Some modular inequalities in Lebesgue spaces with variable exponent

Mitsuo Izuki, Takahiro Noi and Yoshihiro Sawano

November 11, 2019

Abstract

Our aim is to study the modular inequalities for some operators, for example the Bergman projection acting on, in Lebesgue spaces with variable exponent. Under proper assumptions on the variable exponent, we prove that the modular inequalities are hold if and only if the exponent almost everywhere equals to a constant. In order to get the main results, we prove a lemma for a lower pointwise bound for these operators of a characteristic function.
Key words: variable exponent, modular inequality, Lebesgue space
AMS Subject Classification: 42B35.

1 Introduction

The study on variable exponent analysis has been rapidly developed after the work [18] where Kováčik and Rákosník have established fundamental properties of variable Lebesgue spaces (see also [4, 14, 21]). In particular the theory of variable function spaces in connection with the boundedness of the Hardy–Littlewood maximal operator M has been deeply studied. Cruz-Uribe, Fiorenza and Neugebauer [6, 7] and Diening [9] have independently obtained the log-Hölder continuous conditions that guarantee the boundedness of M on variable Lebesgue spaces. We also note that the recent development of variable exponent analysis has the extrapolation theorem from weighted inequalities to norm inequalities on variable Lebesgue spaces ([5, 8]).

In general, the boundedness of M on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ describes that the norm inequality

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{1.1}$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, where C is a positive constant independent of f. Lerner [19] has pointed out the crucial difference between the norm inequality (1.1) and the following modular inequality

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \le C \int_{\mathbb{R}^n} |f(x)|^{p(x)}dx. \tag{1.2}$$

More precisely, Lerner has proved that $p(\cdot)$ must be a constant function whenever $1 < \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \le \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty$ and the modular inequality (1.2) holds. Izuki [11] has considered the difference for some operators arising from the wavelet theory. Izuki, Nakai and Sawano [13, 14] have given an alternative proof of Lerner's result. They have also studied the problem in the weighted case ([15]).

Recently, Izuki, Koyama, Noi and Sawano [12] have considered some modular inequalities for some operators. In this paper, we focus on three operators below. First, we investigate the Bergman projection operator on the unit disc $\mathbb{D}$ in the complex plane. The generalization of holomorphic function spaces in terms of variable exponent and the boundedness of Bergman projection operators on variable exponent spaces have been studied ([1, 2, 3, 16, 17]). Among them we focus on the work [1] due to Chacón and Rafeiro. They defined Bergman spaces $A^{p(\cdot)}(\mathbb{D})$ with variable exponent $p(\cdot)$ on the open unit disk $\mathbb{D}$. Applying the local log-Hölder continuous condition and the extrapolation theorem, they proved the density of the set of polynomials in $A^{p(\cdot)}(\mathbb{D})$ and the boundedness of the Bergman projection $P : L^{p(\cdot)}(\mathbb{D}) \to A^{p(\cdot)}(\mathbb{D})$ and the α-Berezin transform $B_\alpha : L^{p(\cdot)}(\mathbb{D}) \to L^{p(\cdot)}(\mathbb{D})$. In particular, Chacón and Rafeiro [1] have obtained the norm inequality

$$\|Pf\|_{L^{p(\cdot)}(\mathbb{D})} \le C\, \|f\|_{L^{p(\cdot)}(\mathbb{D})} \tag{1.3}$$

for all $f \in L^{p(\cdot)}(\mathbb{D})$.

Second our target operator is

$$B_{\mathbb{R}^2_+} f(z) = \frac{-1}{\pi} \int_{\mathbb{R}^2_+} \frac{f(w)}{(z-\overline{w})^2} dA(w), \quad z = x+iy \in \mathbb{R}^2_+,$$

where $dA(w)$ denotes the Lebesgue measure and $\mathbb{R}^2_+$ is the upper half-space over $\mathbb{R}^2_+ \simeq \mathbb{C}$. Karapetyants and Samko [17] proved that $B_{\mathbb{R}^2_+}$ is a projection from $L^{p(\cdot)}(\mathbb{R}^2_+)$ onto $\mathcal{A}^{p(\cdot)}(\mathbb{R}^2_+)$ if $p(\cdot) \in \mathcal{P}(\mathbb{R}^2_+)$ satisfies the log-Hölder condition and the log-decay condition ([17, Theorem 3.1 (1)]). So they have obtained the norm inequality

$$\|B_{\mathbb{R}^2_+} f\|_{L^{p(\cdot)}(\mathbb{R}^2_+)} \le C \|f\|_{L^{p(\cdot)}(\mathbb{R}^2_+)} \tag{1.4}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^2_+)$.

Finally, we consider $b_{\mathbb{R}^n_+}$, the harmonic projection in $\mathbb{R}^n_+$. Let $\mathbb{R}^n_+$ stand for the upper half-space over $\mathbb{R}^n$ with $n \ge 2$. For $x = (x_1, x_2, \ldots, x_n)$, we write $x' = (x_1, x_2, \ldots, x_{n-1})$ and $\bar{x} = (x', -x_n)$. As usual, $h^p(\mathbb{R}^n_+)$ stands for the harmonic Bergman space of harmonic functions that belong to $L^p(\mathbb{R}^n_+)$. Here and below $dA(x)$ denotes the Lebesgue measure. The corresponding Bergman projection $b_{\mathbb{R}^n_+}$ defined by

$$\begin{aligned} b_{\mathbb{R}^n_+} f(x) &= \int_{\mathbb{R}^n_+} R(x,y) f(y) dA(y) \\ &= \frac{2}{\pi^{\frac{n}{2}}} \Gamma\left(\frac{n}{2}\right) \int_{\mathbb{R}^n_+} \frac{n(x_n+y_n) - |x-\bar{y}|^2}{|x-\bar{y}|^{n+2}} f(y) dA(y), \end{aligned}$$

is bounded from $L^p(\mathbb{R}^n_+)$ onto $h^p(\mathbb{R}^n_+)$ ([22]). Namely $b_{\mathbb{R}^n_+} f \in h^p(\mathbb{R}^n_+)$ and the norm inequality

$$\|b_{\mathbb{R}^n_+} f\|_{L^p(\mathbb{R}^n_+)} \le C\|f\|_{L^p(\mathbb{R}^n)} \tag{1.5}$$

hold for all $f \in L^p(\mathbb{R}^n_+)$. Karapetyants and Samko have extended (1.5) in the variable exponent settings ([17, Theorem 5.1]).

In the present paper, we consider the modular inequalities corresponding to the norm inequalities (1.3), (1.4) and (1.5). More precisely, for example, if $p(\cdot)$ satisfies

$$1 < \operatorname*{ess\,sup}_{z\in\mathbb{D}} p(z) \le \operatorname*{ess\,sup}_{z\in\mathbb{D}} p(z) < \infty$$

and the modular inequality

$$\int_{\mathbb{D}} |Pf(z)|^{p(z)} dA(z) \le C \int_{\mathbb{D}} |f(z)|^{p(z)} dA(z)$$

holds for all $f \in L^{p(\cdot)}(\mathbb{D})$, then the variable exponent $p(\cdot)$ must be a constant function. We can prove similar results for $B_{\mathbb{R}^n_+}$ and $b_{\mathbb{R}^n_+}$. In order to prove them we need a lower bound for the image of the characteristic function of a certain set. We will show a key lemma for the lower bound before the statement of the main results.

In the present paper we will use the following notation.

1. Given a measurable set E, we denote the Lebesgue measure of E by $|E|$. We define the characteristic function of E by χ_E.

2. A symbol C always stands for a positive constant independent of the main parameters.

2 Function spaces with variable exponent

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, that is,

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

Let also $\mathbb{R}^n_+$ be the upper half plane, that is,

$$\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0\}.$$

In the present paper we concentrate on the theory on function spaces defined on $\mathbb{D}$ or $\mathbb{R}^n_+$ with $n \ge 2$. We first define some fundamental notation on variable exponents. Let X denote either $\mathbb{D}$ or $\mathbb{R}^n_+$.

Definition 2.1.

1. *Given a measurable function $p(\cdot) : X \to [1, \infty)$, we define*

$$p_+ := \operatorname*{ess\,sup}_{z\in X} p(z), \quad p_- := \operatorname*{ess\,inf}_{z\in X} p(z).$$

2. *The set $\mathcal{P}(X)$ consists of all measurable functions $p(\cdot) : X \to [1, \infty)$ satisfying $1 < p_-$ and $p_+ < \infty$.*

Chacón and Rafeiro [1] defined generalized Lebesgue spaces and Bergman spaces on $\mathbb{D}$ with variable exponent.

Definition 2.2. *Let $dA(z)$ be the normalized Lebesgue measure on X and $p(\cdot) \in \mathcal{P}(X)$. The Lebesgue space $L^{p(\cdot)}(X)$ consists of all measurable functions f on X satisfying that the modular*

$$\rho_p(f) := \int_X |f(z)|^{p(z)}\, dA(z)$$

is finite. The Bergman space $A^{p(\cdot)}(\mathbb{D})$ is the set of all holomorphic functions f on $\mathbb{D}$ such that $f \in L^{p(\cdot)}(\mathbb{D})$.

We note that $L^{p(\cdot)}(X)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(X)} := \inf\left\{\lambda > 0 \,:\, \rho_p(f/\lambda) \le 1\right\}.$$

The projection $P : L^2(\mathbb{D}) \to A^2(\mathbb{D})$ is called the Bergman projection and given by

$$Pf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \bar{w}z)^2} dA(w).$$

It is known that $P : L^p(\mathbb{D}) \to A^p(\mathbb{D})$ is bounded in the case that $1 < p < \infty$ is a constant exponent ([10, 22]). See also [20] for the case of $p = 2$.

Chacón and Rafeiro [1, Theorem 4.4] proved the following boundedness:

Theorem 2.3. *Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{D})$ satisfies the local log-Hölder continuous condition*

$$|p(z_1) - p(z_2)| \le \frac{C}{\log(\mathrm{e} + 1/|z_1 - z_2|)} \quad (z_1,\, z_2 \in \mathbb{D}).$$

Then the Bergman projection P is bounded from $L^{p(\cdot)}(\mathbb{D})$ to $A^{p(\cdot)}(\mathbb{D})$, in particular, the norm inequality

$$\|Pf\|_{L^{p(\cdot)}(\mathbb{D})} \le C\, \|f\|_{L^{p(\cdot)}(\mathbb{D})}$$

holds for all $f \in L^{p(\cdot)}(\mathbb{D})$.

In the following sections, we consider the modular inequalities corresponding to the norm inequalities (1.3), (1.4) and (1.5).

3 Bergman projection on $\mathbb{D}$

Theorem 3.1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{D})$. If the modular inequality*

$$\int_{\mathbb{D}} |Pf(z)|^{p(z)} dA(z) \le C \int_{\mathbb{D}} |f(z)|^{p(z)} dA(z) \tag{3.1}$$

holds for all $f \in L^{p(\cdot)}(\mathbb{D})$, then $p(z)$ equals to a constant for almost every $z \in \mathbb{D}$.

In order to prove this theorem, we apply the following lower pointwise estimate for the Bergman projection.

Lemma 3.2. *Let $\tau \in \mathbb{D}$. Then there exists a compact neighborhood K_τ of τ such that*

$$\mathrm{Re}(P\chi_E(z)) \geq c_\tau |E|$$

for all measurable sets $E \subset K_\tau$, where c_τ is a positive constant depending only on τ.

Proof. Note that there exists a compact neighborhood K_τ of τ such that

$$c_\tau := \inf_{z,w \in K_\tau} \mathrm{Re}\left(\frac{1}{(1-\bar{w}z)^2}\right) > 0.$$

Thus,

$$\mathrm{Re}(P\chi_E(z)) = \int_E \mathrm{Re}\left(\frac{1}{(1-\bar{w}z)^2}\right) dA(w) \geq c_\tau \int_E dA(w) = c_\tau |E|,$$

as required. □

Now we prove Theorem 3.1.

Proof of Theorem 3.1. Let $\tau \in \mathbb{D}$ and K_τ be the compact neighborhood appearing in Lemma 3.2. Assume that $p(z)$ does not equal to any constant for almost every $z \in K_\tau$. Then we can find subsets $K_\tau^\pm$ of K_τ such that

$$\sup_{z \in K_\tau^-} p(z) < \inf_{z \in K_\tau^+} p(z). \tag{3.2}$$

Using Lemma 3.2 and modular inequality (3.1), we have

$$\int_{K_\tau^+} (kc_\tau |K_\tau^-|)^{p(z)}\, dA(z) \leq \int_{K_\tau^+} |kP\chi_{K_\tau^-}(z)|^{p(z)}\, dA(z) \leq C \int_{\mathbb{D}} (k\chi_{K_\tau^-})^{p(z)}\, dA(z)$$

for all $k > 0$. Consequently, if $kc_\tau |K_\tau^-| > 1$ and $k > 1$, then we obtain

$$|K_\tau^+| (kc_\tau |K_\tau^-|)^{\operatorname{ess\,inf}_{z \in K_\tau^+} p(z)} \leq C |K_\tau^-| k^{\operatorname{ess\,sup}_{z \in K_\tau^-} p(z)}.$$

This contradicts to (3.2). Consequently, it follows that for all $\tau \in \mathbb{D}$ there exists a compact neighborhood K_τ such that $p(z)$ equals to a constant for almost every $z \in K_\tau$. Since $\mathbb{D}$ is connected, it follows that $p(z)$ equals to a constant for almost every $z \in \mathbb{D}$. □

4 Bergman projection onto $\mathbb{R}^2_+$

As the following lemma shows, $B_{\mathbb{R}^2_+}$ is not degenerate.

Lemma 4.1. *Let $\tau \in \mathbb{R}^2_+$. Then there exists a compact neighborhood K_τ of τ such that*

$$\mathrm{Re}\left(B_{\mathbb{R}^2_+}(\chi_E)(z)\right) \geq C_\tau |E|$$

for all measurable sets $E \subset K_\tau$.

Proof. Let $\tau = \alpha + \beta i \in \mathbb{C} \simeq \mathbb{R}^2_+$. Firstly, we prove that there exist a compact neighborhood K_τ of τ such that

$$\mathrm{Re}\left(\frac{1}{(z-\overline{w})^2}\right) < 0$$

holds for any $z, w \in K_\tau$. To do this, we consider the real part of $(\overline{z} - w)^2$ keeping in mind that

$$\mathrm{Re}\left(\frac{1}{(z-\overline{w})^2}\right) = \mathrm{Re}\left(\frac{(\overline{z}-w)^2}{|z-\overline{w}|^4}\right).$$

We can take $\gamma > 0$ so that $\beta - \gamma > 0$ because $\beta > 0$. We learn that

$$K_\tau = \{x + yi : \alpha - (\beta-\gamma)/2 \leq x \leq \alpha + (\beta - \gamma)/2,\ \beta - \gamma \leq y \leq \beta + \gamma\} (\subset \mathbb{R}^2_+)$$

does the job. Let $z = a + bi$, $w = c + di \in K_\tau$. It is easy to see that $\mathrm{Re}(\overline{z} - w)^2 < 0$ since

$$(\overline{z} - w)^2 = (a-c)^2 - (b+d)^2 - 2(a-c)(b+d)i$$

and $|a - c| \leq \beta - \gamma < 2(\beta - \gamma) \leq |b + d|$.

Finally, we have

$$\mathrm{Re}(B_{\mathbb{R}^2_+}(\chi_E(z))) = \frac{-1}{\pi}\int_E \mathrm{Re}\left(\frac{1}{(z-\overline{w})^2}\right) dA(w) \geq C_\gamma \int_E dA(w) = c_\gamma |E|$$

for any $E \subset K_\tau$. □

Using Lemma 4.1 and an argument similar to the proof of Theorem 3.1, we obtain the following theorem. So we omit the proof.

Theorem 4.2. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^2_+)$. If the modular inequality*

$$\int_{\mathbb{R}^2_+} \left|B_{\mathbb{R}^2_+} f(z)\right|^{p(z)} dA(z) \leq C \int_{\mathbb{R}^2_+} |f(z)|^{p(z)}\, dA(z)$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^2_+)$, then $p(z)$ equals to a constant for almost every $z \in \mathbb{R}^2_+$.

5 Harmonic projection in $\mathbb{R}^n_+$

The same technique can be applied to the harmonic projection over $\mathbb{R}^n_+$.

Theorem 5.1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_+)$. If the modular inequality*

$$\int_{\mathbb{R}^n_+} \left|b_{\mathbb{R}^n_+} f(z)\right|^{p(z)} dA(z) \leq C \int_{\mathbb{R}^n_+} |f(z)|^{p(z)}\, dA(z)$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^n_+)$, then $p(z)$ equals to a constant for almost every $z \in \mathbb{R}^n_+$.

Proof. Let $x = (x', x_n) \in \mathbb{R}^n_+$ be fixed. Then we have

$$\frac{n(x_n + z_n) - |x - \bar{z}|^2}{|x - \bar{z}|^{n+2}} = \frac{n-1}{2^{n+2}} {x_n}^{-n}$$

for $z = (z', z_n) = x = (x', x_n)$. Thus we obtain

$$\frac{n(x_n + y_n) - |x - \bar{y}|^2}{|x - \bar{y}|^{n+2}} > \frac{n-1}{2^{n+3}} {x_n}^{-n}$$

as long as $y = (y', y_n)$ belongs to an open neighborhood U of x. Thus, if we go through the same argument as before, we see that $p(z)$ equals to a constant for almost every $z \in \mathbb{R}^n_+$. □

Acknowledgements

Mitsuo Izuki was partially supported by Grand-in-Aid for Scientific Research (C), No. 15K04928, for Japan Society for the Promotion of Science. Takahiro Noi was partially supported by Grand-in-Aid for Young Scientists (B), No. 17K14207, for Japan Society for the Promotion of Science. Yoshihiro Sawano was partially supported by Grand-in-Aid for Scientific Research (C), No. 19K03546, for Japan Society for the Promotion of Science. This work was partly supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics). The authors are thankful to Professor Alexey Karapetyants for letting us know about some literatures in the theory of Bergman spaces.

References

[1] Chacón GR, Rafeiro H. Variable exponent Bergman spaces, Nonlinear Anal 2014; 105:41–49.

[2] Chacón GR, Rafeiro H. Toeplitz operators on variable exponent Bergman spaces, Mediterr J Math 2016; 13: 3525–3536.

[3] Chacón GR, Rafeiro H, Vallejo JC. Carleson measures for variable exponent Bergman spaces, Complex Anal Oper Theory 2017; 11: 1623–1638.

[4] Cruz-Uribe DV, Fiorenza A. Variable Lebesgue Spaces. Foundations and Harmonic Analysis, Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Heidelberg, 2013.

[5] Cruz-Uribe D, Fiorenza A, Martell JM, Pérez C. The boundedness of classical operators on variable L^p spaces, Ann Acad Sci Fenn Math 2006; 31: 239–264.

[6] Cruz-Uribe D, Fiorenza A, Neugebauer CJ. The maximal function on variable L^p spaces, Ann Acad Sci Fenn Math 2003; 28: 223–238.

[7] Cruz-Uribe D, Fiorenza A, Neugebauer CJ. Corrections to: "The maximal function on variable L^p spaces" [Ann. Acad. Sci. Fenn. Math. **28** (1) (2003), 223–238.], Ann Acad Sci Fenn Math 2004; 29: 247–249.

[8] Cruz-Uribe DV, Martell JM, Pérez C. Weights, Extrapolation and the Theory of Rubio de Francia, Operator Theory: Advances and Applications, **215**. Birkhäuser/Springer Basel AG, Basel, 2011.

[9] Diening L. Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, Math Inequal Appl 2004; 7: 245–253.

[10] Hedenmalm H, Korenblum B, Zhu K. Theory of Bergman Spaces, Springer Verlag, 2000.

[11] Izuki M. Wavelets and modular inequalities in variable L^p spaces, Georgian Math J 2008; 15: 281–293.

[12] Izuki M, Koyama T, Noi T, and Sawano Y. Some modular inequalities in Lebesgue spaces with variable exponent on the complex plane, Math. Notes 2019, 106, 229–234.

[13] Izuki M, Nakai E, Sawano Y. The Hardy-Littlewood maximal operator on Lebesgue spaces with variable exponent, RIMS Kokyuroku Bessatsu 2013; B42: 51–94.

[14] Izuki M, Nakai E, Sawano Y. Function spaces with variable exponents –an introduction–, Sci Math Jpn 2014; 77: 187–315.

[15] Izuki M, Nakai E, Sawano Y. Wavelet characterization and modular inequalities for weighted Lebesgue spaces with variable exponent, Ann Acad Sci Fenn Math 2015; 40: 551–571.

[16] Karapetyants A, Samko S. Mixed norm variable exponent Bergman space on the unit disc, Complex Var Elliptic Equ 2016; 8: 1090–1106.

[17] Karapetyants A, Samko S. On boundedness of Bergman projection operators in Banach spaces of holomorphic functions in half-plane and harmonic functions in half-space, Math Sci 2017; 226: 344–354.

[18] Kováčik O, Rákosník J. On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math J 1991; 41: 592–618.

[19] Lerner AK. On modular inequalities in variable L^p spaces, Arch Math (Basel) 2005; 85: 538–543.

[20] Saitoh S. and Sawano Y. Theory of Reproducing Kernel Hilbert Spaces, Development of Mathematics **44**, Springer, Singapore, 2018. xviii+452 pp.

[21] Sawano Y. Theory of Besov Spaces, Development in Mathematics **56**, Springer, Singapore, 2018. xxiii+945 pp.

[22] Zhu K. Operator Theory in Function Spaces. Second edition, Mathematical Surveys and Monographs, **138**. American Mathematical Society, Providence, RI, 2007.

Mitsuo Izuki (Corresponding author),
Faculty of Liberal Arts and Sciences,
Tokyo City University,
1-28-1, Tamadutsumi Setagaya-ku Tokyo 158-8557, Japan.
E-mail: izuki@tcu.ac.jp

Takahiro Noi,
Department of Mathematics and Information Science,
Tokyo Metropolitan University,
Hachioji, 192-0397, Japan.
E-mail: taka.noi.hiro@gmail.com

Yoshihiro Sawano,
Department of Mathematics and Information Science,
Tokyo Metropolitan University,
Hachioji, 192-0397, Japan.
E-mail: yoshihiro-sawano@celery.ocn.ne.jp