H_n}^{\chi_\mathfrak{p}}(f_\mathfrak{p})$$

drops out. Since the subgroups $\mathrm{GU}_{H_n}(\mathbf{F})$ and $\mathcal{P}_{H_n}(F)$ generate $\mathrm{GU}_{H_n}(F)$ by the Bruhat decomposition, Lemma 7.1 concludes by induction that $\mathfrak{Z}_{H_n}^{\chi_\mathfrak{p}} \in \mathrm{Sh}_{H_n}^{\chi_\mathfrak{p}}(A_n^{\chi_\mathfrak{p}}(\pi))$. Remark 4.4(2) now says that $\mathfrak{Z}_{H_n}^{\hat{\chi}} \in \mathrm{Sh}_{H_n}^{\hat{\chi}}(A_n^{\hat{\chi}}(\pi))$ for all the extensions $\hat{\chi}$ of $\hat{\omega}$. □

8. Holomorphic cusp forms on $\mathcal{G}_n$

From now on the ground field F is a totally real number field of degree d and E is its totally imaginary quadratic extension unless otherwise stated. That is, E is a CM-field and F is its maximal real subfield. We denote by F_v the completion of F at a prime v, by $\mathfrak{S}_\infty$ the set of real primes of F, by $\mathbb{A}$ the adèles ring of F, by $F_+^\times$ the group of totally positive elements of F and by $C_F = F^\times\backslash\mathbb{A}^\times$ the idèle class group of F. We do not use $\mathfrak{p}$ to denote archimedean primes. The basic character of $F\backslash\mathbb{A}$ is defined as the product $\boldsymbol{\psi} = \prod_v \boldsymbol{\psi}_v$. We denote the adèle ring of E by $\mathbb{E}$, the idèle class group of E by C_E and the set of totally positive definite Hermitian matrices of size n over E by Her_n^+. It is worth noting that when $n = 1$,

$$\mathrm{Her}_1 = F, \qquad \mathrm{Her}_1^+ = F_+^\times, \qquad \mathfrak{H}_1 = \{Z \in \mathbb{C} \mid \Im Z > 0\}.$$

For any algebraic group $\mathcal{G}$ over F we denote its localization at a place v by $\mathcal{G}(F_v)$ or simply by $\mathcal{G}_v$, its adèlization by $\mathcal{G}(\mathbb{A})$, the direct product of all the archimedean localizations by $\mathcal{G}(\mathbb{A}_\infty)$ and the restricted direct product of all the nonarchimedean localizations by $\mathcal{G}(\mathbb{A}_\mathbf{f})$. Given another F-rational algebraic group $\mathcal{G}'$, an F-rational homomorphism φ of $\mathcal{G}$ into $\mathcal{G}'$ and an F-algebra $\mathcal{A}$, we can extend φ naturally to a homomorphism of $\mathcal{G}(\mathcal{A})$ to $\mathcal{G}'(\mathcal{A})$, which we shall denote by the same letter φ. For example, we employ N_F^E even for the map of $\mathbb{E}^\times$ into $\mathbb{A}^\times$ derived from the map $\mathrm{N}_F^E : E^\times \to F^\times$. For an adèle point $x \in \mathcal{G}(\mathbb{A})$ we denote its projections to $\mathcal{G}(\mathbb{A}_\mathbf{f})$, $\mathcal{G}(\mathbb{A}_\infty)$ and $\mathcal{G}_v$ by $x_\mathbf{f}$, x_∞ and x_v, respectively. Put $\boldsymbol{\psi}_\mathbf{f} = \prod_\mathfrak{p} \boldsymbol{\psi}_\mathfrak{p}$. For $B \in \mathrm{Her}_n(F)$ define a character $\boldsymbol{\psi}_\mathbf{f}^B : \mathrm{Her}_n(\mathbb{A}_\mathbf{f}) \to \mathbb{S}$ by $\boldsymbol{\psi}_\mathbf{f}^B(z) = \boldsymbol{\psi}_\mathbf{f}(\mathrm{tr}(Bz))$. For $\chi \in \Omega(\mathbb{E}^\times)$ we denote its restrictions to $\mathbb{E}_\mathbf{f}^\times$ and $\mathbb{E}_\infty^\times$ by $\chi_\mathbf{f}$ and χ_∞, respectively.

Fix a real place $v \in \mathfrak{S}_\infty$. Put

$$\mathcal{G}_n(F_v)^+ = \{g \in \mathcal{G}_n(F_v) \mid \lambda_n(g_v) > 0\},$$
$$\mathfrak{H}_n = \{Z \in \mathrm{M}_n(\mathbb{C}) \mid \sqrt{-1}({}^t\bar{Z} - Z) > 0\}.$$

Define the action of $\mathcal{G}_n(F_v)^+$ on $\mathfrak{H}_n$ and the automorphy factor $j(g,Z)$ on $\mathcal{G}_n(F_v)^+ \times \mathfrak{H}_n$ by

$$gZ = (\alpha Z + \beta)(\gamma Z + \delta)^{-1}, \qquad j(g,Z) = \lambda_n(g)^{-n/2} \det(\gamma Z + \delta)$$

for $Z \in \mathfrak{H}_n$ and $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathcal{G}_n(F_v)^+$ with matrices $\alpha, \beta, \gamma, \delta$ of size n over $\mathbb{C}$. For $\ell, \varkappa \in \mathbb{Z}$ and $B \in \mathrm{Her}_n^+$ we define a function $W_B^{\ell,\varkappa} : \mathcal{G}_n(F_v)^+ \to \mathbb{C}$ by

$$W_B^{\ell,\varkappa}(g) = (\det B)^{\ell/2} \mathbf{e}(\mathrm{tr}(Bg(\mathbf{i}))) \varepsilon(\det g)^{\varkappa} j(g,\mathbf{i})^{-\ell},$$

where $\mathbf{i} = \sqrt{-1}\mathbf{1}_n \in \mathfrak{H}_n$. Put $K_{n,v} = \{g \in G_n(F_v) \mid g(\mathbf{i}) = \mathbf{i}\}$. We can see that

$$W_B^{\ell,\varkappa}(\mathbf{n}(z)\mathbf{d}(\xi)\mathbf{m}(A)gk) = \frac{\mathbf{e}(\mathrm{tr}(Bz))\varepsilon(\det k)^{\varkappa}}{\varepsilon(\det A)^{\ell-2\varkappa} j(k,\mathbf{i})^{\ell}} W_{\xi^{-1}B\langle A\rangle}^{\ell,\varkappa}(g) \tag{8.1}$$

for all $z \in \mathrm{Her}_n(F_v)$, $A \in \mathrm{GL}_n(E_v)$, $0 < \xi \in F_v^\times$, $g \in \mathcal{G}_n(F_v)^+$ and $k \in K_{n,v}$.

Let $\ell = (\ell_v)_{v \in \mathfrak{S}_\infty}$ be a tuple of d integers. Given $g = (g_v) \in \mathrm{GL}_2(\mathbb{A}_\infty)^+$ and a function $\mathcal{F}$ on $\mathfrak{H}_1^d$, we define another function $\mathcal{F}|_\ell g$ on $\mathfrak{H}_1^d$ by

$$\mathcal{F}|_\ell g(Z) = \mathcal{F}(gZ) J_\ell(g,Z)^{-1}, \quad J_\ell(g,Z) = \prod_{v \in \mathfrak{S}_\infty} (\det g_v)^{-\ell_v/2} (c_v Z_v + d_v)^{\ell_v},$$

where $g_v = \begin{bmatrix} * & * \\ c_v & d_v \end{bmatrix}$. Define a maximal compact subgroup of $\mathrm{SL}_2(\mathbb{A}_\infty)$ by

$$K_\infty = \{g \in \mathrm{SL}_2(\mathbb{A}_\infty) \mid g(\sqrt{-1}, \dots, \sqrt{-1}) = (\sqrt{-1}, \dots, \sqrt{-1})\}.$$

For $\xi \in F_+^\times$ and $x \in \mathbb{R}^d$ we put $|\xi|^x = \prod_{v \in \mathfrak{S}_\infty} |\xi|_{F_v}^{x_v}$. A Hilbert cusp form $\mathcal{F}$ of weight ℓ having central character $\hat{\omega} \in \Omega(C_F)$ is a smooth function on $\mathrm{GL}_2(\mathbb{A})$ satisfying

$$\mathcal{F}(z\gamma g k) = \hat{\omega}(z) \mathcal{F}(g) J_\ell(k, (\sqrt{-1}, \dots, \sqrt{-1}))^{-1}$$

for $z \in \mathbb{A}^\times$, $\gamma \in \mathrm{GL}_2(F)$, $g \in \mathrm{GL}_2(\mathbb{A})$, $k \in K_\infty$, and having a Fourier expansion of the form

$$\mathcal{F}_\Delta(Z) = \sum_{\xi \in F_+^\times} |\xi|^{\ell/2} \mathbf{w}_\xi(\Delta, \mathcal{F}) \mathbf{e}_\infty(\xi Z)$$

for each $\Delta \in \mathrm{GL}_2(\mathbb{A}_\mathbf{f})$, where $\mathbf{w}_\xi(\mathcal{F})$ is a function on $\mathrm{GL}_2(\mathbb{A}_\mathbf{f})$ and the function $\mathcal{F}_\Delta : \mathfrak{H}_1^d \to \mathbb{C}$ is defined by

$$\mathcal{F}_\Delta|_\ell g(\sqrt{-1}, \dots, \sqrt{-1}) = \mathcal{F}(g\Delta), \qquad g \in \mathrm{GL}_2(\mathbb{A}_\infty)^+.$$

We write $\mathfrak{C}_\ell^{\hat{\omega}}$ for the space of such Hilbert cusp forms.

For tuples $\ell, \varkappa$ of d integers and $B \in \mathrm{Her}_n^+$ we define the character $\varepsilon^\varkappa \in \Omega(\mathbb{E}_\infty^\times)$ and a function $W_B^{\ell,\varkappa} : \mathcal{G}_n(\mathbb{A}_\infty)^+ \to \mathbb{C}$ by

$$\varepsilon^\varkappa(a) = \prod_v \varepsilon(a_v)^{\varkappa_v}, \qquad W_B^{\ell,\varkappa}(g) = \prod_v W_B^{\ell_v,\varkappa_v}(g_v),$$

where v runs over all real primes of F. For $g \in \mathcal{G}_n(\mathbb{A}_\infty)^+$ and a function $\mathcal{F} : \mathfrak{H}_n^d \to \mathbb{C}$ we define another function $\mathcal{F}|_\ell^\varkappa g : \mathfrak{H}_n^d \to \mathbb{C}$ by

$$\mathcal{F}|_\ell^\varkappa g(Z) = \mathcal{F}(gZ)\varepsilon^\varkappa(\det g) j_\ell(g, Z)^{-1},$$

where $j_\ell(g, Z) = \prod_{v \in \mathfrak{S}_\infty} j(g_v, Z_v)^{\ell_v}$. Put $\mathcal{G}_n(\mathbb{A})^+ = \mathcal{G}_n(\mathbb{A}_\infty)^+ \mathcal{G}_n(\mathbb{A}_{\mathbf{f}})$.

Definition 8.1. The space $\mathfrak{G}_\ell^{n,\varkappa}$ (resp. $\mathfrak{T}_\ell^{n,\varkappa}$) consists of all smooth functions $\mathcal{F}$ on $\mathcal{G}_n(\mathbb{A})$ that are left invariant under $\mathcal{G}_n(F)$ (resp. $\mathcal{P}_n(F)$) and admit Fourier expansions of the form

$$\mathcal{F}(g) = \sum_{B \in \mathrm{Her}_n^+} \mathbf{w}_B(g_{\mathbf{f}}, \mathcal{F}) W_B^{\ell,\varkappa}(g_\infty)$$

which is absolutely and uniformly convergent on any compact neighborhood of $g = g_\infty g_{\mathbf{f}} \in \mathcal{G}_n(\mathbb{A})^+$.

We call functions in the space $\mathfrak{G}_\ell^{n,\varkappa}$ Hilbert-Hermitian cusp forms on $\mathcal{G}_n(\mathbb{A})$ of weight ℓ with respect to the character $\varepsilon^\varkappa$. For each $\Delta \in \mathcal{G}_n(\mathbb{A}_{\mathbf{f}})$ we associate to $\mathcal{F} \in \mathfrak{T}_\ell^{n,\varkappa}$ a holomorphic function $\mathcal{F}_\Delta$ on $\mathfrak{H}_n^d$ by the condition $\mathcal{F}_\Delta|_\ell^\varkappa g(\mathbf{i}) = \mathcal{F}(g\Delta)$ for $g \in \mathcal{G}_n(\mathbb{A}_\infty)^+$. Since $\mathcal{G}_n(\mathbb{A}) = \mathcal{P}_n(F)\mathcal{G}_n(\mathbb{A})^+$, the function $\mathcal{F}$ is determined by the family of holomorphic functions $\{\mathcal{F}_\Delta\}$.

Remark 8.2.

(1) For the space $\mathfrak{C}_\ell^{\hat{\omega}}$ to be nonzero it is necessary that $\hat{\omega}_\infty = \prod_{v \in \mathfrak{S}_\infty} \mathrm{sgn}^{\ell_v}$, where $\mathrm{sgn}(x) = \frac{x}{\|x\|}$ for $x \in F_v^\times$. If $\hat{\chi} \in \Omega(C_E)$ is an extension of $\hat{\omega}$, then we can extend $\mathcal{F} \in \mathfrak{C}_\ell^{\hat{\omega}}$ to a function on $\mathcal{G}_1(\mathbb{A})$ in such a way that $\mathcal{F}(ag) = \hat{\chi}(a)\mathcal{F}(g)$ for $a \in \mathbb{E}^\times$, using the isomorphism given in Remark 2.1. In this way we view $\mathfrak{C}_\ell^{\hat{\omega}}$ as a subspace of $\mathfrak{G}_\ell^{1,\varkappa}$, where $\hat{\chi}_\infty = \varepsilon^{2\varkappa-\ell}$.

(2) If χ is a unitary character of C_E having trivial restriction to C_F, then $\mathfrak{G}_\ell^{n,\varkappa} \otimes \tilde{\chi} \circ \Lambda_n = \mathfrak{G}_\ell^{n,\varkappa+j}$, recalling the notation in Remark 4.4, where $j \in \mathbb{Z}^d$ is such that $\chi_\infty = \varepsilon^{2j}$.

(3) We sometimes regard the coefficients of the Fourier expansion in Definition 8.1 as functions $g_{\mathbf{f}} \mapsto \mathbf{w}_B(g_{\mathbf{f}}, \mathcal{F})$ on $\mathcal{G}_n(\mathbb{A}_{\mathbf{f}})$. It is noteworthy that $\mathcal{F} \mapsto \mathbf{w}_B(\mathcal{F})$ is an intertwining map from $\mathfrak{G}_\ell^{n,\varkappa}$ or $\mathfrak{T}_\ell^{n,\varkappa}$ to $\mathrm{Ind}_{N_n(\mathbb{A}_{\mathbf{f}})}^{\mathcal{G}_n(\mathbb{A}_{\mathbf{f}})} \boldsymbol{\psi}_{\mathbf{f}}^B$.

We fix, once and for all, a Hecke character $\hat{\omega} : C_F \to \mathbb{S}$, an auxiliary Hecke character $\hat{\chi} : C_E \to \mathbb{S}$ extending $\hat{\omega}$. Fix an irreducible summand $\pi_{\mathbf{f}}$ of $\mathfrak{C}_\ell^{\hat{\omega}}$. For each odd n and $B \in \mathrm{Her}_n^+$ Proposition 7.2 and Definition 3.2 naturally define $\mathfrak{I}_B^{\hat{\chi}_\mathfrak{p}}$ and a nonzero vector $\mathfrak{I}_B^{\hat{\chi}_{\mathbf{f}}} \in \mathrm{Sh}_B^{\hat{\chi}_{\mathbf{f}}}(A_n^{\hat{\chi}_{\mathbf{f}}}(\pi_{\mathbf{f}}))$ by $\mathfrak{I}_B^{\hat{\chi}_{\mathbf{f}}}(f) = \prod_\mathfrak{p} \mathfrak{I}_B^{\hat{\chi}_\mathfrak{p}}(f_\mathfrak{p})$ for all pure tensors $f = \otimes_\mathfrak{p} f_\mathfrak{p} \in A_n^{\hat{\chi}_{\mathbf{f}}}(\pi_{\mathbf{f}}) := \otimes_\mathfrak{p}' A^{\hat{\chi}_\mathfrak{p}}(\pi_\mathfrak{p})$, where, as the proof of Proposition 4.5 shows, almost all the factors are 1.

Theorem 1.1. *The series*

$$J_\kappa^\varkappa(g, f) = \sum_{B \in \mathrm{Her}_n^+} W_B^{\kappa+n-1,\varkappa}(g_\infty) \mathfrak{I}_B^{\hat{\chi}_{\mathbf{f}}}(\Pi_{\mathbf{f}}(g_{\mathbf{f}}) f), \quad \varkappa = \frac{1}{2}(\kappa + n - 1 + \ell(\hat{\chi}))$$

defines a $\mathcal{G}_n(\mathbb{A}_{\mathbf{f}})$-intertwining embedding $A_n^{\hat{\chi}_{\mathbf{f}}}(\pi_{\mathbf{f}}) \hookrightarrow \mathfrak{G}_{\kappa+n-1}^{n,\varkappa}$.

9. Convergence of the Fourier series

Let $\mathfrak{o}$ (resp. $\mathfrak{r}$) be the integer ring of F (resp. E), $\mathfrak{d}$ the different of $F/\mathbb{Q}$ and $\mathfrak{D}$ the discriminant of E/F. The norm and the order of a fractional ideal of $\mathfrak{o}$ are defined by $\mathfrak{N}(\mathfrak{p}^k) = [\mathfrak{o} : \mathfrak{p}]^k = q^k$ and $\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{p}^k = k$ for each prime ideal $\mathfrak{p}$ of $\mathfrak{o}$. Put

$$\mathcal{R}_n = \mathrm{Her}_n(F) \cap \mathrm{M}_n(\mathfrak{r}), \qquad \mathscr{R}_n = \{z \in \mathrm{Her}_n(F) \mid \mathrm{tr}(z\mathcal{R}_n) \subset \mathfrak{o}\}.$$

Denote the closure of $\mathcal{R}_n$ in $\mathrm{Her}_{n,\mathfrak{p}}$ by $\mathcal{R}_{n,\mathfrak{p}}$. Put $\mathscr{R}_n^{\mathrm{nd}} = \mathscr{R}_n \cap \mathrm{GL}_n(E)$.

We fix a finite prime $\mathfrak{p}$ and temporarily suppress $\mathfrak{p}$ from the notation. Thus F is an extension of $\mathbb{Q}_p$ for the moment. For every $B \in \mathrm{Her}_n^{\mathrm{nd}}$ and an irreducible admissible unitary generic representation π of $\mathrm{GL}_2(F)$ with central character $\hat{\omega}$ we can define $\mathfrak{Z}_B^{\hat{\chi}} \in \mathrm{Sh}_B^{\hat{\chi}}(A_n^{\hat{\chi}}(\pi))$, following Definition 3.2 and Proposition 7.2. We first provide a bound of $\mathfrak{Z}_B^{\hat{\chi}}$.

Lemma 9.1. *Let $f \in A_n^{\hat{\chi}}(\pi)$. For any compact subset $\mathcal{C}$ of $\mathcal{G}_n$ there are $0 \le \phi \in \mathcal{S}(\mathrm{Her}_n)$ and $M \in \mathbb{R}_+^\times$ such that for all $\Delta \in \mathcal{C}$ and $B \in \mathrm{Her}_n^{\mathrm{nd}}$*

$$\|\mathfrak{Z}_B^{\hat{\chi}}(\varrho(\Delta)f)\| \le |\det B|_F^{-M}\phi(B).$$

Proof. Since $\{\varrho(\Delta)f \mid \Delta \in \mathcal{C}\}$ is a finite set, we may suppose that $\mathcal{C} = \{\mathbf{1}_{2n}\}$. One can find a compact subset $\mathcal{L}$ of Her_n such that $\mathfrak{Z}_B^{\hat{\chi}}(f) = 0$ if $B \notin \mathcal{L}$. Therefore the claimed estimate is equivalent to saying that there are positive constants c and M which satisfy

$$\|\mathfrak{Z}_{H_n}^{\hat{\chi}}(\varrho(\mathbf{d}(\xi)\mathbf{m}(A))f)\| \le c|\xi^{-n}\mathrm{N}_F^E(\det A)|_F^{-M}$$

for all $\xi \in F^\times$ and $A \in \mathrm{GL}_n(E)$. Since $\mathbf{d}(\mathrm{N}_F^E(E^\times)) \subset \mathcal{Z}_n\mathbf{m}_n(\mathrm{GL}_n(E))$, we have only to vary A. Recall that $\mathcal{Z}_n$ stands for the center of $\mathcal{G}_n$.

For simplicity we here exclude the split case. The split case is none other than Lemma B.4. Let $\mathfrak{q}$ be the maximal ideal of $\mathfrak{r}$. The order of the residue field $\mathfrak{r}/\mathfrak{q}$ is denoted by q_E. Thanks to the Iwasawa decomposition, it suffices to let A vary over the parabolic subgroup $Q_{1,n-2,1}$ of $\mathrm{GL}_E(\mathcal{X}_n)$ defined in Lemma 6.1(2). Define a homomorphism

$$\wp_n : E^\times \times \mathrm{GL}_{n-2}(E) \to \mathbf{m}(Q_{1,n-2,1}), \quad \wp_n(a, A_0) = \mathbf{m}(\mathrm{diag}[a, 1, A_0]).$$

For each $A \in Q_{1,n-2,1}$ there are $u \in E^{n-2}$, $\eta \in E$, $a \in E^\times$ and $A_0 \in \mathrm{GL}_{n-2}(E)$ such that $\mathbf{m}(A) \in \mathbf{m}(P_{H_n})X(u,\eta)\wp_n(a, A_0)$, where $P_{H_n} = \mathrm{U}_{H_n} \cap \mathcal{P}_{H_n}$. Lemma 7.1 therefore allows us to suppose that $A = X(u,\eta)\wp_n(a, A_0)$.

As in the proof of Proposition 4.5, we write $f = \Phi \star h$. From (4.2) we get

$$\begin{aligned}&[\varrho(X(u,\eta)\wp_n(a, A_0))f](Y(y,z))\\ =&h(\wp_n(a, A_0)Y(A_0^{-1}y, z))\overline{\psi(z\mathrm{T}_F^E(\eta) + \mathrm{T}_F^E({}^t u\tau y))}\hat{\Phi}(aA_0^{-1}y, az)\end{aligned}$$

Substituting this expression into (4.3), we see by Lemma 6.1(3) that

$$\|\mathfrak{Z}_{H_n}^{\hat{\chi}}(\varrho(X(u,\eta)\wp_n(a, A_0))f)\| \le |\det A_0|_E \int_F \int_{E^{n-2}} \|\hat{\Phi}(ay, az)\|$$

$$\times \left\| \mathfrak{Z}^{\hat{\chi}}_{H_{n-2}}\left(h\left(\mathfrak{i}_2\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{m}'_2(A_0)\right)Y(y,z)\right)\right)\right\| \mathrm{d}y\mathrm{d}z.$$

Let P_1^{n-1} (resp. P_{n-1}^{n-1}) be the parabolic subgroup of of $G_{n-1} \subset M_1$ which stabilizes the subspace spanned by f_2 (resp. $f_2, e_3, e_4, \dots, e_n$). Let $N_{1,n-2}^{n-1}$ be the unipotent radical unipotent radical of $P_1^{n-1} \cap P_{n-1}^{n-1}$. Since $Y(y,z) \in G_{n-1} \subset M_1$, we can write

$$Y(y,z) = \mathfrak{i}_2\left(\begin{bmatrix} 1 & 0 \\ 0 & a_1 \end{bmatrix}, \mathbf{m}'_2(A_1)\right)u_1 k$$

by the Iwasawa decomposition for G_{n-1} relative to $P_1^{n-1} \cap P_{n-1}^{n-1}$, where

$$a_1 \in E^\times, \qquad A_1 \in \mathrm{GL}_{n-2}(E), \qquad u_1 \in N_{1,n-2}^{n-1}, \qquad k \in \Gamma_{n-1}[\mathfrak{o}].$$

For $(y,z) \in E^{n-2} \oplus F$ let $N(y,z)$ and $N(z)$ be the nonnegative integers defined by $\mathfrak{q}^{-N(y,z)} = \mathfrak{r} + z\mathfrak{r} + \sum_{j=1}^{n-2} y_j \mathfrak{r}$ and $\mathfrak{q}^{-N(z)} = \mathfrak{r} + z\mathfrak{r}$. Since

$$a_1^{-1}k^{-1}f_2 = Y(y,z)^{-1}f_2 = f_2 - ze_2 - y_1e_3 - \cdots - y_{n-2}e_n,$$

we get $|a_1|_E = q_E^{-N(y,z)}$. Since $Y(y,0) \in P_{n-1}^{n-1}$, we can infer from the Iwasawa decomposition relative to P_{n-1}^{n-1} that $|a_1 \det A_1|_E = q_E^{-N(z)}$. Thus $|\det A_1|_E = q_E^{N(y,z)-N(z)} \geq 1$. By induction there are positive integers c', M' and M'' such that

$$\begin{aligned}
&\left\| \mathfrak{Z}^{\hat{\chi}}_{H_{n-2}}\left(h\left(\mathfrak{i}_2\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{m}'_2(A_0)\right)Y(y,z)\right)\right)\right\| \\
=&\left\| \mathfrak{Z}^{\hat{\chi}}_{H_{n-2}}\left(h\left(\mathfrak{i}_2\left(\begin{bmatrix} a & 0 \\ 0 & a_1 \end{bmatrix}, \mathbf{m}'_2(A_0A_1)\right)k\right)\right)\right\| \\
\leq& c'|a_1|_E^{n-1}|aa_1^{-1}|_E^{-M''}|\det(A_0A_1)|_E^{-M'} \leq c'|a|_E^{-M''}|\det A_0|_E^{-M'},
\end{aligned}$$

where we have invoked a gauge estimate for $\mathscr{W}(\pi^E)$. Without loss of generality we may assume $\hat{\Phi}$ to be the characteristic function of $(\mathfrak{p}^{-N}\mathfrak{r})^{\oplus n-2} \oplus \mathfrak{p}^{-N}\mathfrak{r}$. Then $\int_F \int_{E^{n-2}} \|\hat{\Phi}(ay, az)\| \,\mathrm{d}y\mathrm{d}z \leq |a|_E^{1-n} q_E^{2N(n-1)}$. □

The Siegel series associated to $B \in \mathscr{R}_n^{\mathrm{nd}}$ is defined by

$$b(B,s) = \sum_{z \in \mathrm{Her}_n/\mathcal{R}_n} \Psi(-\mathrm{tr}(Bz))\nu[z]^{-s},$$

where $\nu[z] = [z\mathfrak{r}^n + \mathfrak{r}^n : \mathfrak{r}^n]^{1/2}$ and Ψ is a character of F of order zero. As is well-known, this definition is independent of the choice of Ψ. Put

$$\gamma(s) = \prod_{j=0}^{n-1} L(s-j, \epsilon_{E/F}^j)^{-1}, \qquad F(B, q^{-s}) = b(B,s)\gamma(s)^{-1}. \tag{9.1}$$

If $\mathfrak{d} = \mathfrak{o}$ and $\det B \in \mathfrak{o}^\times$, then $F(B,X) = 1$ by [17]. The Siegel series is nothing but the unramified Jacquet integral.

Lemma 9.2. *If $B \in \mathscr{R}_n^{\mathrm{nd}}$, then*

$$w_B^{\boldsymbol{\alpha}_E^s}(\varepsilon_{\mathfrak{d}}^{(2s+n)/2}) = \mathfrak{N}(\mathfrak{d})^{-2ns} |\det B|_F^{n/2} F(B, q^{-2s-n}).$$

Moreover, there is a positive constant M independent of B and F such that $\|F(B, q^{-2s-n})\| \le |\det B|_F^{-M}$ for all $B \in \mathscr{R}_n^{\mathrm{nd}}$ and $-\frac{1}{2} < \Re s < \frac{1}{2}$.

Proof. The proof of the first (resp. second) part is similar to that of [36, Lemma 4.5] (resp. [17, Lemma 2.3]). We omit detailed verifications. □

We go back to the global setting. We have taken an auxiliary Hecke character $\hat{\chi} : C_E \to \mathbb{S}$ whose restriction to C_F coincides with the fixed Hecke character $\hat{\omega}$. The tuple $\ell(\hat{\chi})$ of d integers is defined by $\hat{\chi}_\infty = \varepsilon^{\ell(\hat{\chi})}$.

Lemma 9.3. *Notation being as in Theorem 1.1, the series $J_\kappa^{\varkappa}(f)$ converges absolutely and uniformly on any compact subset of $\mathcal{G}_n(\mathbb{A})$.*

Proof. The proof is similar to the arguments in Section 4 of [16]. It suffices to show that the series

$$J_\kappa^{\varkappa}(f)_{\mathbf{1}_{2n}}(Z) = \sum_{B \in \mathrm{Her}_n^+} |\det B|^{(\kappa+n-1)/2} \mathfrak{Z}_B^{\hat{\chi}_{\mathbf{f}}}(f) \mathbf{e}_\infty(\mathrm{tr}(BZ))$$

is absolutely and uniformly convergent on any compact subset of $\mathfrak{H}_n^d$. One can find a natural number N such that $\mathfrak{Z}_B^{\hat{\chi}_{\mathbf{f}}}(f) = 0$ unless $B \in N^{-1}\mathscr{R}_n$. There is no harm in assuming that $f = \otimes_{\mathfrak{p}} f_{\mathfrak{p}}$ is factorizable. Corollary 6.4 and Lemma 9.2 give a positive constant M such that $\|\mathfrak{Z}_B^{\hat{\chi}_{\mathfrak{p}}}(f_{\mathfrak{p}})\| \le |\det B|_{F_{\mathfrak{p}}}^{-M}$ for almost all $\mathfrak{p}$. The bound of $\mathfrak{Z}_B^{\hat{\chi}_{\mathfrak{p}}}$ given in Lemma 9.1 now says that

$$\|\mathfrak{Z}_B^{\hat{\chi}_{\mathbf{f}}}(f)\| \le C' \mathrm{N}_{\mathbb{Q}}^F(\det B)^{M'}$$

for all $B \in \mathrm{Her}_n^+$ with constants C' and M' depending only on f. The inequality of arithmetic and geometric means gives

$$\mathrm{N}_{\mathbb{Q}}^F(\det B) \le (nd)^{-nd} (\mathrm{T}_{\mathbb{Q}}^F \mathrm{tr}(B))^{nd}.$$

The number of $B \in N^{-1}\mathscr{R}_n \cap \mathrm{Her}_n^+$ such that $\mathrm{T}_{\mathbb{Q}}^F \mathrm{tr}(B) \le T$ is $O(T^{dn^2})$. From these estimates the series converges absolutely and uniformly on $\{Z \in \mathfrak{H}_n^d \mid \Im Z_v > \epsilon \mathbf{1}_n \text{ for all } v \in \mathfrak{S}_\infty\}$ for any positive constant ϵ. □

10. Proof of Theorem 1.1

We begin with the inductive structure for Jacquet integrals on degenerate principal series in §10.1. It is mentioned in (1.3) and a special case of (4.1). When $\pi \simeq I(\mu, \mu^{-1}\hat{\omega})$, the techniques are substantially those of Section 7 of [18], so that we will sometimes omit details. When $\pi \simeq A(\mu, \mu^{-1}\hat{\omega})$ with $\mu^2\hat{\omega}^{-1} = \boldsymbol{\alpha}_F$, the proof is a bit more complicated than the metaplectic case due to (5.7). We here use the functional equation (5.4).

Next we will prove the general case of (4.1) by a global method which uses a certain residual automorphic representation. So as not to interrupt the flow of the section, we will construct this residual automorphic representation

in Appendix C. Since the restriction of $\hat{\chi}^{-1} \boxtimes \pi$ to G_1 may be reducible, it simplifies the proof considerably to extend the notion of Fourier-Jacobi coefficients to $\mathcal{G}_1$ as in Section 3. The global criterion stated in Proposition 10.4 combined with the local result (4.1) achieves Theorem 1.1.

10.1. **Jacquet integrals revisited.** We return to the local situation again, fix a prime v of F and suppress it from the notation. Fix $1 \le i \le n-1$. Put

$$n' = n - i, \qquad \eta_i = \left[\begin{array}{cc|c} & & -\mathbf{1}_i \\ & \mathbf{1}_{n'} & \\ \hline \mathbf{1}_i & & \\ & & \mathbf{1}_{n'} \end{array}\right] \in G_n.$$

Lemma 10.1. *Let $S \in \mathrm{Her}_i^{\mathrm{nd}}$. Put $n' = n - i$. Given $h \in I_n(\chi)$ and $\phi \in \mathcal{S}(X_i^n)$, we define the function $\beta_S^{\hat{\epsilon}}(g'; h \otimes \bar{\phi}) : G_{n'} \to \mathbb{C}$, for $\Re\chi \gg 0$, by the integral*

$$\int_{Y_i^n \backslash N_i^n} h(\eta_i v g') \overline{[\omega_S^{\hat{\epsilon}}(vg')\phi](0)}\, \mathrm{d}v \prod_{j=1}^{i} L(n'+j, \chi^\dagger \epsilon_{E/F}^{i+j}).$$

This function is meaningful for all χ and gives an N_i^n-invariant and $G_{n'}$-intertwining map $\beta_S^{\hat{\epsilon}} : I_n(\chi) \otimes \overline{\omega_S^{\hat{\epsilon}}} \to I_{n'}(\chi\hat{\epsilon}^{-1})$. Moreover, there is a nonzero constant C_S such that for all $\Xi \in \mathrm{Her}_{n'}^{\mathrm{nd}}$

$$w_\Xi^{\chi\hat{\epsilon}^{-1}}(\beta_S^{\hat{\epsilon}}(h \otimes \bar{\phi})) = C_S |\det \Xi|_F^{-i/2} \int_{X_i^n} \overline{\phi(x)} w_{S \oplus \Xi}^{\chi}(\varrho(x)h)\, \mathrm{d}x.$$

Proof. One can prove Lemma 10.1 in the same way as in the proof of Lemmas 7.1 and 7.2 of [18]. □

Corollary 10.2. If F is nonarchimedean and $-\frac{1}{2} \le \Re\chi < \frac{1}{2}$, then

$$\beta_S^{\hat{\epsilon}}(I_n(\chi) \otimes \overline{\omega_S^{\hat{\epsilon}}}) = I_{n'}(\chi\hat{\epsilon}^{-1}).$$

Proof. Fix $\Xi \in \mathrm{Her}_{n'}^{\mathrm{nd}}$. Lemma 5.3(1) enables us to take $h \in I_n(\chi)$ so that $w_{S\oplus\Xi}^{\chi}(h) \neq 0$. Lemma 10.1 shows that if we choose ϕ to be supported in a small neighborhood, then $w_\Xi^{\chi\hat{\epsilon}^{-1}}(\beta_S^{\hat{\epsilon}}(h \otimes \bar{\phi})) \neq 0$. In particular, $\beta_S^{\hat{\epsilon}}(h \otimes \bar{\phi}) \neq 0$. Thus $\beta_S^{\hat{\epsilon}}$ is surjective, provided that $I_{n'}(\chi\hat{\epsilon}^{-1})$ irreducible. By Proposition 5.4(1) we may assume that $\chi^\dagger = \epsilon_{E/F}^{n-1}\boldsymbol{\alpha}_F^{-1}$ or $E \not\simeq F \oplus F$ and $\chi^\dagger = \epsilon_{E/F}^n$.

When $\chi^\dagger = \epsilon_{E/F}^{n-1}\boldsymbol{\alpha}_F^{-1}$, since $w_\Xi^{\chi\hat{\epsilon}^{-1}}$ kills the maximal proper subrepresentation of $I_{n'}(\chi\hat{\epsilon}^{-1})$ in view of (5.6), the vector $\beta_S^{\hat{\epsilon}}(h \otimes \bar{\phi})$ generates $I_{n'}(\chi\hat{\epsilon}^{-1})$. We discuss the latter case. Since $A_{n'}^{\pm}(\chi\hat{\epsilon}^{-1})$ are irreducible and satisfies

$$\mathrm{Her}_{n'}^{\mathrm{nd}}(A_{n'}^{\pm}(\chi\hat{\epsilon}^{-1})) = \{\Xi \in \mathrm{Her}_{n'}^{\mathrm{nd}} \mid \epsilon(\Xi) = \pm 1\}$$

by Proposition 5.4(2), we see that

$$\beta_S^{\hat{\epsilon}}(A_n^{\pm}(\chi) \otimes \overline{\omega_S^{\hat{\epsilon}}}) = A_{n'}^{\pm\varepsilon_{n,i,n'}\epsilon(S)}(\chi\hat{\epsilon}^{-1}),$$

where $\varepsilon_{n,i,n'} \in \{\pm 1\}$ is independent of S. Hence $\beta_S^{\hat\epsilon}(I_n(\chi)\otimes\overline{\omega_S^{\hat\epsilon}}) = A_{n'}^+(\chi\hat\epsilon^{-1})\oplus A_{n'}^-(\chi\hat\epsilon^{-1}) = I_{n'}(\chi\hat\epsilon^{-1})$ as claimed. □

We let $i = n-1$ and suppose that n is odd for the rest of this section. For simplicity we exclude the case in which $F = \mathbb{R}$. Since $\eta_{n-1}\mathbf{m}_S(A) = \mathbf{d}(\xi)\mathbf{m}({}^t(A^{-1})^\tau)\eta_{n-1}$, a simple computation shows that

$$\begin{aligned}\beta_S^{\hat\epsilon}(\varrho(\mathbf{m}_S(A))h\otimes\overline{{}^\theta L(A)\phi}) &= \overline{\theta(\Lambda_S(A))}|\xi|_F^{-(n-1)/2}|\det A|_E\\ &\quad\times\mu(\xi)\chi(\det({}^t(A^{-1})^\tau))|\xi^{-n}\mathrm{N}_F^E(\det({}^t(A^{-1})^\tau))|_F^{n/2}\beta_S^{\hat\epsilon}(h\otimes\bar\phi)\end{aligned}$$

for all $h \in J_n(\chi,\mu)$ and $A \in \mathrm{GU}_S$ with $\xi = \lambda_S(A)$. Observing that

$$\begin{aligned}&\overline{\theta(\Lambda_S(A))} = {}^\tau\theta(\Lambda_S(A)),\quad \chi(\det({}^t(A^{-1})^\tau)) = \chi(\Lambda_S(A))\chi(\xi)^{(n-1)/2},\\ &|\xi|_F^{-(n-1)/2}|\det A|_E|\xi^{-n}\mathrm{N}_F^E(\det({}^t(A^{-1})^\tau))|_F^{n/2} = |\xi|_F^{-1/2},\end{aligned}$$

we get

$$\beta_S^{\hat\epsilon}(\varrho(\mathbf{m}_S(A))h\otimes\overline{{}^\theta L(A)\phi}) = ({}^\tau\chi\theta)(\Lambda_S(A))\mu(\xi)\chi(\xi)^{(n-1)/2}|\xi|_F^{-1/2}\beta_S^{\hat\epsilon}(h\otimes\bar\phi).$$

In particular, the linear map $\beta_S^{\hat\epsilon} : J_n(\mu[\hat\chi],\mu^{-n}\hat\omega^{(n+1)/2})\otimes\overline{{}^{\hat\chi}\Omega_S} \to \mathbb{C}$ is U_S-invariant. We therefore define a function $\beta_S(h\otimes\bar\phi) : \mathrm{GL}_2(F)\to\mathbb{C}$ by

$$\beta_S(g; h\otimes\bar\phi) = \beta_S^{\hat\epsilon}(\varrho(\iota'(A,g))h\otimes\overline{{}^{\hat\chi}\Omega_S(\iota'(A,g))\phi}),$$

where the right hand side does not depend on the choice of $\hat\epsilon$, $\hat\chi$ and $A \in \mathrm{GU}_S$ with $\lambda_S(A) = \det g$. Moreover, if we put $\xi = \lambda_S(A)$, then

$$\beta_S(\mathbf{d}_1(\xi); h\otimes\bar\phi) = \beta_S^{\hat\epsilon}(\varrho(\mathbf{m}_S(A))h\otimes\overline{{}^{\hat\chi}L(A)\phi}) = \mu(\xi)^{-1}\hat\omega(\xi)|\xi|_F^{-1/2}\,\beta_S(h\otimes\bar\phi).$$

It therefore follows that $\beta_S(h\otimes\bar\phi)\in I(\mu,\mu^{-1}\hat\omega)$.

Lemma 10.3. *Suppose either $-\frac12 < \Re\mu < \frac12$ or $\mu^2\hat\omega^{-1} = \boldsymbol{\alpha}_F$. Then*

$$\Gamma_1^S\langle\mathfrak{Z}_{H_n}^{\hat\chi}\rangle(f\otimes\bar\phi)\in\mathscr{W}(A(\mu,\mu^{-1}\hat\omega))$$

for all odd n, $S\in\mathrm{Her}_{n-1}^{\mathrm{nd}}$, $f\in A_n^{\hat\chi}(A(\mu,\mu^{-1}\hat\omega))$ and $\phi\in{}^{\hat\chi}\Omega_S$.

Proof. Corollary 5.5(3) and Proposition 6.3 tell us that

$$A_n^{\hat\chi}(A(\mu,\mu^{-1}\hat\omega)) = \Upsilon(\mu[\hat\chi])\circ M_n((\hat\omega\mu^{-1})[\hat\chi])(J_n((\hat\omega\mu^{-1})[\hat\chi],\mu^n\hat\omega^{(1-n)/2})).$$

Let $h'\in J_n((\hat\omega\mu^{-1})[\hat\chi],\mu^n\hat\omega^{(1-n)/2})$. Put

$$h = M_n((\hat\omega\mu^{-1})[\hat\chi])h',\qquad\qquad f = \Upsilon(\mu[\hat\chi])h.$$

Then

$$\Gamma_1^S\langle\mathfrak{Z}_{H_n}^{\hat\chi}\rangle(g; f\otimes\bar\phi) = \mu(\det S)^{-1}\int_{X'} w_{S\oplus1}^{\mu[\hat\chi]}(\varrho(x\iota'(A,g))h)\overline{[{}^{\hat\chi}\Omega_S(\iota'(A,g))\phi](x)}\,\mathrm{d}x$$

for $(A,g)\in\mathcal{R}_S$ by Corollary 6.4. The right hand side is equal to

$$e_n(\mu[\hat\chi])(\mu\hat\omega^{-1})(\det S)\int_{X'}\mathbf{w}_{S\oplus1}^{(\hat\omega\mu^{-1})[\hat\chi]}(\varrho(x\iota'(A,g))h')\overline{[{}^{\hat\chi}\Omega_S(\iota'(A,g))\phi](x)}\,\mathrm{d}x$$

by (5.4). Next we exploit Lemma 10.1 to see that

$$\Gamma_1^S\langle \mathfrak{J}_{H_n}^{\hat{\chi}}\rangle(g; f\otimes\bar{\phi}) = e_n'(\mu)\mathbf{w}_1^{(\hat{\omega}\mu^{-1})[\hat{\chi}]}(\varrho(g)\beta_S(h'\otimes\bar{\phi})),$$

where $e_n'(\mu)$ is a meromorphic function on $\Omega(C_F)$, which is holomorphic and nonzero for $\Re\mu > -\frac{1}{2}$. We have seen that $\beta_S(h'\otimes\bar{\phi}) \in I(\hat{\omega}\mu^{-1},\mu)$. Therefore $M_1(\hat{\omega}\mu^{-1})\beta_S(h'\otimes\bar{\phi}) \in A(\mu,\mu^{-1}\hat{\omega})$. We finally get

$$\Gamma_1^S\langle \mathfrak{J}_{H_n}^{\hat{\chi}}\rangle(g; f\otimes\bar{\phi}) = e_n'(\mu)e_1(\mu[\hat{\chi}])^{-1}w_1^{\mu[\hat{\chi}]}(\varrho(g)M_1(\hat{\omega}\mu^{-1})\beta_S(h'\otimes\bar{\phi}))$$

by using (5.4) again. □

10.2. **Fourier-Jacobi coefficients of automorphic forms.** Fix $S\in \mathrm{Her}_i^{\mathrm{nd}}$. Put $n'=n-i$. Recall the Schrödinger model of the Weil representation $\omega_S^{\hat{\epsilon}} \simeq \otimes_v'\omega_S^{\hat{\epsilon}_v}$ realized on $\mathcal{S}(X_i^n(\mathbb{A}))$. We associate to $\varphi\in\mathcal{S}(X_i^n(\mathbb{A}))$ the theta function on $G_{n'}(\mathbb{A})\ltimes N_i^n(\mathbb{A})$ given by

$$\Theta(\omega_S^{\hat{\epsilon}}(v)\varphi) = \sum_{x\in X_i^n(F)}[\omega_S^{\hat{\epsilon}}(v)\varphi](x).$$

The Bth Fourier coefficient $W_B(\mathcal{F})$ of a smooth function $\mathcal{F}$ on $\mathcal{P}_n(F)\backslash\mathcal{G}_n(\mathbb{A})$ is defined in Section 7. The (S,φ)th Fourier-Jacobi coefficient of $\mathcal{F}$ is a function on $G_{n'}(\mathbb{A})$ defined by

$$\mathcal{F}_\varphi^S(g') = \int_{N_i^n(F)\backslash N_i^n(\mathbb{A})}\mathcal{F}(vg')\overline{\Theta(\omega_S^{\hat{\epsilon}}(vg')\varphi)}\,\mathrm{d}v.$$

For $\phi\in\mathcal{S}(X_i^n(\mathbb{A}_{\mathbf{f}}))$ we define $\phi_S\in\mathcal{S}(X_i^n(\mathbb{A}))$ by

$$\phi_S(x) = \phi(x_{\mathbf{f}})\varphi_S^\infty(x_\infty),\quad \varphi_S^\infty(x_\infty) = \prod_{v\in\mathfrak{S}_\infty}\varphi_S(x_v),\quad x=(x_v)\in X_i^n(\mathbb{A}).$$

We will denote the action of $\mathcal{G}_n(\mathbb{A}_{\mathbf{f}})$ on $\mathfrak{T}_{\ell+n}^n$ by ρ. The following result is proved in [18, Lemma 7.7] and will be applied with $i=n-1$.

Proposition 10.4. *If $\mathcal{F}$ is a smooth function on $\mathcal{P}_n(F)\backslash\mathcal{G}_n(\mathbb{A})$, then*

$$\mathcal{F}_\varphi^S(g') = \sum_{\Xi\in\mathrm{Her}_{n'}(F)}\int_{X_i^n(\mathbb{A})}W_{S\oplus\Xi}(xg',\mathcal{F})\overline{[\omega_S^{\hat{\epsilon}}(g')\varphi](x)}\,\mathrm{d}x.$$

Put $n'=n-i$ and $\varkappa'=\varkappa-\frac{1}{2}(\ell(\hat{\epsilon})+i)$. Let

$$\mathcal{F}(g) = \sum_{B\in\mathrm{Her}_n^+}\mathbf{w}_B(g_{\mathbf{f}},\mathcal{F})W_B^{\ell,\varkappa}(g_\infty)$$

be the Fourier expansion of $\mathcal{F}\in\mathfrak{T}_\ell^{n,\varkappa}$. Then $\mathcal{F}_{\phi_S}^S(g')$ is equal to

$$\sum_{\Xi\in\mathrm{Her}_{n'}^+}\mathrm{N}_{\mathbb{Q}}^F(\det\Xi)^{i/2}W_\Xi^{\ell-i,\varkappa'}(g_\infty')\int_{X_i^n(\mathbb{A}_{\mathbf{f}})}\mathbf{w}_{S\oplus\Xi}(xg_{\mathbf{f}}',\mathcal{F})\overline{[\omega_S^{\hat{\epsilon}_{\mathbf{f}}}(g_{\mathbf{f}}')\phi](x)}\,\mathrm{d}x$$

for $\phi\in\mathcal{S}(X_i^n(\mathbb{A}_{\mathbf{f}}))$ up to a nonzero constant. Moreover, $\mathcal{F}\in\mathfrak{G}_\ell^{n,\varkappa}$ if and only if $(\rho(\Delta)\mathcal{F})_{\phi_S}^S$ is left invariant under $G_{n'}(F)$ for all $\Delta\in\mathcal{G}_n(\mathbb{A}_{\mathbf{f}})$, $S\in\mathrm{Her}_i^+$ and $\phi\in\mathcal{S}(X_i^n(\mathbb{A}_{\mathbf{f}}))$.

10.3. **End of the proof.** We again switch to the local observation, which is the final ingredient required for the proof of Theorem 1.1.

Lemma 10.5. *If n is odd and $\pi_{\mathfrak{p}}$ is an irreducible admissible unitary generic representation of $\mathrm{GL}_2(F_{\mathfrak{p}})$ whose central character is $\hat{\omega}_{\mathfrak{p}}$, then*

$$\Gamma_1^S\langle \mathfrak{Z}_{H_n}^{\hat{\chi}_{\mathfrak{p}}}\rangle(f\otimes\bar{\phi})\in\mathscr{W}(\pi_{\mathfrak{p}})\qquad (S\in \mathrm{Her}_{n-1}^{\mathrm{nd}},\ f\in A_n^{\hat{\chi}_{\mathfrak{p}}}(\pi_{\mathfrak{p}}),\ \phi\in {}^{\hat{\chi}_{\mathfrak{p}}}\Omega_S).$$

Proof. We may assume that $\pi_{\mathfrak{p}}$ is supercuspidal in view of Lemma 10.3. We take an auxiliary supercuspidal representation π' and embed $\pi_{\mathfrak{p}}$ and π' as local components of an irreducible cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbf{A})$ at primes $\mathfrak{p}$ and $\mathfrak{p}'$ of a totally complex number field $\mathbf{F}$. We can find a quadratic extension $\mathbf{E}$ of $\mathbf{F}$ which splits at $\mathfrak{p}'$ and such that the base change $\sigma^{\mathbf{E}}$ remains cuspidal. Extends the central character of σ to a Hecke character χ of $\mathbf{E}$. We will construct a residual automorphic representation $A_n^{\chi}(\sigma)$ which is equivalent to $\otimes_v' A_n^{\chi_v}(\sigma_v)$ in Appendix C. We may suppose that S is defined with respect to $\mathbf{E}/\mathbf{F}$.

For $\mathcal{F}\in A_n^{\chi}(\sigma)$ and $\varphi\in {}^{\chi}\Omega_S$ we define a function on $\mathcal{F}_{\varphi}^S$ on $\mathcal{R}_S(\mathbf{A})$ by

$$\mathcal{F}_{\varphi}^S(g')=\int_{N'(\mathbf{F})\backslash N'(\mathbf{A})}\mathcal{F}(vg')\overline{\Theta({}^{\chi}\Omega_S(vg')\varphi)}\,\mathrm{d}v.$$

Clearly, $\Theta({}^{\chi}L(A)\varphi)=\Theta(\varphi)$ for all $A\in \mathrm{GU}_S(\mathbf{F})$. Thus the theta distribution Θ is invariant under $\mathcal{R}_S(\mathbf{F})\ltimes N'(\mathbf{F})$ and hence $\mathcal{F}_{\varphi}^S$ is left invariant under $\mathcal{R}_S(\mathbf{F})$. We see by (C.1) and Proposition 10.4 that

$$\begin{aligned}\mathcal{F}_{\varphi}^S(A,g)=&\sum_{\xi\in\mathbf{F}^{\times}}\Gamma_{\xi}^S\langle\mathfrak{Z}_{H_n}^{\chi}\rangle(g;\mathscr{W}_n(\mathcal{F}_{\mathfrak{P}_{\mathbf{e}}})\otimes\bar{\varphi})\\&+\int_{X'(\mathbf{A})}W_{S\oplus 0}(x\iota'(A,g),\mathcal{F})\overline{[{}^{\chi}\Omega_S(A,g)\varphi](x)}\,\mathrm{d}x,\end{aligned}$$

where we define the function $\Gamma_{\xi}^S\langle\mathfrak{Z}_{H_n}^{\chi}\rangle(\mathscr{W}_n(\mathcal{F}_{\mathfrak{P}_{\mathbf{e}}})\otimes\bar{\varphi})$ on $\mathrm{GL}_2(\mathbf{A})$ by setting

$$\Gamma_{\xi}^S\langle\mathfrak{Z}_{H_n}^{\chi}\rangle(g;\mathscr{W}_n(\mathcal{F}_{\mathfrak{P}_{\mathbf{e}}})\otimes\bar{\varphi})=\prod_v\Gamma_{\xi}^S\langle\mathfrak{Z}_{H_n}^{\chi_v}\rangle(g_v;f_v\otimes\bar{\varphi}_v)$$

if $\mathscr{W}_n(\mathcal{F}_{\mathfrak{P}_{\mathbf{e}}})=\otimes_v f_v$ and $\varphi=\otimes_v\varphi_v$ are factorizable.

Since $J_{N_n(\mathbf{F}_{\mathfrak{p}'})}^{\psi_{\mathfrak{p}'}\circ\ell^{S\oplus 0}}(A_n^{\chi_{\mathfrak{p}'}}(\pi'))$ is zero by by Remark B.1 and Proposition B.2(1), Remark 8.2(3) gives $W_{S\oplus 0}(f)=0$ for all $f\in A_n^{\chi}(\sigma)$. It follows that $\mathcal{F}_{\phi}^S(A,g)$ is independent of A and defines a cusp form on $\mathrm{GL}_2(\mathbf{A})$. Since $S\oplus 1\in\mathcal{O}_{\mathbf{F}}(H_n)$ for the choice of $\mathbf{F}$, these cusp forms generate a nonzero cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$, which we denote by σ'. As we have seen in Lemma 10.3, the restriction of σ' to $\mathrm{GL}_2(\mathbf{F}_v)$ is a multiple of σ_v for almost all v, and so by the strong multiplicity one theorem, $\sigma'\simeq\sigma$ and $\Gamma_1^S\langle\mathfrak{Z}_{H_n}^{\chi}\rangle(\mathscr{W}_n(\mathcal{F}_{\mathfrak{P}_{\mathbf{e}}})\otimes\bar{\varphi})$ is a global Whittaker function of σ. Hence $\Gamma_1^S\langle\mathfrak{Z}_{H_n}^{\chi_v}\rangle(f_v\otimes\bar{\varphi}_v)\in\mathscr{W}(\sigma_v)$. □

We are now ready to prove Theorem 1.1. The series $J_\kappa^\varkappa(f)$ is left invariant under $\mathcal{P}_n(F)\cap\mathcal{G}(\mathbb{A})^+$ by (1.2), (8.1) and the choice of $\varkappa$. Since $\mathcal{G}_n(\mathbb{A})=\mathcal{P}_n(F)\mathcal{G}_n(\mathbb{A})^+$, it has a unique extension to a left $\mathcal{P}_n(F)$-invariant function on $\mathcal{G}_n(\mathbb{A})$. Since $\mathfrak{I}_B^{\hat\chi_{\mathbf{f}}}$ is nonzero for every $B\in\mathrm{Her}_n^+$ by Propositions 4.5(2) and 6.3, the map $J_\kappa^\varkappa$ is nonzero and hence injective as $A_n^{\hat\chi_{\mathbf{f}}}(\pi_{\mathbf{f}})$ is irreducible. Thus $J_\kappa^\varkappa$ is a $\mathcal{G}_n(\mathbb{A}_{\mathbf{f}})$-intertwining embedding $A_n^{\hat\chi_{\mathbf{f}}}(\pi_{\mathbf{f}})\hookrightarrow\mathfrak{T}_{\kappa+n-1}^{n,\varkappa}$. The essential point is to show that $J_\kappa^\varkappa(A_n^{\hat\chi_{\mathbf{f}}}(\pi_{\mathbf{f}}))$ is contained in $\mathfrak{G}_{\kappa+n-1}^{n,\varkappa}$

Let $S\in\mathrm{Her}_{n-1}^+$ and $\phi\in\mathcal{S}(X'(\mathbb{A}_{\mathbf{f}}))$. We extend the Fourier-Jacobi coefficient $J_\kappa^\varkappa(f)_{\phi_S}^S$ to a function on $R_S(\mathbb{A})$ by

$$J_\kappa^\varkappa(f)_{\phi_S}^S(g')=\int_{N'(F)\backslash N'(\mathbb{A})}J_\kappa^\varkappa(f)(vg')\overline{\Theta(\hat\chi\omega_S^1(vg')\phi_S)}\,\mathrm{d}v$$

for $g'\in R_S(\mathbb{A})$, where we let $\hat\epsilon$ be the trivial character of C_E. We also define a function on $\mathcal{R}_S(\mathbb{A})$ by

$$\mathcal{F}(g')=\int_{N'(F)\backslash N'(\mathbb{A})}J_\kappa^\varkappa(f)(vg')\overline{\Theta(\hat\chi\Omega_S(vg')\phi_S)}\,\mathrm{d}v$$

Remark 2.1 implies that

$$R_S(\mathbb{A})=\mathcal{Z}_n(\mathbb{A})\mathcal{R}_S(\mathbb{A}).$$

The central character of $J_\kappa^\varkappa(f)$ is $\hat\chi^n(\hat\omega^E)^{(1-n)/2}$ while $\mathcal{Z}_n(\mathbb{A})$ acts on $\hat\chi\omega_S^1$ by the character $\hat\chi^{n-1}(\hat\omega^E)^{(1-n)/2}$. Therefore $J_\kappa^\varkappa(f)_{\phi_S}^S$ is an extension of $\mathcal{F}$ by the character $\hat\chi$. Put $\varkappa'=\varkappa-\frac{n-1}{2}=\frac{\kappa+\ell(\hat\chi)}{2}$ (cf. Remark 8.2(1)).

Proposition 10.4 gives a nonzero constant C_S such that

$$\mathcal{F}(A,g)=C_S\sum_{\xi\in F_+^\times}W_\xi^{\kappa,\varkappa'}(g_\infty)\Gamma_\xi^S\langle\mathfrak{I}_{H_n}^{\hat\chi_{\mathbf{f}}}\rangle(g_{\mathbf{f}};f\otimes\bar\phi).$$

In particular, $\mathcal{F}$ factors through the homomorphism

$$\mathcal{R}_S(\mathbb{A})\twoheadrightarrow\mathrm{GL}_2(\mathbb{A}_\infty)^+\mathrm{GL}_2(\mathbb{A}_{\mathbf{f}}).$$

We can rewrite the equality as

$$\mathcal{F}(g)=C_S\sum_{\xi\in F_+^\times}W_1^{\kappa,\varkappa'}(\mathfrak{m}(1,\xi)g_\infty)\Gamma_1^S\langle\mathfrak{I}_{H_n}^{\hat\chi_{\mathbf{f}}}\rangle(\mathfrak{m}(1,\xi)g_{\mathbf{f}};f\otimes\bar\phi)$$

by (8.1) and Proposition 3.3. Since $\Gamma_1^S\langle\mathfrak{I}_{H_n}^{\hat\chi_{\mathbf{f}}}\rangle(f\otimes\bar\phi)$ is a $\boldsymbol{\psi}_{\mathbf{f}}$-Whittaker function of $\pi_{\mathbf{f}}$ by Lemma 10.5, the series $\mathcal{F}$ belongs to $\mathfrak{C}_\kappa^{\hat\omega}$ as π is automorphic and cuspidal. It therefore follows that $J_\kappa^\varkappa(f)_{\phi_S}^S$ is left invariant under $G_1(F)$. Proposition 10.4 eventually proves that $J_\kappa^\varkappa(f)\in\mathfrak{G}_{\kappa+n-1}^{n,\varkappa}$.

11. Translation to classical language

We shall translate obtained results into more classical terminology. For an ideal $\mathfrak{a}$ of $\mathfrak{o}$ we put $\mathfrak{a}_\mathfrak{p} = \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}\,\mathfrak{a}}$. Take a finite set $\mathfrak{S}$ of prime ideals of $\mathfrak{o}$. Put

$$\mathcal{K} = \prod_{\mathfrak{p}\in\mathfrak{S}} \Gamma_n[\mathfrak{d}_\mathfrak{p}^{-1}, \mathfrak{d}_\mathfrak{p}\mathfrak{p}] \times \prod_{\mathfrak{p}\notin\mathfrak{S}} \Gamma_n[\mathfrak{d}_\mathfrak{p}^{-1}, \mathfrak{d}_\mathfrak{p}].$$

See Section 5 for the definition of the open compact subgroup $\Gamma_n[\mathfrak{d}_\mathfrak{p}^{-1}, \mathfrak{d}_\mathfrak{p}\mathfrak{p}]$ of $\mathcal{G}_n(F_\mathfrak{p})$. We will let n be odd and construct Hilbert-Hermitian cuspidal Hecke eigenforms by making Corollary 1.2 explicit with the test function f invariant under $\mathcal{K}$. We define a function $\hat{\varepsilon}_\mathfrak{p} : \mathcal{G}_n(F_\mathfrak{p}) \to \mathbb{C}$ as follows: If $g \notin \mathcal{P}_n(F_\mathfrak{p}) J_n \Gamma_n[\mathfrak{d}_\mathfrak{p}^{-1}, \mathfrak{d}_\mathfrak{p}\mathfrak{p}]$, then $\hat{\varepsilon}_\mathfrak{p}(g) = 0$. If $g = \mathbf{d}(\lambda)\mathbf{m}(A)\mathbf{n}(z)J_n k$ with $\lambda \in F_\mathfrak{p}^\times$, $A \in \mathrm{GL}_n(E_\mathfrak{p})$, $z \in \mathrm{Her}_n(F_\mathfrak{p})$ and $k \in \Gamma_n[\mathfrak{d}_\mathfrak{p}^{-1}, \mathfrak{d}_\mathfrak{p}\mathfrak{p}]$, then $\hat{\varepsilon}_\mathfrak{p}(g) = |\lambda|_{F_\mathfrak{p}}^{-n} |\det A|_{E_\mathfrak{p}}$. Note that $\hat{\varepsilon}_\mathfrak{p}^{(2s+n)/2} \in J_n(\boldsymbol{\alpha}_{E_\mathfrak{p}}^s, \boldsymbol{\alpha}_{F_\mathfrak{p}}^{-ns})$.

The following result can be proved easily (cf. Lemma 18.13 of [33]).

Lemma 11.1. *Let $A \in \mathrm{GL}_n(E_\mathfrak{p})$ and $z \in \mathrm{Her}_n(F_\mathfrak{p})$. Then*

$$\hat{\varepsilon}_\mathfrak{p}(J_n\mathbf{n}(z)\mathbf{d}(\lambda)\mathbf{m}(A)) = \begin{cases} |\lambda|_{F_\mathfrak{p}}^n |\det A|_{E_\mathfrak{p}}^{-1} & \text{if } \lambda A^{-1} z ({}^t A^\tau)^{-1} \in \mathrm{M}_n(\mathfrak{d}_\mathfrak{p}^{-1}), \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\mathcal{R}_{n,\mathfrak{p}} = \mathrm{Her}_n(F_\mathfrak{p}) \cap \mathrm{M}_n(\mathfrak{r}_\mathfrak{p})$.

Lemma 11.2. *Take $\chi = \boldsymbol{\alpha}_{E_\mathfrak{p}}^{s_\mathfrak{p}}$ so that $\chi^\dagger = \boldsymbol{\alpha}_{F_\mathfrak{p}}$, i.e., $s_\mathfrak{p} - \frac{1}{2} \in \frac{\pi\sqrt{-1}}{\log q_\mathfrak{p}}\mathbb{Z}$. Put*

$$\hat{h}_\mathfrak{p} = M_n(\chi^{-1}, \boldsymbol{\alpha}_{F_\mathfrak{p}}^{ns_\mathfrak{p}}) \hat{\varepsilon}_\mathfrak{p}^{(-2s_\mathfrak{p}+n)/2} \in J_n(\chi, \boldsymbol{\alpha}_{F_\mathfrak{p}}^{-ns_\mathfrak{p}}).$$

Let n be an odd natural number.

(1) $\hat{h}_\mathfrak{p} \in A_n(\chi, \boldsymbol{\alpha}_{F_\mathfrak{p}}^{-ns_\mathfrak{p}})$.

(2) *Let $B \in \mathrm{Her}_n^{\mathrm{nd}}(F_\mathfrak{p})$ and $A \in \mathrm{GL}_n(E_\mathfrak{p})$. Then*

$$\frac{w_B^\chi(\wp(\mathbf{d}(\lambda)\mathbf{m}(A))\hat{h}_\mathfrak{p})}{\mathfrak{N}(\mathfrak{d}_\mathfrak{p})^{n^2} |\det B|_{F_\mathfrak{p}}^{s_\mathfrak{p}}} = \begin{cases} e_n(\chi)|\lambda^{-n} \det B\langle A\rangle|_{F_\mathfrak{p}}^{(2s_\mathfrak{p}+n)/2} & \text{if } B\langle A\rangle \in \mathcal{R}_{n,\mathfrak{p}}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Corollary 5.5(3) proves (1). Lemma 11.1 gives

$$\mathbf{w}_B^{\boldsymbol{\alpha}_{E_\mathfrak{p}}^s}(\wp(\mathbf{d}(\lambda)\mathbf{m}(A))\hat{\varepsilon}_\mathfrak{p}^s) = \begin{cases} \mathfrak{N}(\mathfrak{d}_\mathfrak{p})^{n^2} |\lambda|_{F_\mathfrak{p}}^{ns} |\det A|_{E_\mathfrak{p}}^{n-s} & \text{if } \lambda^{-1} B\langle A\rangle \in \mathcal{R}_{n,\mathfrak{p}}, \\ 0 & \text{otherwise.} \end{cases}$$

The second part follows from (5.4). □

Take elements $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{A}_\mathbf{f}^\times$ and $A_1, A_2, \ldots, A_r \in \mathrm{GL}_n(\mathbb{A}_\mathbf{f})$ so that $\{\Delta_1, \Delta_2, \ldots, \Delta_r\}$ is a complete set of representatives for the double coset $\mathcal{G}_n(F)\backslash\mathcal{G}_n(\mathbb{A})/\mathcal{G}_n(\mathbb{A}_\infty)\mathcal{K}$, where $\Delta_i = \mathbf{d}(\lambda_i)\mathbf{m}(A_i)$. Put

$$\mathscr{R}_{n,i}^+ = \{B \in \mathrm{Her}_n^+ \mid B\langle A_{i,\mathfrak{p}}\rangle \in \mathcal{R}_{n,\mathfrak{p}} \text{ for all } \mathfrak{p}\}.$$

We write $\mathfrak{G}_\ell^{n,\varkappa}(\mathcal{K})$ for the subspace of $\mathfrak{G}_\ell^{n,\varkappa}$ on which $\mathcal{K}$ acts trivially. Put $\Gamma_i = \mathcal{G}_n(F)^+ \cap \Delta_i \mathcal{K} \Delta_i^{-1}$. Let $\mathscr{C}_\ell^\varkappa(\Gamma_i)$ denote the space of holomorphic

functions H on $\mathfrak{H}_n^d$ such that $H|_\ell^\varkappa\gamma = H$ for all $\gamma \in \Gamma_i$ and such that $H|_\ell^\varkappa\gamma$ has a Fourier expansion of the form (1.1) for all $\gamma \in \mathcal{G}_n(F)^+$. Then $\mathcal{F} \mapsto (\mathcal{F}_{\Delta_1}, \mathcal{F}_{\Delta_2}, \dots, \mathcal{F}_{\Delta_r})$ defines a bijection from $\mathfrak{G}_\ell^{n,\varkappa}(\mathcal{K})$ onto $\oplus_{i=1}^r \mathscr{C}_\ell^\varkappa(\Gamma_i)$.

For $B \in \mathcal{R}_{n,\mathfrak{p}} \cap \mathrm{GL}_n(E_\mathfrak{p})$ and a finite prime $\mathfrak{p}$ we put

$$\widetilde{F}_\mathfrak{p}(B, X) = X^{-\operatorname{ord}_\mathfrak{p}(\mathfrak{D}^{(n-1)/2}\det B)} F_\mathfrak{p}(B, q_\mathfrak{p}^{-n} X^2). \tag{11.1}$$

See (9.1) for the definition of $F_\mathfrak{p}(B, X)$. For $a \in \mathbb{E}_\mathbf{f}^\times$ put $|a|_{\mathbb{E}_\mathbf{f}} = \prod_\mathfrak{p} |a_\mathfrak{p}|_{E_\mathfrak{p}}$.

Corollary 11.3. Let $\pi_\mathbf{f}$ be an irreducible cuspidal automorphic representation generated by a Hilbert cusp form. Assume that $\pi_\mathbf{f}$ is equivalent to the unique irreducible subrepresentation of $\otimes'_\mathfrak{p} I(\boldsymbol{\alpha}_{F_\mathfrak{p}}^{s_\mathfrak{p}}, \boldsymbol{\alpha}_{F_\mathfrak{p}}^{-s_\mathfrak{p}})$. Put $\mathfrak{S} = \{\mathfrak{p} \mid \Re s_\mathfrak{p} \neq 0\}$. For $i = 1, 2, \dots, r$ we define a function $H_i : \mathfrak{H}_n^d \to \mathbb{C}$ by the Fourier series

$$H_i(Z) = \sum_{B \in \mathscr{R}_{n,i}^+} |\det B|^{(\kappa-1)/2} C_i(B) \mathbf{e}_\infty(\operatorname{tr}(BZ)),$$

where

$$C_i(B) = |\lambda_i^{-n} \mathrm{N}_F^E(\det A_i)|_{E_\mathbf{f}}^{n/2} \prod_{\mathfrak{p} \in \mathfrak{S}} |\lambda_{i,\mathfrak{p}}^{-n} \det B\langle A_{i,\mathfrak{p}}\rangle|_{F_\mathfrak{p}}^{s_\mathfrak{p}} \prod_{\mathfrak{p} \notin \mathfrak{S}} \widetilde{F}_\mathfrak{p}(\lambda_{i,\mathfrak{p}}^{-1} B\langle A_{i,\mathfrak{p}}\rangle, q_\mathfrak{p}^{-s_\mathfrak{p}}).$$

Then the tuple $(H_1, \dots, H_r)$ defines a nonzero Hilbert-Hermitian cusp form in $\oplus_{i=1}^r \mathscr{C}_{\kappa+n-1}^\varkappa(\Gamma_i)$ whose standard L-function is equal to

$$\prod_{i=1}^n L\left(s + \frac{n+1}{2} - i, \pi\right) L\left(s + \frac{n+1}{2} - i, \pi \otimes \epsilon_{E/F}\right).$$

Proof. We will apply Corollary 1.2 to $\mu_\mathfrak{p} = \boldsymbol{\alpha}_{F_\mathfrak{p}}^{s_\mathfrak{p}}$, $\hat{\omega} = 1$ and $\hat{\chi} = 1$. Define $h \in A_n(\mu_\mathbf{f}^E, \mu_\mathbf{f}^{-n})$ by $h(g) = \prod_{\mathfrak{p} \in \mathfrak{S}} \hat{\varepsilon}_\mathfrak{p}^{(2s_\mathfrak{p}+n)/2}(g_\mathfrak{p}) \prod_{\mathfrak{p} \notin \mathfrak{S}} \varepsilon_{\mathfrak{d}_\mathfrak{p}}(g_\mathfrak{p})^{(2s_\mathfrak{p}+n)/2}$ for $g = (g_\mathfrak{p}) \in \mathcal{G}_n(\mathbb{A}_\mathbf{f})$. For $B \in \mathscr{R}_{n,i}^+$ we rewrite Lemma 9.2 as

$$\begin{aligned}
&|\det B|_{F_\mathfrak{p}}^{-s} w_B^{\boldsymbol{\alpha}_{E_\mathfrak{p}}^s}(\varrho(\mathbf{d}(\lambda_{i,\mathfrak{p}})\mathbf{m}(A_{i,\mathfrak{p}}))\varepsilon_{\mathfrak{d}_\mathfrak{p}}^{(2s+n)/2}) \\
=&|\det B|_{F_\mathfrak{p}}^{-s} |\lambda_{i,\mathfrak{p}}|_{F_\mathfrak{p}}^{ns} |\det A_{i,\mathfrak{p}}|_{E_\mathfrak{p}}^{-s} w_{\lambda_{i,\mathfrak{p}}^{-1} B\langle A_{i,\mathfrak{p}}\rangle}^{\boldsymbol{\alpha}_{E_\mathfrak{p}}^s}(\varepsilon_{\mathfrak{d}_\mathfrak{p}}^{(2s+n)/2}) \\
=&|\lambda_{i,\mathfrak{p}}^{-n} \det B\langle A_{i,\mathfrak{p}}\rangle|_{F_\mathfrak{p}}^{-s} \mathfrak{N}(\mathfrak{d}_\mathfrak{p})^{-2ns} |\lambda_{i,\mathfrak{p}}^{-n} \det B\langle A_{i,\mathfrak{p}}\rangle|_{F_\mathfrak{p}}^{n/2} F_\mathfrak{p}(\lambda_{i,\mathfrak{p}}^{-1} B\langle A_{i,\mathfrak{p}}\rangle, q_\mathfrak{p}^{-2s-n}) \\
=&\mathfrak{N}(\mathfrak{d}_\mathfrak{p})^{-2ns} \mathfrak{N}(\mathfrak{D}_\mathfrak{p})^{-(n-1)s/2} |\det B|_{F_\mathfrak{p}}^{n/2} |\lambda_{i,\mathfrak{p}}^{-n} \det(A_{i,\mathfrak{p}}\, {}^t A_{i,\mathfrak{p}}^\tau)|_{E_\mathfrak{p}}^{n/2} \tilde{F}_\mathfrak{p}(B\langle A_{i,\mathfrak{p}}\rangle, q_\mathfrak{p}^{-s}).
\end{aligned}$$

Correspondingly, we see that

$$H_i = I_\kappa^\varkappa(h)_{\Delta_i} \prod_{\mathfrak{p} \in \mathfrak{S}} \mathfrak{N}(\mathfrak{d}_\mathfrak{p})^{n^2} e_n(\boldsymbol{\alpha}_{E_\mathfrak{p}}^{s_\mathfrak{p}}) \prod_{\mathfrak{p} \notin \mathfrak{S}} \mathfrak{N}(\mathfrak{d}_\mathfrak{p})^{-2ns_\mathfrak{p}} \mathfrak{N}(\mathfrak{D}_\mathfrak{p})^{-(n-1)s_\mathfrak{p}/2}.$$

Since h is fixed by $\mathcal{K}$, the cusp form $I_\kappa^\varkappa(h)$ is a nonzero element in $\mathfrak{G}_{\kappa+n-1}^{n,\varkappa}(\mathcal{K})$ by Corollary 1.2, so that $H_i \in \mathscr{C}_{\kappa+n-1}^\varkappa(\Gamma_i)$. □

Appendix A. Compatibility with Arthur's conjecture

We will see how Arthur's endoscopic classification [2, 3, 29] accounts for Theorem 1.1. This specialized to our current case is discussed in Section 18 of [17]. Let W_F be the Weil group of F. Langlands has conjectured the existence of a locally compact group $\mathcal{L}_F$ such that the equivalence classes of irreducible k-dimensional representations of $\mathcal{L}_F$ is in one-to-one correspondence with the set of irreducible cuspidal automorphic representations of $\mathrm{GL}_k(\mathbb{A})$. There should be an embedding $\iota_v : \mathcal{L}_{F_v} \hookrightarrow \mathcal{L}_F$ for each v, where $\mathcal{L}_{F_v}$ is the Weil group or the Weil-Deligne group of F_v depending on whether v being archimedean or not.

Let G be a connected reductive algebraic group G over F whose complex dual group is denoted by $\hat{G}$. Arthur speculated that every irreducible cuspidal or residual automorphic representation of $G(\mathbb{A})$ is associated to an elliptic A-parameter, by which we mean a $\hat{G}$-conjugacy class of admissible homomorphisms $\phi : \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ such that $\phi(\mathcal{L}_F)$ is bounded and such that $S(\phi)^+$ is contained in $Z(\hat{G})^{W_F}$, where the semi-direct product ${}^L G = \hat{G} \rtimes W_F$ is the L-group of G and $S(\phi)^+$ is the identity component of the centralizer $S(\phi)$ of $\phi(\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}))$ in $\hat{G}$.

A global A-parameter ϕ provides a local A-parameter $\phi_v = \phi \circ (\iota_v \times \mathrm{Id})$ to which one can associate a finite set $\Pi(\phi_v)$ of equivalence classes of unitary admissible representations of G_v according to the local conjecture, among other things. We define a global A-packet $\Pi(\phi)$ as a tensor product of local A-packets, i.e., the set of representations $\otimes'_v \Pi_v$ of $G(\mathbb{A})$ such that $\Pi_v \in \Pi(\phi_v)$ for all v and Π_v is unramified for almost all v. It is generally believed that to each irreducible representation ϵ_v of the finite group $\mathcal{S}(\phi_v) = S(\phi_v)/S(\phi_v)^+ Z(\hat{G})^{W_{F_v}}$ one can attach a subset $\Pi^{\epsilon_v}(\phi_v)$ of $\Pi(\phi_v)$. If ϵ_v is the trivial representation, then $\Pi^{\epsilon_v}(\phi_v)$ should contain the L-packet $\Pi(\phi'_v)$ for the local L-parameter ϕ'_v defined by

$$\phi'_v(\sigma) = \phi_v\left(\sigma, \mathrm{diag}\left[|\sigma|_{F_v}^{1/2}, |\sigma|_{F_v}^{-1/2}\right]\right).$$

Arthur attached to ϕ a quadratic character ϵ_ϕ of $\mathcal{S}(\phi) = S(\phi)/Z(\hat{G})^{W_F}$. For an irreducible character $\epsilon = \prod_v \epsilon_v$ of the compact group $\prod_v \mathcal{S}(\phi_v)$ we set

$$\Pi^\epsilon(\phi) = \{\otimes'_v \Pi_v \mid \Pi_v \in \Pi^{\epsilon_v}(\phi_v)\}, \qquad m_\epsilon = \frac{1}{\#\mathcal{S}(\phi)} \sum_{s \in \mathcal{S}(\phi)} \epsilon_\phi(s)\epsilon(s).$$

Arthur conjectured that the space of square-integrable automorphic forms on $G(\mathbb{A})$ is the direct sum $\oplus_\phi \oplus_\epsilon \oplus_{\Pi \in \Pi^\epsilon(\phi)} m_\epsilon \Pi$ which runs over the elliptic A-parameters ϕ and irreducible characters ϵ of $\prod_v \mathcal{S}(\phi_v)$.

The action of W_F on $\hat{\mathcal{G}}_n = \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ factors through the Galois group Γ_F^E of E over F and its nontrivial element acts via the automorphism

$$(g, \lambda) \mapsto (w_{2n}\, {}^t g^{-1} w_{2n}^{-1}, \lambda \det g), \qquad w_k = \begin{bmatrix} & & & (-1)^{k-1} \\ & & \iddots & \\ & -1 & & \\ 1 & & & \end{bmatrix}.$$

If $\phi(\sigma) = (\varphi(\sigma), \lambda(\sigma))$ is an A-parameter for $\mathcal{G}_n$, then the central character of representations in $\Pi(\phi)$ should correspond to the homomorphism

$$\sigma \mapsto (\lambda(\sigma) \det \varphi(\sigma), \lambda(\sigma)).$$

We normalize the kth symmetric power representation sym^k of $\mathrm{SL}_2(\mathbb{C})$ so that ${}^t\mathrm{sym}^k(x)^{-1} = w_{k+1}\mathrm{sym}^k(x) w_{k+1}^{-1}$ for $x \in \mathrm{SL}_2(\mathbb{C})$. Fix a character $\theta \in \Omega(C_F)$, a character $\hat{\epsilon}$ of C_E whose restriction to C_F is $\epsilon_{E/F}^{n-1}$ and an element $\sigma_0 \in W_F$ whose projection to Γ_F^E is nontrivial. Define homomorphisms

$${}^{\theta,\hat{\epsilon}}\iota_n : {}^L\mathcal{G}_1 \to {}^L\mathcal{G}_n, \qquad\qquad {}^L\mathrm{sym}^{n-1} : \mathrm{SL}_2(\mathbb{C}) \to \hat{\mathcal{G}}_n^{W_F}$$

by

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \lambda\right) \mapsto \left(\begin{bmatrix} a\,\mathbf{1}_n & b\,\mathbf{1}_n \\ c\,\mathbf{1}_n & d\,\mathbf{1}_n \end{bmatrix}, \lambda^n\right), \quad \sigma' \mapsto (\hat{\epsilon}(\sigma')\mathbf{1}_{2n}, \hat{\epsilon}(\sigma')^{-n}\theta(\sigma')) \rtimes \sigma',$$

$$\sigma_0 \mapsto \left(\begin{bmatrix} (-1)^{n-1}\mathbf{1}_n & \\ & \mathbf{1}_n \end{bmatrix}, \theta(\sigma_0)\right) \rtimes \sigma_0, \quad x \mapsto \begin{bmatrix} \mathrm{sym}^{n-1}(x) & \\ & \mathrm{sym}^{n-1}(x) \end{bmatrix}$$

for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$, $\lambda \in \mathrm{GL}_1(\mathbb{C})$, $\sigma' \in W_E$ and $x \in \mathrm{SL}_2(\mathbb{C})$.

The proof of Proposition 6.1 of [8] gives a natural isomorphism

$${}^L\mathcal{G}_1 \simeq \{(g; (\alpha, \beta) \rtimes \sigma) \in \mathrm{GL}_2(\mathbb{C}) \times {}^L\mathrm{R}_F^E\mathrm{GL}_1 \mid \alpha\beta \det g = 1\}$$

(cf. Remark 2.1). Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ whose central character is $\hat{\omega}$. Recall that $\hat{\chi}^\dagger = \hat{\omega}$. The representation $\hat{\chi}\boxtimes\pi$ of $\mathcal{G}_1(\mathbb{A})$ gives an L-parameter $\phi_1^{\hat{\chi}}[\pi]$ with values in ${}^L\mathcal{G}_1$. Define an A-parameter $\phi_n^{\hat{\chi}}[\pi, \hat{\epsilon}, \theta] : \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathcal{G}_n$ by

$$\phi_n^{\hat{\chi}}[\pi, \hat{\epsilon}, \theta](u, x) = {}^L\mathrm{sym}^{n-1}(x)\, {}^{\theta,\hat{\epsilon}}\iota_n(\phi_1^{\hat{\chi}}[\pi](u))$$

for $u \in \mathcal{L}_F$ and $x \in \mathrm{SL}_2(\mathbb{C})$.

Suppose that n is odd. One can easily see that

$$A_n^{\hat{\chi}_v}(\pi_v) \otimes (\tilde{\hat{\epsilon}}_v \circ \Lambda_n) \otimes (\theta_v \circ \lambda_n) \in \Pi(\phi_n^{\hat{\chi}_v}[\pi_v, \hat{\epsilon}_v, \theta_v \hat{\omega}_v^{(1-n)/2}]').$$

If π_v is a discrete series with extremal weight $\pm\kappa$, then the holomorphic discrete series with lowest K-type $(\det)^{\kappa_v+n-1}$ belongs to the A-packet $\Pi(\phi_n^{\hat{\chi}_v}[\pi_v, \hat{\epsilon}_v, \theta_v])$, which should consist of certain cohomologically induced representations (see [1]). Theorem 1.1 is compatible with the fact that both $\mathcal{S}(\phi_n^{\hat{\chi}}[\pi, \hat{\epsilon}, \theta])$ and $\mathcal{S}(\phi_n^{\hat{\chi}_v}[\pi_v, \hat{\epsilon}_v, \theta_v])$ are trivial.

Appendix B. The split case

We discuss the case in which E is the split quadratic algebra $F \oplus F$, though our exposition included this case so far. In the split case one can prove uniqueness of degenerate Whittaker model and reprove Proposition 7.2 via a purely local method. Proposition B.2(1) played an important role in the proof of Lemma 10.5.

The nondegenerate form $\langle\ ,\ \rangle$ identifies the free E-module W with the sum $\mathbf{W} \oplus \mathbf{W}^\vee$, where $\mathbf{W}$ is a vector space over F and $\mathbf{W}^\vee$ is its dual. The restriction to $\mathbf{W}$ gives an isomorphism of G_n onto the group of all F-automorphisms of $\mathbf{W}$. We fix a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{f}_1, \dots, \mathbf{f}_n\}$ for $\mathbf{W}$ and identify G_n with $\mathrm{GL}_{2n}(F)$. For $A_1, A_2 \in \mathrm{GL}_n(F)$ and $z \in \mathrm{M}_n(F)$ we put

$$\mathfrak{m}(A_1, A_2) = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad \mathfrak{n}(z) = \begin{bmatrix} \mathbf{1}_n & z \\ 0 & \mathbf{1}_n \end{bmatrix}.$$

These matrices generate the parabolic subgroup P_n of G_n which stabilizes the subspace spanned by $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. Denote its unipotent radical by N_n. For $B \in \mathrm{M}_n(F)$ we define a homomorphism $\ell^B : N_n \to F$ by $\ell^B(\mathfrak{n}(z)) = \mathrm{tr}(Bz)$. Let $\mathbf{X}_{2i}$ (resp. $\mathbf{X}_{2i-1}$) be the subspace of $\mathbf{W}$ spanned by

$$\mathbf{e}_1, \mathbf{f}_1, \mathbf{e}_2, \mathbf{f}_2, \dots, \mathbf{e}_i, \mathbf{f}_i \qquad (\text{resp. } \mathbf{e}_1, \mathbf{f}_1, \mathbf{e}_2, \mathbf{f}_2, \dots, \mathbf{e}_{i-1}, \mathbf{f}_{i-1}, \mathbf{e}_i),$$

and $P_{\mathbf{e}}$ the stabilizer in G_n of the flag $\mathbf{X}_2 \subset \mathbf{X}_4 \subset \cdots \subset \mathbf{X}_{2n-2}$.

The rest of this section assumes F to be an extension of $\mathbb{Q}_p$. Let π be an irreducible admissible unitary generic representation of $\mathrm{GL}_2(F)$ with central character $\hat{\omega}$ and $A_n(\pi)$ the unique irreducible subrepresentation of

$$I_n(\pi) = \mathrm{Ind}_{P_{\mathbf{e}}}^{G_n} \delta_{P_{\mathbf{e}}}^{-1/4} \otimes \mathscr{W}(\pi)^{\boxtimes n}.$$

Remark B.1. Note that $\hat{\chi} = (\hat{\omega}\mu, \mu^{-1})$ for some $\mu \in \Omega(C_F)$. Then $\pi[\hat{\chi}] = (\pi \otimes \mu) \boxtimes (\pi \otimes \mu)^\vee$. The restriction of $A_n^{\hat{\chi}}(\pi)$ to G_n is equivalent to $A_n(\pi \otimes \mu)$.

Proposition B.2 ([6, 32, 20])**.** (1) *If n is odd,* $\mathrm{rank} B = n-1$ *and π is supercuspidal, then* $J_{N_n}^{\psi \circ \ell^B}(A_n(\pi)) = \{0\}$.

(2) $\dim J_{N_n}^{\psi \circ \ell^B}(A_n(\pi)) = 1$ *for all* $B \in \mathrm{GL}_n(F)$.

Proof. When $n = 1$, there is nothing to prove. We therefore suppose that $n > 1$. We may assume $B = \mathrm{diag}[1, B']$ without loss of generality. For $x, y \in F^{n-1}$; $z \in F$ and $z' \in \mathrm{M}_{n-1}(F)$ we put

$$X(x) = \mathfrak{m}\left(\begin{bmatrix} 1 & {}^t x \\ 0 & \mathbf{1}_{n-1} \end{bmatrix}, \mathbf{1}_n\right), \qquad C(x, y, z, z') = \mathfrak{n}\left(\begin{bmatrix} z & {}^t x \\ y & z' \end{bmatrix}\right).$$

Since the subgroups

$$X = \{X(x) \mid x \in F^{n-1}\}, \qquad Y = \{C(0, y, 0, \mathbf{0}_{n-1}) \mid y \in F^{n-1}\},$$
$$C = \{C(x, 0, z, z') \mid x \in F^{n-1},\ z \in F,\ z' \in \mathrm{M}_{n-1}(F)\},$$
$$C' = \{C(x, 0, z, \mathbf{0}_{n-1}) \mid x \in F^{n-1},\ z \in F\}$$

satisfy all the requirements of Lemma 2.2 of [10], we obtain

$$J_{N_n}^{\psi\circ\ell^B}(A_n(\pi)) \simeq J_{XC}^{\psi\circ\ell_1^{B'}}(A_n(\pi)),$$

where we define a homomorphism $\ell_1^{B'} : XC \to F$ by

$$\ell_1^{B'}(X(x)C(x',0,z,z')) = z + \operatorname{tr}(B'z').$$

Note that XC' is the unipotent radical of the stabilizer of $\mathbf{X}_1$ in G_n. Define the character $\boldsymbol{\psi}^{\mathfrak{U}}$ of the unipotent radical $\mathfrak{U}$ of the stabilizer of the flag $\mathbf{X}_1 \subset \mathbf{X}_2$ by $\boldsymbol{\psi}^{\mathfrak{U}}(u) = \boldsymbol{\psi}(\langle u\mathbf{f}_1, \mathbf{e}_1^\vee\rangle)$, where $\{\mathbf{e}_1^\vee, \dots, \mathbf{e}_n^\vee, \mathbf{f}_1^\vee, \dots, \mathbf{f}_n^\vee\}$ is the dual basis for $\mathbf{W}^\vee$. The restriction of $\boldsymbol{\psi} \circ \ell_1^{B'}$ to XC' coincides with $\boldsymbol{\psi}^{\mathfrak{U}}$. Put

$$\Phi^- = J_{XC'}^{\boldsymbol{\psi}^{\mathfrak{U}}}, \qquad\qquad \mathit{\Pi} = \Phi^-(A_n(\pi)).$$

Recall the filtration given in §3.5 of [6]:

$$0 \subset \mathit{\Pi}_{2n-1} \subset \cdots \subset \mathit{\Pi}_1 = \mathit{\Pi}, \qquad \mathit{\Pi}_k/\mathit{\Pi}_{k+1} = (\Phi^+)^{k-1}\Psi^+(\mathit{\Pi}^{(k)}).$$

Since $\mathit{\Pi}^{(k)}$ vanishes for $k \geq 2$ and $\mathit{\Pi}^{(1)} \simeq A_{n-1}(\pi) \otimes \boldsymbol{\alpha}_F^{-1/2}$ by Lemma 3.6 of [32], we are led to

$$\mathit{\Pi} = \mathit{\Pi}_1 \simeq \mathit{\Pi}_1/\mathit{\Pi}_2 \simeq J_{\mathfrak{U}}^{\boldsymbol{\psi}^{\mathfrak{U}}}(A_n(\pi)) \simeq \Psi^+(A_{n-1}(\pi) \otimes \boldsymbol{\alpha}_F^{-1/2}).$$

We obtain

$$J_{N_n}^{\psi\circ\ell^B}(A_n(\pi)) \simeq J_{N_{n-1}}^{\psi\circ\ell^{B'}}(\Phi^-(A_n(\pi))) \simeq J_{N_{n-1}}^{\psi\circ\ell^{B'}}(A_{n-1}(\pi) \otimes \boldsymbol{\alpha}_F^{-1/2}).$$

Our proof is complete by induction. □

We define the linear map $\beth_n : I_n(\pi) \to \mathbb{C}$ by

$$\beth_n(f) = \int_{P_{\mathbf{e}}\cap N_n\backslash N_n} f(\mathfrak{n}(z))\overline{\boldsymbol{\psi}(\operatorname{tr}(z))}\,\mathrm{d}z.$$

This integral makes sense since the integrand is a Schwartz function on $P_{\mathbf{e}}\cap N_n\backslash N_n$ as the same proof as that of Proposition 4.5 shows. Though the following results were stated in a uniform way, one can bypass the intricate notation or calculation in the split case.

Lemma B.3. (1) *$\beth_n$ is nonzero on $A_n(\pi)$.*

(2) *For all $f \in I_n(\pi)$*

$$\beth_n(f) = \int_{F^{n-1}} \beth_{n-1}(f(C(0,y,0,0)))\,\mathrm{d}y.$$

(3) *$\beth_n(\varrho(\mathfrak{m}(A,A))f) = \hat{\omega}(\det A)\beth_n(f)$ for $A \in \mathrm{GL}_n(F)$ and $f \in A_n(\pi)$.*

Proof. We can prove the first and second assertions by arguing exactly as in the proof of Proposition 4.5. Since

$$\beth_n \circ \varrho(\mathfrak{m}(A,A)) \in \operatorname{Hom}_{\mathrm{M}_n(F)}(A_n(\pi)\circ\mathfrak{n}, \boldsymbol{\psi}\circ\operatorname{tr}),$$

Proposition B.2(2) gives $\mu \in \Omega(C_F)$ such that $\beth_n \circ \varrho(\mathfrak{m}(A,A)) = \mu(\det A)\beth_n$. Letting $A = \operatorname{diag}[\xi, \mathbf{1}_{n-1}]$, we find that μ is the central character of π. □

Lemma B.4. *For $f \in A_n(\pi)$ there are $0 \le \phi \in \mathcal{S}(\mathrm{M}_n(F))$ and $M \in \mathbb{R}_+^\times$ such that for all $A \in \mathrm{GL}_n(F)$*

$$\|\mathfrak{Z}_n(\varrho(\mathfrak{m}(A, \mathbf{1}_n))f)\| \le |\det A|_F^{-M}\phi(A).$$

Proof. The proof is similar to that of Lemma 9.1 and omitted. □

Appendix C. Fourier coefficients of certain residual automorphic forms

To complete the proof of Lemma 10.5 and Theorem 1.1, we will associate to an irreducible cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbb{A})$ the residual automorphic representation $A_n^\chi(\sigma)$ of $\mathcal{G}_n(\mathbb{A})$ and prove the factorization of Fourier coefficients of those residual automorphic forms. Let us give a brief account of Jacquet modules of degenerate principal series representations with respect to Bessel and Fourier-Jacobi characters. Let E be an étale quadratic algebra over a finite algebraic extension F of $\mathbb{Q}_p$. Fix an odd natural number n. Let $\mathcal{W}_i$ be the unipotent radical of the parabolic subgroup $\cap_{k=1}^i \mathcal{P}_k$ of $\mathcal{G}_n$. If $i < n$ and $w \in \mathcal{X}_i^\perp \cap \mathcal{Y}_i^\perp$, we define a homomorphism $\ell_{i,w} : \mathcal{W}_i \to F$ by

$$\ell_{i,w}(u) = \mathrm{T}_F^E(\langle ue_2, f_1\rangle + \langle ue_3, f_2\rangle + \cdots + \langle ue_i, f_{i-1}\rangle + \langle uw, f_i\rangle).$$

When $w = e_{i+1}$, we write $\ell_i = \ell_{i,e_{i+1}}$. For $i < n-1$ we consider a subgroup

$$\mathcal{W}_{i+1}^0 = \mathcal{W}_i \cdot \{\iota_i(\mathbf{1}_i, \mathbf{v}_1^{n-i}(0;0;z)) \mid z \in F\}$$

of $\mathcal{W}_{i+1}$. For $\xi \in F$ we extend ℓ_i to a homomorphism $\ell_i^\xi : \mathcal{W}_{i+1}^0 \to F$ by $\ell_i^\xi(\iota_i(\mathbf{1}_i, \mathbf{v}_1^{n-i}(0;0;z))) = \xi z$. When $i = n-1$, we define $\ell_n^\xi : \mathcal{W}_n \to F$ by

$$\ell_n^\xi(u) = \mathrm{T}_F^E(\langle ue_2, f_1\rangle + \langle ue_3, f_2\rangle + \cdots + \langle ue_n, f_{n-1}\rangle) + \xi\langle uf_n, f_n\rangle.$$

Lemma C.1 ([11])**.** *Let $\chi \in \Omega(C_E)$. Suppose that $n \ge 2$.*

1. *$J_{\mathcal{W}_i}^{\psi\circ\ell_{i,w}}(I_n(\chi)) = \{0\}$ for all $i \ge 1$ and anisotropic $w \in \mathcal{X}_i^\perp \cap \mathcal{Y}_i^\perp$.*
2. *$J_{\mathcal{W}_{i+1}^0}^{\psi\circ\ell_i^\xi}(I_n(\chi)) = \{0\}$ for all $i \ge 1$ and $\xi \in F^\times$.*
3. *If $E \simeq F \oplus F$, then $J_{\mathcal{W}_i}^{\psi\circ\ell_{i,0}}(I_n(\chi)) = \{0\}$ for all $i \ge 3$.*

Proof. The first part is a special case of Theorem 5.4(1) of [11]. The second part is a special case of Theorem 6.3 of [11]. We discuss the case $E \simeq F \oplus F$. Then $G_n \simeq \mathrm{GL}_{2n}(F)$ and $\mathcal{W}_i$ is conjugate to the unipotent radical of a parabolic subgroup stabilizing the flag

$$\mathbf{X}_1 \subset \cdots \subset \mathbf{X}_i \subset \mathbf{X}_{2n-i} \subset \cdots \subset \mathbf{X}_{2n-1}.$$

Theorem 5.7 of [11] includes (1) whereas Theorem 6.5 of [11] includes (2). The module $J_{\mathcal{W}_i}^{\psi\circ\ell_{i,0}}(I_n(\chi))$ is a certain twisted Jacquet module of the ith derivative $I_n(\chi)^{(i)}$ of $I_n(\chi)$. By the Leibniz rule $I_n(\chi)^{(i)}$ is zero for $i \ge 3$. □

From now on E/F is a quadratic extension of an arbitrary number field. The quadratic character of C_F corresponding to E is denoted by $\epsilon_{E/F}$. Fix an irreducible cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbb{A})$ with central character ω. For simplicity the base change σ^E of σ to $\mathrm{GL}_2(\mathbb{E})$ is assumed to be cuspidal. Extend ω to $\chi \in \Omega(C_E)$. Let $\sigma[\chi] = {}^\tau\chi^{-1} \otimes \sigma^E$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{E})$. Let As denote the Asai representation of the L-group of $\mathrm{R}^E_F\mathrm{GL}_2$. Fix an extension $\hat{\gamma}$ of $\epsilon_{E/F}$ to C_E.

Lemma C.2. *Notation being as above, the L-functions $L(s, \sigma[\chi] \otimes \hat{\gamma}, \mathrm{As})$ and $L(s, \sigma[\chi] \times (\chi^{-1} \boxtimes \sigma)^\vee)$ have a pole at $s = 1$.*

Proof. Since $\sigma[\chi]$ is the stable base change of $\chi^{-1} \boxtimes \sigma$ by Lemma 4.2(4), the product L-function has a pole at $s = 1$ and by Theorem 11.2(4) of [11] $L(s, \sigma[\chi] \otimes \hat{\gamma}, \mathrm{As})$ has a pole at $s = 1$. □

We aim to construct a residual automorphic representation in the packet $\phi^\chi_n[\sigma, \hat{\epsilon}, \theta]$. Let $Q_{2,\dots,2}$ be the parabolic subgroup of $\mathrm{R}^E_F\mathrm{GL}_{2m} = \mathrm{GL}_E(\mathfrak{X}_{2m})$ which leaves the flag $\mathfrak{X}_2 \subset \mathfrak{X}_4 \subset \cdots \subset \mathfrak{X}_{2m}$ stable. The Langlands quotient $A_m(\sigma[\chi])$ of the standard module $\mathrm{Ind}^{\mathrm{GL}_{2m}(\mathbb{E})}_{Q_{2,\dots,2}(\mathbb{A})} \delta^{1/4}_{Q_{2,\dots,2}} \otimes \sigma[\chi]^{\boxtimes m}$ appears in the space of square-integrable forms on $\mathrm{GL}_{2m}(\mathbb{E})$ (cf. [30]).

Fix a good maximal compact subgroup $\mathcal{K}_n$ of $\mathcal{G}_n(\mathbb{A})$. Extend the modulus character $\delta_{\mathfrak{P}_{n-1}}$ of $\mathfrak{P}_{n-1}(\mathbb{A})$ to a right $\mathcal{K}_n$-invariant function on $\mathcal{G}_n(\mathbb{A})$. For

$$\phi \in \mathrm{Ind}^{\mathcal{G}_n(\mathbb{A})}_{\mathfrak{P}_{n-1}(\mathbb{A})} A_{(n-1)/2}(\sigma[\chi]) \boxtimes (\chi^{-1} \boxtimes \sigma)$$

we form the Eisenstein series

$$E(g, \phi, s) = \sum_{\gamma \in \mathfrak{P}_{n-1}(F)\backslash \mathcal{G}_n(F)} \phi(\gamma g) \delta_{\mathfrak{P}_{n-1}}(\gamma g)^{s/4}.$$

For any parabolic subgroup $\mathcal{P}$ of $\mathcal{G}_n$ with unipotent radical $\mathcal{N}$ the constant term map on the space of automorphic forms on $\mathcal{G}_n(\mathbb{A})$ is defined by

$$\mathcal{F} \mapsto \mathcal{F}_{\mathcal{P}}(g) = \int_{\mathcal{N}(F)\backslash \mathcal{N}(\mathbb{A})} \mathcal{F}(ug)\, \mathrm{d}u.$$

Lemma C.3 (cf. [19])**.** *The series $E(\phi, s)$ has at most a simple pole at $s = 1$. Let $A^\chi_n(\sigma)$ denote the representation of $\mathcal{G}_n(\mathbb{A})$ generated by those residues. If $\mathcal{F} \in A^\chi_n(\sigma)$, then $\mathcal{F}$ is square-integrable and the function*

$$g \mapsto \mathscr{W}_2(g, \mathcal{F}_{\mathfrak{P}_2}) = |\lambda_{n-2}(g)|^{(n-1)/2} \int_{E\backslash\mathbb{E}} \mathcal{F}_{\mathfrak{P}_2}\left(\mathfrak{i}_2\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, g\right)\right) \overline{\psi(\mathrm{T}^E_F(x))}\, \mathrm{d}x$$

belongs to $A^\chi_{n-2}(\sigma)$. Assume further that $A^\chi_n(\sigma)$ is nonzero. Then

$$A^\chi_n(\sigma) \simeq \otimes'_v A^{\chi_v}_n(\sigma_v).$$

Proof. Theorem 1.2 of [19] determines a set of possible poles of certain Eisenstein series on classical groups. Though they do not treat similitude groups, one can apply their results to constituents of the restriction of the induced

representation to $G_n(\mathbb{A})$. These combined with Lemma C.2 say that $E(\phi, s)$ has at most a simple pole at $s = 1$.

The induction formula (3-8) of [19] implies some inductivity of $\mathcal{F}_{\mathfrak{P}_2}$. The cuspidal support of $\mathcal{F}$ consists only of $\delta_{\mathfrak{P}_\mathbf{e}}^{-1/4} \otimes \{\sigma[\chi]^{\boxtimes(n-1)/2} \boxtimes (\chi^{-1} \boxtimes \sigma)\}$ and hence it is square-integrable by Lemma I.4.11 of [31]. Thus $A_n^\chi(\sigma)$ is a unitary quotient of $\mathrm{Ind}_{\mathfrak{P}_{n-1}(\mathbb{A})}^{\mathcal{G}_n(\mathbb{A})} \delta_{\mathfrak{P}_{n-1}}^{1/4} \otimes \{A_{(n-1)/2}(\sigma[\chi]) \boxtimes (\chi^{-1} \boxtimes \sigma)\}$, which is a quotient of $\otimes_v' {}^\tau J_n^{\chi_v}(\sigma_v)^\vee$. If $A_n^\chi(\sigma)$ is nonzero, then it must be isomorphic to the Langlands quotient $\otimes_v' A_n^{\chi_v}(\sigma_v)$ of the standard module $\otimes_v' {}^\tau J_n^{\chi_v}(\sigma_v)^\vee$ as the Langlands quotient is the unique semisimple quotient. □

Next we will observe that $\mathfrak{Z}_{H_n}^{\chi_v}$ arises as a local component of the H_nth Fourier coefficient of residual automorphic forms in $A_n^\chi(\sigma)$. Our computations are similar to those carried out in the proof of Theorem 1 of [9].

Lemma C.4. *If $n \geq 3$, then for all $\mathcal{F} \in A_n^\chi(\sigma)$*

$$\int_{\mathcal{W}_1(F)\backslash\mathcal{W}_1(\mathbb{A})} \mathcal{F}(u)\overline{\boldsymbol{\psi}(\ell_1(u))}\,\mathrm{d}u = \int_{\mathcal{W}_2(F)\backslash\mathcal{W}_2(\mathbb{A})} \mathcal{F}(u)\overline{\boldsymbol{\psi}(\ell_{2,0}(u))}\,\mathrm{d}u.$$

Proof. Consider the function

$$h_i(x; y; z, \mathcal{F}) = \int_{\mathcal{W}_i(F)\backslash\mathcal{W}_i(\mathbb{A})} \mathcal{F}(u\mathbf{v}_1^{n-i}({}^t x^\tau; {}^t y^\tau; z))\overline{\boldsymbol{\psi}(\ell_i(u))}\,\mathrm{d}u$$

on $(E\backslash\mathbb{E}^{n-i-1})^2 \oplus (F\backslash\mathbb{A})$ for $1 \leq i \leq n-2$. The left hand side of the identity is $h_1(0; 0; 0, \mathcal{F})$. Lemma C.1(2) implies that $h_i(x; y; z, \mathcal{F})$ is independent of z. We expand h_i in a Fourier series along the coordinates $(x; y)$ to get

$$\begin{aligned} h_i(x; y; z, \mathcal{F}) &= \int_{\mathcal{W}_{i+1}^0(F)\backslash\mathcal{W}_{i+1}^0(\mathbb{A})} \mathcal{F}(u\mathbf{v}_1^{n-i}({}^t x^\tau; {}^t y^\tau; 0))\overline{\boldsymbol{\psi}(\ell_i^0(u))}\,\mathrm{d}u \\ &= \sum_{w\in(\mathcal{X}_{i+1}^\perp\cap\mathcal{Y}_{i+1}^\perp)(F)} c_{i+1,w}(\mathcal{F})\boldsymbol{\psi}(\ell_{i+1,w}(\mathbf{v}_1^{n-i}({}^t x^\tau; {}^t y^\tau; 0))), \end{aligned}$$

where

$$c_{i+1,w}(\mathcal{F}) = \int_{\mathcal{W}_{i+1}(F)\backslash\mathcal{W}_{i+1}(\mathbb{A})} \mathcal{F}(u)\overline{\boldsymbol{\psi}(\ell_{i+1,w}(u))}\,\mathrm{d}u.$$

Our goal is to show that $c_{2,w}(\mathcal{F}) = 0$ for all nonzero vectors w. We eventually get $h_1(x; y; z, \mathcal{F}) = c_{2,0}(\mathcal{F})$. This was to be shown.

We may assume that $\mathcal{F}$ corresponds to a decomposable vector $\otimes_v f_v$. Fix a finite prime $\mathfrak{p}$ of F such that $\sigma_\mathfrak{p} \simeq I(\mu, \mu^{-1}\omega_\mathfrak{p})$ for some $\mu \in \Omega(C_{E_\mathfrak{p}})$. The map $h_\mathfrak{p} \mapsto c_{i+1,w}(h_\mathfrak{p} \otimes \otimes_{v\neq\mathfrak{p}} f_v)$ defines a functional on $J_{\mathcal{W}_{i+1}(F_\mathfrak{p})}^{\boldsymbol{\psi}_\mathfrak{p}\circ\ell_{i+1,w}}(I_n(\mu[\chi_\mathfrak{p}]))$ in view of Proposition 6.3 and Lemma C.3. If w is anisotropic, then $c_{i+1,w}(\mathcal{F}) = 0$ by Lemma C.1(1). If $i \geq 2$, then $c_{i+1,0}(\mathcal{F}) = 0$ by Lemma C.1(3).

If w is a nonzero isotropic vector and if we take $\beta \in \mathcal{G}_{n-i-1}(F) \subset \mathcal{M}_{i+1}(F)$ so that $\beta w = e_{i+2}$, then

$$c_{i+1,w}(\mathcal{F}) = \int_{\mathcal{W}_{i+1}(F)\backslash\mathcal{W}_{i+1}(\mathbb{A})} \mathcal{F}(u\beta)\overline{\boldsymbol{\psi}(\ell_{i+1}(u))}\,\mathrm{d}u = h_{i+1}(0, \varrho(\beta)\mathcal{F}),$$

where if $i = n-2$, then we consider the function

$$h_{n-1}(z,\mathcal{F}) = \int_{\mathcal{W}_{n-1}(F)\backslash\mathcal{W}_{n-1}(\mathbb{A})} \mathcal{F}(u\mathbf{n}'_{n-1}(z))\overline{\boldsymbol{\psi}(\ell_{n-1}(u))}\,\mathrm{d}u$$
$$= \sum_{\xi\in F}\boldsymbol{\psi}(\xi z)\int_{\mathcal{W}_n(F)\backslash\mathcal{W}_n(\mathbb{A})}\mathcal{F}(u)\overline{\boldsymbol{\psi}(\ell_n^{\xi}(u))}\,\mathrm{d}u$$

for $z \in F\backslash\mathbb{A}$. The constant term of $h_{n-1}(\mathcal{F})$ must be zero as $\mathcal{F}$ is concentrated on $\mathfrak{P}_{\mathbf{e}}$. Since $A_n^{\chi}(\sigma)$ is not generic by Lemma C.1(2), all the Fourier coefficient vanish and $h_{n-1}(\mathcal{F})$ is identically zero. We conclude that $h_i(\mathcal{F})$ is identically zero for $2 \le i \le n-2$ by descending induction. It follows that $c_{2,w}(\mathcal{F}) = 0$ unless $w = 0$. □

We retain the notation in the proof of Proposition 4.5.

Lemma C.5. *If $\mathcal{F} \in A_n^{\chi}(\sigma)$, then*

$$\int_{\mathrm{M}^2_{n-2}(E)\backslash\mathrm{M}^2_{n-2}(\mathbb{E})}\int_{\mathrm{Her}_2(F)\backslash\mathrm{Her}_2(\mathbb{A})}\mathcal{F}(\mathbf{v}_2(0;y;z))\overline{\boldsymbol{\psi}^{H_2}(z)}\,\mathrm{d}z\mathrm{d}y$$
$$= \int_{\mathbb{A}}\int_{\mathbb{E}^{n-2}}\mathscr{W}_2(Y(y,z),\mathcal{F}_{\mathfrak{P}_2})\,\mathrm{d}y\mathrm{d}z.$$

Proof. Applying Lemma 7.1 of [11] to

$$X = \{X(x,\xi) \mid x \in E^{n-2},\ \xi \in F\}, \quad Y = \{Y(y,z) \mid y \in E^{n-2},\ z \in F\}.$$

and $C = \mathfrak{y}_1$, we see that the left hand side is

$$\int_{Y(F)\backslash Y(\mathbb{A})}\int_{C(F)\backslash C(\mathbb{A})}\mathcal{F}(\mathbf{v}_1^n(0;{}^t\!b^{\tau};c)Y)\overline{\boldsymbol{\psi}(\mathrm{T}_F^E(b_1))}\,\mathrm{d}c\mathrm{d}b\mathrm{d}Y$$
$$= \int_{Y(\mathbb{A})}\int_{X(F)\backslash X(\mathbb{A})}\int_{C(F)\backslash C(\mathbb{A})}\mathcal{F}(\mathbf{v}_1^n(\xi,{}^t\!x^{\tau};{}^t\!b^{\tau};c)Y)\overline{\boldsymbol{\psi}(\mathrm{T}_F^E(b_1))}\,\mathrm{d}c\mathrm{d}b\mathrm{d}\xi\mathrm{d}x\mathrm{d}Y.$$

Recall a trace zero element $\urcorner \in E^{\times}$. Consider the function

$$h(z) = \int_{X(F)\backslash X(\mathbb{A})}\int_{C(F)\backslash C(\mathbb{A})}\mathcal{F}(\mathbf{v}_1^n(\xi+z\urcorner,{}^t\!x^{\tau};\,{}^t\!b^{\tau};c)Y)\overline{\boldsymbol{\psi}(\mathrm{T}_F^E(b_1))}\,\mathrm{d}c\mathrm{d}b\mathrm{d}\xi\mathrm{d}x$$

on $F\backslash\mathbb{A}$. Since

$$2\zeta z + \mathrm{T}_F^E(b_1) = \ell_{1,\zeta\urcorner^{-1}e_2+f_2}(\mathbf{v}_1^n(\xi+z\urcorner,{}^t\!x^{\tau};\,{}^t\!b^{\tau};c)),$$

its ζth Fourier coefficient vanishes by Lemma C.1(1) for each $\zeta \in F^{\times}$. Thus h is a constant function whose value is

$$\int_{\mathcal{W}_1(F)\backslash\mathcal{W}_1(\mathbb{A})}\mathcal{F}(u)\overline{\boldsymbol{\psi}(\ell_{1,f_2}(u))}\,\mathrm{d}u = \int_{\mathcal{W}_1(F)\backslash\mathcal{W}_1(\mathbb{A})}\mathcal{F}(u\gamma)\overline{\boldsymbol{\psi}(\ell_1(u))}\,\mathrm{d}u,$$

where we have taken an element $\gamma \in \mathcal{G}_{n-1}(F) \subset \mathcal{M}_1(F)$ such that $\gamma f_2 = e_2$. The right hand side is equal to

$$\int_{\mathcal{W}_2(F)\backslash\mathcal{W}_2(\mathbb{A})}\mathcal{F}(u\gamma)\overline{\boldsymbol{\psi}(\ell_{2,0}(u))}\,\mathrm{d}u = \mathscr{W}_2(\mathbf{1}_{2n-2},\mathcal{F}_{\mathfrak{P}_2})$$

by Lemma C.4. We have used Lemma C.4 in the second line. □

Put $\mathfrak{M}_{\mathbf{e}} = \cap_{k=1}^{(n-1)/2} \mathfrak{M}_{2k}$. For an automorphic form ϕ on $\mathfrak{M}_{\mathbf{e}}$ we set

$$\mathscr{W}_n(\phi) = \int_{(\mathfrak{M}_{\mathbf{e}} \cap \mathfrak{N}_\varepsilon)(F) \backslash (\mathfrak{M}_{\mathbf{e}} \cap \mathfrak{N}_\varepsilon)(\mathbb{A})} \phi(u) \overline{\boldsymbol{\psi}(\ell^{H_n}(u))} \, \mathrm{d}u.$$

Proposition C.6. *If $\mathcal{F} \in A_n^\chi(\sigma)$, then*

$$W_{H_n}(\mathcal{F}) = \int_{(\mathfrak{P}_{\mathbf{e}} \cap N_n)(\mathbb{A}) \backslash N_n(\mathbb{A})} \mathscr{W}_n(\mathcal{F}_{\mathfrak{P}_{\mathbf{e}}}(u)) \overline{\boldsymbol{\psi}(\ell^{H_n}(u))} \, \mathrm{d}u.$$

In particular, the space $A_n^\chi(\sigma)$ is nonzero.

Proof. By the inductivity stated in Lemma C.3 we can apply Lemma C.5 repeatedly to obtain the formula above. If $\mathcal{F} \in A_n^\chi(\sigma)$ is factorizable, then $\mathcal{F}_{\mathfrak{P}_{\mathbf{e}}}$ remains factorizable by the proof of Lemma C.3, and hence so does $\mathscr{W}_n(\mathcal{F}_{\mathfrak{P}_{\mathbf{e}}}) \in \otimes_v' A_n^{\chi_v}(\sigma_v)$ by uniqueness of the Whittaker model. Choose local factors $\mathfrak{I}_{H_n}^{\chi_v}$ in order that if we write $\mathscr{W}_n(\varrho(g)\mathcal{F}_{\mathfrak{P}_{\mathbf{e}}}) = \prod_v f_v(g_v)$, then the H_nth Fourier coefficient factorizes as $W_{H_n}(\mathcal{F}) = \prod_v \mathfrak{I}_{H_n}^{\chi_v}(f_v)$ and so by (7.1)

$$W_B(\mathcal{F}) = \prod_v \mathfrak{I}_B^{\chi_v}(f_v), \qquad B \in \mathcal{O}_F(H_n). \tag{C.1}$$

Each factor can be made nonzero by Proposition 4.5(2). □

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