Smooth homotopy 4-sphere

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ABSTRACT

Every smooth homotopy 4-sphere is diffeomorphic to the 4-sphere.

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1. Introduction

For a positive integer n, the *stable 4-sphere of genus* n is the connected sum

$$\Sigma_n = S^4 \# n(S^2 \times S^2) = S^4 \#_{i=1}^n S^2 \times S_i^2$$

of the 4-sphere S^4 and the n copies $S^2 \times S_i^2\,(i = 1, 2, \dots, n)$ of the 2-sphere product $S^2 \times S^2$ done by taking n mutually disjoint 4-balls embedded smoothly in S^4, where a choice of the 4-balls is independent of the diffeomorphism type of Σ_n.

A compact connected oriented smooth 4-manifold is simply called a *4-manifold* in this paper. A *compact punctured* 4-manifold of a 4-manifold X is a 4-manifold X^0 obtained from X by removing an interior of a 4-ball D_0^4 embedded smoothly in the interior of the 4-manifold X.

The following result is a main result of this paper.

Theorem 1.1. Let Σ_n^0 be a compact punctured 4-manifold of the stable 4-sphere Σ_n of every positive genus n. Then every smooth embedding $e : \Sigma_n^0 \to \Sigma_n$ extends to a diffeomorphism $e^+ : \Sigma_n \to \Sigma_n$.

A *smooth homotopy 4-sphere* is a smooth 4-manifold M homotopy equivalent to S^4. The following result is obtained from Theorem 1.1:

Corollary 1.2. Every smooth homotopy 4-sphere M is diffeomorphic to the 4-sphere S^4.

Proof of Corollary 1.2. It is known that there is a diffeomorphism

$$k : M\#\Sigma_n \to \Sigma_n$$

from the connected sum $M\#\Sigma_n$ onto Σ_n for a positive integer n (see Wall [9]). Let $D^4_0 = \mathrm{cl}(\Sigma_n \setminus \Sigma^0_n)$ be a 4-ball. By regarding $M\#\Sigma_n = M^0 \cup \Sigma^0_n$, let

$$e : \Sigma^0_n \to \Sigma_n$$

be a smooth embedding which is extended to a diffeomorphism

$$e^+ : \Sigma_n \to \Sigma_n$$

by Theorem 1.1. By the identity

$$M^0 = \mathrm{cl}(\Sigma_n \setminus e(\Sigma^0_n)) = e^+\mathrm{cl}(\Sigma_n \setminus \Sigma^0_n) = e^+(D^4_0),$$

there is an orientation preserving diffeomorphism

$$h : M^0 \to D^4_0$$

defined by the inverse diffeomorphism $(e^+)^{-1}$ of e^+. By $\Gamma_4 = 0$ in Cerf [2], the diffeomorphism h extends to a diffeomorphism $h^+ : M \to S^4$. $\square$

In the topological category, the corresponding result of Corollary 1.2 (namely, every topological 4-manifold homotopy equivalent to the 4-sphere is homeomorphic to the 4-sphere) is well-known by Freedman [3] (see also [4]). In the piecewise-linear category, the corresponding result of Corollary 1.2 (namely, every piecewise-linear 4-manifold homotopy equivalent to the piecewise-linear 4-sphere is piecewise-linearly homeomorphic to the piecewise-linear 4-sphere) can be shown by using the piecewise-linear versions of the techniques used in this paper (see Hudson [6], Rourke-Sanderson [8]).

It is known by Wall in [9] that for every closed smooth signature-zero spin 4-manifold M with second Betti number $\beta_2(M;\mathbf{Z}) = 2m > 0$, there is a diffeomorphism

$$\kappa : M\#\Sigma_n \to \Sigma_{m+n}$$

for a positive integer n and by Freedman [3] (see also [4]) that there is a homeomorphism from M to Σ_m. However, a technique used for the proof of Theorem 1.1 cannot be directly generalized to this case. In fact, it is known by Akhmedov-Park in [1] that there is a smooth closed signature-zero spin 4-manifold M with a large second Betti number $\beta_2(M;\mathbf{Z}) = 2m$ such that M is not diffeomorphic to Σ_m. What can be said in this paper is the following corollary.

Corollary 1.3. Let M and M' be closed connected smooth 4-manifolds with the same second Betti number $\beta_2(M;\mathbf{Z}) = \beta_2(M';\mathbf{Z})$. Then there is a smooth embedding $e : M^0 \to M'$ extending a diffeomorphism $e^+ : M \to M'$ if and only if the embedding $e : M^0 \to M'$ induces a fundamental group isomorphism

$$f_\# : \pi_1(M^0, x) \to \pi_1(M', f(x)).$$

For this corollary, the proof of the "if"part is obtained by noting that the closed complement $\tilde{D}^4_0 = \mathrm{cl}(M' \backslash e(M^0))$ is a smooth homotopy 4-ball with 3-sphere boundary which is confirmed by the van Kampen theorem and an homological argument. By Corollary 1.2. the smooth homotopy 4-ball $\tilde{D}^4_0$ is diffeomorphic to the 4-ball. Thus, by $\Gamma_4 = 0$ in [2], the map e extends to a diffeomorphism $e^+ : M \to M'$. The proof of the "only if"part is obvious.

2. Proof of Theorem 1.1

A *surface-knot* in a 4-manifold X is a closed oriented surface F embedded in the interior of X by a smooth embedding. It is also called a 2-*knot* if F is the 2-sphere S^2. Two surface-knots F and F' in X are *equivalent* by an *equivalence* f if F is sent to F' orientation-preservingly by an orientation-preserving diffeomorphism f of X.

A *trivial* surface-knot is a surface-knot F which is the boundary of a handlebody smoothly embedded into a 4-ball in the interior of X, where a handlebody means a 3-manifold which is a 3-ball, a solid torus or a boundary-disk sum of some number of solid tori. A trivial genus n surface-knot in X for every $n \geq 0$ exists uniquely up to equivalences of X (see [5]).

Let

$$F = S^2 \# nT = F = S^2 \#_{i=1}^n T_i$$

be a trivial genus n surface-knot in S^4 which is the connected sum of a trivial 2-knot (S^4, S^2) and the n copies (S^4, T_i) $(i = 1, 2, \ldots, n)$ of a trivial torus-knot (S^4, T) done by taking mutually disjoint n disks in S^2. The following lemma is a standard result.

Lemma 2.1 The double branched covering space $S^4(F)_2$ of S^4 branched along a trivial genus n surface-knot F is diffeomorphic to the stable 4-sphere Σ_n of genus n.

Proof of Lemma 2.1. The double branched covering covering space $S^4(S^2)_2$ of S^4 branched along a trivial 2-knot S^2 is easily seen to be diffeomorphic to the 4-sphere $\Sigma_0 = S^4$.

Let T be a trivial torus-knot in S^4. Then the pair (S^4, T) is the double of the product pair $(A, o) \times I = (A \times I, o \times I)$ for a trivial loop o in a 3-ball A and the interval $I = [0, 1]$, namely, (S^4, T) is diffeomorphic to the boundary pair

$$\partial((A, o) \times I^2) = (\partial(A \times I^2), \partial(o \times I^2)),$$

where I^m denotes the m-fold product of I for any $m \geq 1$. Thus, the double branched covering space $S^4(T)_2$ of S^4 branched along T is diffeomorphic to the boundary $\partial(A(o)_2 \times I^2)$, where $A(o)_2$ is the double branched covering space over A branched along o which is diffeomorphic to the product $S^2 \times I$. This means that the 5-manifold $A(o)_2 \times I^2$ is the product $S^2 \times I^3$. Hence the 4-manifold $S^4(T)_n$ is diffeomorphic to $S^2 \times S^2$. For $n \geq 2$, a trivial genus n surface-knot (S^4, F_n) is equivalent to the n-fold connected sum of a trivial torus-knot (S^4, T) and hence the double branched covering space $S^4(F_n)_2$ of S^4 branched along F_n is diffeomorphic to the n-fold connected sum of $S^4(T)_2 = S^2 \times S^2$. Hence $S^4(F_n)_2$ is diffeomorphic to the stable 4-sphere Σ_n of genus n. □

A *loop basis* of a closed surface F of genus n is a system of oriented simple loop pairs (e_j, e'_j) $(j = 0, 1, 2, \ldots, n)$ on F representing a basis for $H_1(F; \mathbb{Z})$ such that $e_j \cap e_{j'} = e'_j \cap e'_{j'} = e_j \cap e'_{j'} = \emptyset$ for all distinct j, j' and $e_j \cap e'_j$ is one point with the intersection number $\mathrm{Int}(e_j, e'_j) = +1$ in F for all j. A simple loop ℓ in a surface-knot F is *spin* if the $\mathbb{Z}_2$-quadratic function $q : H_1(F; \mathbb{Z}_2) \to \mathbb{Z}_2$ associated with the surface-knot F has $q([\ell]) = 0$ for the $\mathbb{Z}_2$-homology class $[\ell] \in H_1(F; \mathbb{Z}_2)$ of ℓ.

A 2-*handle* on a surface-knot F in X is a 2-handle $D \times I$ on F embedded smoothly in the interior of X such that $(D \times I) \cap F = (\partial D) \times I$, where I denotes a closed interval with 0 as the center and $D \times 0$ is called the *core* of the 2-handle $D \times I$ and identified with D. An *orthogonal 2-handle pair* or simply, an *O2-handle pair* on a surface-knot in X is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I$, $D' \times I$ on F such that

$$(D \times I) \cap (D' \times I) = (\partial D) \times I \cap (\partial D') \times I$$

and $(\partial D) \times I$ and $(\partial D') \times I$ *meet orthogonally* on F, that is, ∂D and $\partial D'$ meet transversely at one point p and the intersection $(\partial D) \times I \cap (\partial D') \times I$ is diffeomorphic to the square $Q = p \times I \times I$ (see [7]).

An O2-*handle basis* of a trivial genus n surface-knot F is the system $(D_* \times I, D'_* \times I)$ of mutually disjoint O2-handle pairs $(D_i \times I, D'_i \times I)\,(i = 1, 2, \dots, n)$ on F such that the loop system $(\partial D_*, \partial D'_*)$ of $(\partial D_i, \partial D'_i)\,(i = 1, 2, \dots, n)$ forms a spin loop basis of F.

Lemma 2.2. Let $(D_* \times I, D'_* \times I)$ be an O2-handle basis of a trivial genus n surface-knot F in S^4. Then the surface-knot F bounds a genus n handlebody $V(F; D'_* \times I)$ smoothly embedded into a 3-ball $B(V(F; D'_* \times I), D_* \times I)$ smoothly embedded in S^4 such that

$$(\mathrm{Int}D'_i) \times I \subset \mathrm{Int}V(F; D'_* \times I) \quad \text{and} \quad (\mathrm{Int}D_i) \times I \cap \mathrm{Int}V(F; D'_* \times I) = \emptyset$$

for all $i\,(i = 1, 2, \dots, n)$ and

$$B(V(F; D'_* \times I), D_* \times I) = V(F; D'_* \times I) \cup_{i=1}^{n} D_i \times I.$$

Proof of Lemma 2.2. Let S be the 2-sphere obtained from F by the embedded surgery along the 2-handles $D'_i \times I\,(i = 1, 2, \dots, n)$. By uniqueness of an O2-handle pair in [7, Theorem 3.1], the 2-sphere S is a trivial 2-knot in S^4 and hence bounds a 3-ball V_0 in S^4. The union $V_0 \cup_{i=1}^{n} D'_i \times I$, denoted by $V(F; D'_* \times I)$ is a handlebody such that the union $V(F; D'_* \times I) \cup_{i=1}^{n} D_i \times I$, denoted by $B(V(F; D'_* \times I), D_* \times I)$ is a 3-ball smoothly embedded in S^4. □

In Lemma 2.2, the 3-ball $B_i = D_i \times I \cup D'_i \times I$ is called the *bump* associated with the O2-handle pair $(D_i \times I, D'_i \times I)$, and the 3-ball $B(V(F; D'_* \times I), D_* \times I)$ embedded smoothly in S^4 is called the *total bump* of a trivial surface-knot F associated with the O2-handle basis $(D_* \times I, D'_* \times I)$.

A *2-sphere basis* of the stable 4-manifold Σ_n is a system of mutually disjoint smooth unoriented 2-sphere pairs $(S_i, S'_i)\,(i = 1, 2, \dots, n)$ in Σ_n with $S_i \cap S'_i = p_i$ a point such that the closed complement $\mathrm{cl}(\Sigma_n \setminus N(S_*, S'_*))$ is an compact n-punctured 4-sphere embedded smoothly in Σ_n for a regular neighborhood $N(S_*, S'_*)$ of the union $\cup_{i=1}^{n} S_i \cup S'_i$ in Σ_n. In this case, note that S_i and S'_i meet transversely in Σ_n with intersection number ± 1.

The following lemma gives a relationship between an O2-handle basis $(D_* \times I, D'_* \times I)$ of a trivial surface-knot F and a 2-sphere basis $(S(D_*), S(D'_*))$ of the stable 4-sphere $S^4(F)_2 = \Sigma_n$.

Lemma 2.3. The core system $(D_i, D'_i)\,(i = 1, 2, \dots, n)$ of every O2-handle basis $(D_i \times I, D'_i \times I)\,(i = 1, 2, \dots, n)$ of a trivial genus n surface-knot F in S^4 lifts to a 2-sphere basis $(S(D_i), S(D'_i))\,(i = 1, 2, \dots, n)$ of the stable 4-sphere $S^4(F)_2 = \Sigma_n$ of genus n by the double branched covering projection $p : S^4(F)_2 \to S^4$ branched along F.

Proof of Lemma 2.3. Let $(D_i \times I, D'_i \times I)\,(i = 1, 2, \dots, n)$ be a standard O2-handle basis of a trivial genus n surface-knot F in S^4. Let $N_i\,(i = 1, 2, \dots, n)$ be mutually disjoint regular neighborhoods of the 3-balls $D_i \times I \cup D'_i \times I\,(i = 1, 2, \dots, n)$ in the 4-sphere S^4. Since N_i is a 4-ball, the closed complement $X = \mathrm{cl}(S^4 \setminus \cup_{i=1}^n N_i)$ is a compact n-punctured 4-sphere and $F_X = F \cap X$ is a compact proper n-punctured 2-sphere such that the pair (X, F_X) is smoothly embeddable in a trivial 2-knot (S^4, S^2). Using that the double branched covering space $S^4(S^2)_2$ of the 4-sphere S^4 branched along the 2-sphere S^2 is diffeomorphic to the 4-sphere S^4, we see that the double branched covering space $X(F_X)_2$ of X branched along the compact n-punctured 2-sphere F_X is diffeomorphic to a compact n-punctured 4-sphere. This means that a 2-sphere pair system $(S(D_i), S(D'_i))\,(i = 1, 2, \dots, n)$ is a 2-sphere basis of the stable 4-sphere $S^4(F)_2 = \Sigma_n$ of genus n because the double branched covering space $X(F_X)_2$ is diffeomorphic to the closed complement $\mathrm{cl}(\Sigma_n \setminus N(S(D_*), S(D'_*))$ by Lemma 2.1. Let $(E_i \times I, E'_i \times I)\,(i = 1, 2, \dots, n)$ be any O2-handle basis of a trivial genus n surface-knot F in S^4. By uniqueness of an O2-handle pair in [7, Theorem 3.1], there is an orientartion-preserving diffeomorphism g of S^4 such that

$$g(F) = F, \quad (g(E_i) \times I, g(E'_i) \times I) = (D_i \times I, D'_i \times I)\,(i = 1, 2, \dots, n).$$

The diffeomorphism g lifts to an α-invariant orientation-preserving diffeomorphism f of $S^4(F)_2$ sending the 2-sphere pair $(S(E_i), S(E'_i))$ to the 2-sphere pair $(S(D_i), S(D'_i))$ for all i. Thus, the 2-sphere pair system $(S(E_i), S(E'_i))\,(i = 1, 2, \dots, n)$ is a 2-sphere basis of $\Sigma_n = S^4(F)_2$. □

Let A be a smooth bounded 3-submanifold of S^4. The *sutured triple* associated with the pair (S^4, A) is a triplet $(Y; A_+, A_-)$ such that Y is a smooth compact 4-manifold obtained from S^4 by splitting along the interior $\mathrm{Int}A$ of A and the boundary ∂Y of Y is given by the union $A_+ \cup A_-$ for the splitting copies A_+ and A_- of A where A_+ is a copy of A with orientation preserved and A_- is a copy of A with orientation reversed. Note that there is a canonical identification map $A_+ \to A_-$. For a slightly different explanation of the sutured triple $(Y; A_+, A_-)$, consider a *bi-collar* of A in S^4 which is meant by the image $c(A \times [-1, 1])$ of a smooth embedding $c : A \times [-1, 1] \to S^4$ with $c(x, 0) = x$ for all $x \in A$. Then the sutured triple $(Y; A_+, A_-)$ is understood to the triplet

$$(\mathrm{cl}(S^4 \setminus c(A \times [-1, 1]), c(A \times 1 \cup (\partial A) \times [0, 1]), c(A \times (-1) \cup (\partial A) \times [-1, 0]).$$

Let F be a trivial genus n surface-knot in S^4, and $(D_* \times I, D'_* \times I)$ an O2-handle basis of F in S^4. For the genus n handlebody $V = V(F; D'_* \times I)$ constructed in

Lemma 2.2, let

$$B = B(V(F; D'_* \times I), D_* \times I) = V \cup_{i=1}^n D_i \times I$$

be a total bump of F associated with an O2-handle basis $(D_* \times I, D'_* \times I)$. Let $(W; V_+, V_-)$ be the sutured triple associated with (S^4, V). Note that the 2-handle system $D_i \times I\,(i = 1, 2, \ldots, n)$ is in W. Let $D_i \times I^2\,(i = 1, 2, \ldots, n)$ be a 2-handle system attached to V_+ in W thickening the 2-handle system $D_i \times I\,(i = 1, 2, \ldots, n)$.

We have the following lemma.

Lemma 2.4. The 4-manifold $U = \mathrm{cl}(W \setminus \cup_{i=1}^n D_i \times I^2)$ is a 4-ball smoothly embedded in W.

Proof of Lemma 2.4. Since the bi-collar $c(V \times [-1, 1])$ is diffeomorphic to the disk sum of n copies of the product $S^1 \times D^3$ for the 3-ball D^3 and the union $c(V \times [-1, 1]) \cup_{i=1}^n D_i \times I^2$ forms a 4-ball, the 4-manifold U is diffeomorphic to the 4-manifold obtained from S^4 by removing the interior of a 4-ball, which is a 4-ball. □

The double branched covering space $S^4(F)_2$ is constructed from the sutured triple (W, V_+, V_-) of (S^4, V) and the copy $(\overline{W}, \overline{V}_+, \overline{V}_-)$ of (W, V_+, V_-) by identifying V_+ with $\overline{V}_-$ and V_- with $\overline{V}_+$ by the canonical identification maps $V_+ \to \overline{V}_-$ and $V_- \to \overline{V}_+$, respectively.

A *spine* of the stable 4-sphere Σ_n of genus n is the preimage $Y = p^{-1}(B)$ of the total bump $B = B(V(F; D'_* \times I), D_* \times I)$ for an O2-handle basis $(D_* \times I, D'_* \times I)$ of F and the double branched covering projection $p : S^4(F)_2 \to S^4$ branched along a trivial genus n surface-knot F under the identification $S^4(F)_2 = \Sigma_n$ given by Lemma 2.1. In this case, the preimage $Z = p^{-1}(V)$ of $V = V(F; D'_* \times I)$ is called the *backbone* of the spine Y. The backbone Z of a spine Y of Σ_n is diffeomorphic to the *stable 3-sphere*

$$S^3 \# n(S^1 \times S^2) = S^3 \#_{i=1}^n S^1 \times S^2_i$$

of genus n. From the construction of $S^4(F)_2$ from (W, V_+, V_-) and $(\overline{W}, \overline{V}_+, \overline{V}_-)$, it is seen that the backbone Z splits Σ_n into W and $\overline{W}$. The spine Y is obtained from Z by attaching 2-handle system $D_i \times I\,(i = 1, 2, \ldots, n)$ in W and the copy system $\overline{D}_i \times I\,(i = 1, 2, \ldots, n)$ in $\overline{W}$. Then the following lemma is directly obtained from Lemma 2.4.

Corollary 2.5. The closed complement $\mathrm{cl}(\Sigma_n \setminus N(Y))$ for a regular neighborhood $N(Y)$ of a spine Y in Σ_n is a disjoint union of two smoothly embedded 4-balls in Σ_n.

The following lemma is near the argument of [7].

Lemma 2.6. Let F be a surface-knot in a 4-manifold X, and $(D\times I, D'_i\times I)\,(i=1,2))$ O2-handle pairs on F in X such that $(\partial D'_1)\times I=(\partial D'_2)\times I$. Then there is a smooth isotopy $h_t\,(t\in[0,1])$ of X with $h_0=1$ such that $h_1(D\times I, D'_1\times I)=(D\times I, D'_2\times I)$ and $h_1(F)=F$.

Proof of Lemma 2.6. As it is done in [7, Lemma 2.3], let a be an arc obtained from $D\times I$ by shrinking D into a point such that $a\subset\partial D'_i$. Let $a'_i=\mathrm{cl}(\partial D'_i\setminus a)$ for each $i\,(i=1,2)$. Since $\partial D''_1=\partial D'_2$, we can assume that $a'_1=a'_2$ and a boundary collar of $\partial D'_1$ in D'_1 coincides with a boundary collar of $\partial D'_2$ in D'_2. Let $F(D\times I)$ denotes the surface-knot obtained from F by the surgery along the 2-handle $D\times I$. For each $i\,(i=1,2)$, the arc a is deformed into an arc a''_i parallel to the arc a'_i along the disk D'_i by a smooth isotopy $h^i_t\,(t\in[0,1])$ of X with $h^i_0=1$ keeking the surface-knot $F(D\times I)$ fixed. Since a boundary collar of $\partial D'_1$ in D'_1 coincides with a boundary collar of $\partial D'_2$ in D'_2, we may consider that the disk bounded by the loop $a'_1\cup a''_1$ in the disk D'_1 coincides with the disk bounded by the loop $a'_2\cup a''_2$ in the disk D'_2. By regarding $D\times I$ as a thin 1-handle with the core a on the surface-knot $F(D\times I)$, the isotopies $h^i_t\,(t\in[0,1])$ for $i=1$ and 2 constitute a desired isotopy $h_t\,(t\in[0,1])$ of X with $h_0=1$ such that $h_1(D\times I, D'_1\times I)=(D\times I, D'_2\times I)$ and $h_1(F)=F$. □

Let $\Sigma^{0,0}_n$ be a smooth 4-submanifold of Σ_n obtained from Σ_n by removing the interiors of two 4-balls invariant under the covering involution α of $S^4(F)_2=\Sigma_n$. Let $\mathrm{Diff}^+(D^4,\mathrm{rel}\,\partial)$)be the orientation-preserving diffeomorphism group of the 4-ball D^4 keeping the boundary ∂D^4 by the identity. The following lemma is an essential point to the proof of Theorem 1.1.

Lemma 2.7. Every orientation-preserving smooth embedding $u:\Sigma^{0,0}_n\to\Sigma_n$ is smoothly isotopic to a smooth embedding $\tilde u:\Sigma^{0,0}_n\to\Sigma_n$ such that the composition $u'=f\tilde u:\Sigma^{0,0}_n\to\Sigma_n$ for an α-invariant orientation-preserving diffeomorphism f of Σ_n is smoothly isotopic to the inclusion map

$$\mathrm{inc}:\Sigma^{0,0}_n\to\Sigma_n$$

up to local replacements by diffeomorphisms in $\mathrm{Diff}^+(D^4,\mathrm{rel}\,\partial)$.

In the piecewise-linear category, this local replacement is not needed since every orientation-preserving piecewise-linear homeomorphism of D^4 with the identity on ∂D^4 is piecewise-linearly isotopic to the identity 1.

Proof of Lemma 2.7. Let $Z=V\cup_F\overline{V}$ be the backbone of Σ_n where V and $(D_*\times I, D'_*\times I)$ are identified with the orbit handlebody V and the orbit O2-handle pair $(D_*\times I, D'_*\times I)$ in S^4, respectively, and $\overline{V}$ denotes the image $\alpha(V)$ of V by α. Also, let $Y = B\cup_F \overline{B}$ be the spine of Σ_n where B is identifined with the orbit total bump B in S^4 and $\overline{B}$ denotes the image $\alpha(B)$. The backbone Z and the spine Y of Σ_n are assumed to be in $\Sigma_n^{0,0}$. Since the lifting surface-knot F is a trivial surface-knot in Σ_n, the embedding u is smoothly isotopic to an embedding $\tilde{u}:\Sigma_n^{0,0}\to\Sigma_n$ with $\tilde{u}(F)=F$ in Σ_n. By considering the surface-knot F in S^4 as the covering projection image, there is an orientation-preserving diffeomorphism g of S^4 with $g(F)=F$ which sends the spin loop basis $(p\tilde{u}(\partial D_*), p\tilde{u}(\partial D'_*))$ of F to the spin loop basis $(\partial D_*, \partial D'_*)$ of F. The lifting diffeomorphism f of g is an α-invariant orientation-preserving differomorphism of Σ_n such that the composition embedding $u' = f\tilde{u}:\Sigma_n^{0,0}\to\Sigma_n$ with $f(F)=F$ which sends the spin loop basis $(\tilde{u}(\partial D_*, \tilde{u}(\partial D'_*)$ of F to the spin loop basis $(\partial D_*, \partial D'_*)$ of F. By a smooth isotopy, the embedding u' is deformed to send the handlebody V to Videntically and then deformed by Lemma 2.6 to send the total bump B to B identically.

On deformations of the 2-handle systems $\overline{D}_*\times I$ and $\overline{D}'_*\times I$ on F in Σ_n, the following two assertions are observed, where at the present stage note that the 2-handle system images $pu'(\overline{D}_*\times I)$ and $pu'(\overline{D}'_*\times I)$ are in general singular 2-handles on F in S^4.

(2.7.1) The smooth embedding u' is smoothly isotopic to a smooth embedding $u^*:\Sigma_n^{0,0}\to\Sigma_n$ such that

$$u^*(D_1\times I, \overline{D}_1\times I, D'_1\times I, \overline{D}'_1\times I) = (D_1\times I, u^*(\overline{D}_1)\times I, D'_1\times I, \overline{D}'_1\times I).$$

(2.7.2) The smooth embedding u^* in (2.7.1) is smoothly isotopic to a smooth embedding $u_1:\Sigma_n^{0,0}\to\Sigma_n$ such that

$$u_1(D_1\times I, \overline{D}_1\times I, D'_1\times I, \overline{D}'_1\times I) = (D_1\times I, \overline{D}_1\times I, D'_1\times I, \overline{D}'_1\times I).$$

By continuing the same processes of (2.7.1) and (2.7.2) for $i = 2,3,\ldots,n$, the embedding u is smoothly isotopic to a smooth embedding

$$u_n:\Sigma_n^{0,0}\to\Sigma_n$$

such that

$$u_n(D_i\times I, \overline{D}_i\times I, D'_i\times I, \overline{D}'_i\times I) = (D_i\times I, \overline{D}_i\times I, D'_i\times I, \overline{D}'_i\times I)$$

for all i. By Corollary 2.5, u_n is smoothly isotopic to the inclusion map inc after a local replacement of a diffeomorphism in $\mathrm{Diff}^+(D^4, \mathrm{rel}\ \partial)$. This completes the proof of Lemma 2.7 except for the proofs of (2.7.1) and (2.7.2). □

Proof of (2.7.1). By regarding the bump $\overline{B}_1 = \overline{D}_1 \times I \cup \overline{D}_1' \times I \subset \Sigma_n$ as a line bundle over a twisted disk $\overline{d}_1$ associated with $\overline{D}_1 \cup \overline{D}_1'$ (see [7]), move neighborhoods in $u'(\overline{D}_1')$ of the interior intersection double points between the 2-handle core $\overline{D}_1$ and the 2-handle core image $u'(\overline{D}_1')$ into the interior intersection double points of the 2-handle core $\overline{D}_1'$ and the 2-handle core image $u'(\overline{D}_1')$ through the twisted disk $\overline{d}_1$, so that

$$\mathrm{Int}\overline{D}_1 \cap \mathrm{Int}u'(\overline{D}_1') = \emptyset.$$

In a process of this deformation, every disk in the neighborhoods meets F in two points apart from the part of 2-handle attachments, but in the end of this deformation the interior of the 2-handle core image $u'(\overline{D}_1')$ no longer meets F (see the proof of [7, Lemma 3.2]). This deformation does not touch the 2-handles $D_1 \times I$ and $D_1' \times I$. Let $u'' : \Sigma_n^{0,0} \to \Sigma_n$ be the resulting smooth embedding which is smoothly isotopic to u'. Then we have the following four O2-handle pairs:

$$(D_1 \times I, D_1' \times I),\ (\overline{D}_1 \times I, \overline{D}_1' \times I),\ (D_1 \times I, u''(\overline{D}_1' \times I)),\ (\overline{D}_1 \times I, u''(\overline{D}_1') \times I)$$

on F in Σ_n. By the covering projection $p : \Sigma_n \to S^4$, we obtain the two O2-handle pairs

$$(D_1 \times I, D_1' \times I), \quad (D_1 \times I, pu''(\overline{D}_1') \times I)$$

on F in S^4 with $pu''(\overline{D}_1') \times I$ a singular 2-handle. Apply [7, Lemma 2.3] to the O2-handle pair $(D \times I, pu''(\overline{D}_1') \times I)$ to deform $pu''(\overline{D}_1') \times I$ into a smoothly embedded 2-handle, and then apply Lemma 2.6 to deform $pu''(\overline{D}_1') \times I$ into the 2-handle $D_1' \times I$. These deformations are realized by a smooth isotopy of Σ_n, so that there is a smooth embedding $u^* : \Sigma_n^{0,0} \to \Sigma_n$ smoothly isotopic to u'' such that

$$u^*(D_1 \times I, \overline{D}_1 \times I, D_1' \times I, \overline{D}_1' \times I) = (D_1 \times I, u^*(\overline{D}_1) \times I, D_1' \times I, \overline{D}_1' \times I),$$

where $u^*(D_1) \times I$ is taken to be $D_1 \times I$. □

Proof of (2.7.2). In (2.7.1), we obtain two O2-handle pairs $(D_1 \times I, D_1' \times I)$ and $(pu^*(\overline{D}_1) \times I, D_1' \times I)$ on F in S^4 with $pu^*(\overline{D}_1) \times I$ a singular 2-handle by taking the covering projection image. Apply [7, Lemma 2.3] to the O2-handle pair $(pu^*(\overline{D}_1) \times I, D_1' \times I)$ to deform $pu^*(\overline{D}_1) \times I$ into a smoothly embedded 2-handle, and then apply Lemma 2.6 to deform $pu^*(\overline{D}_1) \times I$ into the 2-handle $D_1 \times I$. These deformations are realized by a smooth isotopy of Σ_n, so that there is a smooth embedding $u_1 : \Sigma_n^{0,0} \to \Sigma_n$ smoothly isotopic to u^* such that

$$u_1(D_1 \times I, \overline{D}_1 \times I, D_1' \times I, \overline{D}_1' \times I) = (D_1 \times I, \overline{D}_1 \times I, D_1' \times I, \overline{D}_1' \times I). \quad \square$$

Now the proof of Theorem 1.1 is done as follows.

Proof of Theorem 1.1. If necessary, by changing changing the orientation of Σ_n, assume that the smooth embedding $e : \Sigma_n^0 \to \Sigma_n$ is orientation-preserving. Let $\Sigma_n^{0,0}$ be an α-invariant smooth 4-submanifold of Σ_n^0 by removing the interiors of a 4-ball D_0^4. Let $u : \Sigma_n^{0,0} \to \Sigma_n$ be the smooth embedding defined by e. By Lemma 2.7, the embedding u is smoothly isotopic to a smooth embedding $\tilde{u} : \Sigma_n^{0,0} \to \Sigma_n$ such that the composition $u' = f\tilde{u} : \Sigma_n^{0,0} \to \Sigma_n$ for an α-invariant orientation-preserving diffeomorphism f of Σ_n is smoothly isotopic to the inclusion map

$$\mathrm{inc} : \Sigma_n^{0,0} \to \Sigma_n$$

up to local replacements by diffeomorphisms in $\mathrm{Diff}^+(D^4, \mathrm{rel}\ \partial)$. Since the inclusion map $\mathrm{inc} : \Sigma_n^{0,0} \to \Sigma_n$ extends to the identity map 1 of Σ_n, the isotopy extension theorem says that the smooth embedding u' extends to a diffeomorphism

$$(u')^+ : \Sigma_n \to \Sigma_n.$$

Then the composite diffeomorphism $f^{-1}(u')^+ : \Sigma_n \to \Sigma_n$ is an extension of the embedding $\tilde{u} : \Sigma_n^{0,0} \to \Sigma_n$. By the isotopy extension theorem, the smooth embedding u extends to a diffeomorphism $u^+ :: \Sigma_n \to \Sigma_n$. Since the closed complement $\mathrm{cl}(\Sigma_n \setminus \Sigma_n^{0,0})$ is the disjoint union of 4-balls D_0^4 and $\alpha(D_0^4)$, we have the identities

$$\mathrm{cl}(\Sigma_n \setminus u(\Sigma_n^{0,0})) = u^+\mathrm{cl}(\Sigma_n \setminus \Sigma_n^{0,0}) = u^+(D_0^4) \cup u^+\alpha(D_0^4).$$

Thus, the closed complement $\mathrm{cl}(\Sigma_n \setminus e(\Sigma_n^0))$ is a 4-ball. By $\Gamma_4 = 0$ in [2], we see that the embedding e extends to a diffeomorphism $e^+ : \Sigma_n \to \Sigma_n$. $\square$,

The property that the diffeomorphisms f is α-equivariant in Lemma 2.7 is not used in the proof of Theorem 1.1. This property is used in the proof of the following corollary.

Corollary 2.8. Every orientation-preserving diffeomorphism $h : \Sigma_n \to \Sigma_n$ is smoothly isotopic to an α-equivariant diffeomorphism up to a local replacement by a diffeomorphism in $\mathrm{Diff}^+(D^4, \mathrm{rel}\ \partial)$.

In the piecewise-linear category, this local replacement is not needed.

Proof of Corollary 2.8. Apply the same argument as the proof of Theorem 1.1 for $h : \Sigma_n \to \Sigma_n$ in place of $u : \Sigma_n^{0,0} \to \Sigma_n$. Then every orientation-preserving diffeomorphism $h : \Sigma_n \to \Sigma_n$ is smoothly isotopic to a diffeomorphism $\tilde{h} : \Sigma_n \to \Sigma_n$ such that the composition $h' = f\tilde{h} : \Sigma_n \to \Sigma_n$ for an α-equivariant orientation-preserving diffeomorphism $f : \Sigma_n \to \Sigma_n$ is smoothly isotopic to the identity $1 : \Sigma_n \to \Sigma_n$ after local replacements by diffeomorphisms in $\mathrm{Diff}^+(D^4, \mathrm{rel}\partial)$ needed to apply Corollary 2.5. Then the diffeomorphism h is smoothly isotopic to the α-equivariant orientation-preserving diffeomorphism f^{-1} since the composition $\tilde{h} = f^{-1}h'$ is smoothly isotopic to f^{-1} and h, up to local replacements by diffeomorphisms in $\mathrm{Diff}^+(D^4, \mathrm{rel}\partial)$. □

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References

[1] A. Akhmedov and B. D. Park, Geography of simply connected spin symplectic 4-manifolds, Math. Res. Lett., 17 (2010), 483-492.

[2] J. Cerf, Sur les difféomorphismes de la sph e re de dimension trois ($\Gamma_4 = 0$), Springer Lecture Notes in Math. 53 (1968).

[3] M. Freedman, The disk theorem for four-dimensional manifolds, Proc. Internat. Congr. Math. (Warsaw, Poland)(1983), 647-663.

[4] M. H. Freedman and F. Quinn, Topology of 4-manifolds, Princeton Univ. Press (1990).

[5] F. Hosokawa and A. Kawauchi, Proposals for unknotted surfaces in four-space, Osaka J. Math. 16(1979), 233-248.

[6] J. F. P. Hudson, Piecewise-linear topology, Benjamin (1969).

[7] A. Kawauchi, Ribbonness of a stable-ribbon surface-link, I. A stably trivial surface-link, revised preprint.

http://www.sci.osaka-cu.ac.jp/ kawauchi/SRibbonSLinkI(STrivialCase).pdf

[8] C. Rourke and B. Sanderson, Introduction to piecewise-linear topology, Ergebnesse der Math., 69 (1972), Springer-Verlag.

[9] C. T. C. Wall, On simply-connected 4-manifolds, J. London Math., Soc. 39 (1964), 141-149.