Proceedings of The 22nd International Workshop on
Differential Geometry of Submanifolds in Symmetric Spaces 22(2019) 1-18

Geometry of R-spaces canonically embedded in Kähler C-spaces as Lagrangian submanifolds

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(2010 Mathematics Subject Classification: 53C40, 53C42, 53D12)

This article is dedicated to Professor Young Jin Suh on the occasion of his 65th birthday.

Abstract. An R-space is a compact homogeneous space obtained as an orbit of the isotropy representation of a Riemannian symmetric space. It is known that any R-space has the canonical embedding into a Kähler C-space as a real form and thus it is a compact totally geodesic Lagrangian submanifold. In this article we provide an exposition on such nice properties of R-spaces as Lagrangian submanifolds and our recent work on minimal Maslov number of R-spaces canonically embedded in Einstein-Kähler C-spaces ([20]).

1 Introduction

A smooth immersion (resp. embedding) $\varphi : L \to M$ of a smooth manifold L into a symplectic manifold (M, ω) is called a *Lagrangian immersion* (resp. *Lagrangian embedding*) if $2 \dim L = \dim M$ and $\varphi^* \omega = 0$. For a Lagrangian immersion $\varphi : L \to M$, we have the vector bundle isomorphism $\varphi^{-1}TM/\varphi_* TL \ni v \leftrightarrow \alpha_v := \omega(v, \cdot) \in T^*L$. A smooth family of Lagrangian immersions $\varphi_t : L \to M$ with $\varphi_0 = \varphi$ is called a *Lagrangian deformation* of φ, which is characterized by the closedness of the 1-form $\alpha_{V_t} \in \Omega^1(L)$ corresponding to the variational vector field $V_t := \frac{\partial \varphi_t}{\partial t} \in \varphi^{-1}TM$ for each t. A Lagrangian deformation $\varphi_t : L \to M$ with

Key words and phrases: R-spaces, Einstein-Kähler C-spaces, monotone Lagrangian submanifolds, minimal Maslov number.

* This work was partly supported by JSPS KAKENHI Grant Numbers JP17H06127, JP18K03307, JP18H03668 and by Osaka City University Advanced Mathematical Institute: (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics).

$\varphi_0 = \varphi$ is called a *Hamiltonian deformation* of φ if $\alpha_{V_t} \in \Omega^1(L)$ is exact for each t. Suppose that $[\omega] \in H^2(M, \mathbb{R})$ is an integral class, that is, there is a complex line bundle E over M and a $U(1)$-connection ∇ of E whose curvature form is equal to $2\pi\sqrt{-1}\omega$. It is known that a Lagrangian deformation $\varphi_t : L \to M$ with $\varphi_0 = \varphi$ a Hamiltonian deformation if and if a family of flat connections $\{\varphi_t^{-1}\nabla\}$ has same holonomy homomorphism $\rho : \pi_1(L) \to U(1)$.

Two group homomorphisms $I_{\mu,L} : \pi_2(M, L) \to \mathbb{Z}$ and $I_{\omega,L} : \pi_2(M, L) \to \mathbb{R}$ are defined for any Lagrangian submanifold of a symplectic manifold in general (see Section 3) so that $I_{\mu,L}$ is a symplectic invariant and $I_{\omega,L}$ is not a symplectic invariant but a Hamiltonian invariant. The minimal Maslov number of a Lagrangian submanifold in a symplectic manifold is defined as the positive generator of $\mathrm{Im} I_{\mu,L} \subset \mathbb{Z}$ as additive groups. It is very fundamental to the study of the Floer homology for intersections of Lagrangian submanifolds. The *monotonicity* for a Lagrangian submanifold of a symplectic manifold is defined by the condition that $I_{\mu,L} = \lambda\, I_{\omega,L}$ $(\exists\lambda > 0)$. The Floer homology theory for the intersection of monotone Lagrangian submanifolds was initiated and well-developed by Y.-G. Oh ([15], [16], [17], [18] and so on). It is known that any compact minimal Lagrangian submanifold of an Einstein-Kähler manifold with positive Einstein constant is monotone (Cieliebak-Goldstein [2], Hajime Ono [21]). Moreover he ([21]) gave a nice formula of the minimal Maslov number for a compact monotone Lagrangian submanifold in a simply connected Einstein-Kähler manifold with positive Einstein constant (see the formula (3.1) in Section 3).

An R-space is a compact homogeneous space obtained as an orbit of the isotropy representation of a Riemannian symmetric space. Note that an R-space is not a symmetric space in general and it is called a *symmetric R-space* when it is a symmetric space. It is known that each R-space has the canonical embedding into a Kähler C-space as a real form. A Kähler C-space is a simply connected compact homogeneous complex manifold which admits invariant Kähler metrics, and it is also called a generalized flag manifold. A real form means the fixed point subset by an anti-holomorphic isometry of a Kähler C-space and thus it is a compact embedded totally geodesic Lagrangian submanifold. So R-spaces canonically embedded in Kähler C-spaces constitute a nice class of Lagrangian submanifolds. As explained in Section 2 any R-space can be canonically embedded in an Einstein-Kähler C-space. In this case it is a compact monotone Lagrangian submanifold and so we can use H. Ono's formula in order to study the minimal Maslov number for R-spaces canonically embedded in Einstein-Kähler C-spaces. In [20] we showed a Lie theoretic formula for the minimal Maslov number of such an R-space and some examples of the calculation by that formula.

In this article we provide an exposition on such nice properties of R-spaces as Lagrangian submanifolds and our related recent work ([20]). This article is organized as follows: In Section 2 we review the definitions and elementary properties of R-spaces and their canonical embeddings into Kähler C-spaces and the description of the invariant symplectic structures, invariant complex structures, invariant Kähler metrics and invariant Einstein-Kähler metrics on a Kähler C-space. We also discuss several properties from the viewpoint of geometry of Lagrangian submanifolds such as an anti-symplectic involutive diffeomorphism, the moment maps, Morse theory and related intersection problem. In Section 3 we recall the definitions of two Hamiltonian invariants $I_{\mu,L}$ and $I_{\omega,L}$ and the monotonicity for Lagrangian submanifolds of general symplectic manifolds. Moreover we refer the monotonicity theorem and minimal Maslov number formula by Cieliebak-Goldstein and H. Ono for Lagrangian submanifolds in Einstein-Kähler manifolds. and mention our applications to the case of the Gauss images of isoparametric hypersurfaces. In Section 4 we describe the construction of the Lie theoretic formula for minimal Maslov number for R-spaces canonically embedded in Einstein-Kähler C-spaces.

Throughout this article any manifold is smooth and connected.

2 The canonical embeddings of an R-space into a Kähler C-space

In this section we review the definitions and elementary properties of R-spaces and their canonical embeddings into Käher C-spaces from the viewpoint of geometry of *Lagrangian submanifolds* (cf. [1], [22], [27], [23], [24], [25], [20]).

Let (G, K, θ) be a Riemannian symmetric pair with an involutive automorphism θ. Suppose that G is a connected compact semi-simple Lie group with Lie algebra $\mathfrak{g}$ and K is a connected compact Lie subgroup of G with Lie algebra $\mathfrak{k}$. We choose an AdG- and θ-invariant inner product $\langle\ ,\ \rangle$ of $\mathfrak{g}$ and extend it to the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ by the complex bi-linearity. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{g}$ with respect to (G, K, θ). Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Choose a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing $\mathfrak{a}$. Then we have $\mathfrak{t} = \mathfrak{b} + \mathfrak{a}, \mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}, \mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$ and $\mathfrak{t}$ is invariant by θ. Let $(\ ,\ )$ denote an inner product of $\mathfrak{t}$ defined by a restriction of $\langle\ ,\ \rangle$ to $\mathfrak{t}$. The root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}$ is given as

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Sigma(\mathfrak{g})} \mathfrak{g}^{\alpha},$$

where

$$\mathfrak{g}^{\alpha} := \{X \in \mathfrak{g}^{\mathbb{C}} \mid (\mathrm{ad}\xi)(X) = \sqrt{-1}(\alpha, \xi)X \ (\forall \xi \in \mathfrak{t})\}$$

and $\Sigma(\mathfrak{g}) \subset \mathfrak{t}$ denotes the set of all roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}$. Set $\Sigma_0(\mathfrak{g}) := \Sigma(\mathfrak{g}) \cap \mathfrak{b}$. Define the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ by $\Sigma(\mathfrak{g}, \mathfrak{a}) := \{\gamma = \bar{\alpha} \mid \alpha \in \Sigma(\mathfrak{g})\}$, where $\bar{\alpha}$ denotes the $\mathfrak{a}$-component of $\alpha \in \Sigma(\mathfrak{g}) \subset \mathfrak{t} = \mathfrak{b} \oplus \mathfrak{a}$. We define an involutive orthogonal transformation $\sigma \in O(\mathfrak{t})$ by $\sigma := -\theta|_{\mathfrak{t}}$. We choose a σ-order $>$ on $\mathfrak{t}$ so that if $\alpha \in \Sigma(\mathfrak{g}) \setminus \Sigma_0(\mathfrak{g})$ and $\alpha > 0$, then $\sigma\alpha > 0$ and thus $\theta\alpha = -\sigma\alpha < 0$ ([22]). Set $\Sigma^+(\mathfrak{g}) := \{\alpha \in \Sigma(\mathfrak{g}) \mid \alpha > 0\}$, $\Sigma_0^+(\mathfrak{g}) := \Sigma_0(\mathfrak{g}) \cap \Sigma^+(\mathfrak{g})$ and $\Sigma^+(\mathfrak{g}, \mathfrak{a}) := \{\gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid \gamma > 0\} = \{\bar{\alpha} \mid \alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_0(\mathfrak{g})\}$.

We choose $E_\alpha \in \mathfrak{g}^\alpha$ for $\alpha \in \Sigma(\mathfrak{g})$ such that $[E_\alpha, E_{-\alpha}] = \sqrt{-1}\alpha, \langle E_\alpha, E_{-\alpha}\rangle = 1, \overline{E_\alpha} = E_{-\alpha}$ for each $\alpha \in \Sigma(\mathfrak{g})$ and let $\{\omega^\alpha \mid \alpha \in \Sigma(\mathfrak{g})\}$ be the linear forms on $\mathfrak{g}^{\mathbb{C}}$ dual to $\{E_\alpha \mid \alpha \in \Sigma(\mathfrak{g})\}$ so that $\omega^\alpha(\mathfrak{t}^{\mathbb{C}}) = \{0\}, \omega^\alpha(E_\beta) = \delta_{\alpha\beta}$ for each $\alpha, \beta \in \Sigma(\mathfrak{g})$. The restricted root space decompositions of $\mathfrak{k}$ and $\mathfrak{p}$ with respect to $\mathfrak{a}$ are given as

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\gamma \in \Sigma^+(\mathfrak{g},\mathfrak{a})} \mathfrak{k}_\gamma, \quad \mathfrak{p} = \mathfrak{a} + \sum_{\gamma \in \Sigma^+(\mathfrak{g},\mathfrak{a})} \mathfrak{p}_\gamma,$$

where $\mathfrak{k}_0 := \{X \in \mathfrak{k} \mid (\mathrm{ad}H)X = 0 \ (\forall H \in \mathfrak{a})\}$ and for each $\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$ set

$$\mathfrak{k}_\gamma := \{X \in \mathfrak{k} \mid (\mathrm{ad}H)^2 X = -(\gamma, H)^2 X \ (\forall H \in \mathfrak{a})\},$$
$$\mathfrak{p}_\gamma := \{X \in \mathfrak{p} \mid (\mathrm{ad}H)^2 Y = -(\gamma, H)^2 Y \ (\forall H \in \mathfrak{a})\}.$$

For $\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$, there are an orthonormal basis $\{S_{\gamma,i} \mid i = 1, \cdots, m(\gamma)\}$ of $\mathfrak{k}_\gamma$ and an orthonormal basis $\{T_{\gamma,i} \mid i = 1, \cdots, m(\gamma)\}$ of $\mathfrak{p}_\gamma$, where $m(\gamma) := \dim \mathfrak{k}_\gamma = \dim \mathfrak{p}^\sharp_\gamma$, such that $[H, S_{\gamma,i}] = (\gamma, H)T_{\gamma,i}$, $[H, T_{\gamma,i}] = -(\gamma, H)S_{\gamma,i}$ for each $H \in \mathfrak{a}$.

Now we fix an arbitrary non-zero element Z of $\mathfrak{a}$. Set

$$\Sigma_Z(\mathfrak{g}) := \{\alpha \in \Sigma(\mathfrak{g}) \mid (\alpha, Z) = 0\} \quad \text{and} \quad \Sigma^+_Z(\mathfrak{g}) := \Sigma_Z(\mathfrak{g}) \cap \Sigma^+(\mathfrak{g}).$$

The element Z is called *reqular* if $\Sigma_Z(\mathfrak{g}) = \Sigma_0(\mathfrak{g})$. Define closed subgroups G_Z and K_Z of G by

$$G_Z := \{a \in G \mid \mathrm{Ad}(a)Z = Z\}$$

and

$$K_Z := \{a \in K \mid \mathrm{Ad}(a)Z = Z\} = K \cap G_Z.$$

It is well-known that G_Z is always connected. Denote by $\mathfrak{g}_Z$ and $\mathfrak{k}_Z$ Lie algebras of G_Z and K_Z, respectively. Note that $\theta(G_Z) = G_Z$, $\theta(\mathfrak{g}_Z) = \mathfrak{g}_Z$ and thus $(G_Z, K_H, \theta|_{G_Z})$ is also a compact symmetric pair.

Definition. The compact homogeneous space $L := K/K_Z$ is called an *R-space*, and it has the *standard imbedding* into the vector space $\mathfrak{p}$ defined by

$$\varphi_Z : L = K/K_Z \ni aK_Z \longmapsto \mathrm{Ad}(a)Z \in \mathrm{Ad}(K)Z \subset \mathfrak{p}. \tag{2.1}$$

If Z is a regular element of $\mathfrak{a}$, then $L = K/K_Z$ is called a *regular R-space*. Another compact homogeneous space $M := G/G_Z$ is called a *generalized flag manifold* or a *Kähler C-space*, and it also has the *standard imbedding* into the Lie algebra $\mathfrak{g}$

$$\Phi_Z : M = G/G_Z \ni aG_Z \longmapsto \mathrm{Ad}(a)Z \in \mathrm{Ad}(G)Z \subset \mathfrak{g}. \tag{2.2}$$

The *canonical embedding* of K/K_Z into G/G_Z is a map defined by

$$\iota_Z : L = K/K_Z \ni aK_Z \longmapsto aG_Z \in G/G_Z = M. \tag{2.3}$$

We take the orthogonal direct sum decompositions of $\mathfrak{g}$ and $\mathfrak{k}$ as $\mathfrak{g} = \mathfrak{g}_Z + \mathfrak{m}$, $\mathfrak{m} \cong T_{eG_Z}M$ and $\mathfrak{k} = \mathfrak{k}_Z + \mathfrak{l}$, $\mathfrak{l} \cong T_{eK_Z}L$. Note that $\mathfrak{k}_Z = \mathfrak{k} \cap \mathfrak{g}_Z$. By using the property $\theta(\mathfrak{g}_Z) = \mathfrak{g}_Z$ one can show that ι_Z is an embedding and $2 \dim L = \dim M$.

The author has heard from Professor Masaru Takeuchi that the "R-space"was named first by Jacques Tits ([31]). Here we should notice that an R-space is not a symmetric space in general, and however the R-space can be considered as a class of the most important compact homogeneous spaces related to symmetric spaces. An R-space K/K_Z is called a *symmetric R-space* if K/K_Z is a symmetric space. It is known that an R-space is a symmetric R-space if and only if one of the following conditions is satisfied:

(1) (K, K_Z) is a symmetric pair.

(2) There is an element $Z \in \mathfrak{p}$ satisfying the equation $(\mathrm{ad}Z)^3 + (\mathrm{ad}Z) = 0$ such that $L = K/K_Z$ and $G = G/G_Z$.

(3) (G, G_Z) is a Hermitian symmetric pair.

(4) The standard imbedding φ_Z has the parallel second fundamental form (Dirk Ferus [3]).

(5) $\varphi_Z(L)$ is an (extrinsic) symmetric submanifold in Euclidean space $\mathfrak{p}$ (Dirk Ferus [4]).

For such Z, we can define a G-invariant symplectic form ω_Z on $M = G/G_Z$ by

$$\omega_Z(X, Y) := \langle [Z, X], Y \rangle \quad \text{for each } X, Y \in \mathfrak{g}.$$

and ω_Z can be also expressed as

$$\omega_Z = -\sqrt{-1} \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} (Z, \alpha)\, \omega^{-\alpha} \wedge \overline{\omega^{-\alpha}}.$$

Then the canonical embedding $\iota_Z : L = K/K_Z \to G/G_Z = M$ is a Lagrangian embedding with respect to ω_Z.

The involutive automorphism θ of G induces an involutive diffeomorphism

$$\hat{\theta}_Z : M = G/G_Z \ni aG_Z \longmapsto \theta(a)G_Z \in G/G_Z = M \tag{2.4}$$

which is equivariant with respect to the Lie group automorphism $\theta : G \to G$ Then $\hat{\theta}_Z : G/G_Z \to G/G_Z$ is anti-symplectic with respect to ω_Z, that is, $\hat{\theta}_Z^* \omega_Z = -\omega_Z$.

Define the fixed point subset of M by $\hat{\theta}_Z$ as

$$\mathrm{Fix}(M, \hat{\theta}_Z) := \{p \in M \mid \hat{\theta}_Z(p) = p\}. \tag{2.5}$$

Then $\iota_Z(K/K_Z) \subset \mathrm{Fix}(M, \hat{\theta}_Z) \subset G/G_Z$.

The natural left action of G on a symplectic manifold $(M = G/G_Z, \omega_Z)$ is a Hamiltonian group action with the moment map

$$\mu_G := \Phi_Z : G/G_Z \longrightarrow \mathfrak{g} \cong \mathfrak{g}^*. \tag{2.6}$$

Moreover the natural left action of $K \subset G$ on $(M = G/G_Z, \omega_Z)$ is also a Hamiltonian group action with the moment map

$$\mu_K := \pi_{\mathfrak{k}} \circ \mu_G = \pi_{\mathfrak{k}} \circ \Phi_Z : G/G_Z \longrightarrow \mathfrak{k} \cong \mathfrak{k}^*. \tag{2.7}$$

Here $\pi_{\mathfrak{k}} : \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \longrightarrow \mathfrak{k}$ denotes the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{k}$. Then $\mu_G \circ \hat{\theta}_Z = -\theta \circ \mu_G$ and $\mu_K \circ \hat{\theta}_Z = -\mu_K$. It follows from these formulas that

$$\mathrm{Fix}(M, \hat{\theta}_Z) = \mu_K^{-1}(0). \tag{2.8}$$

Since K and M are compact, by a result of Kirwan ([12, p.549, (3.1)]) $\mu_K^{-1}(0)$ is connected and thus $\mathrm{Fix}(M, \hat{\theta}_Z)$ is also connected. Therefore we obtain

$$\iota_Z(K/K_Z) = \mathrm{Fix}(M, \hat{\theta}_Z) = \mu_K^{-1}(0). \tag{2.9}$$

The Weyl group of (G, K) is defined by $W(G, K) := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. By the action of the Weyl group $W(G, K)$, we may assume that $Z \in \mathfrak{a} \subset \mathfrak{t}$ satisfies $(\alpha, Z) \geq 0$ for $\forall \alpha \in \Sigma^+(\mathfrak{g})$.

Now we describe an invariant complex structure on $M = G/G_Z$ corresponding to Z. Note that $Z \in \mathfrak{c}(\mathfrak{g}_Z) \subset \mathfrak{t} \subset \mathfrak{g}_Z$. Then

$$\mathfrak{g}_Z^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{\alpha},$$

$$T_{eG_Z}(G/G_Z)^{\mathbb{C}} \cong \mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{-\alpha} + \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{\alpha}$$

Note that $\mathfrak{g}^{\alpha} = \overline{\mathfrak{g}^{-\alpha}}$. Thus we can define a G-invariant complex structure J_Z on G/G_Z such that

$$T_{eG_Z}(G/G_Z)^{1,0} \cong \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{-\alpha}, \quad T_{eG_Z}(G/G_Z)^{0,1} \cong \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} .\mathfrak{g}^{\alpha},$$

Since

$$\theta \left(\sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{-\alpha} \right) = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z(\mathfrak{g})} \mathfrak{g}^{\alpha},$$

the involutive diffeomorphism $\hat{\theta}_Z : G/G_Z \to G/G_Z$ is anti-holomorphic with respect to J_Z, that is, $J_Z \circ d\hat{\theta}_Z = -d\hat{\theta}_Z \circ J_Z$.

Moreover the corresponding G-invariant Kähler metric g_Z on $M = G/G_Z$ is defined by

$$\omega_Z(X, Y) = (-1) g_Z(J_Z X, Y) \quad \text{for each } X, Y \in \mathfrak{m} \tag{2.10}$$

or

$$g_Z = \sum_{\alpha \in \Sigma^+(\mathfrak{g}) \setminus \Sigma_Z^+(\mathfrak{g})} (Z, \alpha)\, \omega^{-\alpha} \cdot \overline{\omega^{-\alpha}}.$$

Hence the diffeomorphism $\hat{\theta}_Z : M \to M$ is an isometry of M with respect to g_Z.

Let $\Pi := \Pi(\mathfrak{g}) = \{\alpha_1, \cdots, \alpha_\ell\}$ be the fundamental root system of $\mathfrak{g}$ with respect to the σ-order $<$ of $\mathfrak{t}$. Set $\Pi(\mathfrak{g})_0 := \Pi(\mathfrak{g}) \cap \mathfrak{b}$. For the above Z, set $\Pi_Z := \Pi_Z(\mathfrak{g}) := \{\alpha_i \in \Pi(\mathfrak{g}) \mid (\alpha_i, Z) = 0\}$. Note that $\Pi_0(\mathfrak{g}) \subset \Pi_Z(\mathfrak{g})$ and thus $\Pi(\mathfrak{g})\setminus\Pi_Z(\mathfrak{g}) \subset \Pi(\mathfrak{g})\setminus\Pi_0(\mathfrak{g})$. Let $\{\Lambda_1,\cdots,\Lambda_\ell\}\subset\mathfrak{t}$ be the fundamental weight system of $\mathfrak{g}$ corresponding to $\Pi(\mathfrak{g})$ defined by

$$\frac{2(\Lambda_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij} \quad (i,j=1,\cdots,\ell).$$

Now we set

$$\mathfrak{c}_Z := \bigoplus_{\alpha_i\in\Pi\setminus\Pi_Z} \mathbb{R}\Lambda_i \subset \mathfrak{t}, \quad \mathbb{Z}_{\mathfrak{c}_Z} := \bigoplus_{\alpha_i\in\Pi\setminus\Pi_Z} \mathbb{Z}\Lambda_i \subset \mathfrak{c}_Z.$$

Note that $\mathfrak{c}_Z$ coincides with the center $\mathfrak{c}(\mathfrak{g}_Z)$ of $\mathfrak{g}_Z$. Define

$$\mathfrak{c}_Z^+ := \bigoplus_{\alpha_i\in\Pi\setminus\Pi_Z} \mathbb{R}^+\Lambda_i \subset \mathfrak{c}_Z, \quad \mathbb{Z}^+_{\mathfrak{c}_Z} := \bigoplus_{\alpha_i\in\Pi\setminus\Pi_Z} \mathbb{Z}^+\Lambda_i \subset \mathfrak{c}_Z^+$$

where $\mathbb{R}^+$ and $\mathbb{Z}^+$ denote all positive real numbers and all positive integers, respectively. Obviously we have $Z\in\mathfrak{c}_Z^+$.

For each $\lambda\in\mathfrak{c}_Z^+$, since $\Pi_\lambda=\Pi_Z$, $\Sigma_\lambda(\mathfrak{g})=\Sigma_Z(\mathfrak{g})$, we have $\mathfrak{g}_\lambda^{\mathbb{C}}=\mathfrak{g}_Z^{\mathbb{C}}$ and thus $\mathfrak{g}_\lambda=\mathfrak{g}_Z$. By the connectedness of G_ξ and G_Z, we obtain $G_\lambda=G_Z$ and $G/G_\lambda=G/G_Z=M$. In particular ω_λ is a G-invariant symplectic form on $M=G/G_Z=G/G_\lambda$. However λ and H define the same G-invariant complex structure $J_\lambda=J_H$ on $M=G/G_H=G/G_\lambda$.

Since $\theta(\mathfrak{g}_Z)=\mathfrak{g}_Z$ and thus $\theta(\mathfrak{c}(\mathfrak{g}_Z))=\mathfrak{c}(\mathfrak{g}_Z)$, there is a direct sum decomposition

$$\mathfrak{c}(\mathfrak{g}_Z)=\mathfrak{c}_Z=(\mathfrak{c}_Z\cap\mathfrak{b})+(\mathfrak{c}_Z\cap\mathfrak{a}).$$

For each $H\in\mathfrak{c}_Z^+\cap\mathfrak{a}$, since $G_H=G_Z$ and $G/G_H=G/G_Z$, we have $K_H=K\cap G_H=K\cap G_Z=K_Z$ and thus $K/K_H=K/K_Z=L$. Hence all $H\in\mathfrak{c}_Z^+\cap\mathfrak{a}$ correspond to the same R-space $L=K/K_Z$ and the convex set $\mathfrak{c}_Z^+\cap\mathfrak{a}$ parametrizes orbits of the same type K_Z.

Let $\mathcal{I}^2_G(M)$ denote the real vector space of all G-invariant closed 2-forms on $M=G/G_Z$. Then the natural linear map $\mathfrak{w}:\mathcal{I}^2_G(M)\ni\omega\longmapsto[\omega]\in H^2(M,\mathbb{R})$. is a linear isomorphism and there is a linear isomorphism $\omega:\frac{1}{2\pi\sqrt{-1}}\mathfrak{c}_Z\longrightarrow\mathcal{I}^2_G(M)$ defined by

$$\omega\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)(X,Y):=-\frac{1}{2\pi}\langle[\lambda,X],Y\rangle \quad (X,Y\in\mathfrak{m})$$

for each $\lambda\in\mathfrak{c}_Z$. Moreover the linear isomorphism $\tau=\mathfrak{w}\circ\omega:\frac{1}{2\pi\sqrt{-1}}\mathfrak{c}_Z\longrightarrow\mathcal{I}^2_G(M)\longrightarrow H^2(M,\mathbb{R})$ given by the transgression operator is restricted to a $\mathbb{Z}$-module isomorphism $\tau=\mathfrak{w}\circ\omega:\frac{1}{2\pi\sqrt{-1}}\mathbb{Z}_{\mathfrak{c}_Z}\longrightarrow H^2(M,\mathbb{Z})$. The 2nd cohomology and homology groups of G/G_Z are described as follows:

$$\begin{array}{ccccc}
\mathfrak{c}_Z=\displaystyle\bigoplus_{\alpha_i\in\Pi\setminus\Pi_Z}\mathbb{R}\Lambda_{\alpha_i} & \ni\ \lambda\longleftrightarrow & \left[\frac{1}{2\pi\sqrt{-1}}\omega(\lambda)\right]\in & H^2(G/G_Z,\mathbb{R})\\
\cup & & & \cup\\
\mathbb{Z}_{\mathfrak{c}_Z}=\displaystyle\bigoplus_{\alpha_i\in\Pi\setminus\Pi_H}\mathbb{Z}\Lambda_{\alpha_i} & \longleftrightarrow & & H^2(G/G_Z,\mathbb{Z})
\end{array}$$

and if we set $\alpha_i^* := \dfrac{2\alpha_i}{(\alpha_i,\alpha_i)}$,

$$\begin{array}{llll}
\mathfrak{c}_Z^* := \bigoplus_{\alpha_i\in\Pi\setminus\Pi_Z} \mathbb{R}\alpha_i^* & \ni \sum_i x_i\alpha_i^* \longleftrightarrow \sum_i x_i[S^2(\alpha_i^*)] & \in & H_2(G/G_Z,\mathbb{R}) \\
\cup & & & \cup \\
\mathbb{Z}_{\mathfrak{c}_Z}{}^* := \bigoplus_{\alpha_i\in\Pi\setminus\Pi_Z} \mathbb{Z}\alpha_i^* & \longleftrightarrow & & H_2(G/G_Z,\mathbb{Z}) \cong \pi_2(G/G_Z).
\end{array}$$

For each $\lambda \in \mathfrak{c}_Z^+$, define a G-invariant Kähler metric on a complex manifold $(M = G/G_Z, J_Z)$ by

$$g\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right) := \frac{1}{2\pi}\sum_{\alpha\in\Sigma^+(\mathfrak{g})\setminus\Sigma_Z(\mathfrak{g})} (\lambda,\alpha)\omega^{-\alpha}\cdot\bar{\omega}^{-\alpha}$$

whose Kähler form coincides with $\omega\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)$ as

$$\omega\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)(X,Y) = g\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)(J_Z X, Y).$$

Note that $Z \in \mathfrak{c}_Z^+ \cap \mathfrak{a}$ and $\omega(\frac{1}{2\pi\sqrt{-1}}2\pi Z) = -\omega_Z$. Namely, the convex open set $\mathfrak{c}_Z^+$ of the vector space $\mathfrak{c}_Z$ parametrizes all G-invariant Kähler metrics on $M = G/G_Z$ relative to the complex structure J_Z. So the parameter spaces of all G-inv. Kähler metrics on G/G_Z are given as

$$\begin{array}{llll}
\mathfrak{c}_Z^+ = \bigoplus_{\alpha_i\in\Pi\setminus\Pi_Z} \mathbb{R}^+\Lambda_{\alpha_i} & \ni \lambda \longleftrightarrow \omega\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right) & \in & \{G\text{-inv. Kähler met. on } G/G_Z\} \\
\cup & & & \cup \\
\mathbb{Z}_{\mathfrak{c}_Z}^+ = \bigoplus_{\alpha_i\in\Pi\setminus\Pi_Z} \mathbb{Z}^+\Lambda_{\alpha_i} & \longleftrightarrow & & \{G\text{-inv. Hodge met. on } G/G_Z\}.
\end{array}$$

For each $H \in \mathfrak{c}_Z^+ \cap \mathfrak{a}$, the diffeomorphism $\hat{\theta}_Z : M = G/G_Z \to M = G/G_Z$ preserves a G-invariant Kähler metric $g\left(\frac{1}{2\pi\sqrt{-1}}H\right)$ on M, that is, $\hat{\theta}_Z : M = G/G_Z \to M = G/G_Z$ is an isometry with respect to $g\left(\frac{1}{2\pi\sqrt{-1}}H\right)$. Hence the canonically embedded R-space $\iota_Z(K/K_Z)$ is a *real form*, that is, the fixed point subset of a Käher C-space $M = G/G_Z$ by the anti-holomorphic isometry $\hat{\theta}_Z$ with respect to J_Z and a Kähler metric $g\left(\frac{1}{2\pi\sqrt{-1}}H\right)$ for any $H \in \mathfrak{c}_Z^+ \cap \mathfrak{a}$.

Next we mention the characterization of a G-invariant Einstein-Kähler metric on $M = G/G_Z$. Set

$$\delta_{\mathfrak{m}} := \frac{1}{2}\sum_{\alpha\in\Sigma^+(\mathfrak{g})\setminus\Sigma_Z(\mathfrak{g})} \alpha \in \mathfrak{t}.$$

Lemma 2.1 ([1]).

$$2\delta_{\mathfrak{m}} = \sum_{\alpha\in\Sigma^+(\mathfrak{g})\setminus\Sigma_H(\mathfrak{g})} \alpha \in \mathbb{Z}_{\mathfrak{c}_H}^+ = \bigoplus_{\alpha\in\Pi\setminus\Pi_H} \mathbb{Z}^+\Lambda_\alpha. \tag{2.11}$$

and it corresponds to the first Chern class of the complex manifold (M, J_H)*:*

$$c_1(M) = \left[\omega\left(\frac{1}{2\pi\sqrt{-1}}2\delta_{\mathfrak{m}}\right)\right] = \tau\left(\frac{1}{2\pi\sqrt{-1}}2\delta_{\mathfrak{m}}\right).$$

Proposition 2.2 ([24])**.** *The G-invariant Kähler metric $g = g\left(\frac{1}{2\pi\sqrt{-1}}\lambda\right)$ on M is Einstein if and only if $\lambda = b\,\delta_{\mathfrak{m}}$ for some $b > 0$.*

Then we can show that $2\delta_{\mathfrak{m}} \in \mathfrak{a}$ ([20]). Therefore we obtain

Proposition 2.3. *The element $Z^{ein} := 2\delta_{\mathfrak{m}} \in Z^+_{\mathfrak{c}_Z} \cap \mathfrak{a} \subset \mathfrak{c}^+_Z \cap \mathfrak{a}$ corresponds to the canonical embedding $\iota_{Z^{ein}}$ of the same R-space $L = K/K_Z$ into an Einstein-Kähler C-space $\left(M = G/G_Z, \omega_{Z^{ein}}, J_Z, g\left(\frac{1}{2\pi\sqrt{-1}}Z^{ein}\right)\right)$. Moreover, the element Z^{ein} is such a unique element of $\mathfrak{c}^+_Z \cap \mathfrak{a}$ up to the multiplication by a positive constant.*

Here we shall mention about geometry of R-spaces as homogeneous spaces of noncompact real semisimple Lie groups. Set $\mathfrak{p}^\sharp := \sqrt{-1}\mathfrak{p}$. Then $\mathfrak{g}^\sharp := \mathfrak{k} + \mathfrak{p}^\sharp$ is the Cartan decomposition of a noncompact real semisimple Lie algebra $\mathfrak{g}^\sharp$ with Cartan involution τ. Let $G^{\mathbb{C}}$ be a connected complex Lie group without center with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and then G can be regarded as an analytic subgroup of $G^{\mathbb{C}}$. Let $G^\sharp$ be a connected real semisimple Lie subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^\sharp$. The root space decomposition of $\mathfrak{g}^\sharp$ with respect to $\sqrt{-1}\mathfrak{a}$ is given as

$$\mathfrak{g}^\sharp = \mathfrak{g}^\sharp_0 + \bigoplus_{\gamma\in\Sigma(\mathfrak{g},\mathfrak{a})} \mathfrak{g}^\sharp_\gamma,$$

where $\mathfrak{g}^\sharp_0 := \{X \in \mathfrak{g}^\sharp \mid [\sqrt{-1}H, X] = 0\ (\forall H \in \mathfrak{a})\}$ and for each $\gamma \in \Sigma(\mathfrak{g}, \mathfrak{a})$

$$\mathfrak{g}^\sharp_\gamma := \{X \in \mathfrak{g}^\sharp \mid [\sqrt{-1}H, X] = (\gamma, H)X\ (\forall H \in \mathfrak{a})\}.$$

Then

$$\mathfrak{u} := \mathfrak{g}^\sharp_0 + \bigoplus_{\gamma\in\Sigma(\mathfrak{g},\mathfrak{a}),\gamma(Z)\geq 0} \mathfrak{g}^\sharp_\gamma.$$

is a parabolic subalgebra of $\mathfrak{g}^\sharp$. Let U be a parabolic subgroup of $G^\sharp$ with Lie algebra $\mathfrak{u}$, which is always connected. The complexification of $\mathfrak{u}$

$$\mathfrak{u}^{\mathbb{C}} = (\mathfrak{g}^\sharp_0)^{\mathbb{C}} + \bigoplus_{\gamma\in\Sigma(\mathfrak{g},\mathfrak{a}),\gamma(Z)\geq 0} (\mathfrak{g}^\sharp_\gamma)^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \bigoplus_{\alpha\in\Sigma(\mathfrak{g}),\alpha(Z)\geq 0} \mathfrak{g}^{\mathbb{C}}_\alpha.$$

is a complex parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let $U^{\mathbb{C}}$ be a complex parabolic subgroup of $G^\sharp$ with Lie algebra $\widetilde{\mathfrak{u}}$, which is always connected. Then we know ([23]) that

(2.12) $\quad K\,U = G^\sharp,\ K \cap U = K_Z,$ and thus $L = K/K_Z \cong G^\sharp/U,$

(2.13) $\quad G\,U^{\mathbb{C}} = G^{\mathbb{C}},\ G \cap U^{\mathbb{C}} = G_Z,$ and thus $G/G_Z \cong M = G^{\mathbb{C}}/U^{\mathbb{C}}.$

The induced complex structure of M under the identification of $M = G/G_Z$ with the complex homogeneous space $G^{\mathbb{C}}/U^{\mathbb{C}}$ coincides with the G-invariant complex structure J_Z of M.

Define two subgroups of K and K_Z as

$$N_K(\mathfrak{a}) := \{k \in K \mid \mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}\} \subset K, \quad N_{K_Z}(\mathfrak{a}) := N_K(\mathfrak{a}) \cap K_Z \subset K_Z.$$

Note that $N_{K_Z}(\mathfrak{a})$ is not a normal subgroup of $N_K(\mathfrak{a})$, $C_K(\mathfrak{a}) \subset N_{K_Z}(\mathfrak{a})$, and if $Z \in \mathfrak{a}$ is regular, then $C_K(\mathfrak{a}) = N_{K_Z}(\mathfrak{a})$:

$$1 \longrightarrow N_{K_Z}(\mathfrak{a})/C_K(\mathfrak{a}) \longrightarrow W(G,K) = N_K(\mathfrak{a})/C_K(\mathfrak{a}) \longrightarrow N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a}) \longrightarrow 1$$

Theorem 2.4 (Masaru Takeuchi [23]). *Let $k_1, \cdots, k_b \in N_K(\mathfrak{a})$ be complete representatives of $N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a}) = \{[k_1], \cdots, [k_b]\}$. Then the orbits $Nk_1o, \cdots, Nk_bo$ of N through the origin $o = eU \in G^\sharp/U = L$ provide a cellular decomposition of L as $L = G^\sharp/U = Nk_1o \cup \cdots \cup Nk_bo$ and these cells are all cycles* mod 2 *of L. In particular,* $\dim H_*(L;\mathbb{Z}_2) = \sum_{i=0}^{\dim L} \dim H_i(L;\mathbb{Z}_2) = b$.

We briefly discuss related results of Masaru Takeuchi and Shoshichi Kobayashi ([27]) on perfect Morse functions on R-spaces. For each $X \in \mathfrak{g}$, we define a linear function $u_X : \mathfrak{g} \to \mathbb{R}$ defined by $u_X(\xi) := \langle \xi, X\rangle$ for each $\xi \in \mathfrak{g}$. A smooth function $\tilde{f}_X$ on $M = G/G_Z$ is defined by

$$\tilde{f}_X := u_X \circ \Phi_Z = u_X \circ \mu_G = \langle \mu_G, X\rangle : M = G/G_Z \longrightarrow \mathfrak{g} \longrightarrow \mathbb{R}.$$

Then by the moment map equation and (2.10) we have

$$d\tilde{f}_X = d\langle \mu_G, X\rangle = \omega_Z(\widetilde{X}, \cdot) = -g_Z(J_Z\widetilde{X}, \cdot), \tag{2.14}$$

where $\widetilde{X}$ denotes a vector field on $M = G/G_Z$ induced by a one-parameter subgroup $\{\exp(tX) \mid t \in \mathbb{R}\}$ of G, which is a Killing vector field on M with respect to a Kähler metric g_Z. Hence the gradient vector field $\mathrm{grad}(\tilde{f}_X)$ of the function $\tilde{f}_X$ on $M = G/G_Z$ with respect to the invariant Kähler metric g_Z is equal to $-J_Z\widetilde{X}$:

$$-\mathrm{grad}(\tilde{f}_X) = J_Z\widetilde{X} = (\widetilde{\sqrt{-1}X}). \tag{2.15}$$

Here $J_Z\widetilde{X} = (\widetilde{\sqrt{-1}X})$ is a holomorphic vector field on $M = G^{\mathbb{C}}/U^{\mathbb{C}}$ induced by a one-parameter subgroup $\{\exp(t\sqrt{-1}X) \mid t \in \mathbb{R}\}$ of $G^{\mathbb{C}}$.

Now assume that $X \in \mathfrak{p}$. A smooth function f_X on $L = K/K_Z$ is defined by

$$f_X = \tilde{f}_X \circ \iota_Z = u_X \circ \varphi_Z = \langle \mu_G \circ \iota_Z, X\rangle : L = K/K_Z \longrightarrow \mathfrak{p} \longrightarrow \mathbb{R}$$

By pulling back the equation (2.14) by the canonical embedding ι_Z, we have

$$df_X = \iota_Z^* d\tilde{f}_X = \iota_Z^*\omega_Z(\widetilde{X}, \cdot) = -g_Z((\widetilde{\sqrt{-1}X}) \circ \iota_Z, (\iota_Z)_*(\cdot)). \tag{2.16}$$

Since $\sqrt{-1}X \in \sqrt{-1}\mathfrak{p} = \mathfrak{p}^\sharp \subset \mathfrak{g}^\sharp \subset \mathfrak{g}^{\mathbb{C}}$, $\exp(t\sqrt{-1}X) \mid t \in \mathbb{R}$ is a one-parameter subgroup of $G^\sharp$ and it induces a vector field $(\widehat{\sqrt{-1}X})$ on $L = G^\sharp/U$. Since $(\iota_Z)_*((\widehat{\sqrt{-1}X})) = (\widetilde{\sqrt{-1}X}) \circ \iota_Z = J_Z\widetilde{X} \circ \iota_Z$, the equation (2.16) becomes

$$df_X = -(\iota_Z^* g_Z)((\widehat{\sqrt{-1}X}), \cdot\,) \tag{2.17}$$

Hence the gradient vector field $\mathrm{grad}(f_X)$ on $L = K/K_Z$ with respect to the induced Riemannian metric $\iota_Z^* g_Z$ is equal to a vector field $-(\widehat{\sqrt{-1}X})$ on $L = K/K_Z = G^\sharp/U$ induced by $-\sqrt{-1}X \in \sqrt{-1}\mathfrak{p} \subset \mathfrak{g}^\sharp$. In particular, the critical point set of f_X on L coincides with the zero set $\mathrm{Zero}(\widehat{\sqrt{-1}X}) = \mathrm{Zero}(\widetilde{X} \circ \iota_Z)$ of vector fields $(\widehat{\sqrt{-1}X})$ and $\widetilde{X} \circ \iota_Z$ on L. In [27] they showed that for each $X = \mathrm{Ad}(k)H \in \mathfrak{p}$ $(k \in K, H \in \mathfrak{a})$, it holds $\ell_k(N_K(\mathfrak{a})eK_Z) \subset \mathrm{Zero}(\widehat{\sqrt{-1}X})$ and if X is regular, then $\ell_k(N_K(\mathfrak{a})eK_Z) = \mathrm{Zero}(\widehat{\sqrt{-1}X})$. Here $\ell_k : K/K_Z \to K/K_Z$ denotes the left natural action by $k \in K$ on K/K_Z. Therefore, for each regular $X \in \mathfrak{p}$, the number of critical points of f_X is equal to $\sharp(N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a}))$ and thus $b = \dim H_*(K/K_Z, \mathbb{Z}_2)$. In particular f_X is a perfect Morse function on the R-space L for each regular $X \in \mathfrak{p}$ ([27]). We can also observe that for each regular $X \in \mathfrak{p}$ the equality $\sharp(\exp(tX)\,\iota_Z(L) \cap \iota_Z(L)) = \sharp(N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a}))$ holds for any sufficiently small $t \neq 0$.

Here we recall some fundamental results from the structure theory of a compact symmetric space G/K (cf. [6], [26]). They are necessary to discuss the geometry of R-spaces canonically embedded in Kähler C-spaces. Set $A := \exp\mathfrak{a} \subset G$ and $\hat{A} := A\,(eK) \subset G/K$. Then the K-equivariant map

$$\psi : K/Z_K(\mathfrak{a}) \times \hat{A} \ni (kZ_K(\mathfrak{a}), \hat{a}) \longmapsto k\hat{a} \in G/K \tag{2.18}$$

is a surjective smooth map. Define the diagram of a compact symmetric pair (G, K) by

$$\mathbf{D}(\mathfrak{g}, \mathfrak{a}) := \{H \in \mathfrak{a} \mid (\gamma, H) \in \pi\mathbb{Z}\ (\exists\gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}))\}$$

and thus we have $\mathfrak{a} \setminus \mathbf{D}(\mathfrak{g}, \mathfrak{a}) = \{H \in \mathfrak{a} \mid (\gamma, H) \notin \pi\mathbb{Z}\ (\forall\gamma \in \Sigma(\mathfrak{g}, \mathfrak{a}))\}$. Set $\hat{A}_s := (\exp\mathbf{D}(\mathfrak{g}, \mathfrak{a}))eK \subset \hat{A}$ and $\hat{A}_r := \hat{A} \setminus \hat{A}_s = (\exp(\mathfrak{a} \setminus \mathbf{D}(\mathfrak{g}, \mathfrak{a})))eK$. Each element of $\hat{A}_r$ (resp. $\hat{A}_s$) is called a *regular* (resp. *singular*) element of $\hat{A}$. Then $G/K = (G/K)_r \cup (G/K)_s$ (disjoint union), where $(G/K)_s := \psi(K/Z_K(\mathfrak{a}) \times \hat{A}_s)$ is a closed set of codimension at least 2 and

$$(G/K)_r := \psi(K/Z_K(\mathfrak{a}) \times \hat{A}_r) \tag{2.19}$$

is a connected open dense subset of G/K. Each element of $(G/K)_r$ is called a *regular* element of G/K. The surjective smooth map

$$\psi : K/Z_K(\mathfrak{a}) \times \hat{A}_r \longrightarrow (G/K)_r \tag{2.20}$$

is a covering map whose covering transformation group is the right natural action of $W(G, K)$ on $K/Z_K(\mathfrak{a}) \times \hat{A}_r$, and thus it induces a diffeomorphism $\bar{\psi} : (K/Z_K(\mathfrak{a}) \times \hat{A}_r)/W(G, K) \longrightarrow (G/K)_r$ which is equivariant with the actions of K. Here note that $K/Z_K(\mathfrak{a}) \times \hat{A}_r$ is not connected in general.

Using the geometry of a compact symmetric space G/K, we discuss the intersection property of $a\,\iota_Z(L)$ and $\iota_Z(L)$ under the left group action of $a \in G$ on $M = G/G_Z$.

For any $a \in G$, by the surjectivity of ψ we have $aK = \psi(kZ_K(\mathfrak{a}), \exp(H)eK)$ for some $k \in K$ and some $H \in \mathfrak{a}$ and thus $ak_1 = kk_0 \exp(H)$ for some $k_0 \in Z_K(\mathfrak{a})$ and some $k_1 \in K$. Thus $ak_1 G_Z = kk_0 \exp(H) G_Z = kG_Z \in G/G_Z$ and hence $a\,\iota_Z(k_1 K_Z) = \iota_Z(kK_Z) \in a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ gives an intersection point of $a\iota_Z(K/K_Z)$ and $\iota_Z(K/K_Z)$. Moreover, for any $k' \in N_K(\mathfrak{a})$, we have $ak_1k' = kk'(k'^{-1}k_0k')\exp(\mathrm{Ad}(k'^{-1})H)$ where note that $k'^{-1}k_0k' \in Z_K(A)$ and $\mathrm{Ad}(k'^{-1})H \in \mathfrak{a}$. Thus $ak_1k'G_Z = kk'(k'^{-1}k_0k')\exp(\mathrm{Ad}(k'^{-1})H)G_Z = kk'G_Z \in G/G_Z$ and hence $a\,\iota_Z(k_1k'K_Z) = \iota_Z(kk'K_Z) \in a\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ also gives an intersection point of $a\iota_Z(K/K_Z)$ and $\iota_Z(K/K_Z)$. Note that if $kk'K_Z = kk''K_Z$ for $k', k'' \in N_K(\mathfrak{a})$, then $k'N_{K_Z}(\mathfrak{a}) = k''N_{K_Z}(\mathfrak{a})$. Therefore, combining the above argument with Theorem 2.4, we obtain

Proposition 2.5. *For any $a \in G$, it holds*

$$\sharp(a\,\iota_Z(L) \cap \iota_Z(L)) \geq \sharp(N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a})) = \dim H_*(L, \mathbb{Z}_2).$$

Next we mention about the transversality condition of the intersection $a\,\iota_Z(L) \cap \iota_Z(L)$. Suppose that $p \in a\,\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$. Then $p = a\,\iota_Z(k_2K_Z) = \iota_Z(k_1K_Z)$ $(\exists k_1, k_2 \in K)$. Since $k_1^{-1}ak_2 \in G_Z$, using the symmetric Lie algebra $\mathfrak{g}_Z = \mathfrak{k}_Z + \mathfrak{p}_Z$ of a compact symmetric pair (G_Z, K_Z) so that $\mathfrak{a} \subset \mathfrak{p}_Z$, there are $k_Z, k'_Z \in K_Z$ and $H_p \in \mathfrak{a}$ such that $k_1^{-1}ak_2 = k_Z(\exp H_p){k_Z}^{-1}k'_Z$. Thus $ak_2 = k_1k_Z(\exp H_p){k_Z}^{-1}k'_Z$. Since

$$\begin{aligned}(\Phi_Z)_*(T_p a\,\iota_Z(K/K_Z)) =& \mathrm{Ad}(ak_2)[\mathfrak{k}, Z]\\ =& \mathrm{Ad}(k_1k_Z(\exp H_p){k_Z}^{-1}k'_Z)[\mathfrak{k}, Z]\\ =& \mathrm{Ad}(k_1)\mathrm{Ad}(k_Z)\mathrm{Ad}(\exp H_p)[\mathfrak{k}, Z],\\ (\Phi_Z)_*(T_p\,\iota_Z(K/K_Z)) =& \mathrm{Ad}(k_1)[\mathfrak{k}, Z] = \mathrm{Ad}(k_1)\mathrm{Ad}(k_Z)[\mathfrak{k}, Z],\end{aligned}$$

the transversality of $a\,\iota_Z(L) \cap \iota_Z(L)$ at p is equivalent to the transversality of $\mathrm{Ad}(\exp H_p)[\mathfrak{k}, Z]$ and $[\mathfrak{k}, Z]$: $\mathrm{Ad}(\exp H_p)[\mathfrak{k}, Z] \cap [\mathfrak{k}, Z] = \{0\}$. Then by a simple computation using the basis $\{S_{\gamma,i}, T_{\gamma,i}\}$ we can show

Lemma 2.6. *$a\,\iota_Z(L)$ intersects transversally with $\iota_Z(L)$ in $M = G/G_Z$ if and only if at each intersection point p such an $H_p \in \mathfrak{a}$ satisfies $(\gamma, H) \notin \pi\mathbb{Z}$ for each $\gamma \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$ with $(\gamma, Z) \neq 0$.*

First we suppose that $a \in G$ satisfies $aK \in (G/K)_r$. Then $a\,\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ is transversal at each intersection point. Fix an intersection point $\iota_Z(k_1K_Z) \in a\,\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$. Then

$$\begin{aligned}aK =& k_1k_{Z,1}\exp(H_1)K \quad (\exists H_1 \in \mathfrak{a} \setminus \mathbb{D}(\mathfrak{g}, \mathfrak{a}), \exists k_Z \in K_Z)\\ =& \psi(k_1k_{Z,1}Z_K(\mathfrak{a}), \exp(H_1)K)\end{aligned}$$

For an arbitrary intersection point $\iota_Z(k_2K_Z) \in a\,\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$,

$$\begin{aligned} aK =& k_2 k_{Z,2} \exp(H_2)K \quad (\exists H_2 \in \mathfrak{a} \setminus \mathbb{D}(G,K), k_{Z,2} \in K_Z) \\ =& \psi(k_2 k_{Z,2} Z_K(\mathfrak{a}), \exp(H_2)K) \end{aligned}$$

Then there is $s = [k'] \in W(G,K) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ such that

$$k_2 k_{Z,2} Z_K(\mathfrak{a}) = k_1 k_{Z,1} Z_K(\mathfrak{a}) s = k_1 k_{Z,1} k' Z_K(\mathfrak{a})$$

and

$$(\exp H_2)K = s^{-1}(\exp H_1)K = (\exp s^{-1} H_1)K = (\exp \mathrm{Ad}(k')^{-1} H_1)K.$$

Thus $k_2 K_Z = k_1 k_{Z,1} k' K_Z$ and hence $\iota_Z(k_2 K_Z) = \iota_Z(k_1 k_{Z,1} k' K_Z)$. Therefore we obtain

$$a\,\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z) = \{\iota_Z(k_1 k_{Z,1} k' K_Z) \mid [k'] \in N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a})\}.$$

In particular $\sharp(a\,\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)) = \sharp(N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a}))$.

In general suppose that $a\,\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ is transversal at each intersection point. Particularly $a\,\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ is a finite set. We may assume that $aK \in (G/K)_s$. Since $(G/K)_r$ is open and dense in G/K, we choose a smooth perturbation $a_t \in G$ of $a_0 = a$ such that $a_t K \in (G/K)_r$ $(0 < \forall t < \varepsilon)$. Then $a_t\,\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)$ is also transversal at each intersection point and for sufficiently small $t > 0$ we have

$$\sharp(a\,\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)) = \sharp(a_t\,\iota_Z(K/K_Z) \cap \iota_Z(K/K_Z)) = \sharp(N_K(\mathfrak{a})/N_{K_0}(\mathfrak{a})).$$

Therefore we obtain

Proposition 2.7. *For any $a \in G$ with transversal $a\,\iota_Z(L) \cap \iota_Z(L)$, it holds*

$$\sharp(a\,\iota_Z(L) \cap \iota_Z(L)) = \sharp(N_K(\mathfrak{a})/N_{K_Z}(\mathfrak{a})).$$

Combining it with Theorem 2.4, we see

Corollary 2.8. *For any $a \in G$ with transversal $a\,\iota_Z(L) \cap \iota_Z(L)$, it holds*

$$\sharp(a\,\iota_Z(L) \cap \iota_Z(L)) = \dim H_*(L, \mathbb{Z}_2).$$

Such a property is called the *global tightness* for a Lagrangian submanifolds in a Kähler C-space ([14], [9], [5]). It was proved by [28] in the case when L is a symmetric R-space. It is still an open problem to classify compact globally tight or simply tight Lagrangian submanifolds of Kähler C-spaces. More generally the intersection theory and Floer homology for two real forms in Kähler C-spaces are discussed in [7], [11].

3 Minimal Maslov number and monotonicity of Lagrangian submanifolds in symplectic manifolds and Einstein-Kähler manifolds

Let L be a Lagrangian submanifold of a symplectic manifold (M,ω). Define two kinds of group homomorphisms $I_{\mu,L} : \pi_2(M,L) \to \mathbb{Z}$ and $I_{\omega,L} : \pi_2(M,L) \to \mathbb{R}$. For a smooth map $u : (D^2, \partial D^2) \to (M,L)$ with $A = [u] \in \pi_2(M,L)$, choose a trivialization of the pull-back bundle as a symplectic vector bundle (which is unique up to the homotopy). $u^{-1}TM \cong D^2 \times \mathbb{C}^n$. This gives a smooth map $\tilde{u} : S^1 = \partial D^2 \to \Lambda(\mathbb{C}^n)$. Here $\Lambda(\mathbb{C}^n)$ denotes the Grassmann manifold of Lagrangian vector subspaces of $\mathbb{C}^n$. Using the Moslov class $\mu \in H^1(\Lambda(\mathbb{C}^n), \mathbb{Z}) \cong \mathbb{Z}$, we define $I_{\mu,L}(A) := \mu(\tilde{u})$. Another homomorphism $I_{\omega,L} : \pi_2(M,L) \to \mathbb{R}$ is defined by $I_{\omega,L}(A) := \int_{D^2} u^*\omega$. Note that $I_{\mu,L}$ is invariant under symplectic diffeomorphisms and $I_{\omega,L}$ is invariant under Hamiltonian diffeomorphisms but not under symplectic diffeomorphisms.

A Lagrangian submanifold L of (M,ω) is called *monotone* ([15]) if $I_{\mu,L} = \lambda\, I_{\omega,L}$ $(\exists \lambda > 0)$. If $I_{\mu,L} = 0$, we define $\Sigma_L = 0$. We assume that $I_{\mu,L} \neq 0$, Denote by $\Sigma_L \in \mathbb{Z}^+$ the positive generator of $\mathrm{Im}(I_{\mu,L})$ as an additive subgroup of $\mathbb{Z}$. We call Σ_L is called the *minimal Maslov number* of LD

Theorem 3.1 ([2], [21]). *Suppose that (M,ω,J,g) is an Einstein-Kähler manifold of positive Einstein constant. If L is a compact minimal Lagrangian submanifold of M, then L is monotone.*

Suppose that (M,ω,J,g) is a *simply connected* Einstein-Kähler manifold with positive Einstein constant and L is a compact monotone Lagrangian submanifold of M. Then Hajime Ono ([21]) showed the formula for Σ_L:

$$n_L \Sigma_L = 2\gamma_{c_1}, \tag{3.1}$$

where we set

$$\gamma_{c_1} := \min\{c_1(M)(A) \mid A \in H_2(M;\mathbb{Z}), c_1(M)(A) > 0\},$$
$$n_L := \min\{k \in \mathbb{Z}^+ \mid \otimes^k E \text{ trivial}\}.$$

$$\begin{array}{ccc} E|_L & \longrightarrow & E \ \text{cplx. line bdle.} \\ \pi_L \big\downarrow \text{flat} & & \pi_E \big\downarrow U(1)\text{-connection } \nabla \\ L & \underset{\text{Lag.}}{\longrightarrow} & M \ \text{Einstein-Kähler mfd.} \end{array}$$

Here $\frac{1}{\gamma}\omega = c_1(E,\nabla)$ for some constant $\gamma > 0$.

As an application of that formula (3.1), we mention the minimal Maslov number formula for the Gauss images of isoparametric hypersurfaces in the standard sphere.

Let $N^n \subset S^{n+1}(1) \subset \mathbb{R}^{n+2}$ be an oriented hypersurface of $S^{n+1}(1)$ and let $\hat{N}^n := \{(\mathbf{x}(p), \mathbf{n}(p)) \in V_2(\mathbb{R}^{n+2}) \mid p \in N\}$ be the Legendrian lift of N^n to T^1S^{n+1}. Then we have the following diagram:

$$
\begin{array}{ccccc}
 & N \subset S^{n+1}(1) & & & \text{unit sphere tangent bundle of } S^{n+1} \\
 & \| & & & = \text{Stiefel mfd. of o.n. 2-frames of } \mathbb{R}^{n+2} \\
 & \hat{L} = \hat{N} & \xrightarrow{\text{Leg.}} & & P = T^1 S^{n+1} = V_2(\mathbb{R}^{n+2}) \cong \frac{SO(n+2)}{SO(n)} \\
\text{Gauss map} & \mathcal{G} \Big\downarrow \rho(\pi_1(L)) & & & \pi \Big\downarrow SO(2) \cong U(1) \\
 & L = \mathcal{G}(N) & \xrightarrow[\text{Lag.}]{} & & M = Q_n(\mathbb{C}) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cong \frac{SO(n+2)}{SO(2)\times SO(n)} \\
 & \text{Gauss image} & & & \text{complex hyperquadric} \\
 & & & & = \text{real ori. 2-plane Grassmann mfd.}
\end{array}
$$

The *Gauss Map* is defined by

$$
\begin{array}{lccccc}
\mathcal{G}: & N & \longrightarrow & \hat{N} & \longrightarrow & \mathcal{G}(N) \subset Q_n(\mathbb{C}) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \\
 & p & \longmapsto & (\mathbf{x}(p), \mathbf{n}(p)) & \longmapsto & [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] = \mathbf{x}(p) \wedge \mathbf{n}(p)
\end{array}
$$

Suppose that $N^n \subset S^{n+1}(1)$ is an isoparametric hypersurface with g distinct principal curvatures. Let $\hat{N}^n := \{(\mathbf{x}(p), \mathbf{n}(p)) \in V_2(\mathbb{R}^{n+2}) \mid p \in N\}$ be the Legendrian lift of N^n to $T^1 S^{n+1}$. The following diagram becomes as follows:

$$
\begin{array}{ccccc}
 & N \subset S^{n+1}(1) & & & \text{simply conn.} \\
 & \| & & & \text{Einstein Sasakian homog. sp.} \\
 & \hat{L} = \hat{N} & \xrightarrow{\text{min. Leg. embed.}} & & P = V_2(\mathbb{R}^{n+2}) = \frac{SO(n+2)}{SO(n)} = T^1 S^{n+1} \\
\text{Gauss map} & \mathcal{G} \Big\downarrow \begin{array}{l}\rho(\pi_1(L)) \\ \cong \mathbb{Z}_g\end{array} & & & \pi \Big\downarrow \begin{array}{l} SO(2) \\ \cong U(1)\end{array} \\
 & & & & \text{Einstein-Kähler Herm. sym. sp.} \\
\text{Gauss image} & L = \mathcal{G}(N) & \xrightarrow[\text{min. Lag. embed.}]{} & & M = Q_n(\mathbb{C}) = \frac{SO(n+2)}{SO(2)\times SO(n)} \\
 & & & & \text{complex hyperquadric}
\end{array}
$$

Then it holds $\Sigma_L = \dfrac{2n}{g}$ ([19], [13]). This formula was crucial to the Hamiltonian non-displaceability theorem for Gauss images of isoparametric hypersurfaces ([8]).

4 Minimal Maslov number of R-spaces canonically embedded in Einstein-Kähler C-spaces

We take the universal cover $\widetilde{G} \to G$ of G. Let $(\widetilde{G}, \widetilde{K}, \theta)$ be a Riemannian symmetric pair of compact type with simply connected $\widetilde{G}$ and connected $\widetilde{K}$.

By Proposition 2.3 we can choose $Z = Z^{ein} = 2\delta_{\mathfrak{m}}$. Then $\iota_Z : L = K/K_Z \to M = G/G_Z$ is the canonical embedding of an R-space into an Einstein-Kähler C-space. As in [24] we use expression

$$
2\delta_{\mathfrak{m}} = \sum_{\alpha_i \in \Pi \setminus \Pi_Z} k_i \Lambda_i = \kappa(M) \sum_{\alpha_i \in \Pi \setminus \Pi_Z} \kappa_i \Lambda_i \quad (k_i \in \mathbb{Z}^+),
$$

where $\kappa(M)$ denotes the greatest common divisor of $\{k_i \mid \alpha_i \in \Pi \setminus \Pi_Z\}$ and set $\kappa_i := \frac{k_i}{\kappa(M)}$ for each $\alpha_i \in \Pi \setminus \Pi_Z$. Then the invariant γ_{c_1} in (3.1) is given as $\gamma_{c_1} = \kappa(M)$ (cf. [20]).

$$\begin{array}{ccc} & & \text{simply conn.} \\ & & \text{Einstein Sasakian homog. sp.} \\ \hat{L} = \widetilde{K}/\widetilde{K}'_Z & \xrightarrow{\text{tot.geod.Leg.}} & P = \widetilde{G}/\widetilde{G}'_Z \\ \hat{\pi} \Big\downarrow \rho(\pi_1(L)) & & \pi \Big\downarrow U(1) \cong S^1 \\ L = \widetilde{K}/\widetilde{K}_Z & \xrightarrow[\substack{\text{canon. embed.} \\ \text{tot.geod.Lag.}}]{} & M = \widetilde{G}/\widetilde{G}_Z \\ R\text{-sp.} & & \text{Einstein-Kähler } C\text{-sp.} \end{array}$$

Here we take the orthogonal direct sum decomposition $\mathfrak{g}_Z = \mathbb{R} \cdot 2\delta_{\mathfrak{m}} \oplus \mathfrak{g}'_Z$. Denote by $\widetilde{G}'_Z$ a connected Lie subgroup of $\widetilde{G}_Z$ with Lie algebra $\mathfrak{g}'_Z$ and set $\widetilde{K}'_Z := \widetilde{K} \cap \widetilde{G}'_Z$. Then $n_L = \sharp(\widetilde{K}_Z/\widetilde{K}'_Z)$ ([20]). Therefore by the formula (3.1) we obtain

Theorem 4.1 ([20])**.** *The minimal Maslov number Σ_L of an R-space L canonically embedded in an Einstein-Kähler C-space M is given by the formula*

$$\Sigma_L = \frac{2\,\kappa(M)}{\sharp(\widetilde{K}_H/\widetilde{K}'_H)}. \tag{4.1}$$

Some concrete examples of computations by this formula are given in [20] in the case when (1) $(G, K) = (SU(n+1), SO(n+1))$ and L is $\mathbb{R}P^n$ or a regular R-space, (2) L is a maximal flag manifold of a compact semisimple Lie group K, (3) L is an irreducible symmetric R-space. It is an interesting problem to study the minimal Maslov number for all other R-spaces by this formula.

Acknowledgements. This article is based on the author's talk at the 22nd International Workshop on Differential Geometry of Submanifolds in Symmetric Spaces and Related Problems & the 16th RIRCM-OCAMI Joint Differential Geometry Workshop at Kyungpook National University in Daegu on July 31– August 5 in 2019. The author sincerely would like to thank Professor Young Jin Suh (director of Research Institute of Real and Complex Manifolds, RIRCM) and Dr. Hyunjin Lee for their kind hospitality and and excellent organization of international workshops.

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