A CHARACTERIZATION OF DIFFERENTIABILITY FOR THE BEST TRACE SOBOLEV CONSTANT FUNCTION

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Abstract. Let $1 < p < N$ and let Ω be a smooth bounded domain in $\mathbb{R}^N$. In this paper we show some regularity results for the best constant S_q of the trace Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$, considering that S_q is a function of q. We prove that S_q is absolutely continuous, thus $S_q' = \frac{d}{dq} S_q$ exists a.e. $q \in [1, p_*]$, $p_* = \frac{p(N-1)}{N-p}$. We give a characterization on a set where S_q' exists. These are natural extensions of the recent work by Ercole for the best constant of the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \in [1, p^*]$, $p^* = \frac{Np}{N-p}$.

Key words: Best trace Sobolev constant, Absolute continuity, Differentiability

2010 Mathematics Subject Classification: 46E35, 35J20, 35J25

1. Introduction

Let $1 < p < N$ be fixed and let Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. The well-known trace Sobolev embedding theorem claims that the continuous inclusion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ holds true for $1 \le q \le p_*$, where $p_* = \frac{p(N-1)}{N-p}$ denotes the trace Sobolev critical exponent. Hence the following *trace Sobolev inequality* holds true for any $u \in W^{1,p}(\Omega)$:

$$C \left(\int_{\partial\Omega} |u|^q \, d\mathcal{H}^{N-1} \right)^{\frac{p}{q}} \le \int_\Omega (|\nabla u|^p + |u|^p) \, dx \quad (1 \le q \le p_*) \tag{1.1}$$

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Date: November 12, 2019.

where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure on the hypersurface $\partial\Omega$. The best constant of the trace Sobolev inequality (1.1) (i.e., the largest C such that the above inequality holds for any $u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)$) is defined as

$$S_q = S_q(\Omega) := \inf_{u \in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\int_\Omega (|\nabla u|^p + |u|^p)\, dx}{\left(\int_{\partial\Omega} |u|^q \, d\mathcal{H}^{N-1}\right)^{\frac{p}{q}}} = \inf_{\substack{u \in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega) \\ \|u\|_{L^q(\partial\Omega)}=1}} \int_\Omega (|\nabla u|^p + |u|^p)\, dx. \tag{1.2}$$

It is known that the continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ for $1 \le q \le p_*$ is actually compact when $1 \le q < p_*$, thus a minimizer for S_q exists for $1 \le q < p_*$. A minimizer u_q for S_q with the property $\|u_q\|_{L^q(\partial\Omega)} = 1$ is a weak solution of the Euler-Lagrange equation

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in}\, \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = S_q |u|^{q-2}u & \text{on}\, \partial\Omega, \end{cases} \tag{1.3}$$

where ν is the outer unit normal of $\partial\Omega$. Note that by the strong maximum principle [18], a solution u of (1.3) has a constant sign on Ω, and we may assume $u > 0$ on Ω. Also regularity results (see e.g., [15], [17]) imply that $u \in C^{1,\alpha}_{loc}(\Omega) \cap C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1)$.

For the case $q = p_*$, the existence of a minimizer becomes a subtle problem because of the lack of compactness. Recently it is proved in [14] that S_{p_*} is attained on any smooth bounded domain when $p \in (1, \frac{N+1}{2} + \beta)$, where $\beta = \beta(\Omega) > 0$. See [1], [11], [6], [7] for earlier results on the existence of extremals for $S_{p_*}(\Omega)$ on bounded domains.

This is a striking difference between the best constant for the Sobolev inequality

$$\tilde{S}_q = \tilde{S}_q(\Omega) := \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \not\equiv 0}} \frac{\int_\Omega |\nabla u|^p dx}{\left(\int_\Omega |u|^q dx\right)^{\frac{p}{q}}} \tag{1.4}$$

for $1 \le q \le p^* = \frac{Np}{N-p}$. Indeed, $\tilde{S}_{p^*}(\Omega)$ is never attained on any domain Ω other than $\mathbb{R}^N$ and $\tilde{S}_{p^*}(\Omega)$ does not depend on the domain Ω but depends only on N. More precisely, $\tilde{S}_{p^*}(\Omega) = \tilde{S}_{p^*}(\mathbb{R}^N)$ and the explicit value of $\tilde{S}_{p^*}$ is known, see [16].

Also, the behaviors of both the constants $S_q(\Omega)$ and $\tilde{S}_q(\Omega)$ under the dilations of the domain are different from each other. That is, if we define $\mu\Omega = \{\mu x \mid x \in \Omega\}$ for $\mu > 0$, we have $\tilde{S}_q(\mu\Omega) = \mu^{N-p-\frac{pN}{q}} \tilde{S}_q(\Omega)$.

On the other hand, it is easy to see by using $u_\mu(x) = u(\mu x)$ that

$$S_q(\mu\Omega) = \mu^{N-\frac{p(N-1)}{q}} \inf_{u\in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\int_\Omega (\mu^{-p}|\nabla u_\mu|^p + |u_\mu|^p)\, dx}{\left(\int_{\partial\Omega} |u_\mu|^q \, d\mathcal{H}^{N-1}\right)^{\frac{p}{q}}}.$$

Recently, several regularity properties of $\tilde{S}_q$ as a function of $q \in [1, p^*] = \frac{Np}{N-p}$ are proved by G. Ercole [3], [4]; see also [8] and [2]. In fact, in [3] it is proved that the function $q \mapsto \tilde{S}_q$ is Lipschitz continuous on the interval $[1, p^* - \varepsilon]$ for any $\varepsilon > 0$ small. Also $\tilde{S}_q$ is absolutely continuous on the whole closed interval $[1, p^*]$ and thus its derivative $\frac{d}{dq}\tilde{S}_q = \tilde{S}'_q$ exists almost all $q \in [1, p^*]$. In [4], the author characterizes the point $q \in [1, p^*)$ where $\tilde{S}_q$ is differentiable; $\tilde{S}'_q$ exists if and only if the functional

$$\tilde{I}_q(u) = \int_\Omega |u|^q \log |u| dx$$

takes a constant value on the set $\tilde{E}_q$ of the L^q-normalized extremal functions corresponding to $\tilde{S}_q$:

$$\tilde{E}_q = \{u \in W_0^{1,p}(\Omega) \,|\, \|u\|_{L^q(\Omega)} = 1, \text{ and } \int_\Omega |\nabla u|^p dx = \tilde{S}_q\}.$$

We say that $\tilde{S}_q(\Omega)$ is simple if the extremal functions associated with $\tilde{S}_q$ are scalar multiple one of the other. This is equivalent to say that $\tilde{E}_q = \{\pm u_q\}$ for an L^q-normalized extremal $u_q \in W_0^{1,p}(\Omega)$. Recall that there is a long-standing conjecture that $\tilde{S}_q(\Omega)$ is simple if Ω is a bounded smooth convex domain in $\mathbb{R}^N$ and $1 \le q < p^*$. Up to now, only several partial results are available for this conjecture, however, the complete solution has not been obtained. Ercole's result is interesting since we can disprove the conjecture if we find q such that $\tilde{S}'_q$ does not exist.

Main purpose of this paper is, in spite of the differences between $\tilde{S}_q$ and S_q, to obtain similar regularity results and a characterization of differentiability of the function $[1, p_*] \ni q \mapsto S_q$. In what follows, $|A|$ stands for both the N-dimensional Lebesgue measure $\mathcal{L}^N(A)$ when $A \subseteq \Omega$ and the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}(A)$ when $A \subseteq \partial\Omega$. We hope that this abbreviation causes no ambiguity. $\|u\|_{L^q(\Omega)}$ and $\|u\|_{L^q(\partial\Omega)}$ denotes the L^q-norm of a function $u : \Omega \to \mathbb{R}$ and $u : \partial\Omega \to \mathbb{R}$ respectively. χ_A denotes a characteristic function of a set A.

2. Monotonicity and Bounded pointwise variation

In what follows, we fix $1 < p < N$ and put $p_* = \frac{(N-1)p}{N-p}$.

Concerning the monotonicity of $q \mapsto S_q$, first, we prove the following lemma:

Lemma 2.1. *The function $q \mapsto |\partial\Omega|^{p/q} S_q$ is monotone decreasing on $[1, p_*]$. In particular, the function $q \in [1, p_*] \mapsto S_q$ is monotone decreasing if $|\partial\Omega| \le 1$ and strictly monotone decreasing if $|\partial\Omega| < 1$.*

Proof. let $1 \le q_1 < q_2 \le p_*$. By Hölder's inequality, we have

$$|\partial\Omega|^{p/q_2} \left(\int_{\partial\Omega} |u|^{q_2} \, d\mathcal{H}^{N-1} \right)^{-p/q_2} \le |\partial\Omega|^{p/q_1} \left(\int_{\partial\Omega} |u|^{q_1} \, d\mathcal{H}^{N-1} \right)^{-p/q_1}.$$

Multiplying $\int_\Omega (|\nabla u|^p + |u|^p) \, dx$ to both sides and taking infimum, we see that $q \in [1, p_*] \mapsto |\partial\Omega|^{p/q} S_q$ is a monotone decreasing function. Thus

$$S_{q_1} \ge |\partial\Omega|^{(1/q_2 - 1/q_1)p} S_{q_2} > S_{q_2}$$

if $|\partial\Omega| < 1$. □

In Lemma 2.1, we see that the function $q \mapsto |\partial\Omega|^{p/q} S_q$ is strictly monotone decreasing on $[1, p_*]$ if $|\partial\Omega| < 1$. However, we can say more. In the next lemma, the Rayleigh quotient associated with the trace Sobolev embedding $W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega) \to \mathbb{R}$ is denoted by

$$R_q(u) = \frac{\int_\Omega (|\nabla u|^p + |u|^p) \, dx}{\left(\int_{\partial\Omega} |u|^q \, d\mathcal{H}^{N-1} \right)^{\frac{p}{q}}} = \frac{\|u\|^p_{W^{1,p}(\Omega)}}{\|u\|^p_{L^q(\partial\Omega)}}.$$

Lemma 2.2. *Let $u \in (W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)) \cap L^\infty(\partial\Omega)$, $u \not\equiv$ constant. Then for each $1 \le q_1 < q_2 \le p_*$*

$$|\partial\Omega|^{\frac{p}{q_1}} R_{q_1}(u) = |\partial\Omega|^{\frac{p}{q_2}} R_{q_2}(u) \exp\left(p \int_{q_1}^{q_2} \frac{K(t,u)}{t^2} dt \right) \tag{2.1}$$

where

$$K(t,u) = \frac{\int_{\partial\Omega} |u|^t \log |u|^t \, d\mathcal{H}^{N-1}}{\|u\|^t_{L^t(\partial\Omega)}} + \log\left(\frac{|\partial\Omega|}{\|u\|^t_{L^t(\partial\Omega)}} \right) > 0 \tag{2.2}$$

Before the proof, we remark that the assumption of $u \in L^\infty(\partial\Omega)$ is used to assure the finiteness of the integral $\int_{\partial\Omega} |u|^{p_*} \log |u| \, d\mathcal{H}^{N-1}$.

Proof. The proof will be done by differentiating $\log\left(\frac{|\partial\Omega|^{\frac{1}{t}}}{\|u\|_{L^t(\partial\Omega)}} \right)$ with respect to t.

Fix $q_0 < p_*$ and consider $t \in [1, q_0]$. For $u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)$, we have an estimate

$$\begin{aligned} \big||u|^t \log |u|\big| &= \chi_{[|u|\le 1]}|u|^t\big|\log |u|\big| + \chi_{[|u|>1]}|u|^t\big|\log |u|\big| \\ &\le \chi_{[|u|\le 1]}(te)^{-1} + \chi_{[|u|>1]}\frac{1}{p_* - t}|u|^{p_*} \\ &\le e^{-1} + \frac{1}{p_* - q_0}|u|^{p_*} \in L^1(\partial\Omega), \end{aligned}$$

here we have used $x^t\big|\log x\big| \le (te)^{-1}$ for $0 < x \le 1$ and $\big|\log x\big| \le \beta^{-1}x^\beta$ for any $x \ge 1$ and $\beta > 0$. Thus we see $\big||u|^t \log |u|\big| \in L^1(\partial\Omega)$. Since q_0 can be chosen arbitrarily near to p_*, we may differentiate under the integral symbol to get

$$\frac{d}{dt}\int_{\partial\Omega} |u|^t \, d\mathcal{H}^{N-1} = \int_{\partial\Omega} |u|^t \log |u| \, d\mathcal{H}^{N-1}$$

for any $1 \le t < p_*$ by Lebesgue's dominated convergence theorem. Thus

$$\begin{aligned} \frac{d}{dt}\left(\log \frac{|\partial\Omega|^{\frac{1}{t}}}{\|u\|_{L^t(\partial\Omega)}}\right) &= \frac{d}{dt}\left(\frac{1}{t}\log |\partial\Omega|\right) - \frac{d}{dt}\left(\frac{1}{t}\log \int_{\partial\Omega} |u|^t \, d\mathcal{H}^{N-1}\right) \\ &= -\frac{1}{t^2}\log |\partial\Omega| + \frac{1}{t^2}\log \int_{\partial\Omega} |u|^t \, d\mathcal{H}^{N-1} \\ &\qquad - \frac{1}{t}\frac{\int_{\partial\Omega} |u|^t \log |u| \, d\mathcal{H}^{N-1}}{\int_{\partial\Omega} |u|^t \, d\mathcal{H}^{N-1}} \\ &= -\frac{K(t,u)}{t^2}. \end{aligned}$$

Integrate the above on $[q_1, q_2]$ with respect to t, we obtain

$$\frac{|\partial\Omega|^{\frac{1}{q_1}}}{\|u\|_{L^{q_1}(\partial\Omega)}} = \frac{|\partial\Omega|^{\frac{1}{q_2}}}{\|u\|_{L^{q_2}(\partial\Omega)}} \exp \int_{q_1}^{q_2} \frac{K(t,u)}{t^2} dt$$

Multiplying $\|u\|_{W^{1,p}(\Omega)}$, and taking p-th power, we get (2.1).

Next, we claim $K(t,u) > 0$. Define $h : [0,\infty) \to \mathbb{R}$ as

$$h(\xi) = \begin{cases} \xi \log \xi & (\xi > 0) \\ 0 & (\xi = 0). \end{cases}$$

Then h is convex, and Jensen's inequality implies

$$\begin{aligned}
&h\left(\frac{1}{|\partial\Omega|}\int_{\partial\Omega}|u|^t\ d\mathcal{H}^{N-1}\right)\le\frac{1}{|\partial\Omega|}\int_{\partial\Omega}h(|u|^t)\ d\mathcal{H}^{N-1}\\
\Leftrightarrow&|\partial\Omega|^{-1}\left(\int_{\partial\Omega}|u|^t\ d\mathcal{H}^{N-1}\right)\log\left(|\partial\Omega|^{-1}\int_{\partial\Omega}|u|^t\ d\mathcal{H}^{N-1}\right)\\
&\le|\partial\Omega|^{-1}\int_{\partial\Omega}|u|^t\log|u|^t\ d\mathcal{H}^{N-1}\\
\Leftrightarrow&\frac{\int_{\partial\Omega}|u|^t\log|u|^t\ d\mathcal{H}^{N-1}}{\|u\|^t_{L^t(\partial\Omega)}}+\log\left(\frac{|\partial\Omega|}{\|u\|^t_{L^t(\partial\Omega)}}\right)\ge 0
\end{aligned}$$

By the equality cases for Jensen's inequality (see [12]), if the equality holds for the above inequality, then $|u|^t$ must be a constant, which is excluded. Thus the equalities do not hold and $K(t,u)>0$. □

From Lemma 2.2, we easily see the next corollary:

Corollary 2.3. *The function $q\in[1,p_*]\mapsto|\partial\Omega|^{p/q}S_q$ is strictly monotone decreasing. In particular, The function $q\in[1,p_*]\mapsto S_q$ is strictly monotone decreasing if $|\partial\Omega|\le 1$.*

Proof. Let $1\le q_1<q_2\le p_*$ and let $u_{q_1}\in W^{1,p}(\Omega)\setminus W^{1,p}_0(\Omega)$ denote an extremal function for S_{q_1}. Then the regularity theorem assures that $u_{q_1}\in C^\alpha(\overline{\Omega})$ and u_{q_1} must not be a constant. It follows from Lemma 2.2 that

$$\begin{aligned}
|\partial\Omega|^{p/q_1}S_{q_1}&=|\partial\Omega|^{p/q_2}R_{q_2}(u_{q_1})\exp\left(p\int_{q_1}^{q_2}\frac{K(t,u_{q_1})}{t^2}dt\right)\\
&>|\partial\Omega|^{p/q_2}R_{q_2}(u_{q_1})\\
&\ge|\partial\Omega|^{p/q_2}S_{q_2}.
\end{aligned}$$

The latter claim is trivial. □

Let $I\subset\mathbb{R}$ be an interval. In what follows, a finite set $P=\{x_0,\cdots,x_n\}\subset I$, $x_0<x_1<\cdots<x_n$, is called a partition of I. Following [10] Chapter 2, we say that a function $f:I\to\mathbb{R}$ has *bounded pointwise variation* if

$$\sup\left\{\sum_{i=1}^n|f(x_i)-f(x_{i-1})|\right\}<\infty$$

where the supremum is taken over all partitions $P=\{x_0,\cdots,x_n\}$ of I, $n\in\mathbb{N}$. The space of all functions $f:I\to\mathbb{R}$ with bounded pointwise variation is denoted by $BPV(I)$.

Corollary 2.4. *The function $q\mapsto S_q$ is in $BPV(I)$ where $I=[1,p_*]$.*

Proof. Since a bounded monotone function on I is in $BPV(I)$ ([10] Proposition 2.10), and the product of a bounded function and a function in $BPV(I)$ is again in $BPV(I)$, we have $S_q = (|\partial\Omega|^{p/q} S_q)|\partial\Omega|^{-p/q}$ is in $BPV(I)$. □

3. Some estimates for the extremals

First by utilizing level set techniques, we derive some pointwise estimates for any positive solution to (1.3).

Lemma 3.1. *Let u be a positive weak solution to* (1.3) *with* $1 \le q < p_*$. *Then for any $\sigma \ge 1$, it holds*

$$\left(\frac{1}{2}\right)^{\sigma+N-1} C_q \|u\|_{L^\infty(\partial\Omega)}^{\frac{(N-1)(p-q)+(p-1)\sigma}{p-1}} \le \|u\|^\sigma_{L^\sigma(\partial\Omega)} \tag{3.1}$$

where

$$C_q = \left(\frac{S_{p_*}}{S_q}\right)^{\frac{N-1}{p-1}} N^{-\frac{Np-1}{p-1}}.$$

Proof. As $u > 0$ solves (1.3) weakly, it holds

$$-\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi dx + S_q \int_{\partial\Omega} u^{q-1}\phi \; d\mathcal{H}^{N-1} = \int_\Omega u^{p-1}\phi dx \tag{3.2}$$

for all $\phi \in W^{1,p}(\Omega)$.

By a regularity theory (see [15], [17]), we may assume $u \in C^\alpha(\bar\Omega)$ for some $0 < \alpha < 1$. Fix $t \in \mathbb{R}$ such that $0 < t < \|u\|_{L^\infty(\partial\Omega)}$. Put

$$A_t = \{x \in \Omega \mid u(x) > t\}, \quad a_t = \{x \in \partial\Omega \mid u(x) > t\}.$$

We take the function

$$\phi = (u-t)^+ \in W^{1,p}(\Omega), \quad \phi = \begin{cases} u - t & \text{in } A_t \cup a_t, \\ 0 & \text{otherwise} \end{cases}$$

in (3.2), then we have

$$-\int_{A_t} |\nabla u|^p dx + S_q \int_{a_t} u^{q-1}(u-t) \; d\mathcal{H}^{N-1} = \int_{A_t} u^{p-1}(u-t)dx.$$

Rewriting this, we have

$$\begin{aligned}\int_{A_t} \left(|\nabla u|^p + u^{p-1}(u-t)\right) dx &= S_q \int_{a_t} u^{q-1}(u-t) \; d\mathcal{H}^{N-1} \\ &\le S_q \|u\|^{q-1}_{L^\infty(\partial\Omega)}(\|u\|_{L^\infty(\partial\Omega)} - t)|a_t|. \end{aligned} \tag{3.3}$$

Now, put

$$g(t) = \int_{\partial\Omega} (u-t)^+ \; d\mathcal{H}^{N-1} = \int_{a_t} (u-t) \; d\mathcal{H}^{N-1}$$

and recall the layer cake representation: Let $v \geq 0$ be a $\mathcal{H}^{N-1}$-measurable function on $\partial\Omega$. Then for any $\sigma \geq 1$, it holds

$$\int_{\partial\Omega} v^\sigma \, d\mathcal{H}^{N-1} = \sigma \int_0^\infty s^{\sigma-1} \mathcal{H}^{N-1}(\{x \in \partial\Omega \mid v(x) > s\}) ds.$$

Thus, we see

$$g(t) = \int_0^\infty \mathcal{H}^{N-1} \left(\{x \in \partial\Omega \mid (u-t)^+ > s\} \right) ds = \int_t^\infty |a_s| ds,$$

here the last equality follows from a change of variables $t+s \mapsto s$. This implies $g'(t) = -|a_t|$. By Hölder's inequality, (1.1) and (3.3), we have

$$\begin{aligned}
g(t)^p &= \left(\int_{\partial\Omega} (u-t)^+ \, d\mathcal{H}^{N-1} \right)^p \\
&\leq \left(\int_{\partial\Omega} \{(u-t)^+\}^{p_*} \, d\mathcal{H}^{N-1} \right)^{\frac{p}{p_*}} |a_t|^{p(1-\frac{1}{p_*})} \\
&\leq \frac{1}{S_{p_*}} |a_t|^{p(1-\frac{1}{p_*})} \int_\Omega \left(|\nabla (u-t)^+|^p + \{(u-t)^+\}^p \right) dx \\
&= \frac{1}{S_{p_*}} |a_t|^{p(1-\frac{1}{p_*})} \int_{A_t} \left(|\nabla u|^p + (u-t)^{p-1}(u-t) \right) dx \\
&\leq \frac{1}{S_{p_*}} |a_t|^{p(1-\frac{1}{p_*})} \int_{A_t} \left(|\nabla u|^p + u^{p-1}(u-t) \right) dx \\
&\leq \frac{S_q}{S_{p_*}} \|u\|_{L^\infty(\partial\Omega)}^{q-1} (\|u\|_{L^\infty(\partial\Omega)} - t) |a_t|^{p(1-\frac{1}{p_*})+1} \\
&= \frac{S_q}{S_{p_*}} \|u\|_{L^\infty(\partial\Omega)}^{q-1} (\|u\|_{L^\infty(\partial\Omega)} - t) (-g'(t))^{\frac{Np-1}{N-1}},
\end{aligned}$$

which results in

$$\left[\frac{S_q}{S_{p_*}} \|u\|_{L^\infty(\partial\Omega)}^{q-1} (\|u\|_{L^\infty(\partial\Omega)} - t) \right]^{-\frac{N-1}{Np-1}} \leq -g(t)^{\frac{Np-p}{Np-1}} g'(t). \tag{3.4}$$

Changing a variable from t to s, and integrating the both sides of (3.4) on $\left[t, \|u\|_{L^\infty(\partial\Omega)}\right]$, we get

$$C_q \|u\|_{L^\infty(\partial\Omega)}^{-\frac{(N-1)(q-1)}{p-1}} (\|u\|_{L^\infty(\partial\Omega)} - t)^N \leq g(t). \tag{3.5}$$

Since $g(t) \leq (\|u\|_{L^\infty(\partial\Omega)} - t)|a_t|$, we have from (3.5) that

$$C_q \|u\|_{L^\infty(\partial\Omega)}^{-\frac{(N-1)(q-1)}{p-1}} (\|u\|_{L^\infty(\partial\Omega)} - t)^{N-1} \leq |a_t|. \tag{3.6}$$

We multiply $\sigma t^{\sigma-1}$ to the both sides of (3.6) and integrate them on $\Big[0, \|u\|_{L^\infty(\partial\Omega)}\Big]$. Then the right hand side becomes $\|u\|^\sigma_{L^\sigma(\partial\Omega)}$ by layer cake representation. By changing variables $t \mapsto \|u\|_{L^\infty(\partial\Omega)}s$, we observe

$$\begin{aligned}(LHS) &= C_q\|u\|_{L^\infty(\partial\Omega)}^{-\frac{(N-1)(q-1)}{p-1}}\sigma\int_0^{\|u\|_{L^\infty(\partial\Omega)}} t^{\sigma-1}(\|u\|_{L^\infty(\partial\Omega)}-t)^{N-1}dt\\ &= C_q\|u\|_{L^\infty(\partial\Omega)}^{-\frac{(N-1)(p-q)+(p-1)\sigma}{p-1}}\sigma\int_0^1 s^{\sigma-1}(1-s)^{N-1}ds\\ &\geq C_q\|u\|_{L^\infty(\partial\Omega)}^{-\frac{(N-1)(p-q)+(p-1)\sigma}{p-1}}\sigma\int_0^{\frac{1}{2}} s^{\sigma-1}2^{-(N-1)}ds\\ &= \left(\frac{1}{2}\right)^{\sigma+N-1} C_q\|u\|_{L^\infty(\partial\Omega)}^{\frac{(N-1)(p-q)+(p-1)\sigma}{p-1}}.\end{aligned}$$

Thus we get the conclusion. □

By the Lemma 3.1, we have the uniform boundedness of the extremizers for the subcritical range.

Lemma 3.2. *Let $\varepsilon > 0$ sufficiently small and let u_q be a positive $L^q(\partial\Omega)$-normalized extremal for S_q where $1 \leq q \leq p_* - \varepsilon$. Then we have*

$$|\partial\Omega|^{-1/q} \leq \|u_q\|_{L^\infty(\partial\Omega)} \leq C_\varepsilon$$

where $C_\varepsilon > 0$ is a constant which depends only on $\varepsilon > 0$.

Proof. Hölder's inequality and the fact $\|u_q\|_{L^q(\partial\Omega)} = 1$ yield the first inequality.

Next, suppose $1 \leq q \leq p$. Taking $\sigma = 1$ in (3.1), we have

$$\left(\frac{1}{2}\right)^N C_q\|u\|_{L^\infty(\partial\Omega)}^{\frac{(N-1)(p-q)+(p-1)}{p-1}} \leq \|u\|_{L^1(\partial\Omega)} \leq |\partial\Omega|^{1-1/q}\|u_q\|_{L^q(\partial\Omega)} = |\partial\Omega|^{1-1/q}.$$

Thus

$$\|u_q\|_{L^\infty(\partial\Omega)} \leq \max_{1\leq q\leq p}\left(\frac{2^N|\partial\Omega|^{1-1/q}}{C_q}\right)^{\frac{p-1}{(N-1)(p-q)+(p-1)}} =: A.$$

If $p \leq q \leq p_* - \varepsilon$, then take $\sigma = q$ in (3.1) to obtain

$$\left(\frac{1}{2}\right)^{q+N-1} C_q\|u\|_{L^\infty(\partial\Omega)}^{\frac{(N-1)(p-q)+(p-1)q}{p-1}} \leq \|u\|^q_{L^q(\partial\Omega)} = 1.$$

Thus

$$\|u_q\|_{L^\infty(\partial\Omega)} \leq \max_{p\leq q\leq p_*-\varepsilon}\left(\frac{2^{q+N+1}}{C_q}\right)^{\frac{(N-p)(p_*-q)}{(p-1)}} =: B_\varepsilon$$

since $(N-1)(p-q)+(p-1)q = (N-p)(p_*-q)$. Put $C_\varepsilon = \max\{A, B_\varepsilon\}$. $\square$

By combining Lemma 3.2 and Proposition 2.7 in [7], we have the following fact:

Proposition 3.3. *(Bonder-Rossi* [7] *Proposition 2.8.) The function $q \in [1, p_*] \mapsto S_q$ is continuous.*

For the proof, we refer the readers to [7].

4. Local Lipschitz and absolute continuity

In this section, by combining the arguments in [3] and [2], we prove the local Lipschitz continuity of S_q on $(1, p_*)$ and the absolute continuity of S_q on the whole closed interval $[1, p_*]$.

Theorem 4.1. *The function $q \mapsto S_q$ is locally Lipschitz continuous on the interval $(1, p_*)$.*

Proof. Fix $u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)$. Since $x^t(\log|x|)^2 \le (te)^{-2}$ for $0 < x \le 1$ and $t > 0$, we see for $1 \le t \le q_0 < p_*$,

$$\begin{aligned} |u|^t(\log|u|)^2 &= \left(\chi_{[|u|\le 1]} + \chi_{[|u|>1]}\right)|u|^t\big|\log|u|\big|^2 \\ &= \chi_{[|u|\le 1]}|u|^t\big|\log|u|\big|^2 + \chi_{[|u|>1]}|u|^t\big|\log|u|\big|^2 \\ &\le \chi_{[|u|\le 1]}(te)^{-2} + \chi_{[|u|>1]}\frac{1}{p_*-t}|u|^{p_*} \\ &\le e^{-2} + \frac{1}{p_*-q_0}|u|^{p_*} \in L^1(\partial\Omega). \end{aligned}$$

Since q_0 can be chosen arbitrarily close to p_*, we have $\|u\|^q_{L^q(\partial\Omega)}$ is at least twice differentiable and

$$\frac{d^2}{dq^2}\|u\|^q_{L^q(\Omega)} = \int_\Omega |u|^q(\log|u|)^2\, dx \ge 0$$

for any $q \in (1, p_*)$ by dominated convergence theorem. Thus $q \in (1, p^*) \mapsto \|u\|^q_{L^q(\partial\Omega)}$ is a convex function. Now, set

$$S = \{u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega) \mid \|u\|_{W^{1,p}(\Omega)} = 1\}$$

and define

$$h(q) = \sup_{u\in S}\|u\|^q_{L^q(\partial\Omega)}.$$

Since h is a supremum of convex functions $\|u\|^q_{L^q(\partial\Omega)}$, it is also convex and locally Lipschitz continuous on $(1, p_*)$ (see [5] pp.236), which yields that $|h(q)| < \infty$ and $|h'(q)| < \infty$ a.e.in $q \in (1, p_*)$. Note that $S_q = h(q)^{-\frac{1}{q}} = e^{-\frac{1}{q}\log h(q)}$, so

$$S'_q = S_q \left(-\frac{1}{q} \log h(q) \right)'.$$

It is easy to see that $h(q)$ is bounded from above and below by a positive constant on $q \in (1, p_*)$. Thus

$$\begin{aligned} |S'_q| &= S_q \left| \left(\frac{1}{q} \log h(q) \right)' \right| \\ &\le S_q \left(\frac{1}{q^2} |\log h(q)| + \frac{1}{q} \left| \frac{h'(q)}{h(q)} \right| \right) < \infty \quad \text{a.e. in } (1, p_*) \end{aligned}$$

From this, we have the conclusion. □

Theorem 4.2. *The function $q \mapsto S_q$ is absolutely continuous on the whole interval $[1, p_*]$.*

Proof. Since we know that S_q is of bounded pointwise variation on $[1, p_*]$ by Corollary 2.4, we have

$$S_q = S_1 = \int_1^q S'_t \, dt + S_C(q) + S_J(q)$$

where S_C is the Cantor part of S_q and S_J is the jump part of S_q, see [10] Theorem 3.73. Then the claim that S_q is absolutely continuous on $[1, p_*]$ is equivalent to $S_C \equiv S_J \equiv 0$. Since S_q is continuous on $[1, p_*]$ by Proposition 3.3, we see that the discontinuous part $S_J \equiv 0$. The Cantor part of S_q, that is S_C, is continuous, differentiable a.e., and $S'_C(q) = 0$ a.e. $q \in [1, p_*]$. Since S_q is Lipschitz continuous on any interval of the form $[1, p_* - \varepsilon]$, $\varepsilon > 0$, it is absolutely continuous on the same interval, thus the support of S_C must be concentrated on $\{p_*\}$. Therefore $S_C \equiv 0$ since S_C is continuous at p_*. □

5. A characterization of differentiability

Let us define the functional $I_q : (W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)) \to \mathbb{R}$ as

$$I_q(u) = \int_{\partial\Omega} |u|^q \log |u| \, d\mathcal{H}^{N-1}$$

and the set of $L^q(\partial\Omega)$-normalized extremal functions

$$E_q = \{u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega) \mid \|u\|_{L^q(\partial\Omega)} = 1, \, \|u\|^p_{W^{1,p}(\Omega)} = S_q\}$$

for $q \in [1, p_*]$.

Theorem 5.1. *For each $q \in [1, p_*)$ let u_q be arbitrarily chosen in E_q. Then we have*

$$\limsup_{t \to q+0} \frac{S_q - S_t}{q - t} \le -\frac{p}{q} I_q(u_q) S_q \le \liminf_{t \to q-0} \frac{S_q - S_t}{q - t}.$$

Therefore for $q \in (1, p_*)$ on which S_q' exists, it holds

$$S_q' + \frac{p}{q} I_q(u_q) S_q = 0. \tag{5.1}$$

Proof. Take $q \in (1, p_*)$ and let u_q an extremal for S_q in E_q. Put

$$J(t) = \int_{\partial\Omega} |u_q|^t \, d\mathcal{H}^{N-1}.$$

Then we see $J(q) = 1$ and $J'(t)\Big|_{t=q} = \int_{\partial\Omega} |u_q|^q \log |u_q| \, d\mathcal{H}^{N-1} = I_q(u_q)$. Since

$$\left(J(t)^{p/t}\right)' = J(t)^{p/t} \left(-\frac{p}{t^2} \log J(t) + \frac{p}{t} \frac{J'(t)}{J(t)} \right),$$

we see

$$\frac{d}{dt}\Big|_{t=q} \left(J(t)^{p/t}\right) = \frac{p}{q} J'(t)\Big|_{t=q} = \frac{p}{q} I_q(u_q).$$

Also testing S_t by u_q, we see

$$S_q = \|u_q\|_{W^{1,p}(\Omega)}^p \ge S_t \left(\int_{\partial\Omega} |u_q|^t \, d\mathcal{H}^{N-1} \right)^{p/t} = S_t J(t)^{p/t}.$$

Thus L'Hopital's rule and the continuity of S_q imply that

$$\begin{aligned} \limsup_{t \to q+0} \frac{S_q - S_t}{q - t} &\le \limsup_{t \to q+0} S_t \frac{J(t)^{p/t} - 1}{q - t} \\ &= -S_q \lim_{t \to q-0} \frac{d}{dt}\Big|_{t=q} \left(J(t)^{p/t}\right) \\ &= -\frac{p}{q} I_q(u_q) S_q. \end{aligned}$$

The similar argument yields

$$\liminf_{t \to q-0} \frac{S_q - S_t}{q - t} \ge -\frac{p}{q} I_q(u_q) S_q.$$

□

If S_q' exists for q, the value S_q' is independent of the choice of $u_q \in E_q$. Therefore, the above theorem implies that the value $I_q(u_q)$ is also independent of the choice of $u_q \in E_q$, which proves the next corollary. Indeed, $I_q(u_q) = -\frac{q}{p} \frac{S_q'}{S_q}$ for any choice of u_q in E_q.

Corollary 5.2. *Let $q \in (1, p_*)$ be such that S'_q exists. Then the functional I_q takes a constant value on E_q; $I_q(u_1) = I_q(u_2)$ for any $u_1, u_2 \in E_q$.*

Now, let us define f as

$$f(q) := \begin{cases} \frac{p}{q} I_q(u_q) & \text{when } S'_q \text{ exists,} \\ 0 & \text{when } S'_q \text{ does not exist.} \end{cases} \tag{5.2}$$

f is well-defined on $[1, p_*)$ by Corollary 5.2 and $f(q) = -\frac{S'_q}{S_q}$ when S'_q exists by (5.1).

We have a representation formula for S_q by using f in (5.2).

Theorem 5.3. *It holds*

$$S_q = S_1 \exp\left(-\int_1^q f(t)\ dt\right) \tag{5.3}$$

for $1 \leq q \leq p_*$

Proof. Since the function $q \mapsto S_q$ is absolutely continuous on $[1, p_*]$ by Theorem 4.2, we have also the function $[1, p_*] \ni q \mapsto \log S_q$ is absolutely continuous. Thus by (5.1),

$$\log S_q - \log S_1 = \int_1^q \left(\frac{d}{dt} \log S_t\right) dt = \int_1^q \frac{S'_t}{S_t} dt = -\int_1^q f(t) dt$$

for all $q \in [1, p_*]$, which yields the result. □

Theorem 5.3 implies also

$$\begin{aligned} S_q &= S_1 \exp\left(-\int_1^{p_*} f(t) dt + \int_q^{p_*} f(t) dt\right) \\ &= S_1 \exp\left(-\int_1^{p_*} f(t)\ dt\right) \exp\left(\int_q^{p_*} f(t) dt\right) = S_{p_*} \exp\left(\int_q^{p_*} f(t) dt\right). \end{aligned}$$

As an immediate corollary of Theorem 5.3, we have the following:

Corollary 5.4. *Let $q \in [1, p_*)$ be a point of continuity of f. Then $\frac{d}{dq} S_q$ exists and*

$$S'_q = -S_q f(q)$$

holds.

Proposition 5.5. *Suppose I_q is constant on E_q for some $q \in [1, p_*)$. Then f is continuous on such q. Especially f is continuous on q where S'_q exists.*

Proof. Take $q \in [1, p_*)$ and a sequence $q_n \to q$ as $n \to \infty$. Since $q \mapsto S_q$ is continuous, we see $S_{q_n} \to S_q$. Also by elliptic regularity and the fact that $\|u_{q_n}\|_{L^\infty(\Omega)}$ is uniformly bounded in n, we have a subsequence (again denoted by q_n) and $u \in E_q$ such that $u_{q_n} \to u$ in $C^1(\overline{\Omega})$ and $\|u\|_{L^q(\partial\Omega)} = 1$. Therefore, we have

$$\begin{aligned} f(q_n) &= \frac{p}{q_n} \int_{\partial\Omega} |u_{q_n}|^{q_n} \log |u_{q_n}| \, d\mathcal{H}^{N-1} \to \frac{p}{q} \int_{\partial\Omega} |u|^q \log |u| \, d\mathcal{H}^{N-1} \\ &= \frac{p}{q} I_q(u) = \frac{p}{q} I_q(u_q) = f(q), \end{aligned}$$

since $I_q(u) = I_q(u_q)$ for $u, u_q \in E_q$. □

Now, we obtain a characterization of the differentiability of the function $q \mapsto S_q$.

Theorem 5.6. *The following 3 assertions on a point $q \in [1, p_*)$ are equivalent:*

(i) S'_q exists.
(ii) I_q is constant on E_q.
(iii) The function $t \in [1, p_*] \mapsto I_t(u_t)$ is continuous at $t = q$.

Proof. $(i) \Longrightarrow (ii)$: Corollary 5.2.

$(ii) \Longrightarrow (iii)$: Since the continuity of $f(t)$ at $t = q$ is equivalent to the continuity of $t \mapsto I_t(u_t)$ is continuous at $t = q$, the proof follows from Proposition 5.5.

$(iii) \Longrightarrow (i)$: Corollary 5.4. □

It is known that S_q is simple when $q = p$ and $E_p = \{\pm u_p\}$ for some $u_p \in E_p$ ([13]). Thus we see $S'_p = \frac{d}{dq} S_q\big|_{q=p}$ exists and $t \mapsto I_t(u_t)$ is continuous at $t = p$. Also if Ω is a ball with sufficiently small radius and $p = 2$, then S_q is simple for any $1 \le q < 2_* = \frac{2(N-1)}{N-2}$ and the unique normalized extremizer for S_q is radial (see [6] Theorem 2.1). Thus $q \mapsto S_q$ is differentiable on $1 \le q < 2_*$ on small balls. Moreover the abstract approach using a variational principle in [9] could be applied to obtain the uniqueness of the positive solution of

$$\begin{cases} \Delta_p u = |u|^{p-2} u & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2} u & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$, $1 < p < N$ and $1 \le q < p$. If this is the case, then we see that the function $q \mapsto S_q$ is differentiable for $1 \le q < p$ on any bounded domain. However, the simplicity of S_q for $p < q < p_*$ on a general bounded smooth domain is unknown.

Acknowledgments.

This work was supported by the Research Institute of Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University, and partly by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics).

The second author (F.T.) was supported by JSPS Grant-in-Aid for Scientific Research (B), No.19H01800.

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