TORUS ORBIT CLOSURES IN FLAG VARIETIES AND RETRACTIONS ON WEYL GROUPS

EUNJEONG LEE, MIKIYA MASUDA, AND SEONJEONG PARK

Abstract. In this manuscript, we define three kinds of retractions on Weyl groups or finite Coxeter groups and study the relations among them. The first is what we call a *geometric retraction* $\mathcal{R}^g_Y$ associated to a torus orbit closure Y in the flag variety G/B of a semisimple algebraic group G. The fixed point set $(G/B)^T$ for the action of the maximal torus T can be identified with the Weyl group W of G. We show that the geometric retraction $\mathcal{R}^g_Y\colon W \to Y^T (\subset (G/B)^T = W)$ maps an element in W to its closest element in Y^T with respect to a metric on W. The second kind of retraction is what we call an *algebraic retraction*. For any subset $\mathcal{M}$ of a Weyl group W of classical Lie type, the algebraic retraction $\mathcal{R}^a_{\mathcal{M}}\colon W \to \mathcal{M}$ is defined by taking the locally optimal choice at each step for $u \in W$ with the intent of finding an element in $\mathcal{M}$ closest to u. The last kind of retraction is what we call a *matroid retraction* $\mathcal{R}^m_{\mathcal{M}}$ associated to a Coxeter matroid $\mathcal{M}$ of a finite Coxeter group. We show that these three kinds of retractions are the same for a torus orbit closure Y in G/B, i.e. $\mathcal{R}^g_Y = \mathcal{R}^a_{Y^T} = \mathcal{R}^m_{Y^T}$ when G is of classical Lie type.

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1. Introduction

Let G be a semisimple algebraic group over $\mathbb{C}$, let B be a Borel subgroup of G, and let T be a maximal torus of G contained in B. Then left multiplication by the torus T on G induces the action of T on G/B. The set of T-fixed points in G/B bijectively corresponds to the Weyl group W of G.

Let Y be the closure of the T-orbit of an element of G/B. When Y is *generic*, which means that Y contains all the T-fixed points in G/B, Y is known to be smooth and normal, so is a smooth toric variety. When G is of Lie type A, the

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generic torus orbit closure in G/B is called *the permutohedral variety*. The maximal cones of the fan of Y are given by the closures of the Weyl chambers (see [21] and [24]). We denote these maximal cones as follows:

$$C(w) := \{\lambda \in \mathfrak{t}_{\mathbb{R}} \mid \langle w(\alpha), \lambda \rangle \leq 0 \quad \text{for all simple roots } \alpha\}$$

for each $w \in W$, where $\mathfrak{t}_{\mathbb{R}}$ is the real Lie algebra of T. Then for every generic point $x \in G/B$ and every element $w \in W$, we have

$$\lim_{t \to 0} \lambda(t)x = w$$

for any $\lambda \in \operatorname{Int}(C(w)) \cap \mathfrak{t}_{\mathbb{Z}}$. Here, $\mathfrak{t}_{\mathbb{Z}}$ is the lattice of $\mathfrak{t}_{\mathbb{R}}$ and we think of $\mathfrak{t}_{\mathbb{Z}}$ as $\operatorname{Hom}(\mathbb{C}^*, T)$.

For a non-generic torus orbit closure Y of G/B, one can find the corresponding fan using the Orbit-Cone correspondence. It should be noted that Y is not necessarily normal when G is an arbitrary algebraic group (see [8] and Remark 3.4). Hence the Orbit-Cone correspondence gives the fan of the normalization of Y. When G is of Lie type A, generic torus orbit closures associated to Schubert varieties and Richardson varieties in G/B were studied in [22] and [23], respectively; Gelfand and Serganova [13] study torus orbit closures in homogeneous manifolds G/P in terms of matroids for any Lie type.

By analyzing torus actions on G/B, we show that for a given $Y = \overline{Tx}$ and $\lambda \in \operatorname{Int}(C(w)) \cap \mathfrak{t}_{\mathbb{Z}}$, the limit point $\lim_{t\to 0} \lambda(t)x$ depends only on w and Y. Using this fact, we define a *geometric retraction* $\mathcal{R}^g_Y \colon W \to Y^T \subset W$ such that the restriction of $\mathcal{R}^g_Y$ to Y^T is the identity. Then the maximal cones

$$C_Y(u) = \bigcup_{\mathcal{R}^g_Y(w) = u} C(w), \quad u \in Y^T$$

determine the fan of the normalization of Y. Actually, the maximal cone in the fan corresponding to a fixed point uB in Y is $C_Y(u)$ projected on the quotient vector space $\mathfrak{t}_{\mathbb{R}}/F_Y$, where the linear subspace F_Y of $\mathfrak{t}_{\mathbb{R}}$ is determined by the subtorus of T which fixes Y pointwise (see Remark 3.3 for the definition of F_Y).

The geometric retraction $\mathcal{R}^g_Y$ produces the closest element in Y^T for each $u \in W$ with respect to the metric $d(v,w) := \ell(v^{-1}w)$ on W, where $\ell(\ )$ denotes the length function on W.

Theorem A (Theorem 4.3)**.** *Let Y be a T-orbit closure in G/B and $u \in W$. If y is an element of Y^T such that*

$$d(u,y) = \min\{d(u,w) \mid w \in Y^T\},$$

then $y = \mathcal{R}^g_Y(u)$. In particular, the closest point $y \in Y^T$ to u is unique for each u.

For a Weyl group W of classical Lie type, we can find the geometric retraction image $\mathcal{R}^g_Y(u)$ algebraically. We denote by $[n] = \{1, \dots, n\}$ and $[\bar{n}] = \{\bar{1}, \dots, \bar{n}\}$, and give an ordering on $[n] \cup [\bar{n}]$ by $1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}$. We regard $\bar{\bar{i}} = i$. Then for a subset $\mathcal{M} \subset W$, we set

$$J_k(\mathcal{M}) = \{(w(1), w(2), \dots, w(k)) \mid w \in \mathcal{M}\} \qquad \text{for } 1 \leq k \leq n.$$

Now we define an *algebraic retraction* $\mathcal{R}^a_{\mathcal{M}}$ from W onto $\mathcal{M}$. To each $u \in W$, we define $\mathcal{R}^a_{\mathcal{M}}(u) = u(i_1)u(i_2)\cdots u(i_n)$ by choosing $i_1, i_2, \dots, i_n$ inductively as follows:

$$i_1 := \min\{i \in [n] \cup [\bar{n}] \mid u(i) \in J_1(\mathcal{M})\}$$

and

$$i_k := \min\{i \in [n] \cup [\bar{n}] \setminus \{i_1, \dots, i_{k-1}\} \mid (u(i_1), \dots, u(i_{k-1}), u(i)) \in J_k(\mathcal{M})\}$$

for $2 \leq k \leq n$. Note that the above procedure provides a way to choose a locally optimal choice for each step with the goal of finding a closest element in $\mathcal{M}$ for each $u \in W$.

Theorem B (Theorem 6.5)**.** *Let G be a product of simple algebraic groups of classical Lie type. For a T-orbit closure Y in G/B, the algebraic retraction $\mathcal{R}^a_{Y^T}$ is the same as the geometric retraction $\mathcal{R}^g_Y$.*

By Theorems A and B, the algebraic retraction image $\mathcal{R}^a_{Y^T}(u)$ of u is the closest element to each $u \in W$. In general, for an arbitrary subset $\mathcal{M}$ of W, the algebraic retraction image $\mathcal{R}^a_{\mathcal{M}}(u)$ of u is not necessarily the closest element in $\mathcal{M}$ to u (see Example 6.7(2)). Furthermore, the fact that $\mathcal{R}^a_{\mathcal{M}}(u)$ produces the closest element to each $u \in W$ does not imply that $\mathcal{M}$ corresponds to a torus orbit closure (see Examples 6.7(1), 7.3, and 7.7).

Suppose that W is a finite Coxeter group. Note that the Weyl group of a semisimple algebraic group is a finite Coxeter group. For $u \in W$, we say that $v \leq^u w$ if $u^{-1}v \leq u^{-1}w$. A subset $\mathcal{M} \subset W$ is called a *Coxeter matroid* if for every $u \in W$, there is a unique $v \in \mathcal{M}$ such that $w \leq^u v$ for all $w \in \mathcal{M}$. Note that the existence of the unique maximal element is equivalent to the existence of a unique minimal element. For such a Coxeter matroid $\mathcal{M}$, we define a *matroid retraction* $\mathcal{R}^m_{\mathcal{M}}$ from W onto $\mathcal{M}$ by taking the unique minimal element for each $u \in W$.

On the other hand, for an algebraic group G of classical Lie type and its Weyl group W, consider a torus orbit closure Y in G/B. Then, by the Gelfand–Serganova theorem (see [5, Theorem 6.3.1]), the set Y^T of fixed points forms a Coxeter matroid in W (cf. [19, Theorem 5.7]).

Theorem C (Proposition 7.2 and Theorem 7.6)**.** *Let G be a product of simple algebraic groups of classical Lie type, and W its Weyl group. For a Coxeter matroid $\mathcal{M} \subset W$, the algebraic retraction $\mathcal{R}^a_{\mathcal{M}}$ is the same as the matroid retraction $\mathcal{R}^m_{\mathcal{M}}$. Therefore, for a T-orbit closure Y in G/B, all the retractions are same*

$$\mathcal{R}^g_Y = \mathcal{R}^a_{Y^T} = \mathcal{R}^m_{Y^T}.$$

This paper is organized as follows. In Section 2, we set up notation and terminology, and have compiled some known facts on flag varieties. In Section 3, we define our geometric retraction $\mathcal{R}^g_Y$ from W onto Y^T for a torus orbit closure Y in G/B. We prove Theorem A in Section 4. In Section 5, we review the description of the flag varieties of classical Lie type. In Section 6, we define our algebraic retraction $\mathcal{R}^a_{\mathcal{M}}$ from W onto $\mathcal{M}$ for any subset $\mathcal{M}$ of W, and then prove Theorem B. In Section 7, we relate Coxeter matroids to the geometric retraction and the algebraic retraction, define a matroid retraction $\mathcal{R}^m_{\mathcal{M}}$ from W onto $\mathcal{M}$, and then prove Theorem C.

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2. Flag varieties, moment polytopes and GKM graphs

In this section, we set up notation and review some known facts on torus actions on flag varieties which will be used in this manuscript. Let G be a semisimple algebraic group, B a Borel subgroup of G, and T a maximal torus of B. We think of the real Lie algebra $\mathfrak{t}_{\mathbb{R}}$ of the torus T as $\mathrm{Hom}(\mathbb{C}^*, T) \otimes \mathbb{R}$ and the lattice $\mathfrak{t}_{\mathbb{Z}}$ of $\mathfrak{t}_{\mathbb{R}}$ as $\mathrm{Hom}(\mathbb{C}^*, T)$. The vector space $\mathfrak{t}_{\mathbb{R}}^*$ dual to $\mathfrak{t}_{\mathbb{R}}$ can be thought of as $\mathrm{Hom}(T, \mathbb{C}^*) \otimes \mathbb{R}$ and the set Φ of roots of G is a finite subset of $\mathfrak{t}_{\mathbb{R}}^*$. The Weyl group W of G acts on $\mathfrak{t}_{\mathbb{R}}$ as the adjoint action and on its dual space $\mathfrak{t}_{\mathbb{R}}^*$ as the coadjoint action, i.e.

$$(w \cdot f)(x) := f(\mathrm{Ad}_{w^{-1}}(x)) \quad \text{for } w \in W,\ f \in \mathfrak{t}_{\mathbb{R}}^* \text{ and } x \in \mathfrak{t}_{\mathbb{R}}.$$

The Borel subgroup B determines the set Φ^+ of positive roots and we have

$$T_u(G/B) = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-u(\alpha)} \quad \text{as } T\text{-modules for } u \in W, \tag{2.1}$$

where $\mathfrak{g}_\alpha$ denotes the eigensubspace of $\mathfrak{g}$ for $\alpha \in \Phi$ (see, for example, [15, §3]). We denote by s_α the reflection associated to $\alpha \in \Phi$, so $s_{-\alpha} = s_\alpha$ and the Weyl group W is generated by the reflections s_α for $\alpha \in \Phi^+$.

We choose an element $\nu \in \mathfrak{t}_{\mathbb{R}}^*$ which is not fixed by any reflection s_α for $\alpha \in \Phi^+$, and we let Δ_W be the convex hull of the W-orbit of ν. Since the isotropy subgroup of W at ν is trivial, Δ_W is a convex polytope called the *W-permutohedron* (see [10] and [18]). We identify $w \cdot \nu$ with w for each $w \in W$. Then the vertices of Δ_W are labeled by the elements in W and two vertices v and w are joined by an edge of Δ_W if and only if $v = ws_\alpha$ for some *simple* root α (see [10, Lemma 2.13]).

The left multiplication of T on G induces an action of T on G/B, and the W-permutohedron Δ_W appears as the image of a moment map from G/B to $\mathfrak{t}_{\mathbb{R}}^*$. Let $G_{\mathbb{R}}$ be the maximal compact Lie subgroup of G. The intersection $T_{\mathbb{R}} := G_{\mathbb{R}} \cap B$ is a maximal torus of $G_{\mathbb{R}}$ and we get a homeomorphism $G/B \cong G_{\mathbb{R}}/T_{\mathbb{R}}$. The Lie algebra of $T_{\mathbb{R}}$ is $\mathfrak{t}_{\mathbb{R}}^*$. Let $\mathfrak{g}_{\mathbb{R}}$ denote the Lie algebra of $G_{\mathbb{R}}$ and (through the Killing form) we regard $\mathfrak{t}_{\mathbb{R}}^*$ as a linear subspace of $\mathfrak{g}_{\mathbb{R}}^*$, the dual space to $\mathfrak{g}_{\mathbb{R}}$. Then the element $\nu \in \mathfrak{t}_{\mathbb{R}}^*$ we chose above can be regarded as an element of $\mathfrak{g}_{\mathbb{R}}^*$. The group $G_{\mathbb{R}}$ acts on $\mathfrak{g}_{\mathbb{R}}^*$ as the coadjoint action and the $G_{\mathbb{R}}$-orbit of ν can be identified with $G_{\mathbb{R}}/T_{\mathbb{R}}$ through the correspondence $gT_{\mathbb{R}} \to g \cdot \nu$. The $G_{\mathbb{R}}$-orbit of ν admits a symplectic form called the Kirillov–Kostant–Souriau symplectic form and its moment map is the projection to $\mathfrak{t}_{\mathbb{R}}^*$ (see [3, §2.3]). Thus we obtain a map

$$\mu\colon G/B = G_{\mathbb{R}}/T_{\mathbb{R}} \to G_{\mathbb{R}} \cdot \nu \subset \mathfrak{g}_{\mathbb{R}}^* \to \mathfrak{t}_{\mathbb{R}}^*.$$

The image $\mu(G/B)$ is the convex hull of $\mu((G/B)^T)$, see [2, 16]. Since $(G/B)^T = W$ and $\mu(w) = w \cdot \nu$, we obtain $\mu(G/B) = \Delta_W$ (see Figure 3 for $\Delta_{\mathfrak{S}_3}$).

Let $x \in G/B$. If the T-orbit Tx of x is one-dimensional, then its closure $\overline{Tx}$ is isomorphic to $\mathbb{P}^1$ and contains two T-fixed points, say v and w, and they are related with $v = ws_\alpha$ for some positive root $\alpha \in \Phi^+$. Indeed, the positive root α is characterized by the condition that the kernel of $\alpha\colon T \to \mathbb{C}^*$ is the isotropy subgroup of $x \in G/B$. The image $\mu(\overline{Tx})$ is a segment with v and w as endpoints. Recall from [7] and [27] that the GKM graph of G/B has these segments as edges with labeling where the label on the edge connecting v and w is α (up to sign).

Precisely speaking, in the definition of GKM graph, we consider oriented edges, i.e. we distinguish the edge from v to w and the edge from w to v, and label them with opposite signs but these orientations and signs are unnecessary in our discussion developed in the paper. (See [14, 17] for the definition of GKM manifolds and GKM graphs.) With this understood, we have the following.

Lemma 2.1. *Let $v, w \in W = (G/B)^T$. Then the following three statements are equivalent to each other:*

(1) *v and w are joined by an edge in the GKM graph of the flag variety G/B,*
(2) *$v = ws_\alpha (= s_{w(\alpha)} w)$ for some positive root α of G,*
(3) *v and w are contained in the same connected component of the set which is fixed under the codimension one subgroup $\ker w(\alpha)$ of T.*

We note that the properties of the torus action on semisimple algebraic groups can be explained using simple algebraic groups. Let $G = \prod_{j=1}^m G_j$ for simple algebraic groups $G_1, \dots, G_m$. Then a maximal torus T of G can be expressed as the product $\prod_{j=1}^m T_j$ of maximal tori of G_j's. Furthermore, for a Borel subgroup B containing T, we have $B = \prod_{j=1}^m B_j$, where B_j is a Borel subgroup of G_j containing T_j. Hence $G/B = \prod_{j=1}^m G_j/B_j$, and for $t = (t_1, \dots, t_m) \in \prod_{j=1}^m T_j$ and $gB = (g_1B_1, \dots, g_mB_m)$, we have that

$$t \cdot gB = (t_1g_1B_1, \dots, t_mg_mB_m). \tag{2.2}$$

Moreover, the Weyl group W of G is the product of Weyl groups of G_j's and the set $\Phi(G)$ of roots of G is the disjoint union of those of G_j's:

$$W = \prod_{j=1}^m W(G_j) \quad \text{and} \quad \Phi(G) = \Phi(G_1) \sqcup \cdots \sqcup \Phi(G_m).$$

Hence the W-permutohedron Δ_W is the product of permutohedra $\Delta_{W(G_j)}$:

$$\Delta_W = \prod_{j=1}^m \Delta_{W(G_j)}. \tag{2.3}$$

3. The fan of a torus orbit closure and geometric retractions

For a point x in G/B, the closure $Y := \overline{Tx}$ of the T-orbit is a toric variety in G/B. In this section, we study the fan of the toric variety using the Orbit-Cone correspondence. Moreover, we show that each maximal cone of the fan is obtained by a geometric retraction $\mathcal{R}_Y^g$ from the Weyl group W of G onto the set of fixed points $Y^T \subset W$ of Y.

For each $u \in W$, we define

$$C(u) = \{\lambda \in \mathfrak{t}_\mathbb{R} \mid \langle u(\alpha), \lambda \rangle \le 0 \quad \text{for all simple roots } \alpha\} \tag{3.1}$$

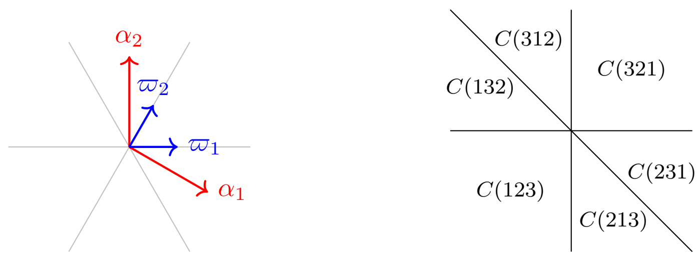
where $\langle \ , \ \rangle$ denotes the natural pairing between $\mathfrak{t}_\mathbb{R}^*$ and $\mathfrak{t}_\mathbb{R}$. Note that the interiors of the cones $C(u)$ above form the Weyl chambers. In Figure 1(a), we draw simple roots $\alpha_1, \alpha_2 \in \mathfrak{t}_\mathbb{R}^*$ and their duals $\varpi_1, \varpi_2 \in \mathfrak{t}_\mathbb{R}$, *fundamental weights*, (we draw them in the same figure even though roots and weights live in different vector spaces), and then draw cones $C(u)$ when $G = \mathrm{SL}_3(\mathbb{C})$ in Figure 1(b). Here, we use the identification $\mathfrak{t}_\mathbb{R}^* = \mathbb{R}^3/\langle(1,1,1)\rangle \xrightarrow{\cong} \mathbb{R}^2$ given by $[a_1, a_2, a_3] \mapsto (a_1 - a_2, a_2 - a_3)$, which is well-defined. For instance, a point $[a_1, a_2, a_3]$ in the cone $C(213)$ satisfies $a_2 \le a_1 \le a_3$. So that this point corresponds to a point (b_1, b_2) satisfying $b_1 \ge 0$ and

$b_1+b_2 \le 0$ under the identification. Since every positive root is a linear combination of simple roots with nonnegative coefficients, the inequality $\langle u(\alpha), \lambda\rangle \le 0$ above is satisfied for every positive root $\alpha \in \Phi^+$ if it is satisfied for all simple roots α.

The identity (2.1) implies that any isotropy subgroup of the action of T on G/B is contained in the kernel of some positive root. Therefore, for any $\lambda \in \mathrm{Int}(C(u)) \cap \mathfrak{t}_{\mathbb{Z}}$, we have

$$(G/B)^{\lambda(\mathbb{C}^*)} = (G/B)^T. \tag{3.2}$$

Here, we use the identification $\mathfrak{t}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^*, T)$.



(a) Simple roots and fundamental weights. (b) Cones $C(u)$.

FIGURE 1. Simple roots, fundamental weights, and Weyl chambers when $G = \mathrm{SL}_3(\mathbb{C})$.

For each $w \in W = (G/B)^T$, we choose an element $\lambda_w \in \mathrm{Int}(C(w)) \cap \mathfrak{t}_{\mathbb{Z}}$ and define

$$S_w := \{x \in G/B \mid \lim_{t\to 0} \lambda_w(t)x = w\},$$

which is independent of the choice of λ_w. Then S_w is a T-invariant affine open subset of G/B and isomorphic to $T_w(G/B)$ as a T-variety (see [4]).

Example 3.1. Suppose that $G = \mathrm{SL}_3(\mathbb{C})$ and B is the subset of upper triangular matrices in G. For $w = 231 \in \mathfrak{S}_3 (= W)$ in one-line notation,

$$S_{231} = \left\{ \begin{pmatrix} x & y & 1 \\ 1 & 0 & 0 \\ z & 1 & 0 \end{pmatrix} B \;\middle|\; x, y, z \in \mathbb{C} \right\}.$$

The T-action on an element of S_{231} is given by

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \begin{pmatrix} x & y & 1 \\ 1 & 0 & 0 \\ z & 1 & 0 \end{pmatrix} B = \begin{pmatrix} t_1x & t_1y & t_1 \\ t_2 & 0 & 0 \\ t_3z & t_3 & 0 \end{pmatrix} B = \begin{pmatrix} t_1t_2^{-1}x & t_1t_3^{-1}y & 1 \\ 1 & 0 & 0 \\ t_2^{-1}t_3z & 1 & 0 \end{pmatrix} B.$$

Hence, as T-varieties, we have the following identification:

$$S_{231} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{-\alpha_2}$$

where α_1 and α_2 are simple roots given by

$$\alpha_1 = e_1 - e_2 \text{ and } \alpha_2 = e_2 - e_3 \text{ in } \mathfrak{t}_{\mathbb{Z}}$$

with the standard basis vectors e_1, e_2, e_3 of $\mathbb{R}^3$ (see §5.1). On the other hand, we have

$$\begin{aligned}\{-(231)(\alpha) \mid \alpha \in \Phi^+\} &= \{-(231)(\alpha_1), -(231)(\alpha_2), -(231)(\alpha_1+\alpha_2)\} \\ &= \{-\alpha_2, \alpha_1+\alpha_2, \alpha_1\}.\end{aligned}$$

Therefore, S_{231} is isomorphic to $T_{231}(G/B)$ as a T-variety by (2.1).

Lemma 3.2. *Let x be a point of G/B and Y the closure of the T-orbit Tx. For any $u \in W$ and $\lambda \in \mathrm{Int}(C(u)) \cap \mathfrak{t}_{\mathbb{Z}}$, $\lim_{t\to 0}\lambda(t)x$ is an element of Y^T depending only on u and Y.*

Proof. Since Y is closed, $\lim_{t\to 0}\lambda(t)x$ exists in Y and clearly remains fixed under the action of $\lambda(\mathbb{C}^*)$. Therefore, the limit point is indeed in Y^T by (3.2). Denote the limit point by w. Then it follows from the definition of S_w that x is in S_w. Since S_w is isomorphic to $T_w(G/B)$ as a T-variety, it follows from (2.1) that $\lim_{t\to 0}\lambda(t)x$ is independent of the choice of λ in $\mathrm{Int}(C(u)) \cap \mathfrak{t}_{\mathbb{Z}}$. $\square$

By Lemma 3.2, we may write

$$\mathcal{R}^g_Y(u) := \lim_{t\to 0}\lambda(t)x. \tag{3.3}$$

It is clear that if u is in Y^T, then $\mathcal{R}^g_Y(u) = u$; hence the map

$$\mathcal{R}^g_Y \colon W \to Y^T \subset W$$

is a retraction of W onto Y^T, which we call a *geometric retraction*.

In the following, we fix Y and denote $u^g := \mathcal{R}^g_Y(u)$ to simplify the notation. For each $y \in Y^T$, we define

$$C_Y(y) := \bigcup_{u^g = y} C(u). \tag{3.4}$$

Then the collection of the cones $C_Y(y)$, $y \in Y^T$, determines the fan of the normalization of Y (see [9, Appendix of Chapter 3] for the Orbit-Cone correspondence of non-normal toric varieties).

Remark 3.3. Since the action of T on Y is not effective, the ambient space of the fan of Y is the quotient of $\mathfrak{t}_{\mathbb{R}}$ by the subspace $\mathrm{Hom}(\mathbb{C}^*, T_Y) \otimes \mathbb{R}$, where T_Y is the toral subgroup of T which fixes Y pointwise. Then F_Y in the introduction is $\mathrm{Hom}(\mathbb{C}^*, T_Y) \otimes \mathbb{R}$. However, for simplicity, we will think of $\mathfrak{t}_{\mathbb{R}}$ as the ambient space of the fan of Y throughout the paper.

Remark 3.4. It has been proven in [8, Proposition 4.8] that if G is of type A_n, D_4, or B_2, then every T-orbit closure in G/B is normal. When $G = G_2$, non-normal torus orbit closures exist in that case (see [8, Example 6.1] and references therein).

Example 3.5. Let us consider two points

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B \quad \text{and} \quad x' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B$$

in $\mathrm{SL}_3(\mathbb{C})/B$.

(1) Suppose that Y is the closure of the torus orbit Tx. Then it is not difficult to show that $Y^T = \{123, 132, 213, 312\}$. Let us choose an element $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Int}(C(123)) \cap \mathfrak{t}_{\mathbb{Z}}$, that is $\lambda_1 < \lambda_2 < \lambda_3$. Since B consists of

u	123	213	231	321	312	132
u^g for Y	123	213	213	312	312	132
u^g for Y'	123	123	321	321	321	123

TABLE 1. u^g for $Y = \overline{Tx}$ and $Y' = \overline{Tx'}$ in Example 3.5.

upper triangular matrices, we have that

$$\begin{aligned}\lambda(t) \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B &= \begin{pmatrix} t^{\lambda_1} & 0 & 0 \\ 0 & t^{\lambda_2} & 0 \\ 0 & 0 & t^{\lambda_3} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} B \\ &= \begin{pmatrix} t^{\lambda_1} & 0 & 0 \\ t^{\lambda_2} & t^{\lambda_2} & 0 \\ t^{\lambda_3} & t^{\lambda_3} & t^{\lambda_3} \end{pmatrix} B \\ &= \begin{pmatrix} 1 & 0 & 0 \\ t^{\lambda_2-\lambda_1} & 1 & 0 \\ t^{\lambda_3-\lambda_1} & t^{\lambda_3-\lambda_2} & 1 \end{pmatrix} B \xrightarrow{t\to 0} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B.\end{aligned}$$

The last step in the above is achieved by the condition $\lambda_1 < \lambda_2 < \lambda_3$. Hence we get $123^g = 123$.

We present one more example. Let us choose an element $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Int}(C(231)) \cap \mathfrak{t}_{\mathbb{Z}}$, that is $\lambda_2 < \lambda_3 < \lambda_1$. Then we have that

$$\begin{aligned}\lambda(t) \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B &= \begin{pmatrix} t^{\lambda_1} & 0 & 0 \\ 0 & t^{\lambda_2} & 0 \\ 0 & 0 & t^{\lambda_3} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} B \\ &= \begin{pmatrix} t^{\lambda_1} & t^{\lambda_1} & t^{\lambda_1} \\ t^{\lambda_2} & 0 & 0 \\ t^{\lambda_3} & 0 & t^{\lambda_3} \end{pmatrix} B \\ &= \begin{pmatrix} t^{\lambda_1-\lambda_2} & 1 & t^{\lambda_1-\lambda_3} \\ 1 & 0 & 0 \\ t^{\lambda_3-\lambda_2} & 0 & 1 \end{pmatrix} B \xrightarrow{t\to 0} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B.\end{aligned}$$

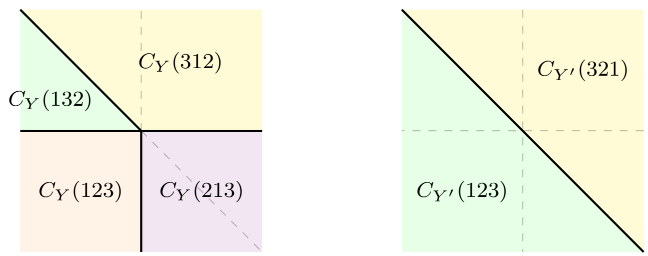
Therefore we get $231^g = 213$.

By a similar computation, one can get the result in Table 1. Hence the fan of Y is as in Figure 2(a). (Also, see Figure 1(b).)

(2) If Y' is the closure of the torus orbit Tx', then $(Y')^T = \{123, 321\}$, and we have the result in Table 1. Then both $C_{Y'}(123)$ and $C_{Y'}(321)$ are not strongly convex, that is, they contain a straight line through the origin. See Figure 2(b). By projecting them to the quotient space $\langle \varpi_1, \varpi_2 \rangle / \langle -\varpi_1 + \varpi_2 \rangle \cong \langle \varpi_1 + \varpi_2 \rangle$, we obtain the fan associated to Y'. (See Figure 1(a) for the fundamental weight vectors ϖ_1 and ϖ_2.)

Note that for an element x in $\mathrm{SL}_n(\mathbb{C})$, one can describe $(\overline{Tx})^T$ using minors of x. For a sequence $\underline{i} = (i_1, \dots, i_d)$ such that $1 \leq i_1 < i_2 < \cdots < i_d \leq n$, we define $p_{\underline{i}}(x)$ to be the $d \times d$ minor of x with row indices $i_1, \dots, i_d$ and the column indices $1, \dots, d$. We define the collection $I_d(x)$ by

$$I_d(x) = \{\underline{i} = (i_1, \dots, i_d) \mid p_{\underline{i}}(x) \neq 0\} \quad 1 \leq d \leq n.$$



(a) C_Y for Y in Example 3.5(1). (b) $C_{Y'}$ for Y' in Example 3.5(2).

Figure 2. Examples of $C_Y(y)$.

Then one can get the following.

Proposition 3.6 ([13, Proposition 1 in §5.2]). *For an element $x \in \mathrm{SL}_n(\mathbb{C})$, we have that*

$$(\overline{Tx})^T = \{w \in \mathfrak{S}_n \mid \{w(1), \dots, w(d)\} \uparrow \in I_d(x) \quad \text{for all } 1 \le d \le n\}.$$

Here, $\{a_1, \dots, a_d\} \uparrow$ is defined by the ordered d-tuple obtained from $\{a_1, \dots, a_d\}$ by arranging its elements in ascending order.

For instance, when x is the element in Example 3.5(1), we have that

$$I_1(x) = \{(1), (2), (3)\}, \quad I_2(x) = \{(1,2), (1,3)\}, \quad I_3(x) = \{(1,2,3)\},$$
$$\{123, 132, 213, 312\} = \{w \in \mathfrak{S}_3 \mid \{w(1), \dots, w(d)\} \uparrow \in I_d(x) \quad \text{for all } 1 \le d \le 3\}.$$

4. Geometric retractions and the closest points

In the previous section, we defined the geometric retraction $\mathcal{R}_Y^g \colon W \to Y^T$ for a torus orbit closure Y in G/B. In this section, we show that for every $u \in W$, the retraction image $\mathcal{R}_Y^g(u) = u^g$ is the closest point in Y^T with respect to the metric d on W defined in (4.1) below.

Note that if $v = ws_\alpha$ for a root α, then $v = s_{w(\alpha)}w$ and $s_{w(\alpha)} = s_{v(\alpha)}$ since $ws_\alpha = s_{w(\alpha)}w$ and $v(\alpha) = -w(\alpha)$.

Lemma 4.1. *If $v = ws_\alpha (= s_{w(\alpha)}w)$ for some simple root α and $v^g \neq w^g$, then $v^g = s_{w(\alpha)}w^g = s_{v(\alpha)}w^g$.*

Proof. First $s_{w(\alpha)} = s_{v(\alpha)}$ as remarked above, so it suffices to prove $v^g = s_{w(\alpha)}w^g$. Since $v = ws_\alpha$ and α is simple, the intersection $C(v) \cap C(w)$ is a common facet of $C(v)$ and $C(w)$ (which determines $\ker w(\alpha)$). Then the cones $C_Y(v^g)$ and $C_Y(w^g)$ defined in (3.4) share a common facet containing $C(v) \cap C(w)$ because $C_Y(v^g)$ and $C_Y(w^g)$ are distinct as $v^g \neq w^g$ and contain $C(v)$ and $C(w)$ respectively. This means that v^g and w^g, which are in Y^T, are contained in the same connected component of $Y^{\ker w(\alpha)}$. Therefore, $v^g = s_{w(\alpha)}w^g$ by Lemma 2.1. □

We define a metric d on W by

$$d(v,w) := \ell(v^{-1}w) = \ell(w^{-1}v) \qquad \text{for } v, w \in W \tag{4.1}$$

where $\ell(\ )$ denotes the length function on W. Note that the metric d is invariant under the left multiplication of W. For a subset $\mathcal{M}$ of W, we define

$$d(v, \mathcal{M}) := \min\{d(v,w) \mid w \in \mathcal{M}\}. \tag{4.2}$$

Remark 4.2. Recall that Δ_W is the convex hull of the W-orbit of a point ν in a Weyl chamber, the vertices of Δ_W are in W, and two vertices v and w are joined by an edge in Δ_W if and only if $v = ws_\alpha$ for some simple root α. Therefore, the distance $d(v,w)$ can be thought of as the minimum length of the paths in Δ_W connecting v and w through edges of Δ_W. In other words, the metric d is the graph metric on the graph obtained as the 1-skeleton of Δ_W.

For every subset $\mathcal{M}$ of W and every $u \in W$, there exists an element $y \in \mathcal{M}$ such that $d(u,y) = d(u,\mathcal{M})$ since $\mathcal{M}$ is a finite set, but such an element y may not be unique. However, the following theorem says that such an element y is unique if $\mathcal{M} = Y^T$ for some T-orbit closure Y in G/B and that the element y can be obtained as the limit point of a one parameter subgroup in $\mathrm{Int}(C(u)) \cap \mathfrak{t}_{\mathbb{Z}}$.

Theorem 4.3. *Let Y be a T-orbit closure in G/B and $u \in W$. If y is an element of Y^T such that $d(u,y) = d(u,Y^T)$, then $y = u^g (= \mathcal{R}^g_Y(u))$.*

Example 4.4. Suppose that $\mathcal{M} = \{123, 213, 132, 312\} \subset \mathfrak{S}_3$. We observed in Example 3.5(1) that $\mathcal{M} = Y^T$ for some torus orbit closure Y. Then one can check that if $d(u,y) = d(u,Y^T)$ then $y = u^g = \mathcal{R}^g_Y(u)$ for every $u \in W$ (see Table 1 and Figure 3).

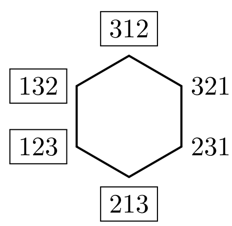


FIGURE 3. The 1-skeleton of $\Delta_{\mathfrak{S}_3}$ and $\mathcal{M} = \{123, 213, 132, 312\}$.

Proof of Theorem 4.3. We set $p = d(u,y)$. Then one can write

$$u = y s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_p}} \tag{4.3}$$

where $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_p}$ are simple roots and the product $s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_p}}$ is reduced. For $0 \le k \le p$, we set

$$u_k = y s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_k}}. \tag{4.4}$$

Note that since $d(u,y) = d(u,Y^T) = p$, it follows from (4.3) that

$$d(u_k, y) = d(u_k, Y^T) = k \qquad \text{for any } 0 \le k \le p. \tag{4.5}$$

We shall prove $u_k^g = y$ by induction on k. When $k = 0$, we have $u_0^g = y$ since $u_0 = y$ and $y \in Y^T$. Suppose $u_{k-1}{}^g = y$ and $u_k^g \neq y$. Then since $u_k = u_{k-1} s_{\alpha_{i_k}}$ and $u_k^g \neq u_{k-1}{}^g$, it follows from Lemma 4.1 and the induction assumption $u_{k-1}{}^g = y$ that

$$u_k^g = s_{u_k(\alpha_{i_k})} u_{k-1}{}^g = s_{u_k(\alpha_{i_k})} y.$$

Using (4.4), the identity above turns into

$$u_k^g = s_{u_k(\alpha_{i_k})} u_k s_{\alpha_{i_k}} s_{\alpha_{i_{k-1}}} \cdots s_{\alpha_{i_1}}.$$

Here $s_{u_k(\alpha_{i_k})}u_k s_{\alpha_{i_k}} = u_k$ since $s_{w(\alpha)}w = ws_\alpha$ and s_α^2 is the identity element for any $w \in W$ and any root α. Therefore, the identity above reduces to

$$u_k^g = u_k s_{\alpha_{i_{k-1}}} \cdots s_{\alpha_{i_1}}$$

and this shows that $d(u_k, u_k^g) = k-1$ which contradicts (4.5) because $u_k^g \in Y^T$. □

Example 4.5. Let v and w be elements in W with $v \leq w$ in Bruhat order. Then we have the Richardson variety $X_w^v := X_w \cap w_0 X_{w_0 v}$ in G/B, where w_0 is the longest element in W. Here, $X_w := \overline{BwB/B}$ is the Schubert variety associated to $w \in W$. Note that X_w^v is T-invariant and the fixed points of T in X_w^v are identified with the elements in the Bruhat interval $[v, w] := \{z \in W \mid v \leq z \leq w\}$. We can take a point $x \in X_w^v$ such that

$$(\overline{Tx})^T = (X_w^v)^T (= [v, w]).$$

We call such a point x *generic* (see [22, Definition 2.1]). Moreover, the existence of generic points in the Richardson variety X_w^v can be proven similarly to [22, Proposition 3.8]. Therefore, every Bruhat interval can appear as the T-fixed point set Y^T of some torus orbit closure Y. We remark that the intersection

$$R_{v,w;>0} := (BwB/B \cap w_0 B w_0 v B/B) \cap (G/B)_{\geq 0}$$

of the cell $BwB/B \cap w_0 B w_0 v B/B$ in G/B with the *totally non-negative part* $(G/B)_{\geq 0}$ of G/B consists of generic points in X_w^v (see [26, Theorem 7.1 and Lemma 7.5]). Moreover, the set $R_{v,w;>0}$ is not empty. Indeed, it is proven in [25, Theorem 2.8] that $R_{v,w;>0}$ is a semi-algebraic cell of dimension $\ell(w) - \ell(v)$. Combining them, one can also prove the existence of generic points in X_w^v.

5. Flag varieties of classical Lie type

When G is a simple classical algebraic group, the flag variety G/B has the well-known description as the set of flags. We shall review it in this section. See [20] and [11] for more details on roots systems, and see [12, §6.1] for the flag varieties G/B of classical algebraic groups G. In what follows, we fix an ordering on the simple roots as in the book of Humphreys [20, §11.4]. Since the Weyl groups of type B_n and C_n are the same and the description of the flag variety of type C as flags is slightly easier to understand than that of type B, we shall explain the case of type C and not the case of type B.

5.1. **Type A_{n-1}.** In this case, $G = \mathrm{SL}_n(\mathbb{C})$ and we take B to be the Borel subgroup consisting of all upper triangular matrices in $\mathrm{SL}_n(\mathbb{C})$. Then the flag variety G/B of type A_{n-1} can be described as the set of all complete flags

$$L_1 \subset L_2 \subset \cdots \subset L_n = \mathbb{C}^n$$

where each L_i is a linear subspace of $\mathbb{C}^n$ of dimension i. Suppose that each vector space L_i is spanned by vectors $v_1, \ldots, v_i$. Then the flag $(L_1 \subset L_2 \subset \cdots \subset L_n)$ is associated to a matrix $[v_1 \ \cdots \ v_n] \in \mathrm{SL}_n(\mathbb{C})$ whose column vectors are $v_1, \ldots, v_n$. We have the Plücker embedding of G/B into the product $\prod_{k=1}^{n-1} \mathbb{P}(\wedge^k \mathbb{C}^n)$ of complex projective spaces induced from the map

$$\mathrm{SL}_n(\mathbb{C}) \to \wedge^k \mathbb{C}^n, \quad [v_1 \ \ldots \ v_n] \mapsto v_1 \wedge \ldots \wedge v_k.$$

Let $(\mathbb{C}^*)^n_0$ be the subgroup of $(\mathbb{C}^*)^n$ such that the product of coordinates is equal to 1. We take the subgroup of $\mathrm{SL}_n(\mathbb{C})$ consisting of all diagonal matrices as the maximal torus T and identify it with $(\mathbb{C}^*)^n_0$ through the map

$$(\mathbb{C}^*)^n_0 \to \mathrm{SL}_n(\mathbb{C}), \quad (t_1, \dots, t_n) \mapsto \mathrm{diag}(t_1, \dots, t_n).$$

The coordinatewise multiplication of T on $\mathbb{C}^n$ induces the action of T on $\wedge^k \mathbb{C}^n$ and on the projective space $\mathbb{P}(\wedge^k \mathbb{C}^n)$. Explicitly, if $p_{\underline{i}}$ denotes the Plücker coordinate of $\mathbb{P}(\wedge^k(\mathbb{C}^n))$, where $\underline{i} = (i_1, \dots, i_k)$ such that $1 \le i_1 < i_2 < \cdots < i_d \le n$, then the action of $(t_1, \dots, t_n) \in T$ on the coordinate $p_{\underline{i}}$ is given by

$$(t_1, \dots, t_n) p_{\underline{i}} = t_{\underline{i}} p_{\underline{i}} \qquad \text{where } t_{\underline{i}} = \prod_{j=1}^{k} t_{i_j}, \tag{5.1}$$

so that the Plücker embedding becomes T-equivariant.

The identification of T with $(\mathbb{C}^*)^n_0$ induces an identification of the real Lie algebra $\mathfrak{t}_{\mathbb{R}}$ of T with the linear subspace $(\mathbb{R}^n)_0$ of $\mathbb{R}^n$ having the sum of coordinates equal to 0. We further identify $\mathfrak{t}^*_{\mathbb{R}}$ with $(\mathbb{R}^n)_0$ using the standard inner product on $\mathbb{R}^n$. Then the roots of $\mathrm{SL}_n(\mathbb{C})$ (with respect to the maximal torus T) are

$$\{\pm(e_i - e_j) \mid 1 \le i < j \le n\},$$

where $e_1, \dots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$, and the positive roots (associated to the Borel subgroup B) are $\{e_i - e_j \mid 1 \le i < j \le n\}$. The Weyl group W is the permutation group $\mathfrak{S}_n$ on $[n]$ and it acts on the roots through $e_i \to e_{u(i)}$ for $u \in \mathfrak{S}_n$. Since the simple roots are

$$\begin{array}{cccc} e_1 - e_2, & e_2 - e_3, & \dots, & e_{n-1} - e_n, \\ \| & \| & & \| \\ \alpha_1 & \alpha_2 & & \alpha_{n-1} \end{array}$$

the cone $C(u)$ in (3.1) is of the form

$$C(u) = \{(a_1, \dots, a_n) \in (\mathbb{R}^n)_0 \mid a_{u(1)} \le \ldots \le a_{u(n)}\} \quad \text{for } u \in \mathfrak{S}_n. \tag{5.2}$$

In the following we will denote the standard basis of $\mathbb{C}^{2n}$ or $\mathbb{R}^{2n}$ by

$$e_1, e_2, \dots, e_n, e_{\bar{n}}, \dots, e_{\bar{2}}, e_{\bar{1}},$$

where $e_{\bar{i}}$ for $i \in [n]$ is e_{2n+1-i} in the standard notation and we consider the following ordering

$$1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}. \tag{5.3}$$

We denote the set $\{\bar{1}, \dots, \bar{n}\}$ by $[\bar{n}]$ and regard $\bar{\bar{i}} = i$.

5.2. **Type C_n.** In this case, the group G is $\mathrm{Sp}_{2n}(\mathbb{C})$ associated to the nondegenerate skew-symmetric form $\langle\ ,\ \rangle$ on $\mathbb{C}^{2n}$ defined by

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i \in [n] \text{ and } j = \bar{i}, \\ -1 & \text{if } j \in [n] \text{ and } i = \bar{j}, \\ 0 & \text{otherwise.} \end{cases}$$

We take B to be the Borel subgroup consisting of all upper triangular matrices in $\mathrm{Sp}_{2n}(\mathbb{C})$. Then the flag variety G/B of type C_n can be described as the set of all complete *isotropic* flags

$$L_1 \subset L_2 \subset \cdots \subset L_n \tag{5.4}$$

where each L_i is an isotropic linear subspace of $\mathbb{C}^{2n}$ of dimension i, that is, the skew-symmetric form $\langle\ ,\ \rangle$ vanishes on L_i (so that $i \le n$). The target space of the Plücker embedding of G/B is the product $\prod_{k=1}^n \mathbb{P}(\wedge^k \mathbb{C}^{2n})$.

We take the subgroup of $\mathrm{Sp}_{2n}(\mathbb{C})$ consisting of all diagonal matrices as the maximal torus T and identify it with $(\mathbb{C}^*)^n$ through the map

$$(\mathbb{C}^*)^n \to \mathrm{Sp}_{2n}(\mathbb{C}), \quad (t_1, \dots, t_n) \mapsto \mathrm{diag}(t_1, \dots, t_n, t_{\bar{n}}, \dots, t_{\bar{1}})$$

where $t_{\bar{j}} = t_j^{-1}$ for $j \in [n]$. As in the case of type A, if $p_{\underline{i}}$ denotes the Plücker coordinate of $\mathbb{P}(\wedge^k(\mathbb{C}^{2n}))$, where $\underline{i} = (i_1, \dots, i_k)$ such that $1 \le i_1 < i_2 < \cdots < i_d \le \bar{1}$, then the action of $(t_1, \dots, t_n) \in T$ on the coordinate $p_{\underline{i}}$ is given by (5.1), so that the Plücker embedding is T-equivariant.

Similarly to the case of type A, the identification of T with $(\mathbb{C}^*)^n$ induces an identification of the real Lie algebra $\mathfrak{t}_{\mathbb{R}}$ of T with $\mathbb{R}^n$ and further $\mathfrak{t}_{\mathbb{R}}^*$ with $\mathbb{R}^n$ through the standard inner product on $\mathbb{R}^n$. Then the roots of $\mathrm{Sp}_{2n}(\mathbb{C})$ (with respect to the maximal torus T) are

$$\{\pm(e_i \pm e_j) \mid 1 \le i < j \le n\} \cup \{\pm 2e_i \mid 1 \le i \le n\}$$

and the positive roots (associated to the Borel subgroup B) are

$$\{e_i \pm e_j \mid 1 \le i < j \le n\} \cup \{2e_i \mid 1 \le i \le n\}.$$

The Weyl group W is the signed permutation group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$ on $[n] \cup [\bar{n}]$ and it acts on the roots through $e_i \to e_{u(i)}$, where $u(\bar{i}) = \overline{u(i)}$ for $u \in (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$ and $e_{\bar{i}} = -e_i$. Since the simple roots are

$$\begin{array}{ccccc} e_1 - e_2, & e_2 - e_3, & \dots, & e_{n-1} - e_n, & 2e_n \\ \| & \| & & \| & \| \\ \alpha_1 & \alpha_2 & & \alpha_{n-1} & \alpha_n \end{array}$$

the cone $C(u)$ in (3.1) is of the form

$$C(u) = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid a_{u(1)} \le \dots \le a_{u(n)} \le 0\} \tag{5.5}$$

for $u \in (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$, where $a_{\bar{i}} = -a_i$ for $i \in [n]$.

5.3. **Type D_n.** In this case, the group G is $\mathrm{SO}_{2n}(\mathbb{C})$ associated to the nondegenerate symmetric form $\langle\ ,\ \rangle$ on $\mathbb{C}^{2n}$ defined by

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i \in [n] \text{ and } j = \bar{i} \text{ or } j \in [n] \text{ and } i = \bar{j}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly to the case of type C, we take B to be the Borel subgroup consisting of all upper triangular matrices in $\mathrm{SO}_{2n}(\mathbb{C})$. Then the flag variety G/B of type D_n can also be described as the set of all complete *isotropic* flags (5.4) but with one requirement that the dimension of the intersection of L_n with the (isotropic) linear subspace spanned by $e_1, \dots, e_n$ is congruent to n modulo 2.

The maximal torus of $\mathrm{SO}_{2n}(\mathbb{C})$ is the same as in the case of type C_n. The Plücker embedding of G/B into the product $\prod_{k=1}^n \mathbb{P}(\wedge^k \mathbb{C}^{2n})$ is also the same as in the case of type C_n and T-equivariant.

The roots of $\mathrm{SO}_{2n}(\mathbb{C})$ (with respect to the maximal torus T) are

$$\{\pm(e_i \pm e_j) \mid 1 \le i < j \le n\}$$

and the positive roots (associated to the Borel subgroup B) are

$$\{e_i \pm e_j \mid 1 \le i < j \le n\}.$$

The Weyl group W is the index two subgroup $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes \mathfrak{S}_n$ of the signed permutation group on $[n] \cup [\bar{n}]$, where the number of bars in $w(1), \dots, w(n)$ is even (possibly zero). Since the simple roots are

$$\begin{array}{ccccc} e_1 - e_2, & e_2 - e_3, & \dots, & e_{n-1} - e_n, & e_{n-1} + e_n, \\ \| & \| & & \| & \| \\ \alpha_1 & \alpha_2 & & \alpha_{n-1} & \alpha_n \end{array}$$

the cone $C(u)$ in (3.1) is of the form

$$C(u) = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid a_{u(1)} \leq \dots \leq a_{u(n-1)} \leq a_{u(n)} \leq -a_{u(n-1)}\} \tag{5.6}$$

for $u \in (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$.

6. Algebraic retractions on the Weyl group of classical Lie type

In this section, for a subset $\mathcal{M} \subset W$, we define a retraction $\mathcal{R}^a_{\mathcal{M}}$ from W onto $\mathcal{M}$ algebraically by taking the locally optimal choice at each step with the intent of finding a closest element. In general, this strategy does not produce a closest element, but we can show that $\mathcal{R}^a_{Y^T} = \mathcal{R}^g_Y$ for every torus orbit closure Y of G/B, that is, $\mathcal{R}^a_{Y^T}$ produces the closest element.

We continue to assume that our group G is of type A_{n-1}, B_n, C_n or D_n, so that the Weyl group W is of the following form:

$$W = \begin{cases} \mathfrak{S}_n & \text{if } W \text{ is of type } A_{n-1}, \\ (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n & \text{if } W \text{ is of type } B_n \text{ or } C_n, \\ (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes \mathfrak{S}_n & \text{if } W \text{ is of type } D_n. \end{cases}$$

In each type we will use one-line notation for $u \in W$, i.e.

$$u = u(1)u(2) \cdots u(n)$$

where $u(i) \in [n] \cup [\bar{n}]$ and $u(1)u(2) \cdots u(n)$ is a permutation on $[n]$ if we forget the bars. There is no bar in type A and the number of bars in $u(1), \dots, u(n)$ is even (possibly zero) in type D. In types B, C and D, we have $u(\bar{i}) = \overline{u(i)}$ and recall the ordering (5.3) on $[n] \cup [\bar{n}]$. With this understood, we make the following definition.

Definition 6.1. For a subset $\mathcal{M}$ of W, we consider the sets

$$J_k(\mathcal{M}) = \{(w(1), w(2), \dots, w(k)) \mid w \in \mathcal{M}\} \qquad \text{for } 1 \leq k \leq n$$

and to each $u \in W$, we first choose

$$i_1 := \min\{i \in [n] \cup [\bar{n}] \mid u(i) \in J_1(\mathcal{M})\}$$

and inductively choose

$$i_k := \min\{i \in [n] \cup [\bar{n}] \setminus \{i_1, \dots, i_{k-1}\} \mid (u(i_1), \dots, u(i_{k-1}), u(i)) \in J_k(\mathcal{M})\}$$

for $2 \leq k \leq n$. We define

$$\mathcal{R}^a_{\mathcal{M}}(u) := u(i_1)u(i_2) \cdots u(i_n) \in \mathcal{M}.$$

Clearly, $\mathcal{R}^a_{\mathcal{M}}(u) = u$ if $u \in \mathcal{M}$, so the map

$$\mathcal{R}^a_{\mathcal{M}} \colon W \to \mathcal{M} \, (\subset W)$$

is a retraction of W onto $\mathcal{M}$, which we call an *algebraic retraction*.

Example 6.2. (1) We take $\mathcal{M} = \{1423, 1432, 2413, 3412\}$ in $\mathfrak{S}_4$. Then

$$J_1(\mathcal{M}) = \{(1), (2), (3)\}, \qquad J_2(\mathcal{M}) = \{(1,4), (2,4), (3,4)\},$$
$$J_3(\mathcal{M}) = \{(1,4,2), (1,4,3), (2,4,1), (3,4,1)\},$$
$$J_4(\mathcal{M}) = \{(1,4,2,3), (1,4,3,2), (2,4,1,3), (3,4,1,2)\}.$$

Therefore, if $u = 2314$, then

$$i_1 = 1, \quad i_2 = 4, \quad i_3 = 3, \quad i_4 = 2,$$

so $\mathcal{R}^a_{\mathcal{M}}(2314) = 2413$.

(2) We take $\mathcal{M} = \{1\bar{4}23, 14\bar{3}\bar{2}, 2413, \bar{3}\bar{4}1\bar{2}\}$ in $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathfrak{S}_4$. Then

$$J_1(\mathcal{M}) = \{(1), (2), (\bar{3})\}, \qquad J_2(\mathcal{M}) = \{(1,\bar{4}), (1,4), (2,4), (\bar{3},\bar{4})\},$$
$$J_3(\mathcal{M}) = \{(1,\bar{4},2), (1,4,\bar{3}), (2,4,1), (\bar{3},\bar{4},1)\},$$
$$J_4(\mathcal{M}) = \{(1,\bar{4},2,3), (1,4,\bar{3},\bar{2}), (2,4,1,3), (\bar{3},\bar{4},1,\bar{2})\}.$$

Therefore, if $u = \bar{2}3\bar{1}4$, then

$$i_1 = \bar{3}, \quad i_2 = 4, \quad i_3 = \bar{2}, \quad i_4 = 1,$$

so $\mathcal{R}^a_{\mathcal{M}}(\bar{2}3\bar{1}4) = 14\bar{3}\bar{2}$.

Remark 6.3. It follows from the definition of $\mathcal{R}^a_{\mathcal{M}}$ that we have $\mathcal{R}^a_{v\mathcal{M}}(vu) = v\mathcal{R}^a_{\mathcal{M}}(u)$ for any $u, v \in W$. Therefore $d(vu, \mathcal{R}^a_{v\mathcal{M}}(vu)) = d(u, \mathcal{R}^a_{\mathcal{M}}(u))$.

If G is a product of simple algebraic groups $G = \prod_{j=1}^m G_j$, and $\mathcal{M}$ is also a product $\mathcal{M} = \prod_{j=1}^m \mathcal{M}_j$ for $\mathcal{M}_j \subset W(G_j)$, then one can define an algebraic retraction by applying the algebraic retractions on $\mathcal{M}_j \subset W(G_j)$, i.e., for $u = (u_1, \dots, u_m) \in W$,

$$\mathcal{R}^a_{\mathcal{M}}(u) := (\mathcal{R}^a_{\mathcal{M}_1}(u_1), \dots, \mathcal{R}^a_{\mathcal{M}_m}(u_m)). \tag{6.1}$$

Remark 6.4. Note that the definition above only works when $\mathcal{M}$ is a product. For example, let $G = \mathrm{SL}_3(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ and $\mathcal{M} = \{(123, 12), (231, 21)\} \subset \mathfrak{S}_3 \times \mathfrak{S}_2$. Consider the projection maps $\pi_1 \colon \mathfrak{S}_3 \times \mathfrak{S}_2 \to \mathfrak{S}_3$ and $\pi_2 \colon \mathfrak{S}_3 \times \mathfrak{S}_2 \to \mathfrak{S}_2$, and take $\mathcal{M}_1 = \pi_1(\mathcal{M})$ and $\mathcal{M}_2 = \pi_2(\mathcal{M})$. Unfortunately, for $u = (213, 12) \in \mathfrak{S}_3 \times \mathfrak{S}_2$, we have that $(\mathcal{R}^a_{\mathcal{M}_1}(213), \mathcal{R}^a_{\mathcal{M}_2}(12)) = (231, 21) \notin \mathcal{M}$.

Note that when $\mathcal{M}$ is the set of T-fixed points of a torus orbit closure in G/B, we have $\mathcal{M} = \prod_{j=1}^m \mathcal{M}_j$ for $\mathcal{M}_j \subset W(G_j)$ since T acts on G/B coordinatewise as in (2.2). Therefore, the algebraic retraction on $\mathcal{M} \subset W$ is obtained by applying the algebraic retractions on $\mathcal{M}_j \subset W(G_j)$ as in (6.1).

Recall that if Y is a T-orbit closure in the flag variety G/B, then the T-fixed point set Y^T of Y is a subset of the Weyl group W of G, so that the algebraic retraction $\mathcal{R}^a_{Y^T}$ is defined on W. On the other hand, we have the geometric retraction $\mathcal{R}^g_Y$ on W defined in (3.3). With this understood, we have

Theorem 6.5. *If G is a product of simple algebraic groups of classical Lie type, then, for every T-orbit closure Y in G/B, we have $\mathcal{R}^a_{Y^T} = \mathcal{R}^g_Y$.*

Proof. We first consider the situation where G is a simple algebraic group of classical Lie type. Let $u \in W$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathrm{Int}(C(u)) \cap \mathfrak{t}_{\mathbb{Z}}$. Using the identification

$\mathfrak{t}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^*, T)$, we have that $\lambda(t) = (t^{\lambda_1}, \dots, t^{\lambda_n})$. Since $\lambda \in \mathrm{Int}(C(u))$, it follows from (5.2), (5.5) and (5.6) that we have strict inequalities

$$\begin{cases} \lambda_{u(1)} < \cdots < \lambda_{u(n)} & \text{in type } A, \\ \lambda_{u(1)} < \cdots < \lambda_{u(n)} < 0 & \text{in type } C, \\ \lambda_{u(1)} < \cdots < \lambda_{u(n-1)} < \lambda_{u(n)} < -\lambda_{u(n-1)} & \text{in type } D. \end{cases}$$

Remember that $u(\bar{i}) = \overline{u(i)}$ and $\lambda_{\bar{i}} = -\lambda_i$. Therefore the inequalities above are equivalent to the following:

$$\begin{cases} \lambda_{u(1)} < \cdots < \lambda_{u(n)} & \text{in type } A, \\ \lambda_{u(1)} < \cdots < \lambda_{u(n)} < \lambda_{u(\bar{n})} < \cdots < \lambda_{u(\bar{1})} & \text{in type } C, \\ \lambda_{u(1)} < \cdots < \lambda_{u(n-1)} < \lambda_{u(n)} < \lambda_{u(\overline{n-1})} < \cdots < \lambda_{u(\bar{1})} & \text{in type } D. \end{cases}$$

In type D, the sign of $\lambda_{u(n)}$ is undetermined and we choose $\lambda \in \mathrm{Int}(C(u)) \cap \mathfrak{t}_{\mathbb{Z}}$ such that $\lambda_{u(n)} < 0$. Therefore we have

$$\begin{cases} \lambda_{u(1)} < \cdots < \lambda_{u(n)} & \text{in type } A, \\ \lambda_{u(1)} < \cdots < \lambda_{u(n)} < \lambda_{u(\bar{n})} < \cdots < \lambda_{u(\bar{1})} & \text{in types } C \text{ and } D. \end{cases} \tag{6.2}$$

Let x be a point of G/B such that $Y = \overline{Tx}$. We denote $u^g := \mathcal{R}_Y^g(u)$ as before. Then

$$\lim_{t \to 0} \lambda(t)x = u^g. \tag{6.3}$$

If the $\underline{i}$ in the Plücker coordinate $p_{\underline{i}}$ is $(w(1), w(2), \dots, w(d))$, then it follows from (5.1) that the action of $t \in \mathbb{C}^*$ on $p_{\underline{i}}$ through λ is multiplication by $t^{\sum_{k=1}^d \lambda_{w(k)}}$. Therefore, through the Plücker embedding $G/B \to \prod_{k=1}^n \mathbb{P}(\wedge^k \mathbb{C}^m)$, where $m = n$ in type A_{n-1}, $m = 2n$ in types C_n and D_n, and $m = 2n+1$ in type B_n, the identity (6.3) means the following statement, which we call $(*)$.

$(*)$ For each $1 \le k \le n$, we have $\displaystyle\sum_{j=1}^k \lambda_{u^g(j)} \le \sum_{j=1}^k \lambda_{w(j)}$ for every $w \in Y^T$.

We denote by $u^a := \mathcal{R}_{\mathcal{M}}^a(u)$ for simplicity. We shall prove $u^g(k) = u^a(k)$ by induction on k for $1 \le k \le n$. Suppose that $u^g(j) = u^a(j)$ for all $j < k$ (note that the assumption is empty when $k = 1$). Then, since $u^a \in Y^T$, we have $\sum_{j=1}^k \lambda_{u^g(j)} \le \sum_{j=1}^k \lambda_{u^a(j)}$ by $(*)$ and hence $\lambda_{u^g(k)} \le \lambda_{u^a(k)}$ since $u^g(j) = u^a(j)$ for $j < k$ by the induction hypothesis. The opposite inequality $\lambda_{u^a(k)} \le \lambda_{u^g(k)}$ can be proved as follows. We define $q \in [n] \cup [\bar{n}]$ by $u(q) = u^g(k)$. Then

$$(u^a(1), \dots, u^a(k-1), u(q)) = (u^g(1), \dots, u^g(k-1), u^g(k)) \in J_k(Y^T).$$

Therefore, we have

$$i_k = \min\{i \in [n] \cup [\bar{n}] \backslash \{i_1, \dots, i_{k-1}\} \mid (u^a(1), \dots, u^a(k-1), u(i)) \in J_k(Y^T)\} \le q,$$

where $u(i_j) = u^a(j)$ for $1 \le j < k$. Since $u^a(k) = u(i_k)$ and $i_k \le q$, we obtain

$$\lambda_{u^a(k)} = \lambda_{u(i_k)} \le \lambda_{u(q)} = \lambda_{u^g(k)},$$

where the middle inequality above follows from (6.2). This proves the desired opposite inequality. Thus $\lambda_{u^g(k)} = \lambda_{u^a(k)}$ and hence $u^g(k) = u^a(k)$ again by (6.2). This completes the induction step and proves the theorem when G is a simple algebraic group of classical Lie type.

Now assume that $G = \prod_{j=1}^m G_j$ for simple algebraic groups G_j of classical Lie type. Then $B = \prod_{j=1}^m B_j$ and $T = \prod_{j=1}^m T_j$ where B_j and T_j denote a Borel subgroup of G_j and a maximal torus of G_j in B_j respectively for $1 \le j \le m$. Moreover, $W = \prod_{j=1}^m W_j$, where W_j is the Weyl group of G_j, and we have

$$d((u_1, \dots, u_m), (v_1, \dots, v_m)) = \sum_{j=1}^m d(u_j, v_j) \tag{6.4}$$

for elements $(u_1, \dots, u_m)$ and $(v_1, \dots, v_m)$ in W. Let $\mathcal{M}$ be the set of T-fixed points of a torus orbit closure Y in G/B. Since $G/B = \prod_{j=1}^m G_j/B_j$ and $T = \prod_{j=1}^m T_j$, there exists a torus orbit closure Y_j in G_j/B_j such that $Y = \prod_{j=1}^m Y_j$. Therefore $\mathcal{M} = \prod_{j=1}^m \mathcal{M}_j$ where $\mathcal{M}_j$ is the set of T_j-fixed points of Y_j. Hence, for $u = (u_1, \dots, u_m)$, we get that

$$\begin{aligned} d(u, \mathcal{M}) &= d((u_1, \dots, u_m), \mathcal{M}_1 \times \cdots \times \mathcal{M}_m) \\ &= \min \left\{ \sum_{j=1}^m d(u_j, m_j) \mid m_j \in \mathcal{M}_j \right\} \qquad \text{(from (6.4))} \\ &= \sum_{j=1}^m \min\{ d(u_j, m_j) \mid m_j \in \mathcal{M}_j \} \\ &= \sum_{j=1}^m d(u_j, \mathcal{R}^a_{\mathcal{M}_j}(u_j)) \\ &= d(u, \mathcal{R}^a_{\mathcal{M}}(u)). \qquad \text{(from (6.1))} \end{aligned}$$

The fourth equality above follows from the fact that $\mathcal{R}^a_{\mathcal{M}_j}(u_j) = \mathcal{R}^g_{\mathcal{M}_j}(u_j)$ and Theorem 4.3. Therefore, again by Theorem 4.3, the result follows. □

Combining Theorem 6.5 with Theorem 4.3, we obtain the following.

Corollary 6.6. *Let G be a product of simple algebraic groups of classical Lie type and W its Weyl group. Let $\mathcal{M}$ be a subset of W. If $\mathcal{M} = Y^T$ for some T-orbit closure Y in the flag variety G/B with W as the Weyl group, then for any $u \in W$, an element in $\mathcal{M}$ closest to u among elements in $\mathcal{M}$ is unique and given by $\mathcal{R}^a_{\mathcal{M}}(u)$.*

The following simple examples show that the conclusion in the corollary above does not hold for an arbitrary subset $\mathcal{M}$ of W.

Example 6.7.
(1) Let $W = \mathfrak{S}_3$ and $\mathcal{M} = \{213, 132\}$. Then $\mathcal{R}^a_{\mathcal{M}}(123) = 132$. But $d(123, 132) = d(123, 213)(= 1)$, so the both elements in $\mathcal{M}$ are closest to 123.
(2) Let $W = \mathfrak{S}_4$ and $\mathcal{M} = \{2143, 4312\}$. Then $\mathcal{R}^a_{\mathcal{M}}(1423) = 4312$. However $d(1423, 4312) = 3$ while $d(1423, 2143) = 2$. Therefore, $\mathcal{R}^a_{\mathcal{M}}(1423)$ does not give an element in $\mathcal{M}$ closest to 1423.

In the next section we will improve Corollary 6.6 (see Proposition 7.10). For that purpose, we prepare a lemma.

Lemma 6.8. *Let G be a product of simple algebraic groups of classical Lie type $G = \prod_{j=1}^m G_j$ and and W its Weyl group. Let $\mathcal{M} = \prod_{j=1}^m \mathcal{M}_j$ for $\mathcal{M}_j \subset W(G_j)$ for $1 \le j \le m$. If $\mathcal{M}$ has a unique minimal element v in the Bruhat order, i.e. $v \le w$ for all $w \in \mathcal{M}$, then $\mathcal{R}^a_{\mathcal{M}}(e) = v$.*

Proof. We first assume that G is simple. Since we observe $\mathcal{R}^a_{\mathcal{M}}(e)$, we take $u=e$ in Definition 6.1. The assumption of the lemma implies that for any $1 \le k \le n$, we have $v(k) \le w(k)$ for all $w \in \mathcal{M}$ with $w(i) = v(i)$ for $1 \le i \le k-1$. Therefore, i_k in Definition 6.1 is $v(k)$ and hence $\mathcal{R}^a_{\mathcal{M}}(e) = v$.

Now we assume that $G = \prod_{j=1}^m G_j$ for simple algebraic groups G_j and $\mathcal{M} = \prod_{j=1}^m \mathcal{M}_j$ for $\mathcal{M}_j \subset W(G_j)$. Note that for $v = (v_1, \dots, v_m)$ and $w = (w_1, \dots, w_m)$ in $W(G)$, we have $v \le w$ if and only if $v_j \le w_j$ in $W(G_j)$ for $1 \le j \le m$. Then, from (6.1), we have $\mathcal{R}^a_{\mathcal{M}}(e) = (\mathcal{R}^a_{\mathcal{M}_1}(e), \dots, \mathcal{R}^a_{\mathcal{M}_m}(e)) = (v_1, \dots, v_m) = v$. □

7. Coxeter matroids and matroid retractions

In this section, we shall observe that what we discussed so far is closely related to Coxeter matroids. Let W be a finite Coxeter group. Similarly to Weyl groups, W has the Bruhat order $\le$, the metric d, Weyl chambers and roots. For $u \in W$, $v \le^u w$ means $u^{-1}v \le u^{-1}w$ in the following.

Lemma 7.1. *For $u, v, w \in W$, if $v \le^u w$, then $d(u,v) \le d(u,w)$.*

Proof. We have

$$\begin{aligned} v \le^u w &\iff u^{-1}v \le u^{-1}w \\ &\implies d(e, u^{-1}v) \le d(e, u^{-1}w) \\ &\iff d(u,v) \le d(u,w), \end{aligned}$$

where the last equivalence follows from the fact that multiplication by u from the left preserves the metric, proving the lemma. □

A subset $\mathcal{M}$ of a finite Coxeter group W is called a *Coxeter matroid* if for any $u \in W$, there is a unique element $v \in \mathcal{M}$ such that $w \le^u v$ for all $w \in \mathcal{M}$ (see [5, §6.1.1] and [28] for more details on Coxeter matroids and matroid theories). Since the multiplication by the longest element w_0 reverses the Bruhat order, the existence of the unique maximal element is equivalent to the existence of a unique minimal element. When $\mathcal{M}$ is a Coxeter matroid, for $u \in W$ we denote the unique minimal element by $\mathcal{R}^m_{\mathcal{M}}(u)$, so

$$\mathcal{R}^m_{\mathcal{M}}(u) \le^u w \quad \text{for all } w \in \mathcal{M}. \tag{7.1}$$

Clearly $\mathcal{R}^m_{\mathcal{M}}(u) = u$ for $u \in \mathcal{M}$, so the map

$$\mathcal{R}^m_{\mathcal{M}} \colon W \to \mathcal{M} \subset W$$

is a retraction of W onto $\mathcal{M}$, which we call a *matroid retraction.*

Proposition 7.2. *Let $\mathcal{M}$ be a Coxeter matroid. Then the unique minimal element $\mathcal{R}^m_{\mathcal{M}}(u)$ satisfies that*

$$d(u, \mathcal{R}^m_{\mathcal{M}}(u)) = d(u, \mathcal{M}) \text{ for all } u \in W.$$

Moreover, for each $u \in W$, if there is an element $u' \in \mathcal{M}$ satisfying $d(u,u') = d(u,\mathcal{M})$, then $u' = \mathcal{R}^m_{\mathcal{M}}(u)$.

Proof. By (7.1) and Lemma 7.1, we have that

$$d(u, \mathcal{R}^m_{\mathcal{M}}(u)) \le d(u,w) \quad \text{for all } w \in \mathcal{M}.$$

Hence $d(u, \mathcal{R}^m_{\mathcal{M}}(u)) = d(u, \mathcal{M})$.

Now assume that for $u \in W$ we have an element u' satisfying $d(u, u') = d(u, \mathcal{M})$. Since $\mathcal{M}$ is a Coxeter matroid and $\mathcal{R}^m_{\mathcal{M}}(u)$ is the unique minimal element in $\mathcal{M}$ with respect to the ordering $\leq^u$, we have that $\mathcal{R}^m_{\mathcal{M}}(u) \leq^u u'$. Hence

$$u^{-1}\mathcal{R}^m_{\mathcal{M}}(u) \leq u^{-1}u'. \tag{7.2}$$

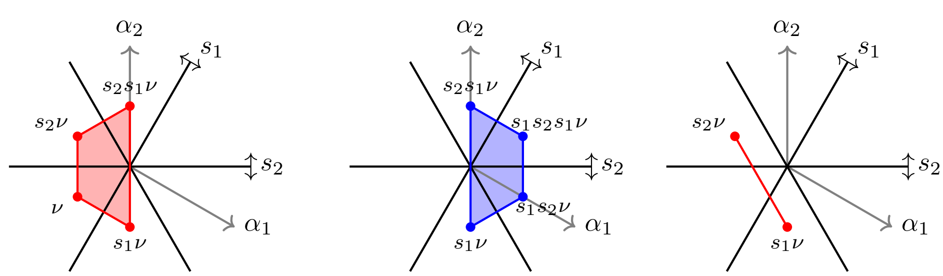
Since $d(u, u') = d(u, \mathcal{M})$ by assumption, we have

$$\ell(u^{-1}u') = \min\{\ell(u^{-1}w) \mid w \in \mathcal{M}\}.$$

Since $\mathcal{R}^m_{\mathcal{M}}(u) \in \mathcal{M}$, the inequality in (7.2) should become an equality, so we get $\mathcal{R}^m_{\mathcal{M}}(u) = u'$. □

Example 7.3. Suppose that $\mathcal{M} = \{213, 132\} \subset \mathfrak{S}_3$. Then we have seen in Example 6.7 that there are two different closest points in $\mathcal{M}$ to 123. Hence $\mathcal{M}$ is not a Coxeter matroid by Proposition 7.2. Indeed, since $213 \not\leq 132$ and $132 \not\leq 213$, there is no element $v \in \mathcal{M}$ such that $w \leq^{123} v$ for all $w \in \mathcal{M}$.

We recall that Coxeter matroids can be characterized in terms of polytopes. Let W be a Coxeter group and let Φ the set of roots of W in a Euclidean space V, where $V = \mathfrak{t}^*_{\mathbb{R}}$ when W is a Weyl group. A convex polytope Δ in V is called a Φ-*polytope* if every edge of Δ is parallel to a root in Φ. The generators of W act on V as reflections. Choose a point $\nu \in V$ which is not fixed by any reflection s_α for $\alpha \in \Phi^+$. Then the isotropy subgroup of W at ν is trivial and for a subset $\mathcal{M}$ of W we define $\Delta_{\mathcal{M}}$ to be the convex hull of the $\mathcal{M}$-orbit $\{u \cdot \nu \mid u \in \mathcal{M}\}$ of the point ν in V. The following is a part of the Gelfand–Serganova theorem (see [5, Theorem 6.3.1]).



(a) $\mathcal{M} = \{123, 213, 132, 312\}$. (b) $\mathcal{M} = \{213, 312, 231, 321\}$. (c) $\mathcal{M} = \{213, 132\}$.

FIGURE 4. Examples of $\Delta_{\mathcal{M}}$.

Theorem 7.4 (Gelfand–Serganova)**.** *A subset $\mathcal{M}$ of a finite Coxeter group W is a Coxeter matroid if and only if $\Delta_{\mathcal{M}}$ is a Φ-polytope.*

The Gelfand–Serganova theorem provides the following corollary.

Corollary 7.5. *Let W be a product of the Weyl groups $W_1, \dots, W_m$. Then a subset $\mathcal{M}$ is a Coxeter matroid of W if and only if there is a Coxeter matroid $\mathcal{M}_j$ of W_j for $1 \leq j \leq m$ such that $\mathcal{M}$ is the product $\prod_{j=1}^m \mathcal{M}_j$.*

Proof. Recall that $\Phi = \Phi(W_1) \sqcup \cdots \sqcup \Phi(W_m)$. Then from the Gelfand–Serganova theorem (Theorem 7.4), we have the following.

$$\begin{aligned}
&\mathcal{M} \text{ is a Coxeter matroid of } W\\
&\iff \Delta_{\mathcal{M}} \text{ is a } \Phi\text{-polytope}\\
&\iff \Delta_{\mathcal{M}} = \Delta_{\mathcal{M}_1} \times \cdots \times \Delta_{\mathcal{M}_m}\\
&\qquad\qquad \text{such that } \mathcal{M}_j \subset W_j \text{ and } \Delta_{\mathcal{M}_j} \text{ is a } \Phi(W_j)\text{-polytope}\\
&\iff \mathcal{M} = \prod_{j=1}^{m} \mathcal{M}_j \text{ and } \mathcal{M}_j \subset W_j \text{ is a Coxeter matroid.}
\end{aligned}$$

The second equivalence above follows from the fact that $\Delta_W = \Delta_{W_1} \times \cdots \times \Delta_{W_m}$. □

If Y is a T-orbit closure in the flag variety G/B, then the fixed point set Y^T is a Coxeter matroid because Δ_{Y^T} can be thought of as the moment map image of Y and it is a Φ-polytope. Therefore Y^T is a Coxeter matroid of W, where W is the Weyl group of G, so that $\mathcal{R}^m_{Y^T}$ is defined on W (see (7.1)). On the other hand, we have the retractions $\mathcal{R}^g_Y$ and $\mathcal{R}^a_{Y^T}$ on W defined in (3.3) and Definition 6.1, respectively. With this understood, we have the following.

Theorem 7.6. *Let G be a product of simple algebraic groups of classical Lie type. For every T-orbit closure Y in G/B, we have*

$$\mathcal{R}^g_Y = \mathcal{R}^a_{Y^T} = \mathcal{R}^m_{Y^T}.$$

Proof. The first identity is nothing but Theorem 6.5 and the second identity follows from Theorem 4.3, Lemma 7.1 and (7.1). □

Example 7.7. It follows from Theorem 7.4 that the subsets $\{123, 213, 132, 312\}$ and $\{213, 312, 231, 321\}$ of $\mathfrak{S}_3$ are Coxeter matroids because every edge of the corresponding polytopes is parallel to a root in Φ (see Figures 4(a) and 4(b)). On the other hand, the subset $\{213, 132\}$ of $\mathfrak{S}_3$ is not a Coxeter matroid since the edge $(213, 132)$ is not parallel to any root in Φ (see Figure 4(c) and Example 7.3).

Remark 7.8. It is natural to ask whether every Coxeter matroid of a Weyl group W can be obtained as the T-fixed point set of a T-orbit closure in G/B. One can check that this is the case when W is the symmetric group $\mathfrak{S}_n$ for $n \leq 4$ but not in general. Indeed, there is such a Coxeter matroid (or a *flag matroid* explained below) of $\mathfrak{S}_7$. More precisely, let

$$A = \{\{1,2,4\}, \{1,3,5\}, \{1,6,7\}, \{2,3,6\}, \{2,5,7\}, \{3,4,7\}, \{4,5,6\}\}$$

be a collection of subsets of $\{1, 2, \ldots, 7\}$. Each subset corresponds to one of the seven lines on the Fano plane in Figure 5. Define a subset $\mathcal{M} \subset \mathfrak{S}_7$ by

$$\mathcal{M} = \{w \in \mathfrak{S}_7 \mid \{w(1), w(2), w(3)\} \notin A\}.$$

Then $\mathcal{M}$ is a Coxeter matroid obtained from the Fano matroid using Higgs lifts (see [5, §1.7] for the definition of Higgs lift). Suppose that $\mathcal{M}$ is the T-fixed point set of a T-orbit closure in $\mathrm{Fl}(\mathbb{C}^7)$. Then, by Proposition 3.6, there is an element x in $\mathrm{SL}_7(\mathbb{C})$ such that

$$\mathcal{M} = \{w \in \mathfrak{S}_7 \mid \{w(1), \ldots, w(d)\} \uparrow\in I_d(x) \text{ for all } 1 \leq d \leq 7\}.$$

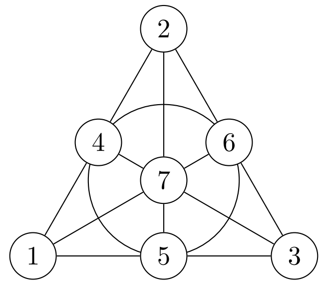


FIGURE 5. The Fano plane.

However, it is known in [29, §16] and easy to check that there is no 7×3 matrix of rank 3 whose three rows $v_{j_1}, v_{j_2}, v_{j_3}$ are linearly independent if and only if $\{j_1, j_2, j_3\} \notin A$. Therefore, $\mathcal{M}$ cannot be obtained as Y^T for a T-orbit closure Y.

When W is a Weyl group of classical Lie type, the notion of Coxeter matroid of W is equivalent to the notion of flag matroid. We shall recall this equivalence (see [5] for details). Suppose that W is a Weyl group of type A_{n-1}, B_n, C_n or D_n. We consider the ordering (5.3) on $J := [n] \cup [\bar{n}]$, that is,

$$1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1} \tag{7.3}$$

(the part $[\bar{n}]$ is unnecessary when W is of type A_{n-1}). Let J_k be the collection of subsets of J with cardinality k. Then for two subsets of J_k

$$A = \{i_1, \ldots, i_k\},\ i_1 < i_2 < \cdots < i_k$$

and

$$B = \{j_1, \ldots, j_k\},\ j_1 < j_2 < \cdots < j_k,$$

we set

$$A \leq B \quad \text{if and only if} \quad i_1 \leq j_1, \ldots, i_k \leq j_k.$$

For $u \in W$, we set $i \leq^u j$ for $i, j \in J$ if and only if $u^{-1}(i) \leq u^{-1}(j)$, which means that i precedes j in

$$u(1)u(2) \cdots u(n)u(\bar{n}) \cdots u(\bar{2})u(\bar{1}),$$

in other words,

$$u(1) <^u u(2) <^u \cdots <^u u(n) <^u u(\bar{n}) <^u \cdots <^u u(\bar{2}) <^u u(\bar{1}).$$

We can associate an ordering of J_k with $u \in W$ by putting

$$A \leq^u B \quad \text{if and only if} \quad u^{-1}A \leq u^{-1}B$$

for $A, B \in J_k$.

Example 7.9. Suppose that $u = 1\bar{4}23 \in (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathfrak{S}_4$. Then we have that

$$1 <^u \bar{4} <^u 2 <^u 3 <^u \bar{3} <^u \bar{2} <^u 4 <^u \bar{1}.$$

For $A = \{2, \bar{4}\}$ and $B = \{\bar{3}, 2\}$ in J_2, we get

$$u^{-1}A = \{2, 3\} \quad \text{and} \quad u^{-1}B = \{3, \bar{4}\}.$$

Hence $A \leq^u B$.

To an element $w \in W$, we associate an increasing sequence of finite sets

$$F(w) := (\{w(1)\} \subset \{w(1), w(2)\} \subset \cdots \subset \{w(1), w(2), \ldots, w(n)\}),$$

which is called a *complete flag*, and this correspondence gives a bijection between W and the set of complete flags. We set $F(v) \leq^u F(w)$ for $v, w \in W$ if and only if

$$\{v(1), \ldots, v(k)\}\!\uparrow \leq^u \{w(1), \ldots, w(k)\}\!\uparrow \quad \text{for any } 1 \leq k \leq n.$$

Here, for a subset $S \subset [n] \cup [\bar{n}]$, we denote by $S\!\uparrow$ the ordered tuple obtained from S by sorting its elements in ascending order with respect to $\leq^u$. A collection $\mathcal{F}$ of complete flags is called a *flag matroid*[1] if and only if it satisfies the Maximality Property, that is,

> for every $u \in W$, the collection $\mathcal{F}$ contains a unique element maximal in $\mathcal{F}$ with respect to the ordering $\leq^u$.

When W is a Weyl group of classical Lie type, the notion of flag matroids is equivalent to that of Coxeter matroids. As remarked above for Coxeter matroids, we may replace "maximal" in the definition of flag matroid by "minimal", namely we may replace the Maximality Property by the "Minimality" Property in the definition of flag matroids. The following is an improvement of Corollary 6.6.

Proposition 7.10. *Let G be a product of simple algebraic groups of classical Lie type $G = \prod_{j=1}^m G_j$ and W its Weyl group. Let $\mathcal{M} = \prod_{j=1}^m \mathcal{M}_j$ for $\mathcal{M}_j \subset W(G_j)$ for $1 \leq j \leq m$ and $\mathcal{R}^a_{\mathcal{M}}$ the algebraic retraction defined in* (6.1). *Then the following are equivalent:*

(1) *The collection of flags corresponding to $\mathcal{M}$ is a flag matroid.*
(2) *For each $u \in W$, $\mathcal{R}^a_{\mathcal{M}}(u)$ gives a unique minimal element in $\mathcal{M}$ with respect to the ordering $\leq^u$.*

Each of the above implies the following:

(3) *For each $u \in W$, $\mathcal{R}^a_{\mathcal{M}}(u)$ gives a unique element in $\mathcal{M}$ such that $d(u, \mathcal{R}^a_{\mathcal{M}}(u)) = d(u, \mathcal{M})$.*

Therefore, if the collection corresponding to $\mathcal{M}$ is a flag matroid, then $\mathcal{R}^a_{\mathcal{M}} = \mathcal{R}^m_{\mathcal{M}}$ (see (7.1) *for the definition of $\mathcal{R}^m_{\mathcal{M}}$).*

Proof. [(1) $\implies$ (2)] Suppose (1) holds. Then, $\mathcal{M}$ contains a unique minimal element in $\mathcal{M}$, say v, with respect to the ordering $\leq^u$, i.e. $u^{-1}v \leq u^{-1}w$ for all $w \in \mathcal{M}$. Therefore we have $\mathcal{R}^a_{u^{-1}\mathcal{M}}(e) = u^{-1}v$ by Lemma 6.8 and hence $\mathcal{R}^a_{\mathcal{M}}(u) = v$ by Remark 6.3, proving that (1) implies (2).

[(2) $\implies$ (1)] Let $\mathcal{F}$ be the collection of complete matroids corresponding to $\mathcal{M}$. Suppose (2) holds. Then, for each $u \in \mathcal{M}$, we have $\mathcal{R}^a_{\mathcal{M}}(u) \leq^u w$ for all $w \in \mathcal{M}$. This means that $\mathcal{F}$ satisfies the Minimality Property and hence the Maximality Property as remarked above, so $\mathcal{F}$ is a flag matroid.

[(2) $\implies$ (3)] Suppose (2) holds. Then we already showed that $\mathcal{M}$ produces a flag matroid. Since W is a Weyl group of classical Lie type, $\mathcal{M}$ is a Coxeter matroid. Hence by Proposition 7.2, (3) follows. □

Remark 7.11. We can further show that statement (3) in Proposition 7.10 implies that for each $u \in W$, $\mathcal{R}^a_{\mathcal{M}}(u)$ is minimal in $\mathcal{M}$ with respect the ordering $\leq^u$, but it may not be a unique minimal element in $\mathcal{M}$ with respect to the ordering $\leq^u$.

[1] An ordering $\bar{n} < \cdots < \bar{2} < \bar{1} < 1 < 2 \cdots < n$ is used in [5] to define flag matroids of type C_n or D_n but the notion of flag matroid does not depend on the choice of the orderings because the permutation on J sending $\bar{i}$ to $n + 1 - i$ maps the ordering above to the ordering (7.3).

It would be natural and interesting to ask whether (3) is equivalent to (2) and (1). In the remaining part of this section, we show that (3) implies (2) in Proposition 7.10 when $\mathcal{M}$ is a two-element subset of $\mathfrak{S}_n$.

Proposition 7.12. *Let $\mathcal{M}$ be a subset of $\mathfrak{S}_n$ consisting of two elements. Then the three statements in Proposition 7.10 are equivalent.*

Proof. Since we have already proven the equivalence (1) $\iff$ (2) and (2) $\implies$ (3), it is enough to prove that (3) $\implies$ (2).

Recall from Remark 7.11 that statement (3) in Proposition 7.10 implies that $\mathcal{R}^a_{\mathcal{M}}(u)$ is minimal in $\mathcal{M}$ with respect to the ordering $\leq^u$. Hence it is enough to prove that such a minimal element is unique. Let $\mathcal{M} = \{x, y\}$. We claim that

Claim. The elements x and y span an edge of the GKM graph of $\mathrm{Fl}(\mathbb{C}^n)$.

If the claim holds, then $\Delta_{\mathcal{M}}$ is a Φ-polytope by Lemma 2.1. Therefore, $\mathcal{M}$ is a flag matroid by Theorem 7.4. Hence there exists a unique minimal element in $\mathcal{M}$ with respect to the ordering $\leq^u$ by the Minimality Property of flag matroids. Thus it is enough to show the above claim to prove the proposition.

Recall that two elements x and y in $\mathfrak{S}_n$ span an edge of the GKM graph if and only if $x = yt$ for some transposition t (see Lemma 2.1). This implies that if x and y span an edge of the GKM graph, then vx and vy span an edge of the GKM graph for every $v \in \mathfrak{S}_n$. Therefore, we may assume that one of the two elements in $\mathcal{M}$ is the identity element e by Remark 6.3 and we denote the other element by x, so that $\mathcal{M} = \{e, x\}$ and $J_1(\mathcal{M}) = \{1, x(1)\}$.

We assume that Proposition 7.10(3) holds, i.e.

$$(**) \qquad \begin{array}{l}\text{For each } u \in W,\ \mathcal{R}^a_{\mathcal{M}}(u) \text{ gives a unique element in } \mathcal{M} \text{ such that} \\ d(u, \mathcal{R}^a_{\mathcal{M}}(u)) = d(u, \mathcal{M}).\end{array}$$

As before, we denote $u^a := \mathcal{R}^a_{\mathcal{M}}(u)$ and use the one-line notation

$$x = x(1)x(2)\cdots x(n).$$

We assume that $x(1) \neq 1$ for simplicity. (One can see that the following argument will work for $x(p)$ instead of $x(1)$ such that $x(q) = q$ for all $q < p$ but $x(p) \neq p$.) Let $x(i) = 1$. Since $x(1) \neq 1$, we have $i > 1$. We consider two elements

$$\begin{aligned} y &= x(i)\ x(1)\ x(2) \cdots x(i-1)\ x(i+1) \cdots x(n), \\ z &= x(1)\ x(i)\ x(2) \cdots x(i-1)\ x(i+1) \cdots x(n). \end{aligned} \tag{7.4}$$

We note that since $\mathcal{M} = \{e, x\}$ and $x(1) \neq 1$, we have $u^a = e$ if $u(1) = 1$ and $u^a = x$ if $u(1) = x(1)$ for $u \in \mathfrak{S}_n$. Therefore

(1) $y^a = e$ since $y(1) = x(i) = 1$ and
(2) $z^a = x$ since $z(1) = x(1)$.

Hence by assumption $(**)$, we can see that

$$d(y, e) < d(y, x) \text{ and } d(z, x) < d(z, e). \tag{7.5}$$

We note that

(3) $d(y, x) = i - 1$ since $y = xs_{i-1}s_{i-2}\cdots s_1$,
(4) $d(z, e) = d(y, e) + 1$ since $z = ys_1$ and $z(1) = x(1) > 1 = x(i) = z(2)$,
(5) $d(z, x) = d(y, x) - 1 = i - 2$.

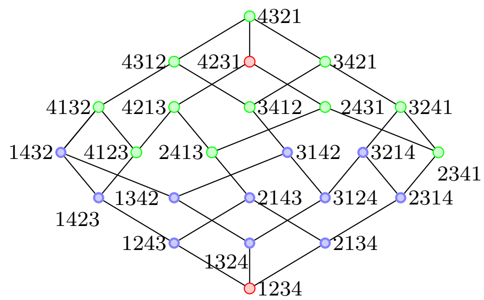


FIGURE 6. $\mathcal{R}^a_{\mathcal{M}}$ on the two-element subset $\mathcal{M} = \{1234, 4231\}$.

Therefore, by applying (4) and (5) above to (7.5), we get

$$d(y, x) = d(y, e) + 1 \text{ and } d(z, e) = d(z, x) + 1. \tag{7.6}$$

It follows from (3) above and (7.6) that

$$d(y, e) = i - 2,\ d(z, e) = i - 1, \text{ and } d(z, x) = i - 2. \tag{7.7}$$

Now we will prove that x is a transposition. Suppose that there exists some j such that $3 \le j < n$ and $z(j) > z(j+1)$. Then for $w = t_{z(j),z(j+1)}z$, we have

$$d(w, e) = d(z, e) - 1 \quad \text{and} \quad d(w, x) = d(z, x) + 1.$$

It follows from these identities and (7.6) that

$$d(w, e) + 1 = d(w, x). \tag{7.8}$$

However, $w^a = x$ since $w(1) = z(1) = x(1)$. Hence $d(w, x) < d(w, e)$ by assumption $(**)$. This is a contradiction to (7.8), and there exists no j such that $3 \le j < n$ and $z(j) > z(j+1)$. It follows from (7.4) that

$$x(2) < x(3) < \cdots < x(i-1) < x(i+1) < \cdots < x(n). \tag{7.9}$$

Recall $x(i) = 1$. Combining (7.7) with (7.4) and (7.9), we can see that $x(1) = i$ and $x(j) = j$ for $j \neq 1, i$. Thus x is the transposition which transposes 1 and i. Hence the result follows. $\square$

For example, if we consider the set $\mathcal{M} = \{1234, 4231\} \subset \mathfrak{S}_4$, then for every element $u \in \mathfrak{S}_4$, $\mathcal{R}^a_{\mathcal{M}}(u)$ gives the unique closest element in $\mathcal{M}$. Figure 6 shows that $\mathcal{R}^a_{\mathcal{M}}$ maps the elements colored by green go to 4231 and the elements colored by blue to 1234. In fact, $\mathcal{M}$ spans an edge of the GKM graph since 4231 is a transposition.

As in Proposition 7.12, one can check that the three statement in Proposition 7.10 are equivalent if we choose a subset $\mathcal{M}$ in $\mathfrak{S}_n$, $n \le 4$. However, we do not know any example which is not a Coxeter matroid but still satisfies statement (3) in Proposition 7.10.

Problem. Can we find an example of a subset $\mathcal{M}$ of a Weyl group W such that for every element $u \in W$, the algebraic retraction image $\mathcal{R}^a_{\mathcal{M}}(u)$ is a unique point in $\mathcal{M}$ closest to u, but $\mathcal{M}$ is not a Coxeter matroid?

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