Representation theory of the general linear groups after Riche and Williamson *

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This is a set of notes for my lecture 代数学特論 IV delivered in the second semester of 2018-19 school year. The lecture was meant to give an introduction/survey of the first 2 parts of a recent monumental work by Riche and Williamson [RW]. Appendix A is a class note for 数学概論 II on July 17, 2018, and Appendix B is a set of notes for my lectures at 東大 during the final week of May 2019.

I am grateful to 山中仁, who attended the entire course at OCU and read these notes, for pointing out numerous inaccuracies, and to OCAMI for the inclusion of the notes in its preprint series. Thanks are also due to the audience at 東大 for their close attention.

We will consider the representation theory of $\mathrm{GL}_n(\Bbbk)$ over an algebraically closed field $\Bbbk$ of positive characteristic p.

1° Preliminaries

(1.1) Set $G = \mathrm{GL}_n(\Bbbk)$. We will consider only algebraic representations of G, that is, group homomorphisms $\phi : G \to \mathrm{GL}(M)$ with M a finite dimensional $\Bbbk$-linear space such that, choosing a basis of M and identifying $\mathrm{GL}(M)$ with $\mathrm{GL}_r(\Bbbk)$, $r = \dim M$, the functions $y_{\nu\mu} \circ \phi$ on G, $\nu, \mu \in [1, r]$, all belong to $\Bbbk[x_{ij}, \det^{-1} | i, j \in [1, n]]$, where $y_{\nu\mu}(g') = g'_{\nu\mu}$ is the (ν, μ)-th element of $g' \in \mathrm{GL}_r(\Bbbk)$ and $x_{ij}(g) = g_{ij}$ is the (i, j)-th element of $g \in \mathrm{GL}_n(\Bbbk)$ [J, I.2.7, 2.9]. Given a representation ϕ we also say that M affords a G-module, and write gm for $\phi(g)m$, $g \in G$,$m \in M$. Set $\Bbbk[G] = \Bbbk[x_{ij}, \det^{-1} | i, j \in [1, n]]$.

A basic problem of the representation theory of G is the determination of simple representations. A nonzero G-module M is called simple/irreducible iff M admits no proper subspace M' such that $gm \in M'$ $\forall g \in G$ $\forall m \in M'$.

(1.2) Let B denote a Borel subgroup of G consisting of the lower triangular matrices and T a maximal torus of B consisting of the diagonals. Let $\Lambda = \mathbf{Grp}_{\Bbbk}(T, \mathrm{GL}_1(\Bbbk))$, called the character group of T. Recall that Λ is a free abelian group of basis $\varepsilon_1, \dots, \varepsilon_n$ such that $\varepsilon_i : \mathrm{diag}(a_1, \dots, a_n) \mapsto a_i$. We write the group operation on Λ additively; for $m_1, \dots, m_n \in \mathbb{Z}$,

*supported in part by JSPS Grants in Aid for Scientific Research 15K04789

$\sum_{i=1}^{n} m_i\varepsilon_i : \operatorname{diag}(a_1,\dots,a_n) \mapsto a_1^{m_1}\dots a_n^{m_n}$. Let $R = \{\varepsilon_i - \varepsilon_j | i,j \in [1,n], i \neq j\}$ be the set of roots, and put $R^+ = \{\varepsilon_i - \varepsilon_j | i,j \in [1,n], i < j\}$, the set of positive roots such that the roots of B are $-R^+$: $B = T \ltimes U$ with $U = \prod_{\alpha\in R^+} U_{-\alpha}$, $U_{-\alpha} = \{x_{-\alpha}(a) | a \in \Bbbk\}$ such that if $-\alpha = \varepsilon_i - \varepsilon_j$, $\forall \nu, \mu \in [1,n]$,

$$x_{-\alpha}(a)_{\nu\mu} = \begin{cases} 1 & \text{if } \nu = \mu, \\ a & \text{if } \nu = i \text{ and } \mu = j, \\ 0 & \text{else.} \end{cases}$$

If $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$, $i \in [1,n[$, $R^{\mathrm{s}} = \{\alpha_1,\dots,\alpha_{n-1}\}$ forms a set of all simple roots of R^+. For $\alpha = \varepsilon_i - \varepsilon_j \in R$ let $\alpha^\vee \in \Lambda^\vee$ denote the coroot of α such that

$$\langle \varepsilon_k, \alpha^\vee \rangle = \begin{cases} 1 & \text{if } k = i, \\ -1 & \text{if } k = j, \\ 0 & \text{else.} \end{cases}$$

Let $\Lambda^+ = \{\lambda \in \Lambda | \langle \lambda, \alpha^\vee \rangle \geq 0 \, \forall \alpha \in R^+\}$, called the set of dominant weights of T. We introduce a partial order on Λ such that $\lambda \geq \mu$ iff $\lambda - \mu \in \sum_{\alpha\in R^+} \mathbb{N}\alpha$.

(1.3) Any T-module M is simultaneously diagonalizable:

$$M = \coprod_{\lambda\in\Lambda} M_\lambda \quad \text{with} \quad M_\lambda = \{m \in M | tm = \lambda(t)m \, \forall t \in T\}.$$

We call M_λ the λ-weight space of M, λ a weight of M iff $M_\lambda \neq 0$, and the coproduct the weight space decomposition of M. Let $\mathbb{Z}[\Lambda]$ be the group ring of Λ with a basis e^λ, $\lambda \in \Lambda$. We call

$$\operatorname{ch} M = \sum_{\lambda\in\Lambda} (\dim M_\lambda) e^\lambda \in \mathbb{Z}[\Lambda]$$

the (formal) character of M; if M is a G-module, for $g \in G$ write $g = g_u g_s$ is the Jordan-Chevalley decomposition of $g \in G$. Then the trace $\operatorname{Tr}(g)$ on M is given by

$$\begin{aligned} \operatorname{Tr}(g) &= \operatorname{Tr}(g_u g_s) = \operatorname{Tr}(g_s) \\ &= \operatorname{Tr}(t) \quad \text{if } g_s \text{ is conjugate to some } t \in T \\ &= \sum_\lambda \lambda(t) \dim M_\lambda, \end{aligned}$$

which does not make much sense in positive characterstic.

(1.4) Assume for the moment that $\Bbbk$ is of characteristic 0. Here the representation theory of G is well-understood. Any G-module is semisimple, i.e., a direct sum of simple G-modules [J, II.5.6.6]. For $\lambda \in \Lambda$ regard λ as a 1-dimensional B-module via the projection $B = T \ltimes U \to T$, and let $\nabla(\lambda) = \{f \in \Bbbk[G] | f(gb) = \lambda(b)^{-1} f(g) \, \forall g \in G \, \forall b \in B\}$ with G-action defined by $g \cdot f = f(g^{-1}?)$. The Borel-Weil theorem asserts that $\nabla(\lambda) \neq 0$ iff $\lambda \in \Lambda^+$ [J, II.2.6]. Any simple G-module is isomorphic to a unique $\nabla(\lambda)$, $\lambda \in \Lambda^+$, and $\operatorname{ch} \nabla(\lambda)$ is given by Weyl's character formula. To describe the formula, we have to recall the Weyl group $\mathcal{W} = \mathrm{N}_G(T)/T$ of G and its action on Λ: $\forall w \in \mathcal{W}$, $\forall \mu \in \Lambda$, we define $w\mu \in \Lambda$ by setting $(w\mu)(t) = \mu(w^{-1}tw)$ $\forall t \in T$. More concretely, identify Λ with $\mathbb{Z}^{\oplus n}$ via $\sum_{i=1}^n \mu_i\varepsilon_i \mapsto (\mu_1,\dots,\mu_n)$. Then $\mathcal{W} \simeq \mathfrak{S}_n$ such that $w\varepsilon_i = \varepsilon_{wi}$, i.e., $w\mu = (\mu_{w^{-1}1},\dots,\mu_{w^{-1}n})$. Let also $\zeta = (0,-1,\dots,-n+1) \in \Lambda$, and set $w \bullet \lambda = w(\lambda+\zeta)-\zeta$; we replace the usual choice of $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$, which may not live in Λ, e.g., in the case of $\mathrm{GL}_2(\Bbbk)$, by ζ. Then [J, II.5.10] for $\lambda \in \Lambda^+$

$$\operatorname{ch}\nabla(\lambda) = \frac{\sum_{w\in\mathcal{W}}\det(w)e^{w(\lambda+\zeta)}}{\sum_{w\in\mathcal{W}}\det(w)e^{w\zeta}} = \frac{\sum_{w\in\mathcal{W}}\det(w)e^{w\bullet\lambda}}{\sum_{w\in\mathcal{W}}\det(w)e^{w\bullet 0}}.$$

In particular, $\nabla(\lambda)$ has highest weight λ of multiplicity 1: any weight of $\nabla(\lambda)$ is $\leq \lambda$, and $\dim\nabla(\lambda)_\lambda = 1$.

(1.5) Back to our original setting, each $\nabla(\lambda)$ in (1.4) is defined over $\mathbb{Z}$ and gives us a standard module, denoted by the same letter, having the same character [J, II.8.8]; this is a highly nontrivial result requiring the universal coefficient theorem [J, I.4.18] on induction and Kempf's vanishing theorem [J, II.4] among other things. In particular, the ambient space V of our G is $\nabla(\varepsilon_1)$; if $v_1,\dots,v_n$ is the standard basis of V, each v_i is of weight ε_i. More generally, let $\mathrm{S}(V) = \Bbbk[v_1,\dots,v_n]$ denote the symmetric algebra of V, and $\mathrm{S}^m(V)$ its homogeneous part of degree m. Then $\mathrm{S}^m(V) \simeq \nabla(m\varepsilon_1)$ [J, II.2.16]. Note, however, that $\mathrm{S}^p(V)$ has a proper G-submodule $\sum_{i=1}^n \Bbbk v_i^p$, and hence $\nabla(\lambda)$ is no longer simple in general; for information on when $\nabla(\lambda)$ remains simple see [J, II.6.24, 8.11]. Nonetheless, each $\nabla(\lambda)$ has a unique simple submodule, which we denote by $L(\lambda)$ [J, II.2.3]. It has highest weight λ, and any simple G-module is isomorphic to a unique $L(\mu)$, $\mu \in \Lambda^+$ [J, II.2.4]. Thus, our basic problem is to find all $\operatorname{ch} L(\mu)$.

For that, as any composition factor of $\nabla(\lambda)$ is of the form $L(\mu)$, $\mu \leq \lambda$, with $L(\lambda)$ appearing just once, the finite matrix $[\![[\nabla(\nu) : L(\mu)]]\!]$ of the composition factor multiplicities for $\nu,\mu \leq \lambda$ is unipotent, from which $\operatorname{ch} L(\lambda)$ can be obtained as a $\mathbb{Z}$-linear combinations of $\operatorname{ch}\nabla(\nu)$'s.

(1.6) To find the irreducible characters, some reductions are in order. First, let $\Lambda_1 = \{\lambda \in \Lambda^+ | \langle\lambda,\alpha^\vee\rangle < p\,\forall\alpha \in R^{\mathrm{s}}\}$. If $\varpi_i := \varepsilon_1+\cdots+\varepsilon_i$, $i \in [1,n]$, $\Lambda = \coprod_{i=1}^n \mathbb{Z}\varpi_i$, $\varpi_n = \det$, and $\Lambda^+ = \mathbb{Z}\det + \sum_{i=1}^{n-1}\mathbb{N}\varpi_i$. Thus, $\Lambda_1 = \mathbb{Z}\det + \{\sum_{i=1}^{n-1} a_i\varpi_i | a_i \in [0,p[\}$. One can write any $\lambda \in \Lambda^+$ in the form $\lambda = \sum_{i=0}^r p^i\lambda^i$, $\lambda^i \in \Lambda_1$. Then

Steinberg's tensor product theorem [J, II.3.17]:

$$L(\lambda) \simeq L(\lambda^0)\otimes L(\lambda^1)^{[1]}\otimes\cdots\otimes L(\lambda^r)^{[r]},$$

where $L(\lambda^k)^{[k]}$ is $L(\lambda^k)$ with G acting through the k-th Frobenius $F^k : G \to G$ via $[\![g_{ij}]\!] \mapsto [\![g_{ij}^{p^k}]\!]$.

Thus, if $\operatorname{ch} L(\lambda^k) = \sum_\mu m_\mu e^\mu$, $\operatorname{ch} L(\lambda^k)^{[k]} = \sum_\mu m_\mu e^{p^k\mu}$, and our problem is reduced to finding $\operatorname{ch} L(\lambda)$ for $\lambda \in \Lambda_1$ or $\operatorname{ch} L(\sum_{i=1}^{n-1}\lambda_i\varpi_i)$ for $\lambda_i \in [0,p[$; $\forall m \in \mathbb{Z}$, $\nabla(m\det + \sum_{i=1}^{n-1}\lambda_i\varpi_i) \simeq \det^{\otimes m}\otimes\nabla(\sum_{i=1}^{n-1}\lambda_i\varpi_i)$ by the tensor identity [J, I.3.6], and hence also $L(m\det+\sum_{i=1}^{n-1}\lambda_i\varpi_i) \simeq \det^{\otimes m}\otimes L(\sum_{i=1}^{n-1}\lambda_i\varpi_i)$.

(1.7) There is a direct way to compute $\operatorname{ch} L(\lambda)$, $\lambda \in \Lambda_1$, due to Burgoyne, which goes as follows [HMR, 4.2]: let $\mu \in \Lambda$ with $L(\lambda)_\mu \neq 0$. Thus $\lambda - \mu \in \sum_{\alpha\in R^+}\mathbb{N}\alpha$. Let $\mathrm{Dist}(G_1)$ (resp. $\mathrm{Dist}(U_1)$, $\mathrm{Dist}(B_1^+)$) be the algebra of distributions of the Frobenius kernel G_1 of G (resp. U, B^+ the Borel subgroup of G opposite to B consisting of the upper triangular matrices) [J, I.7-9]. Then $\mathrm{Dist}(G_1)$ admits a $\Bbbk$-linear triangular decomposition $\mathrm{Dist}(G_1) \simeq \mathrm{Dist}(U_1)\otimes\mathrm{Dist}(B_1^+)$. Regarding λ as a B^+-module by the projection $B^+ = U^+ \rtimes T \to T$ with $U^+ = \prod_{\alpha\in R^+} U_\alpha$, put

$\hat{\Delta}(\lambda) = \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(B_1^+)} \lambda$. It comes equipped with a structure of G_1T-module [J, II.3.6], called the G_1T-Verma module of highest weight λ, and $L(\lambda)$ is the head of $\hat{\Delta}(\lambda)$: $L(\lambda) \simeq \hat{\Delta}(\lambda)/\mathrm{rad}\hat{\Delta}(\lambda)$ [J, II.3.15]. For $\alpha \in R$ let $z_\alpha = x_\alpha(1) - \mathrm{id} = (\mathrm{d}x_\alpha)(1) \in \mathrm{M}_n(\Bbbk) = \mathfrak{g} = \mathrm{Lie}(G)$. Fix an order of the positive roots $\beta_1, \dots, \beta_N$, $N = |R^+|$. For $\mathbf{m} = (m_1, \dots, m_N) \in [0, p[^N$ put $Y_{\mathbf{m}} = z_{-\beta_1}^{m_1} \dots z_{-\beta_N}^{m_N}$ and $X_{\mathbf{m}} = z_{\beta_1}^{m_1} \dots z_{\beta_N}^{m_N}$. Then $Y_{\mathbf{m}}$ (resp. $X_{\mathbf{m}}$), $\mathbf{m} \in [0, p[^N$, forms a $\Bbbk$-linear basis of $\mathrm{Dist}(U_1)$ (resp. $\mathrm{Dist}(U_1^+)$, U_1^+ denoting the Frobenius kernel of U^+). Put $v^+ = 1 \otimes 1 \in \hat{\Delta}(\lambda)$ and $\mathcal{P} = \{\mathbf{m} = (m_1, \dots, m_N) \in [0, p[^N | \sum_{i=1}^N m_i\beta_i = \lambda - \mu\}$. Then $\hat{\Delta}(\lambda)$ admits a basis $Y_{\mathbf{m}}v^+$ of weight $\lambda - \sum_{i=1}^N m_i\beta_i$, $\mathbf{m} \in [0, p[^N$, and hence $Y_{\mathbf{m}}v^+$, $\mathbf{m} \in \mathcal{P}$, forms a $\Bbbk$-linear basis of $\hat{\Delta}(\lambda)_\mu$. Now let $c(\mathbf{m}, \mathbf{m}') \in \Bbbk$ such that $X_{\mathbf{m}'}Y_{\mathbf{m}}v^+ = c(\mathbf{m}, \mathbf{m}')v^+$ for $\mathbf{m}, \mathbf{m}' \in \mathcal{P}$, which one can compute using the commutator relations among the z_β's; $X_{\mathbf{m}'}Y_{\mathbf{m}} \in \mathrm{Dist}(U_1)\mathrm{Dist}(B_1^+)$ and $\mathrm{Dist}(B_1^+)v^+ \in \Bbbk$. In fact, as the structure constants of the commutation lie in $\mathbb{F}_p$, $c(\mathbf{m}, \mathbf{m}') \in \mathbb{F}_p$. Define a $\Bbbk$-linear map $\phi : \hat{\Delta}(\lambda)_\mu \to \Bbbk^{|\mathcal{P}|}$ via $Y_{\mathbf{m}}v^+ \mapsto (c(\mathbf{m}, \mathbf{m}')|\mathbf{m}' \in \mathcal{P})$. As v^+ is a $\mathrm{Dist}(G_1)$-generator of $\hat{\Delta}(\lambda)$,

$$\ker \phi = \{v \in \hat{\Delta}(\lambda)_\mu | (\mathrm{Dist}(U_1^+)v) \cap \Bbbk v^+ = 0\} = \hat{\Delta}(\lambda)_\mu \cap \mathrm{rad}\hat{\Delta}(\lambda),$$

and hence

$$\begin{aligned} \mathrm{im}\phi &\simeq \hat{\Delta}(\lambda)_\mu / \ker \phi = \hat{\Delta}(\lambda)_\mu / \{\hat{\Delta}(\lambda)_\mu \cap \mathrm{rad}\hat{\Delta}(\lambda)\} = \hat{\Delta}(\lambda)_\mu / \mathrm{rad}\hat{\Delta}(\lambda)_\mu \\ &\simeq \{\hat{\Delta}(\lambda)/\mathrm{rad}\hat{\Delta}(\lambda)\}_\mu \simeq L(\lambda)_\mu. \end{aligned}$$

It follows that

$$\dim L(\lambda)_\mu = \mathrm{rk}\, [\![c(\mathbf{m}, \mathbf{m}')]\!]_{\mathcal{P}}.$$

But we want a more systematic description of $\mathrm{ch}\, L(\lambda)$.

(1.8) Just to show how much information V carries, put $\Lambda^{++} = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n | \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\} \subset \Lambda^+$. If $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+ \setminus \Lambda^{++}$, $\lambda - \lambda_n \det \in \Lambda^{++}$. For $\lambda \in \Lambda^{++}$ put $|\lambda| = \lambda_1 + \dots + \lambda_n$. Then

$$(V^{\otimes |\lambda|} : \nabla(\lambda)) \neq 0.$$

To see that, we argue by induction on $|\lambda|$. If $|\lambda| = 1$, $\lambda = \varepsilon_1$, and $\nabla(\varpi_1) = V$. If $|\lambda| > 1$, $\lambda = \mu + \varepsilon_i$ for some $\mu \in \Lambda^{++}$ and $i \in [1, n]$. As $(V^{\otimes |\mu|} : \nabla(\mu)) \neq 0$ by induction, it is enough to show that $(V \otimes \nabla(\mu) : \nabla(\lambda)) \neq 0$. One has $V \otimes \nabla(\mu) \simeq \nabla(\mu \otimes V)$ by the tensor identity [J, I.3.6]; ∇ stands really for the induction functor $\mathrm{ind}_B^G : \mathrm{Rep}(B) \to \mathrm{Rep}(G)$ from the category $\mathrm{Rep}(B)$ of B-modules to $\mathrm{Rep}(G)$ defined by $\nabla(M) = \{f : G \to M | f(gb) = b^{-1}f(g)\ \forall g \in G\ \forall b \in B\} = (M \otimes \Bbbk[G])^B$, $M \in \mathrm{Rep}(B)$. Now $\mu \otimes V$ admits a filtration of B-modules of subquotients $\mu + \varepsilon_j$, $j \in [1, n]$, with all $\mu + \varepsilon_j \in \zeta + \Lambda^+ + \mathbb{Z}\det$; $\forall k \in [1, n[$, $\langle \varepsilon_j, \alpha_k^\vee \rangle \geq -1$. It follows from Bott's theorem [J, II.5.4] that $\mathrm{R}^1\mathrm{ind}_B^G(\mu + \varepsilon_j) = 0$, and hence $V \otimes \nabla(\mu)$ admits a G-filtration with the subquotients $\nabla(\mu + \varepsilon_j)$, $j \in [1, n]$, such that $\mu + \varepsilon_j \in \Lambda^+$. In fact, $(V^{\otimes |\lambda|} : \nabla(\lambda))$ is explicitly known [岡田, Th. 7.6, p. 38]/[J, A.23].

Let us recall also that

Theorem [J, II.4.21+6.20]: $\forall \lambda, \mu \in \Lambda^+$, $\nabla(\lambda) \otimes \nabla(\mu)$ *admits a filtration of G-modules $M^0 = \nabla(\lambda) \otimes \nabla(\mu) > M^1 > \dots > M^r > 0$ such that each M^i/M^{i+1} is isomorphic to some $\nabla(\nu_i)$, $\nu_i \in \Lambda^+$, $i \in [0, r]$, and that $\nu_i \not< \nu_{i+1}$ $\forall i$. In particular, $\nabla(\lambda + \mu)$ appears at the top of such a filtration.*

(1.9) Let $\mathcal{W}_a = \mathcal{W} \ltimes \mathbb{Z}R$, called the affine Weyl group of $\mathcal{W}$, acting on Λ with $\mathbb{Z}R$ by translation. For $\alpha \in R$ let $s_\alpha \in \mathcal{W}$ such that $s_\alpha : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, $\lambda \in \Lambda$, and $s_{\alpha_0,1} : \lambda \mapsto \lambda - \langle \lambda, \alpha_0^\vee \rangle \alpha_0 + \alpha_0$ with $\alpha_0 = \alpha_1 + \cdots + \alpha_{n-1} = \varepsilon_1 - \varepsilon_n$. Under the identification $\mathcal{W} \simeq \mathfrak{S}_n$ one has $s_{\alpha_i} \mapsto (i, i+1)$, $i \in [1, n[$. If $\mathcal{S} = \{s_\alpha | \alpha \in R^s\}$ and $\mathcal{S}_a = \mathcal{S} \cup \{s_{\alpha_0,1}\}$, $(\mathcal{W}_a, \mathcal{S}_a)$ forms a Coxeter system with a subsystem $(\mathcal{W}, \mathcal{S})$ [J, II.6.3]. Let $\ell : \mathcal{W}_a \to \mathbb{N}$ denote the length function on $\mathcal{W}_a$ with respect to $\mathcal{S}_a$, and let $\leq$ denote the Chevalley-Bruhat order on $\mathcal{W}_a$.

We let also $\mathcal{W}_a$ act on Λ by setting

$$x \bullet \lambda = px(\frac{1}{p}(\lambda + \zeta)) - \zeta \quad \forall \lambda \in \Lambda\ \forall x \in \mathcal{W}_a.$$

Let $\mathrm{Rep}(G)$ denote the category of finite dimensional representations of G. By $\mathrm{Ext}^1_G(M, M')$ we will mean the 米田-extension of M by M' in $\mathrm{Rep}(G)$ [Weib, pp. 79-80], [dJ, 27]; $\mathrm{Rep}(G)$ admits no nonzero injectives nor projectives.

The linkage principle [J, II.6.17]: $\forall \lambda, \mu \in \Lambda^+$,

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \neq 0 \Rightarrow \lambda \in \mathcal{W}_a \bullet \mu.$$

In particular, if $L(\lambda)$ is a composition factor of $\nabla(\mu)$, $\lambda \in \mathcal{W}_a \bullet \mu$. By the linkage principle one has a decomposition

$$\mathrm{Rep}(G) = \coprod_{\Omega \in \Lambda / \mathcal{W}_a \bullet} \mathrm{Rep}_\Omega(G),$$

where $\mathrm{Rep}_\Omega(G)$ consists of G-modules whose composition factors are all of the form $L(\lambda)$, $\lambda \in \Omega \cap \Lambda^+$. For $\Omega \ni 0$ we abbreviate $\mathrm{Rep}_\Omega(G)$ as $\mathrm{Rep}_0(G)$ and call it the principal block of G.

(1.10) We extend the $\mathcal{W}_a\bullet$-action on Λ to one on $\Lambda_\mathbb{R} := \Lambda \otimes_\mathbb{Z} \mathbb{R}$. For each $\alpha \in R^+$ and $m \in \mathbb{Z}$ let $H_{\alpha,m} = \{x \in \Lambda_\mathbb{R} | \langle x + \zeta, \alpha^\vee \rangle = mp\}$. We call a connected component of $\Lambda_\mathbb{R} \setminus \cup_{\alpha \in R^+, m \in \mathbb{Z}} H_{\alpha,m}$ an alcove of $\Lambda_\mathbb{R}$. Thus, $\mathcal{W}_a$ acts on the set of alcoves $\mathcal{A}$ in $\Lambda_\mathbb{R}$ simply transitively [J, II.6.2.4]. We call $A^+ = \{x \in \Lambda_\mathbb{R} | \langle x + \zeta, \alpha^\vee \rangle > 0\ \forall \alpha \in R^+, \langle x + \zeta, \alpha_0^\vee \rangle < p\}$ the bottom dominant alcove of $\mathcal{A}$. Thus the action induces a bijection $\mathcal{W}_a \to \mathcal{A}$ via $w \mapsto w \bullet A^+$. The closure $\overline{A^+}$ is a fundamental domain for $\mathcal{W}_a$ on $\Lambda_\mathbb{R}$ [J, II.6.2.4], i.e., $\forall x \in \Lambda_\mathbb{R}$, $(\mathcal{W}_a \bullet x) \cap \overline{A^+}$ is a singleton. For $A = \{x \in \Lambda_\mathbb{R} | p(m_\alpha - 1) < \langle x + \zeta, \alpha^\vee \rangle < pm_\alpha\ \forall \alpha \in R^+\} \in \mathcal{A}$, $m_\alpha \in \mathbb{Z}$, a facet of A is some $\{x \in \overline{A} |\ p | \langle x + \zeta, \alpha^\vee \rangle\ \forall \alpha \in R_0\}$, $R_0 \subseteq R^+$, and a wall of A is a facet with $|R_0| = 1$. Also, we call $\hat{A} = \{x \in \Lambda_\mathbb{R} | p(m_\alpha - 1) < \langle x + \zeta, \alpha^\vee \rangle \leq pm_\alpha\ \forall \alpha \in R^+\}$ the upper closure of A. One has [J, II.6.2.8]

$$\Lambda \cap A \neq \emptyset\ \exists A \in \mathcal{A} \quad \text{iff} \quad 0 \in A^+ \quad \text{iff} \quad p \geq n,$$

in which case each wall of an alcove contains an element of Λ [J, II.6.3]. Assume from now on throughout the rest of §1 that $p \geq n$.

For $\nu \in \Lambda$ let $\mathrm{pr}_\nu = \mathrm{pr}_{\mathcal{W}_a \bullet \nu} : \mathrm{Rep}(G) \to \mathrm{Rep}(G)$ denote the projection onto $\mathrm{Rep}_{\mathcal{W}_a \bullet \nu}(G)$. Now let $\lambda, \mu \in \Lambda \cap \overline{A^+}$. We choose a finite dimensional G-module $V(\lambda, \mu)$ of highest weight $\nu \in \Lambda^+ \cap \mathcal{W}(\mu - \lambda)$ such that $\dim V(\lambda, \mu)_\nu = 1$, e.g., $V(\lambda, \mu) = \nabla(\nu), L(\nu)$. Define the translation functor $T_\lambda^\mu : \mathrm{Rep}(G) \to \mathrm{Rep}(G)$ by setting $T_\lambda^\mu M = \mathrm{pr}_\mu(V(\lambda, \mu) \otimes \mathrm{pr}_\lambda M)\ \forall M \in \mathrm{Rep}(G)$. A different choice of $V(\lambda, \mu)$ yields an isomorphic functor [J, II.7.6 Rmk. 1]. Each T_λ^μ is exact.

As T_μ^λ may be defined with $V(\lambda,\mu)$ replaced by $V(\lambda,\mu)^*$, T_λ^μ and T_μ^λ are adjoint to each other [J, II.7.6]: $\forall M, M' \in \mathrm{Rep}(G)$,

$$\mathrm{Rep}(G)(T_\lambda^\mu M, M') \simeq \mathrm{Rep}(G)(M, T_\mu^\lambda M'). \tag{1}$$

The translation principle: *Let* $\lambda, \mu \in \Lambda \cap \overline{A^+}$.

(i) If λ and μ belong to the same facet, T_λ^μ and T_μ^λ induce a quasi-inverse to each other between $\mathrm{Rep}_{\mathcal{W}_a\bullet\lambda}(G)$ *and* $\mathrm{Rep}_{\mathcal{W}_a\bullet\mu}(G)$ *[J, II.7.9].*

(ii) If λ belongs to a facet F and if $\mu \in \overline{F}$, $\forall x \in \mathcal{W}_a$, $T_\lambda^\mu \nabla(x\bullet\lambda) \simeq \nabla(x\bullet\mu)$ [J, II.7.11].

(iii) If $\lambda \in A^+$ and if $\mu \in \overline{A^+}$ with $\mathrm{C}_{\mathcal{W}_a\bullet}(\mu) = \{1, s\}$ for some $s \in \mathcal{S}_a$, then $\forall x \in \mathcal{W}_a$ with $x\bullet\lambda \in \Lambda^+$ and $xs\bullet\lambda > x\bullet\lambda$, there is an exact sequence [J, II.7.12]

$$0 \to \nabla(x\bullet\lambda) \to T_\mu^\lambda \nabla(x\bullet\mu) \to \nabla(xs\bullet\lambda) \to 0.$$

We note also that the morphisms $\nabla(x\bullet\lambda) \to T_\mu^\lambda\nabla(x\bullet\mu)$ and $T_\mu^\lambda\nabla(x\bullet\mu) \to \nabla(xs\bullet\lambda)$ are unique up to $\Bbbk^\times$;

$$\mathrm{Rep}(G)(\nabla(x\bullet\lambda), T_\mu^\lambda\nabla(x\bullet\mu)) \simeq \mathrm{Rep}(G)(T_\lambda^\mu\nabla(x\bullet\lambda), \nabla(x\bullet\mu)) \simeq \mathrm{Rep}(G)(\nabla(x\bullet\mu), \nabla(x\bullet\mu)) \simeq \Bbbk.$$

(iv) If $\lambda \in A^+$ and if $\mu \in \overline{A^+}$, then $\forall x \in \mathcal{W}_a$ with $x\bullet\lambda \in \Lambda^+$ [J, II.7.13, 7.15],

$$T_\lambda^\mu L(x\bullet\lambda) \simeq \begin{cases} L(x\bullet\mu) & \text{if } x\bullet\mu \in \widehat{x\bullet A^+}, \\ 0 & \text{else.} \end{cases}$$

(1.11) For $M, L \in \mathrm{Rep}(G)$ with L simple let $[M : L]$ denote the multiplicity of L in a composition series of M. Recall that each $\nabla(\lambda)$, $\lambda \in \Lambda^+$, is of highest weight λ of multiplicity 1, and has the simple socle $L(\lambda)$. It follows from the linkage principle that

$$\mathrm{ch}\, L(\lambda) \in \sum_{\substack{\mu\in\mathcal{W}_a\bullet\lambda \\ \mu\leq\lambda}} \mathbb{Z}\,\mathrm{ch}\nabla(\mu).$$

Moreover, to find all $\mathrm{ch}\, L(\lambda)$, one may assume $\lambda \in \mathcal{W}_a\bullet 0$ by the translation principle. In 1978 Lusztig proposed a formula for $\mathrm{ch}\, L(x\bullet 0)$ with $x\bullet 0 \in \Lambda^+$ and such that $\langle x\bullet 0 + \zeta, \alpha_0^\vee\rangle < p(p-n+2)$. If $p \geq 2n-3$, all $x\bullet 0 \in \Lambda_1$ satisfy the condition, and hence all the irreducible characters should be obtained from the conjectured formula by Steinberg's tensor product theorem. To explain the conjecture, let $\mathcal{H}$ be the 岩堀-Hecke algebra of $(\mathcal{W}_a, \mathcal{S}_a)$ over the Laurent polynomial ring $\mathbb{Z}[v, v^{-1}]$. This is a free $\mathbb{Z}[v, v^{-1}]$-module of basis H_x, $x \in \mathcal{W}_a$, subject to the relations $H_e = 1$, e denoting the unity of $\mathcal{W}_a$, $H_xH_y = H_{xy}$ if $\ell(x) + \ell(y) = \ell(xy)$, and $H_s^2 = 1 + (v^{-1} - v)H_s$ $\forall s \in \mathcal{S}_a$ [S97]. For this and other reasons we will often denote the unity e of $\mathcal{W}_a$ by 1. Under the specialization $v \rightsquigarrow 1$ one has an isomorphism of rings $\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H} \simeq \mathbb{Z}[\mathcal{W}_a]$. Thus, $\mathcal{H}$ is a quantization of $\mathbb{Z}[\mathcal{W}_a]$.

As $(H_s)^{-1} = H_s + (v - v^{-1})$ $\forall s \in \mathcal{S}_a$, every H_x is a unit of $\mathcal{H}$. There is a unique ring endomorphism $\overline{?}$ of $\mathcal{H}$ such that $v \mapsto v^{-1}$ and $H_x \mapsto (H_{x^{-1}})^{-1}$ $\forall x \in \mathcal{W}_a$. Then $\forall x \in \mathcal{W}_a$, there is unique $\underline{H}_x \in \mathcal{H}$ with $\overline{\underline{H}_x} = \underline{H}_x$ and such that $\underline{H}_x \in H_x + \sum_{y \in \mathcal{W}_a} v\mathbb{Z}[v]H_y$, in which case $\underline{H}_x \in H_x + \sum_{y<x} v\mathbb{Z}[v]H_y$ [S97, Th. 2.1]. In particular, $\underline{H}_s = H_s + v\ \forall s \in \mathcal{S}_a$. For $x, y \in \mathcal{W}_a$ define $h_{x,y} \in \mathbb{Z}[v]$ by the equality $\underline{H}_x = \sum_{y \in \mathcal{W}_a} h_{y,x}H_y$. The $h_{y,x}$ are the celebrated Kazhdan-Lusztig polynomials of $\mathcal{H}$. Let $w_0 = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$ denote the longest element of $\mathcal{W}$. Then Lusztig's conjecture reads [S97, Prop. 3.7], [F, 2.4], [RW, 1.9] that $\forall x \in \mathcal{W}_a$ with $x \bullet 0 \in \Lambda^+$ and such that $\langle x \bullet 0 + \zeta, \alpha_0^\vee \rangle < p(p-n+2)$,

$$\operatorname{ch} L(x \bullet 0) = \sum_{y \in \mathcal{W}_a} (-1)^{\ell(x)-\ell(y)} h_{w_0 y, w_0 x}(1) \operatorname{ch} \nabla(y \bullet 0), \tag{1}$$

which should hold for any simple algebraic group as long as $p \geq h$ the Coxeter number of the group. Lusztig formulated his conjecture with respect to the Coxeter system $(\mathcal{W}_a, w_0 \mathcal{S}_a w_0)$ [L]/[W17, 1.12]. Let h'_{xy} be the KL-polynomial associated to $x, y \in \mathcal{W}_a$ with respect to $w_0 \mathcal{S}_a w_0$. The original conjecture was for $x \bullet 0 \in \Lambda^+$ as in (1)

$$\begin{aligned} \operatorname{ch} L(x \bullet 0) = \operatorname{ch} L(xw_0 \bullet (w_0 \bullet 0)) &= \sum_{y \in \mathcal{W}_a} (-1)^{\ell(xw_0)-\ell(yw_0)} h'_{yw_0, xw_0}(1) \operatorname{ch} \nabla(yw_0 \bullet (w_0 \bullet 0)) \\ &= \sum_{y \in \mathcal{W}_a} (-1)^{\ell(x)-\ell(y)} h'_{yw_0, xw_0}(1) \operatorname{ch} \nabla(y \bullet 0). \end{aligned} \tag{2}$$

There is a $\mathbb{Z}[v, v^{-1}]$-algebra automorphism of $\mathcal{H}$ via $H_x \mapsto H_{w_0 x w_0}\ \forall x \in \mathcal{W}_a$, which exchanges $\mathcal{S}_a$ with $w_0 \mathcal{S}_a w_0$ and is compatible with $\bar{?}$. Then $\forall x, y \in \mathcal{W}_a$, $h'_{xy} = h_{w_0 x w_0, w_0 y w_0}$, and hence (2) reads

$$\operatorname{ch} L(x \bullet 0) = \sum_{y \in \mathcal{W}_a} (-1)^{\ell(x)-\ell(y)} h_{w_0 y w_0 w_0, w_0 x w_0 w_0}(1) \operatorname{ch} \nabla(y \bullet 0),$$

which is (1).

The bound on $x \bullet 0$ was called Jantzen's condition, introduced as follows [J, II.8.22]: it was expected that the irreducible character should be independent of p for large enough p, dependent only the type of G, i.e., on $\mathcal{W}_a$. Assume thus that for $z \in \mathcal{W}_a$ with $z \bullet 0 \in \Lambda_1$ there are $a_{yz} \in \mathbb{Z}$, $y \in \mathcal{W}_a$ with $y \bullet 0 \leq z \bullet 0$, independent of p such that $\operatorname{ch} L(z \bullet 0) = \sum_y a_{yz} \operatorname{ch} \nabla(y \bullet 0)$. Note that there may be y appearing in the sum with $y \bullet 0 \notin \Lambda_1$ such that $a_{yz} \neq 0$. Let now $x \in \mathcal{W}_a$ with $x \bullet 0 \in \Lambda^+ \setminus \Lambda_1$ and write $x \bullet 0 = 0_x^0 + p 0_x^1$ with $0_x^0 \in \Lambda_1$. If $0_x^1 \in \Lambda_1$, we should have $\operatorname{ch} L(x \bullet 0)$ independent of p by Steinberg's tensor product theorem; $L(x \bullet 0) \simeq L(0_x^0) \otimes L(0_x^1)^{[1]}$. Meanwhile, $\operatorname{ch} L(0_x^0) \otimes \nabla(0_x^1)^{[1]}$ would also be independent of p as $\operatorname{ch} \nabla(0_x^1)$ is given by Weyl's formula. Then whether or not $\operatorname{ch} L(0_x^0) \otimes L(0_x^1)^{[1]} = \operatorname{ch} L(0_x^0) \otimes \nabla(0_x^1)^{[1]}$ should be independent of p. We have, however, $L(0_x^0) < \nabla(0_x^1)$, in general, which is dependent on p. If $0_x^1 \in \overline{A^+}$, $L(0_x^0) = \nabla(0_x^1)$ by the linkage principle, and hence $\operatorname{ch} L(0_x^0) \otimes L(0_x^1)^{[1]} = \operatorname{ch} L(0_x^0) \otimes \nabla(0_x^1)^{[1]}$. Jantzen's condition on x was imposed to assure that $0_x^1 \in \overline{A^+}$. But then for small $p \geq h$ not all $z \bullet 0 \in \Lambda^1$ satisfies Jantzen's condition, e.g., if $p = 3$ for $\mathrm{GL}_3(\Bbbk)$, which raised a question about the initial assumption that all $\operatorname{ch} L(z \bullet 0)$ with $z \bullet 0 \in \Lambda_1$ should be independent of p. If $p \geq 2h - 3$, such a problem dissappears. Subsequently, 加藤 [Kat] showed that if (1) holds for all x with $x \bullet 0 \in \Lambda_1$, then (1) will also hold for all $y \in \mathcal{W}_a$ with $y \bullet 0$ satisfying the Jantzen condition. Based on that he conjectured for $p \geq h$ that (1) should hold for all $x \in \mathcal{W}_a$ with $x \bullet 0 \in \Lambda_1$.

Lusztig's conjecture was then solved for $p \gg 0$ by the combined work of Andersen, Jantzen and Soergel [AJS], Kazhdan and Lusztig [KL], [L94], and 柏原 and 谷崎 [KT]; [AJS] reduced the G_1T-version of the conjecture to one for the quantum algebras at a p-th root of unity for $p \gg 0$, the conjecture for the quantum algebras was related by [KL] and [L] to the one for the affine Lie algebras, where the conjecture was solved in [KT]. In the case of quantum algebras Jantzen's condition is irrelevant as Lusztig's quantum version of Steinberg's tensor product theorem says for any simple module $L_q(x \bullet 0)$ of dominant highest weight $x \bullet 0$, $L_q(x \bullet 0) \simeq L_q(0_x^0) \otimes \nabla(0_x^1)^{[1]}$, where $\nabla(0_x^1)^{[1]}$ is the old simple module $\nabla(0_x^1)$ for the corresponding G over the base field of characteristic 0 twisted by the quantized Frobenius, c.f. [J08], [Ta] for more details. Fiebig [F11] showed the G_1T-version of Lusztig's conjecture for $p \gg 0$ without appealing to [KL], [L], [KT], using the moment graphs on the affine flag varieties. Fiebig [F, Th. 3.5] also shows for $p > h$ that 加藤's conjecture is equivalent to its G_1T-version in terms of periodic Kazhdan-Lusztig polynomials. Then, Williamson [W] has come up with counterexamples to the conjecture; the bound on p for Lusztig's conjecture to hold must be much larger than n. The subsequent sections of the present lecture is then an introduction/survey of Riche's and Williamson's effort to remedy the situation and to give a new irreducible character formula for $p \geq 2(n-1)$.

2° **Overview**

We will assume from now on throughout the rest of the lecture that $p > n$, unless otherwise specified, which comes partly from the requirement to have well-behaved diagrammatic Soergel bimodules.

(2.1) For an abelian category $\mathcal{C}$ let $[\mathcal{C}]$ denote the Grothendieck group of $\mathcal{C}$, which is the free $\mathbb{Z}$-module of basis (M), $M \in \mathrm{Ob}(\mathcal{C})$, modulo a submodule generated by all $(M) + (M') - (M'')$ whenever there is an exact sequence $0 \to M \to M'' \to M' \to 0$ in $\mathcal{C}$. We write $[M]$ for the image of (M) in $[\mathcal{C}]$. If M and M' are isomorphic in $\mathcal{C}$, $[M] = [M']$ in $[\mathcal{C}]$. Thus, $[\mathrm{Rep}_0(G)]$ has a $\mathbb{Z}$-linear basis $[L(x \bullet 0)]$, $x \in (\mathcal{W}_a \bullet 0) \cap \Lambda^+$. As each $\nabla(\lambda)$, $\lambda \in \Lambda^+$, has highest weight λ of multiplicity 1 and has simple socle $L(\lambda)$, the $[\nabla(x \bullet 0)]$, $x \in (\mathcal{W}_a \bullet 0) \cap \Lambda^+$, also form a $\mathbb{Z}$-linear basis of $[\mathrm{Rep}_0(G)]$.

On the other hand, let $\mathbb{Z}[\mathcal{W}_a]$ (resp. $\mathbb{Z}[\mathcal{W}]$) be the group ring of $\mathcal{W}_a$ (resp. $\mathcal{W}$), and let ${}^f\mathcal{W} = \{x \in \mathcal{W}_a | \ell(wx) \geq \ell(x)\ \forall w \in \mathcal{W}\}$. Then, $\mathbb{Z}[\mathcal{W}_a]$ is a free left $\mathbb{Z}[\mathcal{W}]$-module of basis w, $w \in {}^f\mathcal{W}$, and there is a bijection ${}^f\mathcal{W} \to (\mathcal{W}_a \bullet 0) \cap \Lambda^+$ via $w \mapsto w \bullet 0$. Let $\mathrm{sgn}_{\mathbb{Z}} = \mathbb{Z}$ be the sign representation of $\mathcal{W}$, defining a right $\mathbb{Z}[\mathcal{W}]$-module such that $s \mapsto -1\ \forall s \in \mathcal{S}$. There follows an isomorphism of $\mathbb{Z}$-modules

$$(1) \qquad \mathrm{sgn}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a] \to [\mathrm{Rep}_0(G)] \quad \text{via} \quad 1 \otimes w \mapsto [\nabla(w \bullet 0)].$$

We call $\mathrm{sgn}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a]$ the antispherical module of $\mathbb{Z}[\mathcal{W}_a]$. Thus, $\mathrm{Rep}_0(G)$ gives a "categorification" of $\mathrm{sgn}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a]$; by categorification we näively mean that the Grothendieck group of the category $\mathrm{Rep}_0(G)$ recovers the abelian group $\mathrm{sgn}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a]$, c.f. [Maz] for a more sophisticated notion.

For each $s \in \mathcal{S}_a$ choose $\mu \in \Lambda \cap \overline{A^+}$ such that $\mathrm{C}_{\mathcal{W}_a}(\mu) = \{1, s\}$, and let $\mathrm{T}^s = T_0^\mu$ be a translation functor into the s-wall of A^+ and $\mathrm{T}_s = T_\mu^0$ a translation functor out of the s-wall. We call $\Theta_s = \mathrm{T}_s\mathrm{T}^s$ an s-wall crossing functor. If we let $1 + s$, $s \in \mathcal{S}_a$, act on $[\mathrm{Rep}_0(G)]$ by

Θ_s, the isomorphism (1) is made into an isomorphism of right $\mathbb{Z}[\mathcal{W}_a]$-modules by (1.10.iii); if $w \in {}^f\mathcal{W}$ and $ws \notin {}^f\mathcal{W}$, $s \in \mathcal{S}_a$, then there is $t \in S$ such that $ws = tw$ [S97, p. 86]: $\forall s, s' \in \mathcal{S}_a$,

$$(1 \otimes w)(1+s)(1+s') \mapsto [\nabla(w \bullet 0)]\Theta_s\Theta_{s'} = [\Theta_{s'}\Theta_s\nabla(w \bullet 0)].$$

A main theorem of [RW] categorifies the $\mathcal{W}_a$-action on $[\mathrm{Rep}_0(G)]$ by the right action of the diagrammatic Bott-Samelson Hecke category $\mathcal{D}_{\mathrm{BS}}$ of the affine Weyl group $\mathcal{W}_a$ on the principal block $\mathrm{Rep}_0(G)$.

Theorem [RW, Th. 8.1.1]: *For $p > n \geq 3$ there is a strict monoidal functor*

$$\mathcal{D}_{\mathrm{BS}} \to \mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G))^{\mathrm{op}} \quad \textit{such that} \quad B_s\langle m\rangle \mapsto \Theta_s \ \forall s \in \mathcal{S}_a \ \forall m \in \mathbb{Z}.$$

Here $\mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G))$ denotes the category of functors from $\mathrm{Rep}_0(G)$ to itself with the morphisms given by the natural transformations, and $^{\mathrm{op}}$ signifies the right action of $\mathcal{D}_{\mathrm{BS}}$. To describe $\mathcal{D}_{\mathrm{BS}}$, for $w \in \mathcal{W}_a$ we mean by $\underline{w} = \underline{s_1 \dots s_r}$ a sequence of simple reflections $s_1, \dots, s_r \in \mathcal{S}_a$ such that the product $s_1 \dots s_r$ yields w, in which case we call $\underline{w}$ an expression of w. Then $\mathcal{D}_{\mathrm{BS}}$ is a category equipped with a shift of the grading autoequivalence $\langle 1\rangle$, whose objects are $B_{\underline{w}}\langle m\rangle$, parametrized by pairs of an expression $\underline{w}$, $w \in \mathcal{W}_a$, and $m \in \mathbb{Z}$, such that $(B_{\underline{w}}\langle m\rangle)\langle 1\rangle = B_{\underline{w}}\langle m+1\rangle$. It is also equipped with a product such that $B_{\underline{w}}\langle m\rangle \bullet B_{\underline{v}}\langle m'\rangle = B_{\underline{wv}}\langle m+m'\rangle$.

Definition [中岡, Def. 3.5.2, p. 211]/[Bor, Def. II.6.1.1, p. 292]: A strict monoidal category is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $I \in \mathrm{Ob}(\mathcal{C})$, and a natural "associativity" identity $\alpha_{A,B,C} : (A \otimes B) \otimes C = A \otimes (B \otimes C)$, a natural "left unital" identity $\lambda_A : I \otimes A = A$, and a natural "right unital" identity $\rho_A : A \otimes I = A$.

Thus, $\mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G))$ is a strict monoidal category under the composition of functors while $\mathcal{D}_{\mathrm{BS}}$ is a strict monoidal category with respect to the product.

Definition [Mac, pp. 255-256]: Given two strict monoidal categories $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{C}', \otimes', I', \alpha', \lambda', \rho')$ a strict monoidal functor $(F, F_2, F_0) : \mathcal{C} \to \mathcal{C}'$ consists of the following data

(M1) $F : \mathcal{C} \to \mathcal{C}'$ is a functor,

(M2) $\forall A, B \in \mathrm{Ob}(\mathcal{C})$, bifunctorial identity $F_2(A, B) \in \mathcal{C}'(F(A) \otimes' F(B), F(A \otimes B))$,

(M3) an identity $F_0 \in \mathcal{C}'(I', F(I))$.

Thus the strict monoidal functor in the theorem is really just a homomorphism of monoids.

(2.2) The proof of the theorem is given, using the theory of 2-representations of 2-Kac-Moody algebras $\mathfrak{U}(\widehat{\mathfrak{gl}_n})$, $\mathfrak{U}(\widehat{\mathfrak{gl}_p})$: one constructs 3 strict monoidal functors, first $\mathcal{U}_+(\widehat{\mathfrak{gl}_p})(\varpi, \varpi) \to \mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G))$ with a quotient $\mathcal{U}_+(\widehat{\mathfrak{gl}_p})$ of the Khovanov-Lauda-Rouquier 2-category $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ associated to Kac-Moody Lie algebra $\widehat{\mathfrak{gl}_p}$, secondly its "restriction" $\mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi, \varpi) \to \mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G))$ to a quotient $\mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})$ of the Khovanov-Lauda-Rouquier 2-category $\mathcal{U}(\widehat{\mathfrak{gl}_n})$ associated to Kac-Moody Lie algebra $\widehat{\mathfrak{gl}_n}$, and finally $\mathcal{D}_{\mathrm{BS}} \to \mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi, \varpi)$. In fact, all 3 indivisual steps were known and available for use. The basic strategy follows one for the category of $\mathfrak{g}_n(\mathbb{C})$-modules locally finite over a Borel subalgebra and of integral weights due to Mackaay, Stočić, and Vas [MSV].

Definition [中岡, Def. 3.5.22, p. 220]/[Bor, I.7]: A strict 2-category $\mathcal{C}$ consists of the data

(i) a class $|\mathcal{C}|$, whose elements are called objects,

(ii) $\forall A, B \in |\mathcal{C}|$, a small category $\mathcal{C}(A, B)$, whose elements are called 1-morphisms and written as $f : A \to B$ with the morphisms in $\mathcal{C}(A, B)$ denoted as $\alpha : f \Rightarrow g$ and their compositions written

$$\begin{tikzcd} f \arrow[r, Rightarrow, "\alpha"] \arrow[dr, dotted, "\beta\odot\alpha"'] & g \arrow[d, Rightarrow, "\beta"] \\ & h, \end{tikzcd}$$

(iii) $\forall A, B, C \in |\mathcal{C}|$, a bifunctor $c_{A,B,C} : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$, written

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, phantom, "\alpha\Downarrow"] & B \arrow[r, "\ell"] \arrow[d, phantom, "\beta\Downarrow"] & C & & A \arrow[r, "\ell\circ f"] \arrow[d, phantom, "\Downarrow\beta*\alpha"] & C \\ A \arrow[r, "f'"'] & B \arrow[r, "\ell'"'] & C \arrow[r, phantom, "\xmapsto{c_{A,B,C}}"] & & A \arrow[r, "\ell'\circ f'"'] & C, \end{tikzcd}$$

(iv) $\forall A \in |\mathcal{C}|$, there is a 1-morphism $1_A \in \mathcal{C}(A, A)$,

subject to the axioms that $\forall A, B, C, D \in |\mathcal{C}|$,

$$\begin{tikzcd} \mathcal{C}(A,B)\times\mathcal{C}(B,C)\times\mathcal{C}(C,D) \arrow[r, "\mathcal{C}(A,B)\times c_{B,C,D}"] \arrow[d, "c_{A,B,C}\times\mathcal{C}(C,D)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{C}(A,B)\times\mathcal{C}(B,D) \arrow[d, "c_{A,B,D}"] \\ \mathcal{C}(A,C)\times\mathcal{C}(C,D) \arrow[r, "c_{A,C,D}"'] & \mathcal{C}(A,D) \end{tikzcd}$$

and $c_{A,A,B}(1_A, ?) = \mathrm{id}_{\mathcal{C}(A,B)} = c_{A,B,B}(?, 1_B)$. We will denote $\mathrm{id}_{1_A} \in \mathcal{C}(A, A)(1_A, 1_A)$ by ι_A.

Then, $\forall \alpha, \beta \in \mathrm{Mor}(\mathcal{C}(A, B))$, $\forall \mu, \nu \in \mathrm{Mor}(\mathcal{C}(B, C))$, the "interchange law" holds:

$$\begin{aligned} (\nu * \beta) \odot (\mu * \alpha) &= c_{A,B,C}(\beta, \nu) \odot c_{A,B,C}(\alpha, \mu) \quad \text{by definition} \\ &= c_{A,B,C}((\beta, \nu) \odot (\alpha, \mu)) \quad \text{by the functoriality of } c_{A,B,C} \\ &= c_{A,B,C}(\beta \odot \alpha, \nu \odot \mu) \\ &= (\nu \odot \mu) * (\beta \odot \alpha); \end{aligned}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, phantom, "\alpha\Downarrow"] & B \arrow[r, "\ell"] \arrow[d, phantom, "\Downarrow\mu"] & C & A \arrow[r, "\ell\circ f"] \arrow[d, phantom, "\Downarrow\mu*\alpha"] & C & & & & \\ A \arrow[r, "f'"'] \arrow[d, phantom, "\beta\Downarrow"] & B \arrow[r, "\ell'"'] \arrow[d, phantom, "\Downarrow\nu"] & C \arrow[r, phantom, "\xmapsto{c_{A,B,C}}"] & A \arrow[r, "\ell'\circ f'"'] \arrow[d, phantom, "\Downarrow\nu*\beta"] & C \arrow[r, phantom, "="] & A \arrow[r, "f"] \arrow[d, phantom, "\mu\odot\alpha\Downarrow"] & B \arrow[r, "\ell"] \arrow[d, phantom, "\Downarrow\nu\odot\beta"] & C \\ A \arrow[r, "f''"'] & B \arrow[r, "\ell''"'] & C & A \arrow[r, "\ell''\circ f''"'] & C & A \arrow[r, "f''"'] & B \arrow[r, "\ell''"'] & C. \end{tikzcd}$$

Thus, a strict 2-category $\mathcal{C}$ is just a category enriched in the category Cat of small categories. A strict monoidal category $\mathcal{C}$ is just a 2-category with one object pt and $\mathcal{C}(\mathrm{pt}, \mathrm{pt}) = \mathcal{C}$.

The KLR 2-category $\mathcal{U}(\widehat{\mathfrak{gl}_n})$ is a strict $\Bbbk$-linear additive 2-category, which is a strict 2-category enriched in the category of $\Bbbk$-linear additive categories.

(2.3) The proofs of Lusztig's conjecture (1.10) by [AJS] and [F] were actually done in terms of G_1T-modules, an analogue of the representation theory of the Lie algebra of G, G_1 denoting the Frobenius kernel of G. There the standard (resp. simple) G_1T-modules are parametrized by the whole of Λ, which we denote by $\hat{\nabla}(\lambda)$ (resp. $\hat{L}(\lambda)$), $\lambda \in \Lambda$, with $\hat{L}(\lambda) = L(\lambda)$ as long as $\lambda \in \Lambda_1$. Each $\hat{\nabla}(\lambda)$ has highest weight λ of multiplicity 1 and $\hat{L}(\lambda)$ is a unique simple submodule of $\hat{\nabla}(\lambda)$;

$$\operatorname{ch} \hat{\nabla}(\lambda) = e^{\lambda} \prod_{\alpha \in R^+} \frac{1 - e^{-p\alpha}}{1 - e^{-\alpha}}.$$

Thus, to find the irreducible characters $\operatorname{ch} \hat{L}(\lambda)$, it is enough to compute the composition factor multiplicities $[\hat{\nabla}(\lambda) : \hat{L}(\mu)]$, $\lambda, \mu \in \Lambda$, [J, II.9.9]. Moreover, this category admits enough injectives/projectives. If we let $\hat{Q}(\lambda)$ denote the injective hull of $\hat{L}(\lambda)$, it is also the projective cover of $\hat{L}(\lambda)$ [J, II.11.5.4], and admits a filtration with subquotients of the form $\hat{\nabla}(\mu)$, $\mu \in \Lambda$, and of the form $\hat{\Delta}(\mu)$ from (1.7). As $\operatorname{Ext}^i_G(\hat{\Delta}(\lambda), \hat{\nabla}(\mu)) = \delta_{i,0}\delta_{\lambda,\mu}\Bbbk$ $\forall \lambda, \mu \in \lambda$ $\forall i \in \mathbb{N}$ [J, II.9.9], the multiplicity $(\hat{Q}(\lambda) : \hat{\nabla}(\mu))$ of $\hat{\nabla}(\mu)$ appearing in such a filtration of $\hat{Q}(\lambda)$ is given by $(\hat{Q}(\lambda) : \hat{\nabla}(\mu)) = [\hat{\nabla}(\mu) : \hat{L}(\lambda)]$ [J, II.11.4]. What Andersen, Jantzen and Soergel (resp. Fiebig) did is to compute $(\hat{Q}(x \bullet 0) : \hat{\nabla}(y \bullet 0))$, $x, y \in \mathcal{W}_a$, for $p \gg 0$, by relating to the correponding multiplicity in the quantum group at a p-th root of 1 (resp. by using the moment graph of $\mathcal{W}_a$); Fiebig provides an explicit bound for the first time though it is enormous above which Lusztig's conjecture holds. There is now an algorithm to compute the weight space multiplicities of $T(x \bullet 0)$, $x \in {}^f\mathcal{W}$, for $p > h + 1$ by Fiebig and Williamson using the Braden-MacPherson algorithm [FW, Th. 9.1], and a proof of Lusztig's conjecture for $p \gg 0$ without using G_1T by Achar and Riche [AR].

Now, in $\operatorname{Rep}(G)$ the modules corresponding to G_1T-injectives are tilting modules. We call the dual G-module $\nabla(\lambda)^*$ of $\nabla(\lambda)$, $\lambda \in \Lambda^+$, a Weyl module, which has highest weight $-w_0\lambda$ and simple head $L(-w_0\lambda) = L(\lambda)^*$. We denote $\nabla(\lambda)^*$ by $\Delta(-w_0\lambda)$. We say a G-module is tilting iff it admits a filtration with subquotients of the form $\nabla(\lambda)$ and also a filtration with subquotients of the form $\Delta(\lambda)$, $\lambda \in \Lambda^+$; in fact, $\hat{\nabla}(\nu)^* \simeq \hat{\nabla}(2(p-1)\rho - \nu)$ $\forall \nu \in \Lambda$ with $2\rho = \sum_{\alpha \in R^+} \alpha$ [J, II.9.2]. To get $\hat{\Delta}(\nu)$ from $\hat{\nabla}(\nu)$ by dualization one needs Chevalley involution [J, II.9.3]. One has [J, II.4.13], $\forall \lambda, \mu \in \Lambda^+$, $\forall i \in \mathbb{N}$,

$$\operatorname{Ext}^i_G(\Delta(\lambda), \nabla(\mu)) = \delta_{i,0}\delta_{\lambda,\mu}\Bbbk,$$

and hence a tilting module has no higher self-extension, which should explain the term tilting. It is a theorem of Donkin [J, E.6] that for each $\lambda \in \Lambda^+$ there is up to isomorphism a unique indecomposable tilting module of highest weight λ, which we denote by $T(\lambda)$. In $T(\lambda)$ the multiplicity of λ is 1. Note that the use of tilting modules in [RW] is also influenced by Soergel's success in the quantum case [S98], c.f. also [Maz, 8.1].

(2.4) Let $\operatorname{Tilt}_0(G)$ denote the principal tilting block of G, the full subcategory of $\operatorname{Rep}_0(G)$

consisting of tilting modules. If $\lambda \in \Lambda^+$ belongs to the bottom dominant alcove A^+, one has by the linkage principle $\nabla(\lambda) = L(\lambda) = \Delta(\lambda)$, and hence also $T(\lambda) = L(\lambda)$. It follows by induction on the partial order on Λ^+ that $[\mathrm{Rep}_0(G)]$ admits a basis $[T(w \bullet 0)]$, $w \in {}^f\mathcal{W}$. The $\mathcal{D}_{\mathrm{BS}}$-action on $\mathrm{Rep}_0(G)$ induces an action on $\mathrm{Tilt}_0(G)$, which in turn will imply a character formula of all $T(w \bullet 0)$, $w \in {}^f\mathcal{W}$, in terms of the p-canonical basis of $\mathcal{H}$ in place of the Kazhdan-Lusztig basis for Lusztig's conjecture as we describe in the next subsection.

For $\lambda, \mu \in \Lambda$ we write $\mu \uparrow \lambda$ iff there is a sequence of reflections $s_{\beta_1,m_1}, \dots, s_{\beta_r,m_r}$, $\beta_i \in R^+$, $m_i \in \mathbb{Z}$, such that $\lambda \geq s_{\beta_1,m_1} \bullet \lambda = s_{\beta_1} \bullet \lambda + pm_1\beta_1 \geq s_{\beta_2,m_2} s_{\beta_1,m_1} \bullet \lambda \geq \dots \geq s_{\beta_r,m_r} \dots s_{\beta_1,m_1} \bullet \lambda = \mu$. For $p \geq 2(n-1)$ each $\hat{Q}(\lambda)$, $\lambda \in \Lambda_1$, lifts to a tilting module $T(2(p-1)\rho + w_0\lambda)$ [J, E.9.1]. Using the lifting, one can show [J, E.10.2] $\forall \lambda \in \Lambda_1$, $\forall \mu \uparrow 2(p-1)\rho + w_0\lambda$,

$$(T(2(p-1)\rho + w_0\lambda) : \nabla(\mu)) = [\nabla(\mu) : L(\lambda)],$$

which then yields the irreducible characters of the principal block of G.

(2.5) Recall from (1.11) the 岩堀-Hecke algebra $\mathcal{H}$ of the Coxeter system $(\mathcal{W}_a, \mathcal{S}_a)$. Let $\mathcal{H}_f$ be the 岩堀-Hecke algebra of the Coxeter subsystem $(\mathcal{W}, \mathcal{S})$. Thus, $\mathcal{H}_f$ is a $\mathbb{Z}[v, v^{-1}]$-subalgebra of $\mathcal{H}$, having the standard basis H_w, $w \in \mathcal{W}$. Let $\mathrm{sgn} = \mathbb{Z}[v, v^{-1}]$ be a right $\mathcal{H}_f$-module such that $H_s \mapsto -v$ $\forall s \in \mathcal{S}$. We set $\mathcal{M}^{\mathrm{asph}} = \mathrm{sgn} \otimes_{\mathcal{H}_f} \mathcal{H}$ and call it the antipherical right module of $\mathcal{H}$, denoted $\mathcal{N} = \mathcal{N}^f$ in [S97, p. 86] and $\mathcal{N}^0 = \mathcal{N}^f$ in [S97, line -3, p. 98]. Then $\mathcal{M}^{\mathrm{asph}}$ has a standard $\mathbb{Z}[v, v^{-1}]$-linear basis $1 \otimes H_w$, $w \in {}^f\mathcal{W}$, and from [S97, line -2, p. 88] a Kazhdan-Lusztig $\mathbb{Z}[v, v^{-1}]$-linear basis $1 \otimes \underline{H}_w$, $w \in {}^f\mathcal{W}$. Thus, $\mathcal{M}^{\mathrm{asph}}$ is a quantization of the antispherical $\mathbb{Z}[\mathcal{W}_a]$-module $\mathrm{sgn}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a]$: under the specialization $v \mapsto 1$

$$\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{asph}} \simeq \mathrm{sgn}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a]. \tag{1}$$

Now let $\mathcal{D}$ be the diagrammatic Hecke category over $\Bbbk$, which is the Karoubian envelope of the additive hull of $\mathcal{D}_{\mathrm{BS}}$ [Bor, Prop. 6.5.9, p. 274]. We will call $\mathcal{D}$ the Elias-Williamson category after their introduction in [EW]. It is defined by diagrammatic generators and relations, graded with shift functor $\langle 1 \rangle$. It is generated as a graded monoidal category by objects B_s, $s \in \mathcal{S}_a$, and is Krull-Schmidt. The indecomposables of $\mathcal{D}$ are the $B_x\langle m \rangle$, parametrized by $(x, m) \in \mathcal{W}_a \times \mathbb{Z}$. We will write B_x for $B_x\langle 0 \rangle$. As $\mathcal{D}$ is only additive, we consider the split Grothendieck group $[\mathcal{D}]$ of $\mathcal{D}$ [中岡, Def. 3.3.35, p. 170]: it is a free $\mathbb{Z}$-module of basis consisting of $\mathrm{Ob}(\mathcal{D})$ subject to the relations $M_1 + M_2 = M_3$ iff there is a split exact sequence $0 \to M_1 \to M_3 \to M_2 \to 0$. We denote the image of $M \in \mathcal{D}$ in $[\mathcal{D}]$ by $[M]$. Then, $[\mathcal{D}]$ comes equipped with a structure of $\mathbb{Z}[v, v^{-1}]$-module such that $v \bullet [M] = [M\langle 1 \rangle]$, and there is a natural isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras [EW]

$$\mathcal{H} \simeq [\mathcal{D}] \quad \text{such that} \quad \underline{H}_s \mapsto [B_s] \ \forall s \in \mathcal{S}_a, \tag{2}$$

under which we define the p-canonical basis of $\mathcal{H}$ to be the preimage of $[B_x]$, $x \in \mathcal{W}_a$: ${}^p\underline{H}_x \mapsto [B_x]$. In $\mathcal{M}^{\mathrm{asph}}$ put $N_w = 1 \otimes H_w$ and ${}^p\underline{N}_w = 1 \otimes {}^p\underline{H}_w$, $w \in {}^f\mathcal{W}$, and write ${}^p\underline{N}_w = \sum_{y \in {}^f\mathcal{W}} {}^p n_{yw} N_y$, ${}^p n_{yw} \in \mathbb{Z}[v, v^{-1}]$. The ${}^p n_{yw}$ are called antispherical p-Kazhdan-Lusztig polynomials. If we define $n_{yw} \in \mathbb{Z}[v.v^{-1}]$ likewise from $\underline{N}_w = 1 \otimes \underline{H}_w$, we have from [S97, Prop. 3.1 and 3.4] that $n_{yw} = 0$ unless $y \leq w$, $n_{ww} = 1$, $n_{yw} \in v\mathbb{Z}[v]$, and

$$n_{yw} = \sum_{z \in \mathcal{W}} (-1)^{\ell(z)} h_{zy,w}.$$

Let $\mathcal{I}^{\text{asph}}$ be the full additive subcategory of $\mathcal{D}$ generated by the $B_w\langle m\rangle$, $w \in \mathcal{W}_a \backslash {}^f\mathcal{W}, m \in \mathbb{Z}$, and let $\mathcal{D}^{\text{asph}} = \mathcal{D}//\mathcal{I}^{\text{asph}}$ be the quotient of $\mathcal{D}$ by $\mathcal{I}^{\text{asph}}$ [中岡, Prop. 3.2.51, p. 150]: $\forall X, Y \in \mathcal{D}$, let $\mathcal{I}(X,Y) = \{f \in \mathcal{D}(X,Y) | f$ factors through some $Z \in \mathcal{I}^{\text{asph}}\}$. Then $\mathcal{D}^{\text{asph}}$ is the category with objects $\text{Ob}(\mathcal{D})$ and $\forall X, Y \in \mathcal{D}$, $\mathcal{D}^{\text{asph}}(X,Y) = \mathcal{D}(X,Y)/\mathcal{I}(X,Y)$. Then $\mathcal{D}^{\text{asph}}$ is a graded category inheriting shift functor $\langle 1\rangle$, and the indecomposables of $\mathcal{D}^{\text{asph}}$ are the images $\bar{B}_w\langle m\rangle$ of $B_w\langle m\rangle$, $w \in {}^f\mathcal{W}$, $m \in \mathbb{Z}$. Let $[\mathcal{D}^{\text{asph}}]$ denote the split Grothendieck group of $\mathcal{D}^{\text{asph}}$ with a $\mathbb{Z}[v, v^{-1}]$-action $v[X] = [X\langle 1\rangle]$. By the natural right $\mathcal{D}$-module structure on $\mathcal{D}^{\text{asph}}$ it comes equipped with a structure of right $\mathcal{H}$-module under (2). As such there follows an isomorphism of right $\mathcal{H}$-modules

$$\mathcal{M}^{\text{asph}} \to [\mathcal{D}^{\text{asph}}] \quad \text{via} \quad {}^p\underline{N}_w \mapsto [\bar{B}_w] \ \forall w \in {}^f\mathcal{W}. \tag{3}$$

Thus, the ${}^p\underline{N}_w$, $w \in {}^f\mathcal{W}$, form a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{M}^{\text{asph}}$, and $\mathcal{D}^{\text{asph}}$ provides a categorification of $\mathcal{M}^{\text{asph}}$.

Finally, let $\mathcal{D}^{\text{asph}}_{\text{deg}}$ be the degrading of $\mathcal{D}^{\text{asph}}$: $\text{Ob}(\mathcal{D}^{\text{asph}}_{\text{deg}}) = \text{Ob}(\mathcal{D}^{\text{asph}})$ but $\forall X, Y \in \text{Ob}(\mathcal{D}^{\text{asph}}_{\text{deg}})$, $\mathcal{D}^{\text{asph}}_{\text{deg}}(X,Y) = \coprod_{m\in\mathbb{Z}} \mathcal{D}^{\text{asph}}(X, Y\langle m\rangle)$. In particular, $\forall m \in \mathbb{Z}$, $X \simeq X\langle m\rangle$ in $\mathcal{D}^{\text{asph}}_{\text{deg}}$; $\text{id}_X \in \mathcal{D}^{\text{asph}}(X,X) \leq \mathcal{D}^{\text{asph}}_{\text{deg}}(X, X\langle m\rangle)$ admits an inverse $\text{id}_{X\langle m\rangle} \in \mathcal{D}^{\text{asph}}(X\langle m\rangle, X\langle m\rangle) \leq \mathcal{D}^{\text{asph}}_{\text{deg}}(X\langle m\rangle, X)$. Thus, under the specialization $v \mapsto 1$

$$\begin{tikzcd}
\mathcal{M}^{\text{asph}} \arrow[rr, "(3)", "\sim"'] \arrow[d] & & {[\mathcal{D}^{\text{asph}}]} \arrow[d] \\
\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\text{asph}} \arrow[d, "(1)"', "\sim"] & & \mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} [\mathcal{D}^{\text{asph}}] \arrow[d, "\sim"] \\
\text{sgn}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a] \arrow[d, "\sim"'] & & {[\mathcal{D}^{\text{asph}}_{\text{deg}}].} \\
{[\text{Rep}_0(G)]} \arrow[urr, dotted, "\sim"'] & &
\end{tikzcd}$$

By the categorical action of $\mathcal{D}_{\text{BS}}$ it will turn out that $\text{Tilt}_0(G)$ is equivalent, as a right "module" of $\mathcal{D}_{\text{BS}}$, to the degraded categorification $\mathcal{D}^{\text{asph}}_{\text{deg}}$ of $\mathcal{M}^{\text{asph}}$ via

$$T(w \bullet 0) \leftarrow\!\shortmid \bar{B}_w \quad \forall w \in {}^f\mathcal{W}.$$

Thus, $\mathrm{Tilt}_0(G)$ gives a categorification of the antispherical module $\mathrm{sgn}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a]$: $\forall w \in {}^f\mathcal{W}$,

$$\begin{tikzcd}
1 \otimes H_w = N_w & & & & & \\
1 \otimes {}^p\underline{H}_w = {}^p\underline{N}_w & & {[\bar{B}_w]} \arrow[ll, maps to] \arrow[dd, maps to] & & & \\
\mathrm{sgn} \otimes_{\mathcal{H}_f} \mathcal{H} = \mathcal{M}^{\mathrm{asph}} \arrow[dd] \arrow[ddr, phantom, "\circlearrowright"] & {[\mathcal{D}^{\mathrm{asph}}]} \arrow[l, "\sim"'] \arrow[dd] & & & & \\
 & & {[\bar{B}_w]} & & & T(w \bullet 0) \arrow[lll, maps to] \arrow[ddd, maps to] \\
\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{asph}} \arrow[d, "\sim"'] & \mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} [\mathcal{D}^{\mathrm{asph}}] \arrow[l, "\sim"'] & {[\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}]} \arrow[l, "\sim"'] & \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}} \arrow[l] & \mathrm{Tilt}_0(G) \arrow[l, dotted, "\sim"'] \arrow[d] & \\
\mathrm{sgn}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a] & & & & {[\mathrm{Rep}_0(G)]} \arrow[llll, "\sim"'] & \\
1 \otimes {}^p\underline{N}_w & & & & & {[T(w \bullet 0)]} \arrow[lllll, maps to] \\
1 \otimes N_w & & & & & {[\nabla(w \bullet 0)]}. \arrow[lllll, maps to]
\end{tikzcd}$$

In particular, the character formula for the indecomposable tilting modules in the principal block will be given by

$$\mathrm{ch}\, T(x \bullet 0) = \sum_{y \in {}^f\mathcal{W}} {}^p n_{yx}(1) \mathrm{ch}\, \nabla(y \bullet 0) \quad \forall x \in {}^f\mathcal{W}.$$

(2.6) It is now a theorem of Achar, Makisumi, Riche and Williamson [AMRW] that the character formula for the indecomposable tiltings in (2.5) holds for general reductive groups as long as $p \geq 2(h-1)$, h the Coxeter number of the group.

3° The affine Lie algebra $\widehat{\mathfrak{gl}}_N$

We start by showing that the complexified Grothendieck group $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)]$ of $\mathrm{Rep}(G)$ admits an action of the affine Lie algebra $\widehat{\mathfrak{gl}_p}$, due to Chuang and Rouquier [ChR]. We will also show that the same holds for G_1T. We will assume $n \geq 3$, see e.g., (3.8).

(3.1) Let $N > 2$. We define the affine Lie algebra $\widehat{\mathfrak{gl}}_N$ associated to $\mathfrak{gl}_N(\mathbb{C})$ as follows. Consider first the Lie algebra $\widehat{\mathfrak{sl}}_N = \mathfrak{sl}_N(\mathbb{C}[t,t^{-1}]) \oplus \mathbb{C}K \oplus \mathbb{C}d$ with $\mathfrak{sl}_N(\mathbb{C}[t,t^{-1}]) = \mathfrak{sl}_N(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}]$ and the Lie bracket defined, for $x, y \in \mathfrak{sl}_N(\mathbb{C})$ and $k, m \in \mathbb{Z}$, by

$$[x \otimes t^k, y \otimes t^m] = [x,y] \otimes t^{k+m} + k\delta_{k+m,0}\mathrm{Tr}(xy)K,$$
$$[d, x \otimes t^m] = mx \otimes t^m, \quad [K, \widehat{\mathfrak{sl}}_N] = 0,$$

which is the affine Lie algebra of type $\mathrm{A}^{(1)}_{N-1}$ in [谷崎, p. 164]. Then $\widehat{\mathfrak{gl}}_N = \widehat{\mathfrak{sl}}_N \oplus \mathbb{C}$ with $(0,1) = \mathrm{diag}(1,\dots,1)$ central in $\widehat{\mathfrak{gl}}_N$, so $\mathfrak{gl}_N(\mathbb{C}) = \mathfrak{sl}_N(\mathbb{C}) \oplus \mathbb{C} \leq \widehat{\mathfrak{gl}}_N$.

Let $e(i,j) \in \mathfrak{gl}_N(\mathbb{C})$, $i, j \in [1, N]$, denote a matrix unit such that $e(i,j)_{ab} = \delta_{a,i}\delta_{b,j}$ $\forall a, b \in$

$[1, N]$. $\forall i \in [0, N[$, let

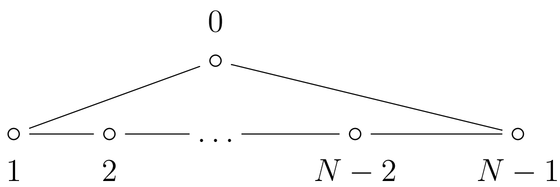
$$\hat{e}_i = \begin{cases} te(1,N) & \text{if } i = 0, \\ e(i+1, i) & \text{else,} \end{cases} \qquad \hat{f}_i = \begin{cases} t^{-1}e(N,1) & \text{if } i = 0, \\ e(i, i+1) & \text{else,} \end{cases}$$

$$\hat{h}_i = [\hat{e}_i, \hat{f}_i] = \begin{cases} e(1,1) - e(N,N) + K & \text{if } i = 0, \\ e(i+1, i+1) - e(i,i) & \text{else.} \end{cases}$$

Set $\mathfrak{h} = \mathfrak{h}_f \oplus \mathbb{C}K \oplus \mathbb{C}d < \widehat{\mathfrak{gl}}_N$ with $\mathfrak{h}_f$ denoting the CSA of $\mathfrak{gl}_N(\mathbb{C})$ consisting of the diagonals. Define $(\hat{\varepsilon}_i, K^*, \delta | i \in [1, N])$ to be the dual basis of $(e(i,i), K, d | i \in [1, N])$ in $\mathfrak{h}^*$. Let $P = \{\lambda \in \mathfrak{h}^* | \lambda(\hat{h}_i) \in \mathbb{Z}\ \forall i \in [0, N[\}$. The simple roots of $\mathfrak{h}^*$ are defined by $\hat{\alpha}_0 = \delta - (\hat{\varepsilon}_N - \hat{\varepsilon}_1)$ and $\hat{\alpha}_i = \hat{\varepsilon}_{i+1} - \hat{\varepsilon}_i$, $i \in [1, N[$. Thus, $\forall i, j \in [0, N[$,

$$\hat{\alpha}_i(\hat{h}_j) = \begin{cases} 0 & \text{if } |i-j| \geq 2, \\ -1 & \text{if } |i-j| = 1 \text{ or } (i,j) \in \{(0, N-1), (N-1, 0)\}, \\ 2 & \text{if } i = j, \end{cases}$$

$$[\hat{h}_i, \hat{e}_j] = \hat{\alpha}_j(\hat{h}_i)\hat{e}_j, \quad [\hat{h}_i, \hat{f}_j] = -\hat{\alpha}_j(\hat{h}_i)\hat{f}_j.$$



(3.2) Let $A = \coprod_{i=1}^N \mathbb{C}a_i$ denote the natural module for $\mathfrak{gl}_N(\mathbb{C})$. Then $A \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ affords a module for $\mathfrak{sl}_N(\mathbb{C}[t, t^{-1}])$ such that $(x \otimes t^k) \cdot (a \otimes t^m) = (xa) \otimes t^{k+m}\ \forall x \in \mathfrak{sl}_N(\mathbb{C})$, $\forall a \in A\ \forall k, m \in \mathbb{Z}$. One may extend it to a representation of $\widehat{\mathfrak{gl}}_N$ by letting K act by 0, $\mathrm{diag}(1, \dots, 1)$ by the identity, and d by the formula $d \cdot (a \otimes t^m) = ma \otimes t^m\ \forall a \in A$, $\forall m \in \mathbb{Z}$. We call the resulting $\widehat{\mathfrak{gl}}_N$-module the natural module and denote it by nat_N.

For $\lambda \in \mathbb{Z}$ write $\lambda = \lambda_0 + N\lambda_1$ with $\lambda_0 \in [1, N]$ and $\lambda_1 \in \mathbb{Z}$. Put $m_\lambda = a_{\lambda_0} \otimes t^{\lambda_1}$. Then $\mathrm{nat}_N = \coprod_{\lambda \in \mathbb{Z}} \mathbb{C}m_\lambda$: $\forall \mu \in \mathbb{Z}$, $a_1 \otimes t^\mu = m_{1+N\mu}, a_2 \otimes t^\mu = m_{2+N\mu}, \dots, a_N \otimes t^\mu = m_{N+N\mu}$, and $\hat{e}_0 a_N = te(1, N)a_N = ta_1 = a_1 \otimes t = m_{1+N}$. $\forall i \in [0, N[$,

(1)
$$\hat{e}_i m_\lambda = \begin{cases} m_{\lambda+1} & \text{if } i \equiv \lambda \bmod N, \\ 0 & \text{else,} \end{cases}$$

(2)
$$\hat{f}_i m_\lambda = \begin{cases} m_{\lambda-1} & \text{if } i \equiv \lambda - 1 \bmod N, \\ 0 & \text{else,} \end{cases}$$

and $\forall h \in \mathfrak{h}$,

(3)
$$hm_\lambda = (\hat{\varepsilon}_{\lambda_0} + \lambda_1\delta)(h)m_\lambda.$$

In particular, all $\mathfrak{h}$-weight spaces of nat_N are 1-dimensional.

(3.3) Recall the natural module $V = \Bbbk^{\oplus n}$ for G with the standard basis $v_1, \dots, v_n$, and its dual V^* with the dual basis $v_1^*, \dots, v_n^*$. Thus, $V = L(\varepsilon_1) = \nabla(\varepsilon_1) = \Delta(\varepsilon_1) = T(\varepsilon_1)$ and

$V^* = L(-w_0\varepsilon_1) = L(-\varepsilon_n) = \nabla(-\varepsilon_n) = \Delta(-\varepsilon_n) = T(-\varepsilon_n)$. Define 2 exact endofunctors E and F of $\mathrm{Rep}_0(G)$ by $E = V\otimes ?$ and $F = V^*\otimes ?$, resp. Define $\eta_{\Bbbk} \in \mathrm{Rep}(G)(\Bbbk, V^*\otimes V)$ such that $\eta_{\Bbbk}(1) = \sum_i v_i^* \otimes v_i$ and $\varepsilon_{\Bbbk} \in \mathrm{Rep}(G)(V\otimes V^*, \Bbbk)$ such that $v\otimes\mu \mapsto \mu(v)$; under a $\Bbbk$-linear isomorphism $V^*\otimes V \simeq \mathrm{Mod}_{\Bbbk}(V,V)$ via $f\otimes v \mapsto f(?)v$ with inverse $\sum_i v_i^*\otimes\phi(v_i) \leftarrow\!\shortmid \phi$, $\sum_i v_i^*\otimes v_i$ corresponds to id_V, and hence fixed by G. In turn, $\eta_{\Bbbk}$ defines a natural transformation $\eta : \mathrm{id}_{\mathrm{Rep}(G)} \Rightarrow FE$ via

$$\begin{tikzcd} M \arrow[r, dotted, "\eta_M"] \arrow[d, "\sim"'] & FE(M) \arrow[d, equal] \\ \Bbbk\otimes M \arrow[r, "\eta_{\Bbbk}\otimes M"'] & V^*\otimes V\otimes M, \end{tikzcd}$$

while $\varepsilon_{\Bbbk}$ defines a natural transformation $\varepsilon : EF \Rightarrow \mathrm{id}_{\mathrm{Rep}(G)}$ via

$$\begin{tikzcd} EF(M) \arrow[r, dotted, "\varepsilon_M"] \arrow[d, equal] & M \arrow[d, "\sim"] \\ V\otimes V^*\otimes M \arrow[r, "\varepsilon_{\Bbbk}\otimes M"'] & \Bbbk\otimes M \end{tikzcd}$$

to make η (resp. ε) into the unit (resp. counit) of an adjunction (E, F) [中岡, Cor. 2.2.9, pp. 65-66] such that

(1) $\quad \mathrm{Rep}(G)(M, FM') \xrightarrow{\sim} \mathrm{Rep}(G)(EM, M')$ via $\psi \mapsto \varepsilon_{M'}\circ E\psi$ with inverse $F\phi\circ\eta_M \leftarrow\!\shortmid \phi$.

Explicitly, $\forall m\in M$,

$$(F\phi\circ\eta_M)(m) = \sum_i v_i^*\otimes\phi(v_i\otimes m),$$

while, if we write $\psi(m) = \sum_i v_i^*\otimes\psi(m)_i$, $\forall v\in V$,

$$(\varepsilon_{M'}\circ E\psi)(v\otimes m) = \sum_i v_i^*(v)\psi(m)_i.$$

Now, let $\mathfrak{g} = \mathfrak{gl}_n(\Bbbk)$ equipped with the structure of G-module Ad: $g\bullet x = gxg^{-1}$ $\forall g\in G$ $\forall x\in\mathfrak{g}$; we identify $\mathfrak{g}$ with $\mathrm{Lie}(G) = \mathrm{Mod}_{\Bbbk}(\mathfrak{m}/\mathfrak{m}^2, \Bbbk)$, $\mathfrak{m} = (x_{ij}, x_{ii}-1 | i,j\in[1,n], i\neq j) \lhd \Bbbk[G]$. $\forall M\in\mathrm{Rep}(G)$, the $\mathfrak{g}$-action on M given by differentiating the G-action $\Delta_M : M\to M\otimes\Bbbk[G]$

$$\begin{tikzcd} \mathfrak{g}\otimes M \arrow[r, dotted, "\mathfrak{a}"] \arrow[d, "\mathfrak{g}\otimes\Delta_M"'] & M & x(f)m \\ \mathfrak{g}\otimes M\otimes\Bbbk[G] \arrow[ur] & x\otimes m\otimes f \arrow[ur, mapsto] & \end{tikzcd}$$

is G-equivariant [J, I.7.18.1]. Let $\eta'_{\Bbbk} : \Bbbk \to V\otimes V^*$ via $1\mapsto \sum_i v_i\otimes v_i^*$ to define the unit of an adjunction (F, E) as above. Using a natural isomorphism $\mathfrak{g}\simeq V^*\otimes V$ via $\mu(?)v \leftarrow\!\shortmid \mu\otimes v$, define for $M\in\mathrm{Rep}(G)$

$$\begin{tikzcd} V\otimes M \arrow[r, dotted, "\mathbb{X}_M"] \arrow[d, "\eta'_{\Bbbk}\otimes V\otimes M"'] & V\otimes M \\ V\otimes V^*\otimes V\otimes M \arrow[r, "\sim"'] & V\otimes\mathfrak{g}\otimes M, \arrow[u, "V\otimes\mathfrak{a}"'] \end{tikzcd}$$

which is functorial in M. Thus, one obtains an endomorphism $\mathbb{X} \in \mathrm{Cat}(\mathrm{Rep}(G), \mathrm{Rep}(G))(E, E)$ of E, i.e., a natural transformation from E to itself. In particular, each $\mathbb{X}_M$ is G-equivariant. In turn, $\mathbb{X}$ induces by adjunction (E, F) an endomorphism $\mathbb{X}'$ of F:

$$\begin{tikzcd}
V^* \otimes M \arrow[rr, dotted, "\mathbb{X}'_M"] \arrow[d, "\eta_{\Bbbk} \otimes V^* \otimes M"'] & & V^* \otimes M \\
V^* \otimes V \otimes V^* \otimes M \arrow[rr, "V^* \otimes \mathbb{X}_{V^* \otimes M}"'] & & V^* \otimes V \otimes V^* \otimes M. \arrow[u, "V^* \otimes \varepsilon_{\Bbbk} \otimes M"']
\end{tikzcd}$$

Thus, $\forall M' \in \mathrm{Rep}(G)$,

(2)
$$\begin{tikzcd}
\mathrm{Rep}(G)(EM, M') & & \mathrm{Rep}(G)(EM, M') \arrow[ll, "\mathrm{Rep}(G)(\mathbb{X}_M, M')"'] \\
\mathrm{Rep}(G)(M, FM') \arrow[u, "\varepsilon_{M'} \circ E", "\sim"'] \arrow[urr, phantom, "\circlearrowleft"] & & \mathrm{Rep}(G)(M, FM'). \arrow[ll, "\mathrm{Rep}(G)(M, \mathbb{X}'_{M'})"] \arrow[u, "\sim", "\varepsilon_{M'} \circ E"']
\end{tikzcd}$$

Let $\mathrm{Dist}(G)$ denote the algebra of distributions on G. As G is defined over $\mathbb{Z}$, $\mathrm{Dist}(G)$ has a $\mathbb{Z}$-form $\mathrm{Dist}(G_{\mathbb{Z}})$ which coincides with Kostant's $\mathbb{Z}$-form of the universal enveloping algebra $\mathbb{U}(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$. Put $\Omega = \sum_{i,j=1}^{n} e(i,j) \otimes e(j,i) \in \mathfrak{g} \otimes \mathfrak{g}$; $\mathrm{Tr}(e(i,j)e(k,l)) = \delta_{jk}\mathrm{Tr}(e(i,l)) = \delta_{jk}\delta_{il}$. For $x \in \mathfrak{g}$ put $\Delta(x) = x \otimes 1 + 1 \otimes x$. If M and M' are G-modules, recall that $\mathrm{Dist}(G)$ acts on the G-module $M \otimes M'$ via $x \mapsto \Delta(x)$, $x \in \mathfrak{g}$.

Lemma: *(i) $\forall v, v' \in V$, $\Omega \cdot (v \otimes v') = v' \otimes v$.*

(ii) $\forall x \in \mathfrak{g}$, $\Omega\Delta(x) = \Delta(x)\Omega$ in $\mathrm{Dist}(G) \otimes \mathrm{Dist}(G)$, and hence the action of Ω on $M \otimes M'$ for $M, M' \in \mathrm{Rep}(G)$ commutes with $\mathrm{Dist}(G)$.

Proof: (i) Let $k, l \in [1, n]$. One has

$$\Omega \cdot (v_k \otimes v_l) = \sum_{i,j=1}^{n} e(i,j)v_k \otimes e(j,i)v_l = \sum_{i,j=1}^{n} \delta_{jk}v_i \otimes \delta_{il}v_j = v_l \otimes v_k.$$

(ii) We may assume $x = e(k,l)$, $k, l \in [1, n]$. One has

$$\begin{aligned}
\Omega\Delta(e(k,l)) &= \sum_{i,j}\{e(i,j)e(k,l) \otimes e(j,i) + e(i,j) \otimes e(j,i)e(k,l)\} \\
&= \sum_{i} e(i,l) \otimes e(k,i) + \sum_{j} e(k,j) \otimes e(j,l)
\end{aligned}$$

while

$$\begin{aligned}
\Delta(e(k,l))\Omega &= \sum_{i,j}\{e(k,l)e(i,j) \otimes e(j,i) + e(i,j) \otimes e(k,l)e(j,i)\} \\
&= \sum_{j} e(k,j) \otimes e(j,l) + \sum_{i} e(i,l) \otimes e(k,i).
\end{aligned}$$

(3.4) We now describe $\mathbb{X}$ and $\mathbb{X}'$ using Ω.

Lemma [RW, 6.3]: *Let $M \in \mathrm{Rep}(G)$.*

(i) $\mathbb{X}_M : EM = V \otimes M \to V \otimes M = EM$ is given by the action of Ω.

(ii) $\mathbb{X}'_M : FM = V^* \otimes M \to V^* \otimes M = FM$ is given by the action of $-n\mathrm{id}_{V^*\otimes M} - \Omega$.

(iii) $(V \otimes \mathbb{X}_M) \circ \mathbb{X}_{V\otimes M} = \mathbb{X}_{V\otimes M} \circ (V \otimes \mathbb{X}_M)$.

(iv) $(V^{\otimes_2} \otimes \mathbb{X}_M) \circ \mathbb{X}_{V^{\otimes_2}\otimes M} = \mathbb{X}_{V^{\otimes_2}\otimes M} \circ (V^{\otimes_2} \otimes \mathbb{X}_M)$.

(v) $\mathbb{X}_{FM} \circ (V \otimes \mathbb{X}'_M) = (V \otimes \mathbb{X}'_M) \circ \mathbb{X}_{FM}$.

(vi) $\mathbb{X}'_{EM} \circ (V^* \otimes \mathbb{X}_M) = (V^* \otimes \mathbb{X}_M) \circ \mathbb{X}'_{EM}$.

Proof: (i) $\forall m \in M$, $\forall s \in [1, n]$,

$$v_s \otimes m \xmapsto{\eta'_{\Bbbk}\otimes V\otimes M} \sum_i v_i \otimes v_i^* \otimes v_s \otimes m \mapsto \sum_i v_i \otimes v_i^*(?)v_s \otimes m = \sum_i v_i \otimes e(s,i) \otimes m$$
$$\xmapsto[V\otimes \mathfrak{a}]{} \sum_i v_i \otimes e(s,i)m$$

while

$$\begin{aligned}\Omega \cdot (v_s \otimes m) &= \sum_{i,j} (e(i,j) \otimes e(j,i))(v_s \otimes m) = \sum_{i,j} (e(i,j)v_s) \otimes (e(j,i)m) \\ &= \sum_{i,j} \delta_{js} v_i \otimes e(j,i)m = \sum_i v_i \otimes e(s,i)m.\end{aligned}$$

Thus, $\mathbb{X}_M$ is given by the multiplication by $\Omega\cdot$.

(ii) Recall first from [HLA, 10.7, p. 76] that $\forall x \in \mathfrak{g}$ $\forall f \in V^*$ $\forall m \in M$,

$$x \cdot (f \otimes m) = (xf) \otimes m + f \otimes xm = -f(x?) \otimes m + f \otimes xm.$$

In particular, x acts on V^* via $-x^{\mathrm{t}}$ with respect to the dual basis:

$$e(i,j)v_k^* = -\delta_{ik} v_j^*. \tag{1}$$

Now,

$$
\begin{aligned}
v_s^* \otimes m &\xmapsto{\eta_{\Bbbk}\otimes V^*\otimes M} \sum_i v_i^* \otimes v_i \otimes v_s^* \otimes m \\
&\xmapsto{V^*\otimes \mathbb{X}_{V^*\otimes M}} \sum_i v_i^* \otimes \Omega \cdot (v_i \otimes v_s^* \otimes m) \quad \text{by (i)} \\
&= \sum_i v_i^* \otimes \sum_{j,k} (e(j,k)v_i) \otimes e(k,j)(v_s^* \otimes m) \\
&= \sum_{i,j,k} v_i^* \otimes \delta_{ki} v_j \otimes \{(e(k,j)v_s^*) \otimes m + v_s^* \otimes e(k,j)m\} \\
&= \sum_{i,j} v_i^* \otimes v_j \otimes \{-\delta_{is} v_j^* \otimes m + v_s^* \otimes e(i,j)m\} \quad \text{by (1)} \\
&= -\sum_j v_s^* \otimes v_j \otimes v_j^* \otimes m + \sum_{i,j} v_i^* \otimes v_j \otimes v_s^* \otimes e(i,j)m \\
&\xmapsto[V^*\otimes \varepsilon_{\Bbbk}\otimes M]{} -n(v_s^* \otimes m) + \sum_i v_i^* \otimes e(i,s)m
\end{aligned}
$$

while

$$
\begin{aligned}
\Omega \cdot (v_s^* \otimes m) = \sum_{i,j} e(i,j)v_s^* \otimes e(j,i)m &= \sum_{i,j} -\delta_{is} v_j^* \otimes e(j,i)m \quad \text{by (1) again} \\
&= -\sum_j v_j^* \otimes e(j,s)m.
\end{aligned}
$$

Thus, $\mathbb{X}'_M$ is given by the action of $-n\mathrm{id}_{V^*\otimes M} - \Omega$.

(iii) $\forall v, v' \in V$, $\forall m \in M$,

$$
\begin{aligned}
\{(V \otimes \mathbb{X}_M) \circ \mathbb{X}_{V\otimes M}\}(v \otimes v' \otimes m) &= (V \otimes \mathbb{X}_M) \sum_{i,j=1}^n \{e(i,j)v \otimes \Delta(e(j,i))(v' \otimes m)\} \\
= \sum_{i,j} e(i,j)v \otimes \Omega\Delta(e(j,i))(v' \otimes m) &= \sum_{i,j} \{e(i,j) \otimes \Omega\Delta(e(j,i))\}(v \otimes v' \otimes m)
\end{aligned}
$$

while

$$
\begin{aligned}
&\{\mathbb{X}_{V\otimes M} \circ (V \otimes \mathbb{X}_M)\}(v \otimes v' \otimes m) = \mathbb{X}_{V\otimes M}\{v \otimes \Omega(v' \otimes m)\} \\
&\quad = \sum_{i,j=1}^n \{e(i,j)v \otimes \Delta(e(j,i))\Omega(v' \otimes m)\} = \sum_{i,j} \{e(i,j) \otimes \Delta(e(j,i))\Omega\}(v \otimes v' \otimes m) \\
&\quad = \sum_{i,j} \{e(i,j) \otimes \Omega\Delta(e(j,i))\}(v \otimes v' \otimes m) \quad \text{by (3.3.ii)}.
\end{aligned}
$$

(iv) Let $x \in V \otimes M$. Then

$$
\begin{aligned}
v_s \otimes v_t \otimes x &\xmapsto{\mathbb{X}_{V^{\otimes 2}\otimes M}} \sum_{i,j} e(i,j)v_s \otimes e(j,i)(v_t \otimes x) = \sum_i v_i \otimes e(s,i)(v_t \otimes x) \\
&= \sum_i v_i \otimes \{e(s,i)v_t \otimes x + v_t \otimes e(s,i)x\} = v_t \otimes v_s \otimes x + \sum_i v_i \otimes v_t \otimes e(s,i)x \\
&\xmapsto{V^{\otimes 2}\otimes \mathbb{X}_M} v_t \otimes v_s \otimes \Omega x + \sum_i v_i \otimes v_t \otimes \Omega e(s,i)x
\end{aligned}
$$

while

$$
\begin{aligned}
v_s \otimes v_t \otimes x &\xmapsto{V^{\otimes 2}\otimes \mathbb{X}_M} v_s \otimes v_t \otimes \Omega x \\
&\xmapsto{\mathbb{X}_{V^{\otimes 2}\otimes M}} \sum_{i,j} e(i,j)v_s \otimes e(j,i)(v_t \otimes \Omega x) = \sum_i v_i \otimes e(s,i)(v_t \otimes \Omega x) \\
&= \sum_i v_i \otimes \{e(s,i)v_t \otimes \Omega x + v_t \otimes e(s,i)\Omega x\} \\
&= \sum_i v_i \otimes e(s,i)v_t \otimes \Omega x + \sum_i v_i \otimes v_t \otimes e(s,i)\Omega x \\
&= v_t \otimes v_s \otimes \Omega x + \sum_i v_i \otimes v_t \otimes e(s,i)\Omega x.
\end{aligned}
$$

The assertion now follows from (3.3.ii).

(v) One has

$$
\begin{aligned}
v_s \otimes v_t^* \otimes m \xmapsto{\mathbb{X}_{FM}} & \sum_{i,j} e(i,j) v_s \otimes \{e(j,i) v_t^* \otimes m + v_t^* \otimes e(j,i)m\} \\
&= \sum_i v_i \otimes \{e(s,i) v_t^* \otimes m + v_t^* \otimes e(s,i)m\} \\
&= \sum_i v_i \otimes \{-\delta_{st} v_i^* \otimes m + v_t^* \otimes e(s,i)m\} \\
&= -\delta_{st} \sum_i v_i \otimes v_i^* \otimes m + \sum_i v_i \otimes v_t^* \otimes e(s,i)m \\
& \xmapsto{V\otimes \mathbb{X}'_M} -\delta_{st} \sum_i v_i \otimes (-n\mathrm{id} - \Omega)(v_i^* \otimes m) + \sum_i v_i \otimes (-n\mathrm{id} - \Omega)(v_t^* \otimes e(s,i)m) \\
&= n\delta_{st} \sum_i v_i \otimes v_i^* \otimes m + \delta_{st} \sum_i v_i \otimes \sum_{k,l} e(k,l) v_i^* \otimes e(l,k)m \\
&\qquad - n \sum_i v_i \otimes v_t^* \otimes e(s,i)m - \sum_i v_i \otimes \sum_{k,l} e(k,l) v_t^* \otimes e(l,k) e(s.i)m \\
&= n\delta_{st} \sum_i v_i \otimes v_i^* \otimes m + \delta_{st} \sum_i v_i \otimes \sum_l (-v_l^*) \otimes e(l,i)m \\
&\qquad - n \sum_i v_i \otimes v_t^* \otimes e(s,i)m - \sum_i v_i \otimes \sum_l e(s,l) v_t^* \otimes e(l,i)m \\
&= n\delta_{st} \sum_i v_i \otimes v_i^* \otimes m - \delta_{st} \sum_i v_i \otimes \sum_l v_l^* \otimes e(l,i)m \\
&\qquad - n \sum_i v_i \otimes v_t^* \otimes e(s,i)m + \sum_i v_i \otimes \sum_l \delta_{st} v_l^* \otimes e(l,i)m \\
&= n\delta_{st} \sum_i v_i \otimes v_i^* \otimes m - n \sum_i v_i \otimes v_t^* \otimes e(s,i)m
\end{aligned}
$$

while

$$
\begin{aligned}
v_s \otimes v_t^* \otimes m &\xmapsto{V\otimes\mathbb{X}'_M = V\otimes(-n\mathrm{id}-\Omega)} -nv_s \otimes v_t^* \otimes m - v_s \otimes \sum_{i,j} e(i,j)v_t^* \otimes e(j,i)m \\
&= -nv_s \otimes v_t^* \otimes m + v_s \otimes \sum_j v_j^* \otimes e(j,t)m \\
&\xmapsto{\mathbb{X}_{FM}} -n\sum_{k,l} e(k,l)v_s \otimes \{e(l,k)v_t^* \otimes m + v_t^* \otimes e(l,k)m\} \\
&\qquad + \sum_{j,k,l} e(k.l)v_s \otimes \{e(l,k)v_j^* \otimes e(j,t)m + v_j^* \otimes e(l,k)e(j,t)m\} \\
&= -n\sum_k v_k \otimes \{e(s,k)v_t^* \otimes m + v_t^* \otimes e(s,k)m\} \\
&\qquad + \sum_{j,k} v_k \otimes \{e(s,k)v_j^* \otimes e(j,t)m + v_j^* \otimes e(s,k)e(j,t)m\} \\
&= -n\sum_k v_k \otimes \{-\delta st v_k^* \otimes m + v_t^* \otimes e(s,k)m\} \\
&\qquad - \sum_k v_k \otimes v_k^* \otimes e(s,t)m + \sum_k v_k \otimes v_k^* \otimes e(s,t)m \\
&= n\delta_{st}\sum_k v_k \otimes v_k^* \otimes m - n\sum_k v_k \otimes v_t^* \otimes e(s,k)m,
\end{aligned}
$$

(vi) One has

$$
\begin{aligned}
&(V^* \otimes \mathbb{X}_M) \circ \mathbb{X}'_{EM}(v_k^* \otimes v_l \otimes m) = (V^* \otimes \mathbb{X}_M)(-n\mathrm{id} - \Omega_{EM})(v_k^* \otimes v_l \otimes m) \\
&\quad = -nv_k^* \otimes \Omega_{EM}(v_l \otimes m) - (V^* \otimes \mathbb{X}_M)\sum_{i,j} e(i,j)v_k^* \otimes (e(j,i)v_l \otimes m + v_l \otimes e(j,i)m) \\
&\quad = -nv_k^* \otimes \sum_{i,j} e(i,j)v_l \otimes e(j,i)m - (V^* \otimes \mathbb{X}_M)\sum_j -v_j^* \otimes (e(j,k)v_l \otimes m + v_l \otimes e(j,k)m) \\
&\quad = -nv_k^* \otimes \sum_i v_i \otimes e(l,i)m + (V^* \otimes \mathbb{X}_M)\sum_j v_j^* \otimes (\delta_{kl}v_j \otimes m + v_l \otimes e(j,k)m) \\
&\quad = -nv_k^* \otimes \sum_i v_i \otimes e(l,i)m + \delta_{kl}\sum_j v_j^* \otimes \sum_{s,t} e(s,t)v_j \otimes e(t,s)m \\
&\qquad\quad + \sum_j v_j^* \otimes \sum_{s,t} e(s,t)v_l \otimes e(t,s)e(j,k)m \\
&\quad = -nv_k^* \otimes \sum_i v_i \otimes e(l,i)m + \delta_{kl}\sum_j v_j^* \otimes \sum_s v_s \otimes e(j,s)m \\
&\qquad\quad + \sum_j v_j^* \otimes \sum_s v_s \otimes e(l,s)e(j,k)m \\
&\quad = -nv_k^* \otimes \sum_i v_i \otimes e(l,i)m + \delta_{kl}\sum_j v_j^* \otimes \sum_s v_s \otimes e(j,s)m + \sum_j v_j^* \otimes v_j \otimes e(l,k)m
\end{aligned}
$$

while

$$
\begin{aligned}
\mathbb{X}'_{EM} \circ (V^* \otimes \mathbb{X}_M)(v_k^* \otimes v_l \otimes m) &= \mathbb{X}'_{EM}(v_k^* \otimes \sum_{i,j} e(i,j)v_l \otimes e(j,i)m) \\
&= \mathbb{X}'_{EM}(v_k^* \otimes \sum_i v_i \otimes e(l,i)m) = (-n\mathrm{id} - \Omega_{EM})(v_k^* \otimes \sum_i v_i \otimes e(l,i)m) \\
&= -nv_k^* \otimes \sum_i v_i \otimes e(l,i)m - \sum_{s,t} e(s,t)v_k^* \otimes \sum_i (e(t,s)v_i \otimes e(l,i)m + v_i \otimes e(t,s)e(l,i)m) \\
&= -nv_k^* \otimes \sum_i v_i \otimes e(l,i)m + \sum_t v_t^* \otimes \sum_i (e(t,k)v_i \otimes e(l,i)m + v_i \otimes e(t,k)e(l,i)m) \\
&= -nv_k^* \otimes \sum_i v_i \otimes e(l,i)m + \sum_t v_t^* \otimes v_t \otimes e(l,k)m + \sum_t v_t^* \otimes \sum_i \delta_{kl} v_i \otimes e(t,i)m.
\end{aligned}
$$

(3.5) Recall from (3.3) the unit η and the counit ε of an adjoint pair (E, F), and also the unit η' and the counit ε' of an adjoint pair (F, E) induced by $\eta'_{\Bbbk} : \Bbbk \to V \otimes V^*$ via $1 \mapsto \sum_i v_i \otimes v_i^*$ and $\varepsilon'_{\Bbbk} : V^* \otimes V \to \Bbbk$ via $\xi \otimes v \mapsto \xi(v)$.

Lemma: *Let $M \in \mathrm{Rep}(G)$ and $r \in \mathbb{N}$.*

(i) $(\mathbb{X}'_{EM})^r \circ \eta_M = (V^* \otimes \mathbb{X}_M)^r \circ \eta_M, \quad \varepsilon_M \circ (\mathbb{X}_{FM})^r = \varepsilon_M \circ (V \otimes \mathbb{X}'_M)^r.$

(ii) $(\mathbb{X}_{FM})^r \circ \eta'_M = (V \otimes \mathbb{X}'_M)^r \circ \eta'_M, \quad \varepsilon'_M \circ (\mathbb{X}'_{EM})^r = \varepsilon'_M \circ (V^* \otimes \mathbb{X}_M)^r.$

Proof: Let $m \in M$.

(i) By definition $\eta_M : M \to FEM = V^* \otimes V \otimes M$ reads $m \mapsto \sum_i v_i^* \otimes v_i \otimes m$. Then

$$
\begin{aligned}
(\mathbb{X}'_{EM} \circ \eta_M)(m) &= (-n\mathrm{id} - \Omega_{V^* \otimes EM}) \sum_{k=1}^n v_k^* \otimes (v_k \otimes m) \quad \text{by (3.4.ii)} \\
&= -n\eta_M(m) - \sum_{i,j,k} e(i,j)v_k^* \otimes (e(j,i)v_k \otimes m + v_k \otimes e(j,i)m) \\
&= -n\eta_M(m) - \sum_{i,j,k} (-\delta_{ik} v_j^*) \otimes (\delta_{ik} v_j \otimes m + v_k \otimes e(j,i)m) \\
&= -n\eta_M(m) - \sum_{i,j} (-v_j^*) \otimes (v_j \otimes m + v_i \otimes e(j,i)m) \\
&= -n\eta_M(m) + n\eta_M(m) + \sum_{i,j} v_j^* \otimes v_i \otimes e(j,i)m = \sum_{i,j} v_j^* \otimes v_i \otimes e(j,i)m \\
&= \sum_j v_j^* \otimes \sum_i v_i \otimes e(j,i)m = \sum_i v_j^* \otimes \Omega \cdot (v_j \otimes m) = (V^* \otimes \Omega_{V \otimes M})(\eta_M(m)) \\
&= (V^* \otimes \mathbb{X}_M) \circ \eta_M(m) \quad \text{by (3.4.ii)},
\end{aligned}
$$

and hence $\mathbb{X}'_{EM} \circ \eta_M = (V^* \otimes \mathbb{X}_M) \circ \eta_M$. Then $(\mathbb{X}'_{EM})^r \circ \eta_M = (V^* \otimes \mathbb{X}_M)^r \circ \eta_M$ by (3.4.vi).

One has

$$
\begin{aligned}
(\varepsilon_M \circ \mathbb{X}_{FM})(v_k \otimes v_l^* \otimes m) &= \varepsilon_M(\sum_{i,j} e(i,j)v_k \otimes e(j,i)(v_l^* \otimes m)) \quad \text{by (3.4.i)} \\
&= \varepsilon_M(\sum_i v_i \otimes e(k,i)(v_l^* \otimes m)) = \varepsilon_M(\sum_i v_i \otimes (e(k,i)v_l^* \otimes m + v_l^* \otimes e(k,i)m)) \\
&= \varepsilon_M(\sum_i v_i \otimes (-\delta_{kl} v_i^* \otimes m + v_l^* \otimes e(k,i)m)) = -\delta_{kl} nm + e(k,l)m
\end{aligned}
$$

while

$$
\begin{aligned}
&\varepsilon_M((V \otimes \mathbb{X}'_M)(v_k \otimes v_l^* \otimes m)) = \varepsilon_M(v_k \otimes (-n\mathrm{id} - \Omega_{V^* \otimes M})(v_l^* \otimes m)) \quad \text{by (3.4.i)} \\
&\quad = \varepsilon_M(-n(v_k \otimes v_l^* \otimes m) - v_k \otimes \sum_{i,j} e(i,j)v_l^* \otimes e(j,i)m) \\
&\quad = -n\delta_{kl} m - \varepsilon_M(v_k \otimes \sum_{i,j}(-\delta_{il} v_j^* \otimes e(j,i)m)) = -n\delta_{kl} m + \varepsilon_M(v_k \otimes \sum_j v_j^* \otimes e(j,l)m) \\
&\quad = -\delta_{kl} nm + e(k,l)m.
\end{aligned}
$$

Thus $\varepsilon_M \circ \mathbb{X}_{FM} = \varepsilon_M \circ (V \otimes \mathbb{X}'_M)$, and hence $\varepsilon_M \circ (\mathbb{X}_{FM})^r = \varepsilon_M \circ (V \otimes \mathbb{X}'_M)^r$ by (3.4.v).

(ii) Likewise,

$$
\begin{aligned}
(\mathbb{X}_{FM} \circ \eta'_M)(m) &= \Omega_{V \otimes FM} \cdot \sum_k v_k \otimes (v_k^* \otimes m) \\
&= \sum_{i,j,k} e(i,j)v_k \otimes (e(j,i)v_k^* \otimes m + v_k^* \otimes e(j,i)m) \\
&= \sum_{i,k} v_i \otimes (e(k,i)v_k^* \otimes m + v_k^* \otimes e(k,i)m) \\
&= \sum_{i,k} v_i \otimes (-v_i^* \otimes m + v_k^* \otimes e(k,i)m) = -\eta'_M(m) + \sum_{i,k} v_i \otimes v_k^* \otimes e(k,i)m \\
&= -n\eta'_M(m) - \sum_i v_i \otimes \Omega_{V^* \otimes M} \cdot (v_i^* \otimes m) = -n\eta'_M(m) - (V \otimes \Omega_{V^* \otimes M})\eta'_M(m) \\
&= \{-n\mathrm{id}_{EFM} - (V \otimes \Omega_{V^* \otimes M})\}\eta'_M(m) = \{V \otimes (-n\mathrm{id} - \Omega_{V^* \otimes M})\}\eta'_M(m),
\end{aligned}
$$

and hence $\mathbb{X}_{FM} \circ \eta'_M = (V \otimes \mathbb{X}'_M) \circ \eta'_M$. Then $(\mathbb{X}_{FM})^r \circ \eta'_M = (V \otimes \mathbb{X}'_M)^r \circ \eta'_M$ by (3.4.v).

Finally, ε'_M reads $\xi \otimes v \otimes m \mapsto \xi(v)m$. Then

$$\begin{aligned}
(\varepsilon'_M \circ \mathbb{X}'_{EM})(v_k^* \otimes v_l \otimes m) &= \varepsilon'_M(-n\mathrm{id} - \Omega_{V\otimes M})(v_k^* \otimes v_l \otimes m) \\
&= -nv_k^*(v_l)m - \varepsilon'_M \circ \Omega_{V\otimes M})(v_k^* \otimes v_l \otimes m) \\
&= -n\delta_{kl}m - \varepsilon'_M\{\sum_{i,j} e(i,j)v_k^* \otimes (e(j,i)v_l \otimes m + v_l \otimes e(j,i)m)\} \\
&= -n\delta_{kl}m - \varepsilon'_M\{\sum_{j} -v_j^* \otimes (e(j,k)v_l \otimes m + v_l \otimes e(j,k)m)\} \\
&= -n\delta_{kl}m - \varepsilon'_M\{\sum_{j} -v_j^* \otimes (\delta_{kl}v_j \otimes m + v_l \otimes e(j,k)m)\} \\
&= -n\delta_{kl}m + n\delta_{kl}m + e(l,k)m = e(l,k)m
\end{aligned}$$

while

$$\begin{aligned}
\{\varepsilon'_M \circ (V^* \otimes \mathbb{X}_M)\}(v_k^* \otimes v_l \otimes m) &= \varepsilon'_M(v_k^* \otimes \sum_{i,j} e(i,j)v_l \otimes e(j,i)m) \\
= \varepsilon'_M(v_k^* \otimes \sum_i v_i \otimes e(l,i)m) = e(l,k)m.
\end{aligned}$$

Thus $\varepsilon'_M \circ \mathbb{X}'_{EM} = \varepsilon'_M \circ (V^* \otimes \mathbb{X}_M)$, and hence the assertion by (3.4.vi).

(3.6) $\forall a \in \Bbbk$, let E_a (resp. F_a) denote the direct summand of E (resp. F) given by the generalized a-eigenspace of $\mathbb{X}$ (resp. $\mathbb{X}'$) acting on E (resp. F): $\forall M \in \mathrm{Rep}(G)$,

$$\begin{aligned}
EM &= \coprod_{a\in\Bbbk}(E_aM) \quad \text{with} \quad E_aM = \cup_{r\in\mathbb{N}} \ker(\mathbb{X}_M - a\mathrm{id}_{EM})^r, \\
FM &= \coprod_{a\in\Bbbk}(F_aM) \quad \text{with} \quad F_aM = \cup_{r\in\mathbb{N}} \ker(\mathbb{X}'_M - a\mathrm{id}_{FM})^r.
\end{aligned}$$

As $\mathbb{X}_M$ and $\mathbb{X}'_M$ are G-equivariant, each E_a (resp. F_a) is a direct summand of E (resp. F) as an endofunctor on $\mathrm{Rep}(G)$.

Lemma [RW, 6.3]: *Let $a \in \Bbbk$.*

(i) The unit η and the counit ε of the adjunction (E, F) induce a unit $\eta_a : \mathrm{id} \to F_aE_a$ and a counit $\varepsilon_a : E_aF_a \to \mathrm{id}$, resp., making (E_a, F_a) into an adjoint pair.

(ii) The unit η' and the counit ε' induce a unit $\eta'_a : \mathrm{id} \to E_aF_a$ and a counit $\varepsilon'_a : F_aE_a \to \mathrm{id}$ of an adjunction (F_a, E_a).

Proof: (i) We first show that η (resp. ε) factors through $\coprod_{a\in\Bbbk}\eta_a : \mathrm{id} \to \coprod_{a\in\Bbbk} F_aE_a$ (resp. $\coprod_{a\in\Bbbk}\varepsilon_a : \coprod_{a\in\Bbbk} E_aF_a \to \mathrm{id}$)

(1)
$$\begin{tikzcd}
\mathrm{id} \arrow[r, "\eta"] \arrow[dr, dotted, "\coprod_{a\in\Bbbk}\eta_a"'] & FE \\
& \coprod_{a\in\Bbbk} F_aE_a \arrow[u, hook]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
EF \arrow[r, "\varepsilon"] \arrow[d] & \mathrm{id} \\
\coprod_{a\in\Bbbk} E_aF_a \arrow[ur, dotted, "\coprod_{a\in\Bbbk}\varepsilon_a"'] &
\end{tikzcd}$$

Let $M \in \mathrm{Rep}(G)$, $m \in M$ and $d = \dim FEM$. Let $\eta(m)_{ab}$ be the F_aE_bM component of $\eta_M(m)$. Then

$$\begin{aligned}
0 &= (\mathbb{X}'_{EM} - a\mathrm{id})^d \eta(m)_{ab} \quad \text{as } \eta(m)_{ab} \in F_a(E_bM) \\
&= ((V^* \otimes \mathbb{X}_M) - a\mathrm{id})^d \eta(m)_{ab} \quad \text{by (3.5.i)} \\
&= (V^* \otimes (\mathbb{X}_M - a\mathrm{id}))^d \eta(m)_{ab}.
\end{aligned}$$

On the other hand, $0 = (V^* \otimes (\mathbb{X}_M - b\mathrm{id}))^d \eta(m)_{ab}$ as $\eta(m)_{ab} \in V^* \otimes (E_bM)$. It follows that $\eta(m)_{ab} = 0$ unless $a = b$, and hence $\mathrm{im}(\eta_M) \leq \coprod_{a \in \Bbbk} F_aE_aM$.

Let next $x \in E_aF_bM$ with $a \neq b$. Take polynomials $\phi, \psi \in \Bbbk[t]$ with $(t-a)^d\phi + (t-b)^d\psi = 1$. Then

$$\begin{aligned}
\varepsilon_M(x) &= \varepsilon_M(\{\phi(\mathbb{X}_{FM})(\mathbb{X}_{FM} - a\mathrm{id})^d + \psi(\mathbb{X}_{FM})(\mathbb{X}_{FM} - b\mathrm{id})^d\}x) \\
&= \varepsilon_M(\psi(\mathbb{X}_{FM})(\mathbb{X}_{FM} - b\mathrm{id})^d x) \quad \text{as } x \in E_a(FM) \\
&= \varepsilon_M(\psi(\mathbb{X}_{FM})(V \otimes \mathbb{X}'_M - b\mathrm{id})^d x) \quad \text{by (3.5.i)} \\
&= \varepsilon_M(\psi(\mathbb{X}_{FM})(V \otimes (\mathbb{X}'_M - b\mathrm{id})^d)x) \\
&= 0 \quad \text{as } x \in E(F_bM),
\end{aligned}$$

and hence (1) holds.

Recall from (3.3.1) the adjunction $\mathrm{Rep}(G)(EM, M') \simeq \mathrm{Rep}(G)(M, FM')$ given by $f \mapsto (Ff) \circ \eta_M$ with inverse $g \mapsto \varepsilon_{M'} \circ Eg$. As each E_a (resp. F_a) is a direct summand of E (resp. F), one obtains commutative diagrams

$$\begin{tikzcd}
\mathrm{Rep}(G)(EM, M') \arrow[r, "F"] \arrow[d, equal] & \mathrm{Rep}(G)(FEM, FM') \arrow[r, "\mathrm{Rep}(G)(\eta_M, FM')"] \arrow[d, equal] & \mathrm{Rep}(G)(M, FM') \arrow[dd, equal] \\
\prod_a \mathrm{Rep}(G)(E_aM, M') \arrow[r, "\prod_a F"'] & \prod_a \mathrm{Rep}(G)(FE_aM, FM') \arrow[d, equal] & \\
& \prod_a \prod_b \mathrm{Rep}(G)(F_bE_aM, F_bM') \arrow[d] & \prod_a \mathrm{Rep}(G)(M, F_aM') \\
& \prod_a \mathrm{Rep}(G)(F_aE_aM, F_aM') \arrow[ur, "\prod_a \mathrm{Rep}(G)(\eta_{a,M}, F_aM')"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Rep}(G)(M, FM') \arrow[r, "E"] \arrow[d, equal] & \mathrm{Rep}(G)(EM, EFM') \arrow[r, "\mathrm{Rep}(G)(EM, \varepsilon_{M'})"] \arrow[d, equal] & \mathrm{Rep}(G)(EM, M') \arrow[dd, equal] \\
\prod_a \mathrm{Rep}(G)(M, F_aM') \arrow[r, dotted, "\prod_a E"'] & \prod_a \mathrm{Rep}(G)(EM, EF_aM') \arrow[d, equal] & \\
& \prod_a \prod_b \mathrm{Rep}(G)(E_bM, E_bF_aM') \arrow[d] & \prod_b \mathrm{Rep}(G)(E_bM, M'). \\
& \prod_b \mathrm{Rep}(G)(E_bM, E_bF_bM') \arrow[ur, "\prod_b \mathrm{Rep}(G)(E_bM, \varepsilon_{b,M'})"'] &
\end{tikzcd}$$

One thus obtains for each $a \in \Bbbk$ isomorphisms $\mathrm{Rep}(G)(E_aM, M') \simeq \mathrm{Rep}(G)(M, F_aM')$ via $f \mapsto F_a(f) \circ \eta_{a,M}$ and $\varepsilon_{a,M'} \circ E_a(g) \leftarrow\!\shortmid g$ inverse to each other.

(ii) As in (i) it suffices to show that the induced counit $\eta' : \mathrm{id} \to EF$ (resp. unit $\varepsilon' : FE \to \mathrm{id}$) factors through $\coprod_{a\in\Bbbk} E_aF_a$ (resp. $\coprod_{a\in\Bbbk} F_aE_a$)

$$\begin{tikzcd} \mathrm{id} \arrow[r, "\eta'"] \arrow[dr, dotted] & EF \\ & \coprod_a E_aF_a \arrow[u, hook] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FE \arrow[r, "\varepsilon'"] \arrow[d] & \mathrm{id}. \\ \coprod_a F_aE_a \arrow[ur, dotted] & \end{tikzcd}$$

Let $\eta'(m)_{ab}$ be the E_aF_bM-component of $\eta'_M(m)$. One has

$$0 = (\mathbb{X}_{FM} - a\mathrm{id})^d \eta'(m)_{ab} = ((V \otimes \mathbb{X}'_M) - a\mathrm{id})^d \eta'(m)_{ab} \quad \text{by (3.5.ii)}$$

while $0 = \{V \otimes (\mathbb{X}'_M - b\mathrm{id})\}^d \eta'_M(m)_{ab}$, and hence $\eta'_M(m) = 0$ unless $n + a = n + b$. Thus, $\mathrm{im}(\eta'_M) \leq \coprod_a E_aF_aM$.

Let finally $y \in F_aE_bM$ with $a \neq b$. Then, with $\phi, \psi \in \Bbbk[t]$ as above,

$$\begin{aligned} \varepsilon'_M(y) &= \varepsilon'_M(\{\phi(\mathbb{X}'_{EM})(\mathbb{X}'_{EM} - a\mathrm{id})^d + \psi(\mathbb{X}'_{EM})(\mathbb{X}'_{EM} - b\mathrm{id})^d\}y) = \varepsilon'_M(\psi(\mathbb{X}'_{EM})(\mathbb{X}'_{EM} - b\mathrm{id})^d y) \\ &= \varepsilon'_M(\psi(\mathbb{X}'_{EM})(V^* \otimes \mathbb{X}_M - b\mathrm{id})^d y) \quad \text{by (3.5.ii)} \\ &= 0, \quad \text{as desired.} \end{aligned}$$

(3.7) Recall now from (3.1) the affine Lie algebra $\widehat{\mathfrak{g}}_p$ over $\mathbb{C}$ and from (3.2) its natural representation nat_p.

Proposition [RW, 6.3]: *(i) $\forall a \in \Bbbk \setminus \mathbb{F}_p$, $E_a = 0 = F_a$, and hence $E = \coprod_{a\in\mathbb{F}_p} E_a$, $F = \coprod_{a\in\mathbb{F}_p} F_a$.*

(ii) Let $\phi : \mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)] \to \wedge^n(\mathrm{nat}_p)$ be a $\mathbb{C}$-linear isomorphism via

$$1 \otimes [\Delta(\lambda)] \mapsto m_{\lambda_1} \wedge m_{\lambda_2 - 1} \wedge \cdots \wedge m_{\lambda_n - n + 1} \quad \forall \lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+.$$

$\forall a \in \mathbb{F}_p$, regarding it as an element of $[0, p[$, one has a commutative diagram

$$\begin{tikzcd}[column sep=large] \wedge^n(\mathrm{nat}_p) \arrow[d, "\hat{e}_a"'] & \mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)] \arrow[l, "\phi", "\sim"'] \arrow[r, "\phi", "\sim"'] \arrow[d, shift right, "\mathbb{C}\otimes_{\mathbb{Z}}[E_a]"'] \arrow[d, shift left, "\mathbb{C}\otimes_{\mathbb{Z}}[F_a]"] & \wedge^n(\mathrm{nat}_p) \arrow[d, "\hat{f}_a"] \\ \wedge^n(\mathrm{nat}_p) & \mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)] \arrow[l, "\sim", "\phi"'] \arrow[r, "\sim", "\phi"'] & \wedge^n(\mathrm{nat}_p). \end{tikzcd}$$

Thus, we may regard the exact functors E_a, F_a, $a \in [0, p[$, as part of an action of $\widehat{\mathfrak{gl}_p}$ on $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)]$ through ϕ.

(iii) The "block" decomposition $\mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}(G)] = \coprod_{b\in\Lambda/\mathcal{W}_{a\bullet}} \mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}_b(G)]$ reads as the weight space decomposition of $\wedge^n(\mathrm{nat}_p)$ under ϕ; each $\phi(\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}_b(G)])$ provides a distinct weight

space on $\wedge^n(\mathrm{nat}_p)$ *of weight* $\sum_{i=1}^n (\lambda_i - i + 1)_1 \delta + \sum_{j=1}^p n_j \hat{\varepsilon}_j$ *with* $n_j = |\{k \in [1,n] | \lambda_k - k + 1 \equiv j \mod p\}|$ *if* $\lambda = (\lambda_1, \dots, \lambda_n) \in b$.

Proof: See (3.9) below.

(3.8) From (3.7.iii) we see that the set of weights of $\wedge^n(\mathrm{nat}_p)$ is

$$P(\wedge^n(\mathrm{nat}_p)) = \{k\delta + \sum_{i=1}^p n_i \hat{\varepsilon}_i | k \in \mathbb{Z}, n_i \in \mathbb{N}, \sum_{i=1}^p n_i = n\}.$$

We will denote the bijection $P(\wedge^n(\mathrm{nat}_p)) \to \Lambda/(\mathcal{W}_a \bullet)$ by ι_n. Note that $\Lambda/(\mathcal{W}_a \bullet)$ is infinite; $\Lambda = \mathbb{Z}\det \oplus \coprod_{i=1}^{n-1} \mathbb{Z}\varpi_i$ with $\mathcal{W}_a$ acting trivially on the $\mathbb{Z}\det$-component.

Let now $\varpi = \hat{\varepsilon}_1 + \cdots + \hat{\varepsilon}_n$. As $\phi([\Delta(n, \dots, n)])$ has weight ϖ, $\iota_n(\varpi) = \mathcal{W}_a \bullet (n, \dots, n) = \mathcal{W}_a \bullet n\det$ with $n\det \in A^+$. $\forall i \in [1, n[$, $\phi([\Delta(\underbrace{n, \dots, n}_{n-i}, n+1, n, \dots, n)])$ has weight $\varpi + \hat{\alpha}_i$, and hence $\iota_n(\varpi + \hat{\alpha}_i) = \mathcal{W}_a \bullet (\underbrace{n, \dots, n}_{n-i}, n+1, n, \dots, n) = \mathcal{W}_a \bullet (n\det + \varepsilon_{n-i+1})$. Put $\mu_{s_j} = n\det + \varepsilon_{j+1}$, $j \in [1, n[$. $\forall k \in [0, n[$,

$$\langle \mu_{s_j} + \zeta, \alpha_k^\vee \rangle = \begin{cases} 1 + \langle \varepsilon_{j+1}, \alpha_k^\vee \rangle & \text{if } k \neq 0, \\ n - 1 + \langle \varepsilon_{j+1}, \varepsilon_1^\vee - \varepsilon_n^\vee \rangle & \text{if } k = 0 \end{cases}$$
$$= \begin{cases} 0 & \text{if } k = j, \\ 2 & \text{if } k = j+1, \\ n-1 & \text{if } k = 0 \text{ and } j \neq n-1, \\ n-2 & \text{if } k = 0 \text{ and } j = n-1, \\ 1 & \text{else,} \end{cases}$$

and hence μ_{s_j} lies in the s_{α_j}-wall of A^+. For $\lambda \in \Lambda$, let us abbreviate $\mathcal{W}_a \bullet \lambda$ as $[\lambda]$, and write $i_{[\lambda]} : \mathrm{Rep}_{[\lambda]}(G) \hookrightarrow \mathrm{Rep}(G)$. Then

$$\begin{aligned} E_{n-j}|_{\mathrm{Rep}_{[n\det]}(G)} &= E_{n-j}|_{\mathrm{Rep}_{\iota_n(\varpi)}(G)} = \mathrm{pr}_{\iota_n(\varpi + \hat{\alpha}_{n-j})}(V \otimes ?) \quad \text{by (3.7)} \\ &= \mathrm{pr}_{[\mu_{s_j}]}(V \otimes \mathrm{pr}_{[n\det]}?) \circ i_{[n\det]} = \mathrm{pr}_{[\mu_{s_j}]}(\nabla(\varepsilon_1) \otimes \mathrm{pr}_{[n\det]}?) \circ i_{[n\det]}. \end{aligned}$$

We could abbreviate $\mathrm{pr}_{[\mu_{s_j}]}$ as $\mathrm{pr}_{\mu_{s_j}}$ after the convention in (1.10). As $\mu_{s_j} - n\det = \varepsilon_{j+1} \in \mathcal{W}\varepsilon_1$, $\mathrm{pr}_{[\mu_{s_j}]}(V \otimes \mathrm{pr}_{[n\det]}?)$ may be taken to be the translation functor $\mathrm{T}_{n\det}^{\mu_{s_j}}$ by (1.10), and hence

$$E_{n-j}|_{\mathrm{Rep}_{[n\det]}(G)} = \mathrm{T}_{n\det}^{\mu_{s_j}}|_{\mathrm{Rep}_{[n\det]}(G)}.$$

Likewise, as $n\det - \mu_{s_j} = -\varepsilon_{j+1} \in \mathcal{W}(-\varepsilon_n) = \mathcal{W}(-w_0\varepsilon_1)$ and as $V^* \simeq \nabla(-w_0\varepsilon_1)$, one may regard $F_{n-j}|_{\mathrm{Rep}_{[\mu_{s_j}]}(G)}$ as the translation functor $\mathrm{T}_{\mu_{s_j}}^{n\det}|_{\mathrm{Rep}_{[\mu_{s_j}]}(G)}$.

Consider next $\mu_{s_0} = (p+1, n, \dots, n) \in \Lambda^+$. $\forall k \in [0, n[$,

$$\langle \mu_{s_0} + \zeta, \alpha_k^\vee \rangle = \begin{cases} p & \text{if } k = 0, \\ p - n + 2 & \text{if } k = 1, \\ 1 & \text{else,} \end{cases}$$

and hence μ_{s_0} lies in the $s_{\alpha_0,1}$-wall of A^+.

Corollary [RW, Rmk. 6.4.7]: *(i) $\forall j \in [1,n[$, one may regard E_{n-j} (resp., F_{n-j}) as the translation functor $\mathrm{T}_{n\det}^{\mu_{s_j}}$ (resp. $\mathrm{T}_{\mu_{s_j}}^{n\det}$) restricted to $\mathrm{Rep}_{[n\det]}(G)$ (resp. $\mathrm{Rep}_{[\mu_{s_j}]}(G)$).*

(ii) One may take $E_0E_{p-1}\dots E_{n+1}E_n|_{\mathrm{Rep}_{[n\det]}(G)}$ (resp. $F_nF_{n+1}\dots F_{p-1}F_0|_{\mathrm{Rep}_{[\mu_{s_0}]}(G)}$) to be the translation functor $\mathrm{T}_{n\det}^{\mu_{s_0}}$ (resp. $\mathrm{T}_{\mu_{s_0}}^{n\det}$) restricted to $\mathrm{Rep}_{[n\det]}(G)$ (resp. $\mathrm{Rep}_{[\mu_{s_0}]}(G)$).

Proof: We have only to show (ii). One checks first that $\phi([\Delta(\varepsilon_1+n\det)])$ has weight $\hat{\varepsilon}_1+\cdots+\hat{\varepsilon}_{n-1}+\hat{\varepsilon}_{n+1}=\varpi+\hat{\alpha}_n$, and that $\forall i\in[0,n[$,

$$\langle \varepsilon_1+n\det+\zeta,\alpha_i^\vee\rangle=\begin{cases} n & \text{if } i=0\\ 2 & \text{if } i=1\\ 1 & \text{else.}\end{cases}$$

Thus, $\iota_n(\varpi+\hat{\alpha}_n)\ni\varepsilon_1+n\det=(n+1,n,\dots,n)\in A^+$. Then

$$\begin{aligned} E_n|_{\mathrm{Rep}_{\iota_n(\varpi)}(G)} &= \mathrm{pr}_{\iota_n(\varpi+\hat{\alpha}_n)}(V\otimes ?) \quad \text{by (3.7)}\\ &= \mathrm{pr}_{[\varepsilon_1+n\det]}(\nabla(\varepsilon_1)\otimes \mathrm{pr}_{[n\det]}?)\circ i_{[n\det]}. \end{aligned}$$

As $\varepsilon_1=(\varepsilon_1+n\det)-n\det$, one may take $\mathrm{pr}_{[\varepsilon_1+n\det]}(\nabla(\varepsilon_1)\otimes\mathrm{pr}_{n\det}?)$ to be $\mathrm{T}_{n\det}^{\varepsilon_1+n\det}$ by (1.10).

One checks next that $\phi([\Delta(n+2,n,\dots,n)])$ has weight $\varpi+\hat{\alpha}_n+\hat{\alpha}_{n+1}$ and that $(n+2,n,\dots,n)=2\varepsilon_1+n\det\in\overline{A^+}$. Then

$$E_{n+1}|_{\mathrm{Rep}_{\iota_n(\varpi+\hat{\alpha}_n)}(G)}=\mathrm{pr}_{\iota_n(\varpi+\hat{\alpha}_n+\hat{\alpha}_{n+1})}(V\otimes ?)=\mathrm{pr}_{[2\varepsilon_1+n\det]}(\nabla(\varepsilon_1)\otimes\mathrm{pr}_{[\varepsilon_1+n\det]}?)\circ i_{[\varepsilon_1+n\det]}.$$

As $\varepsilon_1=(2\varepsilon_1+n\det)-(\varepsilon_1+n\det)$, one may take $\mathrm{pr}_{[2\varepsilon_1+n\det]}(\nabla(\varepsilon_1)\otimes\mathrm{pr}_{[\varepsilon_1+n\det]}?)$ to be $\mathrm{T}_{\varepsilon_1+n\det}^{2\varepsilon_1+n\det}$ by (1.10) again. If $2\varepsilon_1+n\det\notin A^+$, repeat the argument to find $\iota_n(\varpi+\hat{\alpha}_n+\hat{\alpha}_{n+1}+\cdots+\hat{\alpha}_{p-1})\ni(p-n)\varepsilon_1+n\det=(p,n,\dots,n)\in A^+$, and that

$$\begin{aligned} E_{p-1}|_{\mathrm{Rep}_{\iota_n(\varpi+\hat{\alpha}_n+\cdots+\hat{\alpha}_{p-2})}(G)} &= \mathrm{pr}_{\iota_n(\varpi+\hat{\alpha}_n+\cdots+\hat{\alpha}_{p-1}))}(V\otimes ?)\\ &= \mathrm{pr}_{[(p-n)\varepsilon_1+n\det]}(\nabla(\varepsilon_1)\otimes\mathrm{pr}_{[(p-n-1)\varepsilon_1+n\det]}?)\circ i_{[(p-n-1)\varepsilon_1+n\det]} \end{aligned}$$

with $\mathrm{pr}_{[(p-n)\varepsilon_1+n\det]}(\nabla(\varepsilon_1)\otimes\mathrm{pr}_{(p-n-1)\varepsilon_1+n\det}?)$ one may take to be $\mathrm{T}_{(p-n-1)\varepsilon_1+n\det}^{(p-n)\varepsilon_1+n\det}$.

Finally, $\phi([\Delta(p+1,n,\dots,n)])$ has weight $\delta+2\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_{n-1}=\varpi+\hat{\alpha}_n+\hat{\alpha}_{n+1}+\cdots+\hat{\alpha}_{p-1}+\hat{\alpha}_0$ with $(p+1,n,\dots,n)=(p+1-n)\varepsilon_1+n\det=\mu_{s_0}$. Then

$$E_0|_{\mathrm{Rep}_{\iota_n(\varpi+\hat{\alpha}_n+\cdots+\hat{\alpha}_{p-1})}(G)}=\mathrm{pr}_{[(p+1-n)\varepsilon_1+n\det]}(\nabla(\varepsilon_1)\otimes\mathrm{pr}_{(p-n)\varepsilon_1+n\det}?)\circ i_{[(p-n)\varepsilon_1+n\det]}$$

with $\mathrm{pr}_{[(p+1-n)\varepsilon_1+n\det]}(\nabla(\varepsilon_1)\otimes\mathrm{pr}_{[(p-n)\varepsilon_1+n\det]}?)$ one may take to be $\mathrm{T}_{(p-n)\varepsilon_1+n\det}^{(p+1-n)\varepsilon_1+n\det}=\mathrm{T}_{(p-n)\varepsilon_1+n\det}^{\mu_{s_0}}$.

Put $E'=\mathrm{T}_{(p-n)\varepsilon_1+n\det}^{\mu_{s_0}}\mathrm{T}_{(p-n-1)\varepsilon_1+n\det}^{(p-n)\varepsilon_1+n\det}\dots\mathrm{T}_{n\det}^{\varepsilon_1+n\det}$. Thus, E' is a direct summand of $\mathrm{pr}_{\mu_{s_0}}\circ(V^{\otimes_{p-n+1}}\otimes\mathrm{pr}_{n\det}?)$ while $\mathrm{T}_{n\det}^{\mu_{s_0}}\simeq\mathrm{pr}_{\mu_{s_0}}\circ(\nabla((p-n+1)\varepsilon_1)\otimes\mathrm{pr}_{n\det}?)$ as $\mu_{s_0}-n\det=(p-n+1)\varepsilon_1$. Recall from (1.8) that one has an epi $V^{\otimes_{p-n+1}}=\nabla(\varepsilon_1)^{\otimes_{p-n+1}}\to\nabla((p-n+1)\varepsilon_1)$. There follows a morphism of functors $E'\to\mathrm{T}_{n\det}^{\mu_{s_0}}$. As $i\varepsilon_1+n\det\in A^+$ $\forall i\in[0,p-n[$, and as μ_{s_0} is lying on the $s_{\alpha_0,1}$-face of A^+, $\forall \xi \in \Lambda^+ \cap \{\mathcal{W}_a \bullet (n\det)\}$, $\forall x \in {}^f\mathcal{W}$ with $x \bullet n\det < xs_{\alpha_0,1} \bullet n\det$, chasing a highest weight vector yields a nonzero morphism $E'\nabla(x \bullet n\det) \to \mathrm{T}^{\mu_{s_0}}_{n\det}\nabla(x \bullet n\det)$:

$$\begin{tikzcd}
E'\nabla(x \bullet n\det) \arrow[rr] \arrow[dr, "\sim"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathrm{T}^{\mu_{s_0}}_{n\det}\nabla(x \bullet n\det), \arrow[dl, "\sim"] \\
& \nabla(x \bullet \mu_{s_0}) &
\end{tikzcd}$$

which is therefore invertible; $\mathrm{Rep}(G)(\nabla(x \bullet \mu_{s_0}), \nabla(x \bullet \mu_{s_0})) \simeq \Bbbk$. In turn, the isomorphism $E'\nabla(x \bullet n\det) \to \mathrm{T}^{\mu_{s_0}}_{n\det}\nabla(x \bullet n\det)$ induces an isomorphism $E'L(x \bullet n\det) \to \mathrm{T}^{\mu_{s_0}}_{n\det}L(x \bullet n\det)$. As $E'L(xs_{\alpha_0,1} \bullet n\det) = 0 = \mathrm{T}^{\mu_{s_0}}_{n\det}L(xs_{\alpha_0,1} \bullet n\det)$, the morphism $E' \to \mathrm{T}^{\mu_{s_0}}_{n\det}$ induces an isomorphism $E'L(y \bullet n\det) \to \mathrm{T}^{\mu_{s_0}}_{n\det}L(y \bullet n\det)$ $\forall y \in {}^f\mathcal{W}$, and hence $\mathrm{T}^{\mu_{s_0}}_{n\det} \simeq E'$ by the 5-lemma.

Likewise, if we put $F' = \mathrm{T}^{n\det}_{\varepsilon_1+n\det} \dots \mathrm{T}^{(p-n-1)\varepsilon_1+n\det}_{(p-n)\varepsilon_1+n\det}\mathrm{T}^{(p-n)\varepsilon_1+n\det}_{\mu_{s_0}}$, there is a morphism of functors

$$\begin{tikzcd}
F' \arrow[r, dotted] \arrow[d, hook] & \mathrm{T}^{n\det}_{\mu_{s_0}} \arrow[d, dash, "\sim"] \\
\mathrm{pr}_{n\det} \circ (\nabla(-\varepsilon_n)^{\otimes p-n+1} \otimes \mathrm{pr}_{\mu_{s_0}}?) \arrow[r] & \mathrm{pr}_{n\det} \circ (\nabla(-(p-n+1)\varepsilon_n) \otimes \mathrm{pr}_{\mu_{s_0}}?).
\end{tikzcd}$$

For each $x \in \mathcal{W}_a$ with $x \bullet \mu_{s_0} \in \Lambda^+$ we may assume $x \bullet n\det < xs_{\alpha_0,1} \bullet n\det$. Chasing a highest weight vector again yields a commutative diagram

$$\begin{tikzcd}
F'\nabla(x \bullet \mu_{s_0}) \arrow[r, two heads] \arrow[d] & \nabla(xs_{\alpha_0,1} \bullet n\det), \\
\mathrm{T}^{n\det}_{\mu_{s_0}}\nabla(x \bullet \mu_{s_0}) \arrow[ur, two heads] &
\end{tikzcd}$$

and hence a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \nabla(x \bullet n\det) \arrow[r] \arrow[d, equal] & F'\nabla(x \bullet \mu_{s_0}) \arrow[r] \arrow[d] & \nabla(xs_{\alpha_0,1} \bullet n\det) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \nabla(x \bullet n\det) \arrow[r] & \mathrm{T}^{n\det}_{\mu_{s_0}}\nabla(x \bullet \mu_{s_0}) \arrow[r] & \nabla(xs_{\alpha_0,1} \bullet n\det). \arrow[r] & 0
\end{tikzcd}$$

Then the middle vertical arrow is invertible by the 5-lemma. There follows an isomorphism $F' \to \mathrm{T}^{n\det}_{\mu_{s_0}}$ by the 5-lemma.

(3.9) Analogous assertions hold for G_1T-modules with $\wedge^n$ replaced by $\otimes^n$ and $\Delta(\lambda)$, $\lambda \in \Lambda^+$, by $\hat{\Delta}(\lambda) = \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(B_1^+)} \lambda$, $\lambda \in \Lambda$. As the $[\hat{\Delta}(\lambda)]$, $\lambda \in \Lambda$, do not span the whole of $\mathrm{Rep}(G_1T)$ [J, II.9.9], we consider the additive full subcategory $\mathrm{Rep}'(G_1T)$ of $\mathrm{Rep}(G_1T)$ consisting of those admitting a filtration with subquotients $\hat{\Delta}(\lambda)$, $\lambda \in \Lambda$, and hence the Grothendieck group $[\mathrm{Rep}'(G_1T)]$ of $\mathrm{Rep}'(G_1T)$ has $\mathbb{Z}$-basis $[\hat{\Delta}(\lambda)]$, $\lambda \in \Lambda$; although $\mathrm{Rep}'(G_1T)$ does not form a Serre subcategory of $\mathrm{Rep}(G_1T)$ we may talk about its Grothendieck group [CR, 16.3].

Note that, as $\eta'_{\Bbbk}$ and $\mathfrak{a}$ are both G-equivariant, $\mathbb{X}_M$ is G_1T-equivariant $\forall M \in \mathrm{Rep}(G_1T)$, and hence all E_a, $a \in \Bbbk$, are G_1T-equivariant on $\mathrm{Rep}(G_1T)$. Likewise for the F_a's. One could also argue with (3.3.ii).

Proposition: *(i) $\forall a \in \Bbbk \setminus \mathbb{F}_p$, $E_a = 0 = F_a$, and hence $E = \coprod_{a\in\mathbb{F}_p} E_a$, $F = \coprod_{a\in\mathbb{F}_p} F_a$.*

(ii) Let $\phi' : \mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}'(G_1T)] \to \otimes^n(\mathrm{nat}_p)$ be a $\mathbb{C}$-linear isomorphism via

$$[\hat{\Delta}(\lambda)] \mapsto m_{\lambda_1} \otimes m_{\lambda_2-1} \otimes \cdots \otimes m_{\lambda_n-n+1} \quad \forall \lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda.$$

$\forall a \in \mathbb{F}_p$, regarding it as an element of $[0, p[$, one has a commutative diagram

$$\begin{tikzcd}
\otimes^n(\mathrm{nat}_p) \arrow[d, "\hat{e}_a"'] & \mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}'(G_1T)] \arrow[l, "\phi'"', "\sim"] \arrow[r, "\phi'", "\sim"'] \arrow[d, shift right, "\mathbb{C}\otimes_{\mathbb{Z}}[E_a]"'] \arrow[d, shift left, "\mathbb{C}\otimes_{\mathbb{Z}}[F_a]"] & \otimes^n(\mathrm{nat}_p) \arrow[d, "\hat{f}_a"] \\
\otimes^n(\mathrm{nat}_p) & \mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}'(G_1T)] \arrow[l, "\sim", "\phi'"'] \arrow[r, "\sim", "\phi'"'] & \otimes^n(\mathrm{nat}_p).
\end{tikzcd}$$

Thus, we may regard the exact functors E_a, F_a, $a \in [0, p[$, as part of an action of $\widehat{\mathfrak{gl}_p}$ on $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}'(G_1T)]$ through ϕ'.

(iii) The "block" decomposition $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}'(G_1T)] = \coprod_{b\in\Lambda/\mathcal{W}_a\bullet} \mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}'_b(G_1T)]$ reads as the weight space decomposition of $\otimes^n(\mathrm{nat}_p)$ under ϕ'; each $\phi'(\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}'_b(G_1T)])$ provides a distinct weight space on $\otimes^n(\mathrm{nat}_p)$ of weight $\sum_{i=1}^n (\lambda_i - i + 1)_1\delta + \sum_{j=1}^p n_j\hat{\varepsilon}_j$ with $n_j = |\{k \in [1, n] | \lambda_k - k + 1 \equiv j \mod p\}|$ if $\lambda = (\lambda_1, \dots, \lambda_n) \in b$.

Proof: Let $\mathbb{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let $C = \sum_{i,j=1}^n e(i,j)e(j,i) \in \mathbb{U}(\mathfrak{g})$ be the Casimir element with respect to the trace form on V: $\mathrm{Tr}(e(j,i)e(k,l)) = \delta_{ik}\delta_{jl}$. Then

$$(1) \qquad C \text{ is central in } \mathbb{U}(\mathfrak{g}).$$

For let $x \in \mathfrak{g}$. Enumerate the $e(i,j)$ as $x_1, \dots, x_N$, $N = n^2$, and let $y_1, \dots, y_N$ be their dual basis with respect to the trace form. In $\mathbb{U}(\mathfrak{g})$

$$Cx = \sum_{i=1}^N x_iy_ix = \sum_{i=1}^N ([x_iy_i, x] + xx_iy_i) = xC + \sum_{i=1}^N [x_iy_i, x]$$

with $[x_iy_i, x] = [x_i, x]y_i + x_i[y_i, x]$. Write $[x_i, x] = \sum_{j=1}^N \xi_{ij}x_i$ and $[y_i, x] = \sum_{j=1}^N \xi'_{ij}y_i$ for some $\xi_{ij}, \xi'_{ij} \in \Bbbk$. Then $\xi_{ij} = \mathrm{Tr}([x_i, x]y_j) = \mathrm{Tr}(x_i[x, y_j]) = -\xi'_{ji}$, and hence $[x_i, x]y_i = \sum_{j=1}^N \xi_{ji}x_jy_i = -\sum_{j=1}^N \xi'_{ji}x_jy_i$ while $x_i[y_i, x] = \sum_{j=1}^N \xi'_{ij}x_iy_j$. It follows that

$$\sum_{i=1}^N [x_iy_i, x] = \sum_{i=1}^N ([x_i, x]y_i + x_i[y_i, x]) = \sum_{i=1}^N (-\sum_{j=1}^N \xi'_{ji}x_jy_i + \sum_{j=1}^N \xi'_{ij}x_iy_j) = 0,$$

and hence $Cx = xC$. As $\mathrm{Dist}(G) = \mathrm{Dist}(G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \Bbbk$, C is central in $\mathrm{Dist}(G)$ also.

Let us denote by $\Delta : \mathbb{U}(\mathfrak{g}) \to \mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{g})$ the comultiplication on $\mathbb{U}(\mathfrak{g})$. Then in $\mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{g})$ one has

$$\begin{aligned}\Delta(C) &= \sum_{i,j} (e(j,i) \otimes 1 + 1 \otimes e(j,i))(e(i,j) \otimes 1 + 1 \otimes e(i,j)) \\ &= \sum_{i,j} (e(j,i)e(i,j) \otimes 1 + e(i,j) \otimes e(j,i) + e(j,i) \otimes e(i,j) + 1 \otimes e(j,i)e(i,j)),\end{aligned}$$

and hence

$$\Omega = \frac{1}{2}\{\Delta(C) - C \otimes 1 - 1 \otimes C\}, \tag{2}$$

which also explains (3.3.ii) at least when $p \neq 2$. Write $C = 2\sum_{i<j} e(j,i)e(i,j) + \sum_{i=1}^n e(i,i)^2 + \sum_{i<j}(e(i,i) - e(j,j))$.

Let $\lambda = (\lambda_1, \dots, \lambda_n) = \sum_{i=1}^n \lambda_i \varepsilon_i \in \Lambda$. As $\hat{\Delta}(\lambda) = \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(B_1^+)} \lambda$ and as $\mathbb{U}(\mathfrak{g}) \twoheadrightarrow \mathrm{Dist}(G_1)$, C acts on $\hat{\Delta}(\lambda)$ by the scalar

$$b_\lambda := \sum_{i=1}^n \lambda_i^2 + \sum_{i<j} (\lambda_i - \lambda_j). \tag{3}$$

For $e(i,i)$ acts on $1 \otimes 1$ by scalar $\lambda(e(i,i)) = \lambda_i$. If $i > j$, $e(j,i) \in \mathrm{Dist}(U_1^+)$ annihilates $1 \otimes 1$, and hence, for $i < j$, $e(i,j)e(j,i) = e(j,i)e(i,j) + [e(i,j), e(j,i)] = e(j,i)e(i,j) + e(i,i) - e(j,j)$ acts on $1 \otimes 1$ by scalar $\lambda(e(i,i) - e(j,j)) = \lambda_i - \lambda_j$.

One has

$$\begin{aligned}E\hat{\Delta}(\lambda) = V \otimes \hat{\Delta}(\lambda) &= V \otimes \mathrm{ind}_{B^+}^{G_1B^+}(\lambda - 2(p-1)\rho) \quad [\text{J, II.9.2}] \\ &\simeq \mathrm{ind}_{B^+}^{G_1B^+}(V \otimes (\lambda - 2(p-1)\rho)) \quad \text{by the tensor identity [J, I.3.6]},\end{aligned}$$

and hence $E\hat{\Delta}(\lambda)$ admits a filtration with the subquotients $\hat{\Delta}(\lambda + \varepsilon_i)$, $i \in [1, n]$. As C acts on $V \otimes \hat{\Delta}(\lambda)$ through the comultiplication and as $V = \Delta(\varepsilon_1)$, we see from (2) and (3) that Ω acts on $\hat{\Delta}(\varepsilon_i + \lambda)$ by scalar

$$\frac{1}{2}(b_{\lambda+\varepsilon_i} - b_{\varepsilon_1} - b_\lambda) = \lambda_i - i + 1. \tag{4}$$

It follows from (3.4.i) that all the eigenvalues of $\mathbb{X}_{\hat{\Delta}(\lambda)}$ on $E\hat{\Delta}(\lambda)$ belong to $\mathbb{F}_p$. Thus, $\prod_{a \in \mathbb{F}_p} (\mathbb{X}_M - a)^{\dim M}$ annihilates any $M \in \mathrm{Rep}'(G_1T)$. Then $E_a = 0$ unless $a \in \mathbb{F}_p$, and hence $E = \coprod_{a \in \mathbb{F}_p} E_a$.

By (4) $\forall a \in \mathbb{F}_p$ $\forall \lambda \in \Lambda$,

$$[E_a][\hat{\Delta}(\lambda)] = \sum_{\substack{i \in [1,n] \\ \lambda_i - i + 1 \equiv a \bmod p}} [\hat{\Delta}(\lambda + \varepsilon_i)]. \tag{5}$$

For $\mu \in \Lambda$ write $\lambda \to_a \mu$ iff there is $i \in [1, n]$ with $\lambda_i - i + 1 \equiv a \mod p$ such that $\mu = \lambda + \varepsilon_i$. Then (5) reads

$$[E_a][\hat{\Delta}(\lambda)] = \sum_{\substack{\mu \in \Lambda \\ \lambda \to_a \mu}} [\hat{\Delta}(\mu)]. \tag{6}$$

Turning to F, as $F\hat{\Delta}(\lambda) = V^* \otimes \hat{\Delta}(\lambda) \simeq \mathrm{ind}_{B^+}^{G_1B^+}(V^* \otimes (\lambda - 2(p-1)\rho))$, the subquotients of $F\hat{\Delta}(\lambda)$ in its $\hat{\Delta}$-filtration are $\hat{\Delta}(\lambda - \varepsilon_i)$, $i \in [1, n]$. It follows that the eigenvalues of $\mathbb{X}_{\hat{\Delta}(\lambda)}$ on $F\hat{\Delta}(\lambda)$ are, as $V^* = \Delta(-\varepsilon_n)$, $-n - \frac{1}{2}(b_{\lambda-\varepsilon_i} - b_{-\varepsilon_n} - b_\lambda) = \lambda_i - i$ by (3.4). Then $F_a = 0$ unless $a \in \mathbb{F}_p$, and hence $F = \coprod_{a\in\mathbb{F}_p} F_a$. $\forall a \in \mathbb{F}_p$ $\forall \lambda \in \Lambda$,

(7) $$[F_a][\hat{\Delta}(\lambda)] = \sum_{\substack{i\in[1,n] \\ \lambda_i - i \equiv a \bmod p}} [\hat{\Delta}(\lambda - \varepsilon_i)] = \sum_{\substack{\mu\in\Lambda \\ \mu\to_a\lambda}} [\hat{\Delta}(\mu)].$$

Now,

$$(\phi' \circ [E_a])[\hat{\Delta}(\lambda)] = \phi'(\sum_{\substack{\mu\in\Lambda \\ \lambda\to_a\mu}} [\hat{\Delta}(\mu)]) = \sum_{\substack{\mu\in\Lambda \\ \lambda\to_a\mu}} m_{\mu_1} \otimes m_{\mu_2-1} \otimes \cdots \otimes m_{\mu_n-n+1}$$

while

$$\begin{aligned}(\hat{e}_a \circ \phi')[\hat{\Delta}(\lambda)] &= \hat{e}_a(m_{\lambda_1} \otimes m_{\lambda_2-1} \otimes \cdots \otimes m_{\lambda_n-n+1}) \\ &= (\hat{e}_a m_{\lambda_1}) \otimes m_{\lambda_2-1} \otimes \cdots \otimes m_{\lambda_n-n+1} \\ &\quad + m_{\lambda_1} \otimes (\hat{e}_a m_{\lambda_2-1}) \otimes m_{\lambda_3-2} \otimes \cdots \otimes m_{\lambda_n-n+1} + \dots \\ &\quad + m_{\lambda_1} \otimes \cdots \otimes m_{\lambda_{n-1}-n+2} \otimes (\hat{e}_a m_{\lambda_n-n+1}).\end{aligned}$$

For $\mu \in \Lambda$ with $\lambda \to_a \mu$ there is $j \in [1, n]$ with $\lambda_j - j + 1 \equiv a \mod p$ such that $\forall k \in [1, n]$,

$$\mu_k = \begin{cases} \lambda_k + 1 & \text{if } k = j, \\ \lambda_k & \text{else.} \end{cases}$$

On the other hand, by (3.2.1)

$$\hat{e}_a m_{\lambda_i - i+1} = \begin{cases} m_{\lambda_i-i+2} & \text{if } \lambda_i - i + 1 \equiv a \mod p, \\ 0 & \text{else.} \end{cases}$$

Thus,

$$\begin{aligned}(\hat{e}_a \circ \phi')[\hat{\Delta}(\lambda)] &= \sum_{\substack{i \\ \lambda_i-i+1\equiv a \bmod p}} m_{\lambda_1} \otimes m_{\lambda_2-1} \otimes \cdots \otimes m_{\lambda_{i-1}-i+2} \otimes m_{\lambda_i-i+2} \otimes m_{\lambda_{i+1}-i} \otimes \\ &\qquad\qquad \cdots \otimes m_{\lambda_n-n+1} \\ &= (\phi' \circ [E_a])[\hat{\Delta}(\lambda)].\end{aligned}$$

Likewise, $\hat{f}_a \circ \phi' = \phi' \circ [F_a]$ $\forall a \in [0, p[$.

(iii) The weight of $m_{\nu_1} \otimes \cdots \otimes m_{\nu_n} \in \otimes^n(\mathrm{nat}_p)$ is, writing $\nu_i = \nu_{i0} + \nu_{i1}p$ with $\nu_{i0} \in [1, p]$,

$$(\hat{\varepsilon}_{\nu_{10}} + \nu_{11}\delta) + \cdots + (\hat{\varepsilon}_{\nu_{n0}} + \nu_{n1}\delta) = (\sum_{i=1}^n \nu_{i1})\delta + \sum_{i=1}^n \hat{\varepsilon}_{\nu_{i0}} = (\sum_{i=1}^n \nu_{i1})\delta + \sum_{j=1}^p n_j \hat{\varepsilon}_j$$

with $n_j = |\{i \in [1, n] | \nu_{i0} = j\}| = |\{i \in [1, n] | \nu_i \equiv j \mod p\}|$; in particular, $\sum_j n_j = n$. It follows $\forall \lambda, \mu \in \Lambda$ that $\phi'([\hat{\Delta}(\lambda)]) = m_{\lambda_1} \otimes m_{\lambda_2-1} \otimes \cdots \otimes m_{\lambda_n-n+1}$ and $\phi'([\hat{\Delta}(\mu)]) = m_{\mu_1} \otimes$

$m_{\mu_2-1}\otimes\cdots\otimes m_{\mu_n-n+1}$ have the same weight iff
$\sum_{i=1}^n(\lambda_i-i+1)_1=\sum_{i=1}^n(\mu_i-i+1)_1$ and $\forall j\in[1,p]$, $|\{i\in[1,n]|\lambda_i-i+1\equiv j \mod p\}|=|\{i\in[1,n]|\mu_i-i+1\equiv j \mod p\}|$
iff $\sum_{i=1}^n(\lambda+\zeta)_{i1}=\sum_{i=1}^n(\mu+\zeta)_{i1}$ and $\forall j\in[1,p]$, $|\{i\in[1,n]|(\lambda+\zeta)_i\equiv j \mod p\}|=|\{i\in[1,n]|(\mu+\zeta)_i\equiv j \mod p\}|$ as $\zeta=(0,-1,\dots,-n+1)$
iff $\exists\sigma\in\mathfrak{S}_n$ and $\nu_1,\dots,\nu_n\in\mathbb{Z}$ with $\nu_1+\cdots+\nu_n=0$: $(\lambda+\zeta)-\sigma(\mu+\zeta)=p(\nu_1,\dots,\nu_n)$
iff $\lambda+\zeta\in\mathcal{W}_a(\mu+\zeta)$ as $\{(\nu_1,\dots,\nu_n)\in\mathbb{Z}^{\oplus n}|\nu_1+\cdots+\nu_n=0\}=\mathbb{Z}R$
iff $\lambda\in\mathcal{W}_a\bullet\mu$, as desired.

(3.10) Let $a\in[0,p[$. We have seen above that $\mathbb{C}\otimes[\mathrm{Rep}'(G_1T)]$ admits a structure of $\mathfrak{sl}_2(\mathbb{C})$-module such that

$$x=\begin{pmatrix}0&1\\0&0\end{pmatrix}\mapsto\mathbb{C}\otimes[E_a]\quad\text{and}\quad y=\begin{pmatrix}0&0\\1&0\end{pmatrix}\mapsto\mathbb{C}\otimes[F_a].$$

We show that the action extends to $\mathbb{C}\otimes[\mathrm{Rep}(G_1T)]$.

Corollary: *(i) There is a structure of $\mathfrak{sl}_2(\mathbb{C})$-module on $\mathbb{C}\otimes[\mathrm{Rep}(G_1T)]$ such that $x\mapsto\mathbb{C}\otimes[E_a]$ and $y\mapsto\mathbb{C}\otimes[F_a]$. As such, each $1\otimes[\hat{L}(\lambda)]$, $\lambda\in\Lambda$, has weight $\{\sum_{i=1}^n(\lambda_i-i+1)_1\delta+\sum_{j=1}^p n_j\hat{\varepsilon}_j\}(\hat{h}_a)$ with respect to $[x,y]$. Thus, $\mathrm{Rep}(G_1T)$ provides an $\mathfrak{sl}_2$-categorification of $\mathbb{C}\otimes\mathbb{Z}[\mathrm{Rep}(G_1T)]$ in the sense of [ChR]/[Ro].*

(ii) $\forall j\in[1,n[$, one may regard E_{n-j} (resp., F_{n-j}) as the translation functor $\mathrm{T}_{n\det}^{\mu_{s_j}}$ (resp. $\mathrm{T}^{n\det}_{\mu_{s_j}}$) restricted to $\mathrm{Rep}_{[n\det]}(G_1T)$ (resp. $\mathrm{Rep}_{[\mu_{s_j}]}(G_1T)$). Also, one may take $E_0E_{p-1}\dots E_{n+1}E_n|_{\mathrm{Rep}_{[n\det]}(G_1T)}$ (resp. $F_nF_{n+1}\dots F_{p-1}F_0|_{\mathrm{Rep}_{[\mu_{s_0}]}(G_1T)}$) to be the translation functor $\mathrm{T}_{n\det}^{\mu_{s_0}}$ (resp. $\mathrm{T}^{n\det}_{\mu_{s_0}}$) restricted to $\mathrm{Rep}_{[n\det]}(G_1T)$ (resp. $\mathrm{Rep}_{[\mu_{s_0}]}(G_1T)$).

Proof: (i) As E_a and F_a are exact on $\mathrm{Rep}(G_1T)$, they define

$$[E_a],[F_a]\in\mathrm{Mod}_{\mathbb{Z}}([\mathrm{Rep}(G_1T)],[\mathrm{Rep}(G_1T)]),$$

and hence also $\mathbb{C}\otimes_{\mathbb{Z}}[E_a],\mathbb{C}\otimes_{\mathbb{Z}}[F_a]\in\mathrm{Mod}_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}(G_1T)],\mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}(G_1T)])$, which we will abbreviate as $[E_a]$ and $[F_a]$, resp. We thus get a $\mathbb{C}$-algebra homomorphism $\theta:T_{\mathbb{C}}(x,y)\to\mathrm{Mod}_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}(G_1T)],\mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}(G_1T)])$ such that $x\mapsto[E_a]$ and $y\mapsto[F_a]$, where $T_{\mathbb{C}}(x,y)$ denotes the tensor algebra of 2-dimensional $\mathbb{C}$-linear space $\mathbb{C}x\oplus\mathbb{C}y$. Put $z=x\otimes y-y\otimes x$. We show that

$$z\otimes x-x\otimes z-2x,\,z\otimes y-y\otimes z+2y\in\ker\theta,$$

and hence $\mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}(G_1T)]$ is equipped with a structure of $\mathbb{U}(\mathfrak{sl}_2(\mathbb{C}))$-module.

Now, we know from (3.9) that both $z\otimes x-x\otimes z-2x$ and $z\otimes y-x\otimes z+2y$ annihilate $\mathbb{C}$-linear subspace $\mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}'(G_1T)]$ of $\mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}(G_1T)]$. We are to show that they both annihilate $[\hat{L}(\lambda)]$ $\forall\lambda\in\Lambda$. We have an exact sequence of G_1T-modules

$$0\to M'\to M_r\to\cdots\to M_1\to\hat{L}(\lambda)\to 0$$

such that all $M_i\in\mathrm{Rep}'(G)$ and that all of the composition factors $\hat{L}(\mu)$ of M' have $\mu\ll\lambda$. As $\hat{\Delta}(\mu)\twoheadrightarrow\hat{L}(\mu)$, the composition factors of $E_a\hat{L}(\mu)$ (resp. $F_a\hat{L}(\mu)$) are among those of

$E_a\hat{\Delta}(\mu)$ (resp. $F_a\hat{\Delta}(\mu)$). For $X \in [\mathrm{Rep}(G_1T)]$ write $X = \sum_{\nu\in\Lambda} X_\nu[\hat{L}(\nu)]$ with $X_\nu \in \mathbb{Z}$ and set $\mathrm{supp}(X) = \{\hat{L}(\nu) | X_\nu \neq 0\}$. Thus,

$$\begin{aligned}&\mathrm{supp}((zx - xz - 2x)[\hat{L}(\mu)]) \subseteq \\ &\mathrm{supp}(xyx)[\hat{\Delta}(\mu)]) \cup \mathrm{supp}(yxx)[\hat{\Delta}(\mu)]) \cup \mathrm{supp}(xxy)[\hat{\Delta}(\mu)]) \cup \mathrm{supp}(xyx)[\hat{\Delta}(\mu)]) \cup \mathrm{supp}(x[\hat{\Delta}(\mu)]).\end{aligned}$$

$\forall \nu \in \Lambda$, we have

$$\begin{aligned}\mathrm{supp}(x[\hat{\Delta}(\nu)]) &= \bigcup_{\substack{i\in[1,n] \\ \nu_i - i + 1 \equiv a \mod p}} \mathrm{supp}([\hat{\Delta}(\nu + \varepsilon_i)]), \\ \mathrm{supp}(y[\hat{\Delta}(\nu)]) &= \bigcup_{\substack{i\in[1,n] \\ \nu_i - i \equiv a \mod p}} \mathrm{supp}([\hat{\Delta}(\nu - \varepsilon_i)]).\end{aligned}$$

It follows, as μ is far from λ, that

$$\mathrm{supp}((zx - xz - 2x)[\hat{L}(\mu)]) \cap \mathrm{supp}((zx - xz - 2x)[\hat{L}(\lambda)]) = \emptyset.$$

As $(zx - xz - 2x)[M_i] = 0$ $\forall i \in [1, r]$, we must then have $(zx - xz - 2x)[\hat{L}(\lambda)] = 0 = (zx - xz - 2x)[M'])$. Likewise, $(zy - yz + 2y)[\hat{L}(\lambda)] = 0$.

As all $[M_i]$'s have weight $\sum_{i=1}^n (\lambda_i - i + 1)_1\delta + \sum_{j=1}^p n_j\hat{\varepsilon}_j$, so does $[\hat{L}(\lambda)]$; again $\theta(z) - (\sum_{i=1}^n (\lambda_i - i + 1)_1\delta + \sum_{j=1}^p n_j\hat{\varepsilon}_j)(\hat{h}_a)$ annihilates $[\hat{L}(\lambda)]$.

(ii) The assertion holds on the $[n\,\det]$-block of $\mathrm{Rep}'(G_1T)$ by (3.8) and (3.9). Let $\lambda \in \mathcal{W}_a \bullet (n\,\det)$. As $\hat{L}(\lambda)$ is a quotient of $\hat{\Delta}(\lambda)$, $E_a\hat{L}(\lambda)$ is a quotient of $E_a\hat{\Delta}(\lambda)$, and hence $E_a\hat{L}(\lambda)$ belongs to the same block in the whole of $\mathrm{Rep}(G_1T)$ as $E_a\hat{\Delta}(\lambda)$ does. Likewise for $F_a\hat{L}(\lambda)$. The assertion follows from the construction.

(3.11) **Remark:** The same argument as in (3.10) yields that $\mathbb{C}\otimes[\mathrm{Rep}(G_1T)]$ admits a structure of $\widehat{\mathfrak{gl}_p}$-module; $\forall i \in [0, p[$, $\forall m \in \mathbb{Z}$, if $\hat{e}_i \cdot [\hat{\nabla}(\lambda)] = \sum_\mu [\hat{\nabla}(\mu)]$, $(\hat{e}_i \otimes t^m) \cdot [\hat{\nabla}(\lambda)] = \sum_\mu [\hat{\nabla}(\mu + pm\,\det)] = \sum_\mu [\hat{\nabla}(\mu) \otimes pm\,\det]$. Accordingly, we define $(\hat{e}_i \otimes t^m) \bullet [\hat{L}(\lambda)] = \sum_\mu [\hat{L}(\mu) \otimes pm\,\det]$. Likewise for $\hat{f}_i \otimes t^m$. We let d act on $[\hat{L}(\lambda)]$, $\lambda \in \Lambda$, by the scalar $(\sum_{i=1}^n (\lambda_i - i + 1)_1\delta + \sum_{j=1}^p n_j\hat{\varepsilon}_j)(d) = \sum_{i=1}^n (\lambda_i - i + 1)_1$. We let K annihilate the whole $[\mathrm{Rep}(G_1T)]$ and $(0, 1) = \mathrm{diag}(1, \ldots, 1)$ act as the identity on $[\mathrm{Rep}(G_1T)]$.

4° **2-Kac-Moody action on** $\mathrm{Rep}(G)$

We now wish to upgrade the $\widehat{\mathfrak{gl}_p}$-action on $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)]$ to a categorical action of the Khovanov-Lauda-Rouquier, KLR for short, 2-category $\mathcal{U}(\widehat{\mathfrak{gl}_n})$ on $\mathrm{Rep}(G)$ in such a way that $\mathbb{C} \otimes [E_a]$ and $\mathbb{C} \otimes [F_a]$, $a \in [0, p[$, are upgraded to form translation functors on $\mathrm{Rep}(G)$ as in (3.8). The $\mathcal{U}(\widehat{\mathfrak{gl}_n})$-action on $\mathrm{Rep}(G)$ will provide ample 2-morphisms to realize an action of the Bott-Samelson diagrammatic category $\mathcal{D}_{\mathrm{BS}}$. We will see that exactly the same argument gives an upgrading of $\widehat{\mathfrak{gl}_p}$-action on $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G_1T)]$ in (3.10) to a $\mathcal{U}(\widehat{\mathfrak{gl}_n})$-action on $\mathrm{Rep}(G_1T)$. We first take $N = p$ in §3 to consider $\widehat{\mathfrak{gl}_p}$.

(4.1) We recall the definition of Rouquier's strict $\Bbbk$-linear additive 2-category $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ categorifying the enveloping algebra of $\widehat{\mathfrak{gl}_p}$ after Brundan [Br, Def. 1.1]. First, a $\Bbbk$-linear category is a category $\mathcal{C}$ such that $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}(X, Y)$ is a $\Bbbk$-linear space and that the compositions $\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$ are $\Bbbk$-bilinear [中岡, Def. 3.1.11]. It is a $\Bbbk$-linear additive category iff it has, in addition, a zero object and admits a direct sum of any 2 objects [中岡, Def. 3.2.3, p. 130].

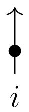
Definition [RW, 6.4.5]: A strict $\Bbbk$-linear additive 2-category $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ consists of the following data:

(i) $\forall i, j \in \mathbb{F}_p$ with $i \neq j$, $t_{ij} = \begin{cases} -1 & \text{if } j = i+1, \\ 1 & \text{else }, \end{cases}$

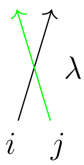
(ii) the objects of $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ are $P = \{\lambda \in \mathfrak{h}^* | \lambda(\hat{h}_i) \in \mathbb{Z}\ \forall i \in [0, p[\}$ from (3.1),

(iii) $\forall \lambda \in P$, $\forall i \in [0, p[$, generating 1-morphisms $E_i 1_\lambda \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda + \hat{\alpha}_i)$, $F_i 1_\lambda \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda - \hat{\alpha}_i)$,

(iv) $\forall \lambda \in P$, $\forall i, j \in [0, p[$, generating 2-morphisms

$x_{\lambda,i} =$  $\lambda \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda + \hat{\alpha}_i)\}(E_i 1_\lambda, E_i 1_\lambda)$

$$\begin{array}{ccc} \lambda & \xrightarrow{E_i 1_\lambda} & \lambda + \hat{\alpha}_i \\ & \Big\Downarrow x_{\lambda,i} & \\ \lambda & \xrightarrow[E_i 1_\lambda]{} & \lambda + \hat{\alpha}_i, \end{array}$$

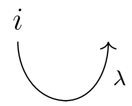
$\tau_{\lambda,(j,i)} =$  $\lambda \ \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda + \hat{\alpha}_i + \hat{\alpha}_j)(E_i E_j 1_\lambda, E_j E_i 1_\lambda)$

$$\begin{array}{ccc} \lambda & \xrightarrow{E_i E_j 1_\lambda} & \lambda + \hat{\alpha}_j + \hat{\alpha}_i \\ & \Big\Downarrow \tau_{\lambda,(j,i)} & \\ \lambda & \xrightarrow[E_j E_i 1_\lambda]{} & \lambda + \hat{\alpha}_i + \hat{\alpha}_j, \end{array}$$

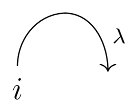
where $E_i E_j 1_\lambda = (E_i 1_{\lambda + \hat{\alpha}_j}) \circ (E_j 1_\lambda) = c_{\lambda, \lambda+\hat{\alpha}_j, \lambda+\hat{\alpha}_j+\hat{\alpha}_i}(E_j 1_\lambda, E_i 1_{\lambda+\hat{\alpha}_j})$ and $E_j E_i 1_\lambda = (E_j 1_{\lambda+\hat{\alpha}_i}) \circ (E_i 1_\lambda) = c_{\lambda, \lambda+\hat{\alpha}_i, \lambda+\hat{\alpha}_i+\hat{\alpha}_j}(E_i 1_\lambda, E_j 1_{\lambda+\hat{\alpha}_i})$:

$$\begin{tikzcd} \lambda \arrow[r, "E_j 1_\lambda"] \arrow[dr, dotted, "E_i E_j 1_\lambda"'] & \lambda + \hat{\alpha}_j \arrow[d, "E_i 1_{\lambda+\hat{\alpha}_j}"] \\ & \lambda + \hat{\alpha}_j + \hat{\alpha}_i, \end{tikzcd} \qquad \begin{tikzcd} \lambda \arrow[r, "E_i 1_\lambda"] \arrow[dr, dotted, "E_j E_i 1_\lambda"'] & \lambda + \hat{\alpha}_i \arrow[d, "E_j 1_{\lambda+\hat{\alpha}_i}"] \\ & \lambda + \hat{\alpha}_i + \hat{\alpha}_j, \end{tikzcd}$$

and

$\eta_{\lambda,i} =$  $\in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda)(1_\lambda, F_i E_i 1_\lambda)$

with 1_λ denoting the unital object of $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda)$ from (2.2.iv), and $F_i E_i 1_\lambda = (F_i 1_{\lambda+\hat{\alpha}_i}) \circ (E_i 1_\lambda)$, and finally

$\varepsilon_{\lambda,i} =$  $\in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda)(E_i F_i 1_\lambda, 1_\lambda)$

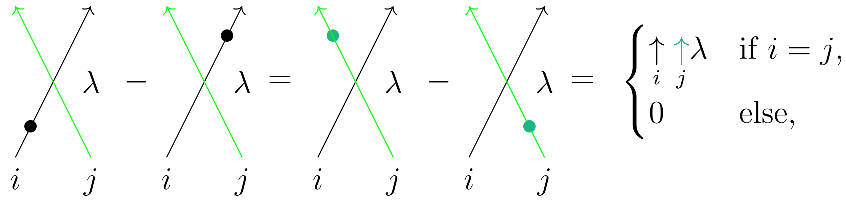
with $E_iF_i1_\lambda = (E_i1_{\lambda-\hat{\alpha}_i}) \circ (F_i1_\lambda)$. In the notation $\tau_{\lambda,(j,i)}$ we follow [RW, p. 90] to write (j,i) instead of (i,j) in accordance to the order of composition reading from the right.

By (2.2.iv) one has $\forall f \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\mu)$, $f \circ 1_\lambda = f$ and $1_\mu \circ f = f$. We will denote the identity 2-morphism $\iota_{E_i1_\lambda}$ of E_i1_λ in $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\lambda+\hat{\alpha}_i)(E_i1_\lambda, E_i1_\lambda)$ (resp. F_i1_λ in $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\lambda-\hat{\alpha}_i)(F_i1_\lambda, F_i1_\lambda)$) by $\underset{i}{\uparrow}\lambda$ (resp. $\overset{i}{\downarrow}\lambda$):

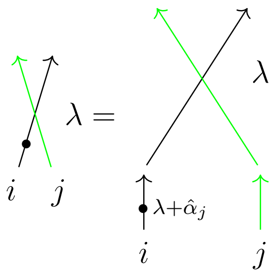
$$\iota_{E_i1_\lambda} = \mathrm{id}_{E_i1_\lambda} = \underset{i}{\uparrow}\lambda, \qquad \iota_{F_i1_\lambda} = \mathrm{id}_{F_i1_\lambda} = \overset{i}{\downarrow}\lambda.$$

Those 2-morphisms are subject to the relations in [Br, Def.1.1], e.g.,

(1)



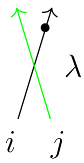
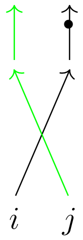
where



$$= \tau_{\lambda,(j,i)} \odot (x_{\lambda+\hat{\alpha}_j,i} * \iota_{E_j1_\lambda}) \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\lambda+\hat{\alpha}_j+\hat{\alpha}_i)(E_iE_j1_\lambda, E_jE_i1_\lambda),$$

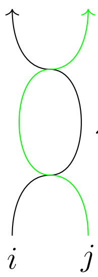
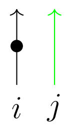
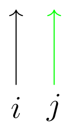
$$\begin{tikzcd}
\lambda \arrow[r, "E_j1_\lambda"] \arrow[d, phantom, "\Downarrow \iota_{E_j1_\lambda}"] & \lambda+\hat{\alpha}_j \arrow[r, "E_i1_{\lambda+\hat{\alpha}_j}"] \arrow[d, phantom, "\Downarrow x_{\lambda+\hat{\alpha}_j,i}"] & \lambda+\hat{\alpha}_j+\hat{\alpha}_i \arrow[rr, phantom, "\xmapsto{c_{\lambda,\lambda+\hat{\alpha}_j,\lambda+\hat{\alpha}_j+\hat{\alpha}_i}}"] & & \lambda \arrow[r, "E_iE_j1_\lambda"] \arrow[d, phantom, "\Downarrow x_{\lambda+\hat{\alpha}_j,i}*\iota_{E_j1_\lambda}"] & \lambda+\hat{\alpha}_j+\hat{\alpha}_i \\
\lambda \arrow[r, "E_j1_\lambda"'] & \lambda+\hat{\alpha}_j \arrow[r, "E_i1_{\lambda+\hat{\alpha}_j}"'] & \lambda+\hat{\alpha}_j+\hat{\alpha}_i & & \lambda \arrow[r, "E_iE_j1_\lambda"'] & \lambda+\hat{\alpha}_j+\hat{\alpha}_i,
\end{tikzcd}$$

$$\begin{tikzcd}
\lambda \arrow[r, "E_iE_j1_\lambda"] \arrow[d, phantom, "\Downarrow x_{\lambda+\hat{\alpha}_j,i}*\iota_{E_j1_\lambda}"] & \lambda+\hat{\alpha}_j+\hat{\alpha}_i \arrow[dd, bend left, dotted, "\tau_{\lambda,(j,i)}\odot(x_{\lambda+\hat{\alpha}_j,i}*\iota_{E_j1_\lambda})"] \\
\lambda \arrow[r, "E_iE_j1_\lambda"'] \arrow[d, phantom, "\Downarrow \tau_{\lambda,(j,i)}"] & \lambda+\hat{\alpha}_j+\hat{\alpha}_i \\
\lambda \arrow[r, "E_jE_i1_\lambda"'] & \lambda+\hat{\alpha}_i+\hat{\alpha}_j,
\end{tikzcd}$$

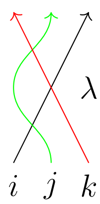
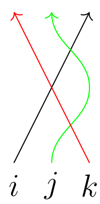
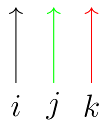
 $\lambda =$  $\lambda = (\iota_{E_j 1_{\lambda+\hat{\alpha}_i}} * x_{\lambda,i}) \odot \tau_{\lambda,(j,i)} \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda+\hat{\alpha}_j+\hat{\alpha}_i)(E_iE_j1_\lambda, E_jE_i1_\lambda),$

 $\lambda \quad = \quad \iota_{E_i 1_{\lambda+\hat{\alpha}_j}} * \iota_{E_j 1_\lambda} = \iota_{E_iE_j1_\lambda} \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda+\hat{\alpha}_j+\hat{\alpha}_i)(E_iE_j1_\lambda, E_jE_i1_\lambda),$

etc. We also impose, among others,

(2)  $\lambda = \begin{cases} 0 & \text{if } i=j, \\ t_{ij}$  $\lambda + t_{ji}$  $\lambda & \text{if } i-j \equiv \pm 1 \mod p, \\$  $\lambda & \text{else,} \end{cases}$

the left hand side of which reads $\tau_{\lambda,(i,j)} \odot \tau_{\lambda,(j,i)}$, and

(3)  $\lambda \quad -$  $\lambda = \begin{cases} t_{ij}$  $\lambda & \text{if } i=j \text{ and } k-j \equiv \pm 1 \mod p, \\ 0 & \text{else,} \end{cases}$

etc. On the LHS of (3) the first (resp. second) term reads $(\tau_{\lambda+\hat{\alpha}_i,(k,j)} * \iota_{E_i 1_\lambda}) \odot (\iota_{E_j 1_{\lambda+\hat{\alpha}_k+\hat{\alpha}_i}} * \tau_{\lambda,(k,i)}) \odot (\tau_{\lambda+\hat{\alpha}_k,(j,i)} * \iota_{E_k 1_\lambda})$ (resp. $(\tau_{\lambda,(j,i)} * \iota_{E_k 1_\lambda}) \odot (\tau_{\lambda+\hat{\alpha}_j,(k,i)} * \iota_{E_j 1_\lambda}) \odot (\iota_{E_i 1_{\lambda+\hat{\alpha}_k+\hat{\alpha}_j}} * \tau_{\lambda,(k,j)})$).

Recall from (2.2.ii) that each $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\mu)$ forms a $\Bbbk$-linear additive category, and hence $\forall X, Y \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\mu)$, $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\mu)(X,Y)$ carries a structure of $\Bbbk$-linear space. The 1-morphisms belonging to $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\mu)$ are direct sums of those

$$E_{i_m}^{a_m} F_{j_m}^{b_m} \dots E_{i_1}^{a_1} F_{j_1}^{b_1} 1_\lambda, \quad i_k, j_k \in [0,p[,\ a_k, b_k \in \mathbb{N} \text{ with } \mu = \lambda + \sum_{k=1}^m (a_k\hat{\alpha}_{i_k} - b_k\hat{\alpha}_{j_k})$$

[Ro12, 4.2.3]. In case $\mu=\lambda$, $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\lambda)$ forms a strict monoidal category with $\otimes$ in (2.1) given by the "composition" $\odot$ of 1-morphisms from (2.2) and $I \in \mathrm{Ob}(\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\lambda))$ given by 1_λ.

If $E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda = E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_i^a E_i E_i^c \dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda$ with $a = a_i - 1$, $c = b_j - 1$, $\nu = \lambda - b_1\hat{\alpha}_{j_1} + a_1\hat{\alpha}_{i_1} - \dots + c\hat{\alpha}_i$,

$$x_{\nu,i} = $$

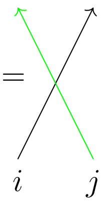
$$\nu \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\nu, \nu + \hat{\alpha}_i)(E_i 1_\nu, E_i 1_\nu)$$

induces a 2-morphism $\iota_{E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_i^a 1_{\nu+\hat{\alpha}_i}} * x_{\nu,i} * \iota_{E_i^c\dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda} \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda + \sum_{k=1}^m (a_k\hat{\alpha}_{i_k} - b_k\hat{\alpha}_{j_k}))$ $(E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda, E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda)$:

$$\begin{tikzcd}
\lambda \arrow[r, "E_i^c\dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda"] \arrow[d, phantom, "\iota_{E_i^c\dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda}\Downarrow"] & \nu \arrow[r, "E_i 1_\nu"] \arrow[d, phantom, "x_{\nu,i}\Downarrow"] & \nu + \hat{\alpha}_i \arrow[r, "E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_i^a 1_{\nu+\hat{\alpha}_i}"] \arrow[d, phantom, "\Downarrow\iota_{E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_i^a 1_{\nu+\hat{\alpha}_i}}"] & \lambda + \sum_{k=1}^m (a_k\hat{\alpha}_{i_k} - b_k\hat{\alpha}_{j_k}) \\
\lambda \arrow[r, "E_i^c\dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda"'] & \nu \arrow[r, "E_i 1_\nu"'] & \nu + \hat{\alpha}_i \arrow[r, "E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_i^a 1_{\nu+\hat{\alpha}_i}"'] & \lambda + \sum_{k=1}^m (a_k\hat{\alpha}_{i_k} - b_k\hat{\alpha}_{j_k}).
\end{tikzcd}$$

If $E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda = E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_i^a E_i E_j E_j^c \dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda$ with $a = a_i - 1$, $c = b_j - 1$, $\nu = \lambda - b_1\hat{\alpha}_{j_1} + a_1\hat{\alpha}_{i_1} - \dots + c\hat{\alpha}_j$,

$$\tau_{\nu,(j,i)} = $$



$$\nu \quad \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\nu, \nu + \hat{\alpha}_i + \hat{\alpha}_j)(E_i E_j 1_\nu, E_j E_i 1_\nu),$$

induces a 2-morphism $\iota_{E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_i^a 1_{\nu+\hat{\alpha}_i+\hat{\alpha}_j}} * \tau_{\nu,(j,i)} * \iota_{E_j^c\dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda} \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda + \sum_{k=1}^m (a_k\hat{\alpha}_{i_k} - b_k\hat{\alpha}_{j_k}))(E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_i^a E_i E_j E_j^c \dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda, E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_i^a E_j E_i E_j^c \dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda)$:

$$\begin{tikzcd}
\lambda \arrow[r, "E_j^c\dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda"] \arrow[d, phantom, "\iota_{E_j^c\dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda}\Downarrow"] & \nu \arrow[r, "E_i E_j 1_\nu"] \arrow[d, phantom, "\tau_{\nu,\nu+\hat{\alpha}_i+\hat{\alpha}_j}\Downarrow"] & \nu + \hat{\alpha}_i + \hat{\alpha}_j \arrow[r, "E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_i^a 1_{\nu+\hat{\alpha}_i+\hat{\alpha}_j}"] \arrow[d, phantom, "\Downarrow\iota_{E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_i^a 1_{\nu+\hat{\alpha}_i+\hat{\alpha}_j}}"] & \lambda + \sum_{k=1}^m (a_k\hat{\alpha}_{i_k} - b_k\hat{\alpha}_{j_k}) \\
\lambda \arrow[r, "E_j^c\dots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda"'] & \nu \arrow[r, "E_j E_i 1_\nu"'] & \nu + \hat{\alpha}_i + \hat{\alpha}_j \arrow[r, "E_{i_m}^{a_m}F_{j_m}^{b_m}\dots E_i^a 1_{\nu+\hat{\alpha}_i+\hat{\alpha}_j}"'] & \lambda + \sum_{k=1}^m (a_k\hat{\alpha}_{i_k} - b_k\hat{\alpha}_{j_k}).
\end{tikzcd}$$

(4.2) **Definition [RW, 6.4.5]:** A 2-representation of $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ is a $\Bbbk$-linear functor from $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ to the 2-category of $\Bbbk$-linear additive categories, i.e., it consists of the following data:

(i) $\forall \lambda \in P$, a $\Bbbk$-linear additive category $\mathcal{C}_\lambda$,

(ii) $\forall \lambda \in P$, $\forall i \in [0, p[$, $\Bbbk$-linear functors $E_i 1_\lambda \in \mathrm{Cat}(\mathcal{C}_\lambda, \mathcal{C}_{\lambda+\hat{\alpha}_i})$ and $F_i 1_\lambda \in \mathrm{Cat}(\mathcal{C}_\lambda, \mathcal{C}_{\lambda-\hat{\alpha}_i})$,

(iii) $\forall \lambda \in P$, $\forall i, j \in [0, p[$,

$$x_{\lambda,i} \in \mathrm{Cat}(\mathcal{C}_\lambda, \mathcal{C}_{\lambda+\hat{\alpha}_i})(E_i 1_\lambda, E_i 1_\lambda),$$
$$\tau_{\lambda,(j,i)} \in \mathrm{Cat}(\mathcal{C}_\lambda, \mathcal{C}_{\lambda+\hat{\alpha}_i+\alpha_j})(E_i E_j 1_\lambda, E_j E_i 1_\lambda) \text{ with } E_i E_j 1_\lambda = (E_i 1_{\lambda+\hat{\alpha}_j}) \circ (E_j 1_\lambda) \text{ and } E_j E_i 1_\lambda = (E_j 1_{\lambda+\hat{\alpha}_i}) \circ (E_i 1_\lambda),$$
$$\eta_{\lambda,i} \in \mathrm{Cat}(\mathcal{C}_\lambda, \mathcal{C}_\lambda)(\mathrm{id}_{\mathcal{C}_\lambda}, F_i E_i 1_\lambda) \text{ with } F_i E_i 1_\lambda = (F_i 1_{\lambda+\hat{\alpha}_i}) \circ (E_i 1_\lambda),$$
$$\varepsilon_{\lambda,i} \in \mathrm{Cat}(\mathcal{C}_\lambda, \mathcal{C}_\lambda)(E_i F_i 1_\lambda, \mathrm{id}_{\mathcal{C}_\lambda}) \text{ with } E_i F_i 1_\lambda = (E_i 1_{\lambda-\hat{\alpha}_i}) \circ (F_i 1_\lambda),$$

subject to the same relations as $x_{\lambda,i}, \tau_{\lambda,(j,i)}, \eta_{\lambda,i}, \varepsilon_{\lambda,i}$ for $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ from (4.1).

(4.3) We now define a 2-representation of $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ on $\mathrm{Rep}(G)$.

Let $\mathbb{T} \in \mathrm{Cat}(\mathrm{Rep}(G), \mathrm{Rep}(G))(E^2, E^2)$ be a natural transformation defined by associating to each $M \in \mathrm{Rep}(G)$ a $\Bbbk$-linear map $\mathbb{T}_M : E^2 M = V \otimes V \otimes M \to E^2 M$ such that $v \otimes v' \otimes m \mapsto v' \otimes v \otimes m$ $\forall v, v' \in V$ $\forall m \in M$. Then

$$(V \otimes \mathbb{T}_M) \circ \mathbb{X}_{V^{\otimes 2} \otimes M} = \mathbb{X}_{V^{\otimes 2} \otimes M} \circ (V \otimes \mathbb{T}_M). \tag{1}$$

Using (3.3.i), one also checks

$$\mathbb{T}_M \circ (V \otimes \mathbb{X}_M) - \mathbb{X}_{V \otimes M} \circ \mathbb{T}_M = -\mathrm{id}_{E^2 M}. \tag{2}$$

Recall from (3.8) the bijection $\iota_n : P(\wedge^n \mathrm{nat}_p) \to \Lambda/(\mathcal{W}_a \bullet)$. For $\lambda \in P$ let us write

$$\mathrm{R}_{\iota_n(\lambda)}(G) = \begin{cases} \mathrm{Rep}_{\iota_n(\lambda)}(G) & \text{if } \lambda \in P(\wedge^n \mathrm{nat}_p), \\ 0 & \text{else.} \end{cases}$$

Consider the following data:

(i) $\forall \lambda \in P$, let $\mathcal{C}_\lambda = \mathrm{R}_{\iota_n(\lambda)}(G)$.

(ii) $\forall \lambda \in P$, $\forall i \in [0, p[$, let $E_i 1_\lambda = E_i|_{\mathrm{R}_{\iota_n(\lambda)}(G)} : \mathrm{R}_{\iota_n(\lambda)}(G) \to \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_i)}(G)$ and $F_i 1_\lambda = F_i|_{\mathrm{R}_{\iota_n(\lambda)}(G)} : \mathrm{R}_{\iota_n(\lambda)}(G) \to \mathrm{R}_{\iota_n(\lambda-\hat{\alpha}_i)}(G)$ from (3.7). In particular, $E_i 1_\lambda = 0$ (resp. $F_i 1_\lambda = 0$) unless λ and $\lambda + \hat{\alpha}_i$ (resp. λ and $\lambda - \hat{\alpha}_i$) $\in P(\wedge^n \mathrm{nat}_p)$. Put for simplicity $E_i^\lambda = E_i|_{\mathrm{R}_{\iota_n(\lambda)}(G)}$ and $F_i^\lambda = F_i|_{\mathrm{R}_{\iota_n(\lambda)}(G)}$.

(iii) $\forall \lambda \in P$, $\forall i, j \in [0, p[$, define $x_{\lambda,i} \in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_i)}(G))(E_i^\lambda, E_i^\lambda)$ by associating to each $M \in \mathrm{R}_{\iota_n(\lambda)}(G)$ a $\Bbbk$-linear map $x_{M,i} = \mathbb{X}_M - i\mathrm{id}_{V \otimes M}$:

$$\begin{tikzcd} V \otimes M \arrow[r, "\mathbb{X}_M - i\mathrm{id}"] & V \otimes M \\ E_i^\lambda M \arrow[u, hook] \arrow[r, dotted] & E_i^\lambda M. \arrow[u, hook] \end{tikzcd} \tag{3}$$

Define $\tau_{\lambda,(j,i)} \in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_i+\hat{\alpha}_j)}(G))(E_i^{\lambda+\hat{\alpha}_j} E_j^\lambda, E_j^{\lambda+\hat{\alpha}_i} E_i^\lambda)$ by associating to each $M \in$

$\mathrm{R}_{\iota_n(\lambda)}(G)$ a $\Bbbk$-linear map $\tau_{M,(j,i)} : E_i^{\lambda+\hat\alpha_j} E_j^\lambda M \to E_j^{\lambda+\hat\alpha_i} E_i^\lambda M$ such that

(4) $\tau_{M,(j,i)} =$
$$\begin{cases} \{\mathrm{id} + (V \otimes \mathbb{X}_M) - \mathbb{X}_{V\otimes M}\}^{-1}(\mathbb{T}_M - \mathrm{id}) & \text{if } j = i, \\ (V \otimes \mathbb{X}_M - \mathbb{X}_{V\otimes M})\mathbb{T}_M + \mathrm{id}_{V\otimes V\otimes M} & \text{if } j \equiv i-1 \bmod p, \\ (V \otimes \mathbb{X}_M - \mathbb{X}_{V\otimes M})\{\mathrm{id} + (V \otimes \mathbb{X}_M) - \mathbb{X}_{V\otimes M}\}^{-1}(\mathbb{T}_M - \mathrm{id}) + \mathrm{id} & \text{else,} \end{cases}$$

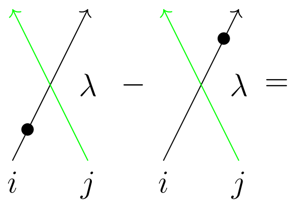
which is well-defined by [Ro, Th. 3.16]/[RW, Th. 6.4.2]; a verification will formally be done using (4.6) and (4.7). In case $j = i$, $E_i^{\lambda+\hat\alpha_i} E_i^\lambda M$ is a generalized i-eigenspace of both $V \otimes \mathbb{X}_M$ and $\mathbb{X}_{V\otimes M}$. As $V \otimes \mathbb{X}_M$ and $\mathbb{X}_{V\otimes M}$ commute by (3.4.iii), $(V \otimes \mathbb{X}_M) - \mathbb{X}_{V\otimes M}$ is nilpotent on $E_i^{\lambda+\hat\alpha_i} E_i^\lambda M$, and hence $\mathrm{id} + (V \otimes \mathbb{X}_M) - \mathbb{X}_{V\otimes M}$ is invertible on $E_i^{\lambda+\hat\alpha_i} E_i^\lambda M$. Likewise the 3rd case.

Define $\eta_{\lambda,i}$ to be the unit $\eta_i \in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda)}(G))(\mathrm{id}, F_i^{\lambda+\hat\alpha_i} E_i^\lambda)$ of the adjunction (E_i, F_i) on $\mathrm{R}_{\iota_n(\lambda)}(G)$ from (3.6). Define finally $\varepsilon_{\lambda,i}$ to be the counit $\varepsilon_i \in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda)}(G))$ $(E_i^{\lambda-\hat\alpha_i} F_i^\lambda, \mathrm{id})$ of the adjunction (E_i, F_i) on $\mathrm{R}_{\iota_n(\lambda)}(G)$ from (3.6) also.

Theorem [RW, Th. 6.4.6]: *The data above constitutes a 2-representation of* $\mathcal{U}(\widehat{\mathfrak{gl}_p})$.

(4.4) To see that the theorem holds, we must check that the 2-morphisms in (4.3.iii) satisfy the relations of those for $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ as given in (4.1).

Consider for example the relation from (4.1.1)



$$= \begin{cases} \underset{i}{\uparrow}\ \underset{j}{\uparrow}\lambda & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

Accordingly, we must verify

(1) $$\tau_{\lambda,(j,i)} \odot (x_{\lambda+\hat\alpha_j,i} * \iota_{E_j^\lambda}) - (\iota_{E_j^{\lambda+\hat\alpha_i}} * x_{\lambda,i}) \odot \tau_{\lambda,(j,i)} = \begin{cases} \mathrm{id} & \text{if } i = j \\ 0 & \text{else,} \end{cases}$$

i.e., in case $i = j$, for example, one must show on $E_i^{\lambda+\hat\alpha_i} E_i^\lambda M$ for $M \in \mathrm{R}_{\iota_n(\lambda)}(G)$ that

$$\{\mathrm{id} + (V \otimes \mathbb{X}_M) - \mathbb{X}_{V\otimes M}\}^{-1}(\mathbb{T}_M - \mathrm{id}) \circ (\mathbb{X}_{E_iM} - i\mathrm{id}) - \\ \{V \otimes (\mathbb{X}_M - i\mathrm{id})\} \circ \{\mathrm{id} + (V \otimes \mathbb{X}_M) - \mathbb{X}_{V\otimes M}\}^{-1}(\mathbb{T}_M - \mathrm{id}) = \mathrm{id}.$$

For that the KLR-algebra $\mathrm{H}_3(\mathbb{F}_p)$ and the degenerate affine Hecke algebra $\bar{\mathrm{H}}_3$ of degree 3 come to rescue.

(4.5) To define the KLR-algebra, recall first $t_{ij} \in \{\pm 1\}$ from (4.1) for $i, j \in \mathbb{F}_p$ with $i \neq j$. Let $\mathfrak{S}_3$ act on $\mathbb{F}_p^3$ such that $\sigma\nu = (\nu_{\sigma^{-1}1}, \nu_{\sigma^{-1}2}, \nu_{\sigma^{-1}3})$ for $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{F}_p^3$. Put $\sigma_k = (k, k+1) \in \mathfrak{S}_3$, $k \in \{1, 2\}$. The algebra $\mathrm{H}_3(\mathbb{F}_p)$ is really a $\Bbbk$-linear additive category with objects $\mathbb{F}_p^3$ and morphisms generated by $x_{z,\nu} \in \mathrm{H}_3(\mathbb{F}_p)(\nu,\nu)$ and $\tau_{c,\nu} \in \mathrm{H}_3(\mathbb{F}_p)(\nu,\sigma_c\nu)$, $z \in [1,3]$, $c \in [1,2]$, $\nu \in \mathbb{F}_p^3$, subject to the relations

(KLR1) $\qquad x_{z,\nu}x_{z',\nu'} = x_{z',\nu}x_{z,\nu'}$,

(KLR2) $$\tau_{c,\sigma_c\nu}\tau_{c,\nu} = \begin{cases} 0 & \text{if } \nu_c = \nu_{c+1}, \\ t_{\nu_c,\nu_{c+1}}x_{c,\nu} + t_{\nu_{c+1},\nu_c}x_{c+1,\nu} & \text{if either } \nu_{c+1} \equiv \nu_c + 1 \text{ or } \nu_c \equiv \nu_{c+1} + 1, \\ \mathrm{id}_\nu & \text{else,} \end{cases}$$

(KLR3)
$$\tau_{c,\nu}x_{z,\nu} - x_{\sigma_c z,\sigma_c\nu}\tau_\nu = \begin{cases} -\mathrm{id}_\nu & \text{if } c = z \text{ and } \nu_c = \nu_{c+1}, \\ \mathrm{id}_\nu & \text{if } z = c+1 \text{ and } \nu_c = \nu_{c+1}, \\ 0 & \text{else.} \end{cases}$$

We do not care what $x_{z,\nu} : \nu \to \nu$ and $\tau_{c,\nu} : \nu \to \sigma\nu$ are as maps.

A representation of $\mathrm{H}_3(\mathbb{F}_p)$ consists of the data

(i) $\forall \nu \in \mathbb{F}_p^3$, a $\Bbbk$-linear space V_ν,

(ii) $\forall \nu \in \mathbb{F}_p^3$, $\forall z \in [1,3]$, a $\Bbbk$-linear map $x_{z,\nu} : V_\nu \to V_\nu$,

(iii) $\forall \nu \in \mathbb{F}_p^3$, $\forall c \in [1,2]$, a $\Bbbk$-linear map $\tau_{c,\nu} : V_\nu \to V_{\sigma_c\nu}$

satisfying the relations (KLR1-3). For $\mathrm{H}_3(\mathbb{F}_p)$ the conditions [RW, (6.5.3) and (6.5.5), p. 86] are irrelevant.

(4.6) Recall next the degenerate affine Hecke algebra, daHa for short, $\bar{\mathrm{H}}_m$ of degree m; DAHA already stands for "double affine Hecke algebra". Thus, let $\Bbbk[X] = \Bbbk[X_1,\dots,X_m]$ be the polynomial $\Bbbk$-algebra in indeterminates $X_1,\dots,X_m$ with a natural $\mathfrak{S}_m$-action: $\sigma : X_i \mapsto X_{\sigma(i)}$. For transposition $\sigma_c = (c, c+1) \in \mathfrak{S}_m$, $c \in [1,m[$, let ∂_c denote the Demazure operator on $\Bbbk[X]$ defined by
$$f \mapsto \frac{f - \sigma_c f}{X_{c+1} - X_c},$$
which differs from the standard one by sign. The daHa $\bar{\mathrm{H}}_m$ is a $\Bbbk$-algebra with the ambient $\Bbbk$-linear space $\Bbbk\mathfrak{S}_m \otimes_\Bbbk \Bbbk[X]$ having $\Bbbk\mathfrak{S}_m$ and $\Bbbk[X]$ as $\Bbbk$-subalgebras such that, letting T_c denote $\sigma_c \in \mathfrak{S}_m$ in $\bar{\mathrm{H}}_m$,

(1) $$fT_c = T_c\sigma_c(f) + \partial_c(f)T_c \quad \forall f \in \Bbbk[X], \forall c \in [1,m[.$$

If $r \leq m$, one has naturally $\bar{\mathrm{H}}_r \leq \bar{\mathrm{H}}_m$.

Lemma [RW, Lem. 6.4.5]: *There is a $\Bbbk$-algebra homomorphism*
$$\bar{\mathrm{H}}_m \to \mathrm{Cat}(\mathrm{Rep}(G),\mathrm{Rep}(G))(E^m,E^m)$$
such that $\forall M \in \mathrm{Rep}(G)$, $X_z \mapsto V^{\otimes_{m-z}} \otimes \mathbb{X}_{V^{\otimes_{z-1}} \otimes M}$, $z \in [1,m]$ *and* $T_c \mapsto V^{\otimes_{m-c-1}} \otimes \mathbb{T}_{V^{\otimes_{c-1}} \otimes M}$, $c \in [1,m[$.

Proof: One checks that the relations $T_c^2 = 1$ $\forall c \in [1,m[$, and the braid relations $T_cT_b = T_bT_c$ for b,c with $|b-c| \geq 2$, $T_cT_{c+1}T_c = T_{c+1}T_cT_{c+1}$ on $\mathrm{Cat}(\mathrm{Rep}(G),\mathrm{Rep}(G))(E^m,E^m)$. Also, the relations $X_z X_y = X_y X_z$, $z, y \in [1, m]$, hold on the RHS by generalizing (3.4). To check (1), we may assume $f \in \{X_1, \dots, X_m\}$ as $\forall g \in \Bbbk[X]$, $(fg)T_c = f(T_c g)$. Then the relations hold on the RHS by generalizing (4.3.1, 2).

(4.6′) The lemma carrie over to $\mathrm{Rep}'(G_1T)$ and to $\mathrm{Rep}(G_1T)$.

(4.7) It follows for $M \in \mathrm{Rep}(G)$ that E^3M comes equipped with a structure of $\bar{\mathrm{H}}_3$-module. By (3.7)

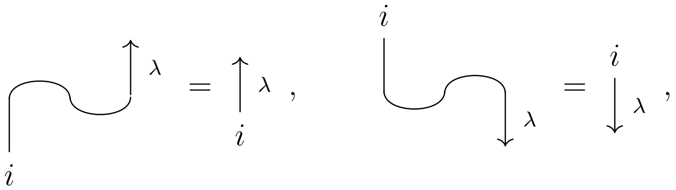
$$E^3M = \coprod_{\nu \in \mathbb{F}_p^3} E^3_\nu M$$

with $E^3_\nu M = E_{\nu_3} E_{\nu_2} E_{\nu_1} M$ and $E_{\nu_i}(V^{\otimes_{i-1}} \otimes M)$ forming a generalized eigenspace of eigenvalue ν_i for $\mathbb{X}_{V^{\otimes_{i-1}} \otimes M}$, $i \in [1, 3]$. Thus, $E^3_\nu M$ affords a generalized eigenspace of eigenvalue ν_i for each X_i by (4.6). As such, it follows from a theorem of Brundan and Kleschev [BrK] and Rouquier [Ro], cf. [RW, Th. 6.4.2], that E^3M affords a representation of $\mathrm{H}_3(\mathbb{F}_p)$ with $x_{z\nu} = X_z - \nu_z$ and

$$\tau_{c\nu} = \begin{cases} (1 + X_c - X_{c+1})^{-1}(T_c - 1) & \text{if } \nu_c = \nu_{c+1}, \\ (X_c - X_{c+1})T_c + 1 & \text{if } \nu_{c+1} = \nu_c + 1, \\ (1 + X_c - X_{c+1})^{-1}(X_c - X_{c+1})(T_c - 1) + 1 & \text{else.} \end{cases}$$

Then (4.4.1) follows from the middle case of (KLR3) with $c = 1$.

(4.8) We have yet to verify [Br, (1.5), (1.7)-(1.9)]:

(1) 

$$E_j F_i 1_\lambda \simeq F_i E_j 1_\lambda \quad \text{if } \lambda(\hat{h}_i) = 0, \tag{2}$$

$$E_i F_i 1_\lambda \simeq F_i E_i 1_\lambda \oplus 1_\lambda^{\oplus \lambda(\hat{h}_i)} \quad \text{if } \lambda(\hat{h}_i) > 0, \tag{3}$$

$$E_i F_i 1_\lambda \simeq F_i E_i 1_\lambda \oplus 1_\lambda^{\oplus -\lambda(\hat{h}_i)} \quad \text{if } \lambda(\hat{h}_i) < 0, \tag{4}$$

respectively.

Now, the LHS of the first relation in (1) should read

$$\begin{aligned} (5) \quad & (\varepsilon_{\lambda+\hat{\alpha}_i,i} * \iota_{E_i 1_\lambda}) \odot (\iota_{E_i 1_\lambda} * \eta_{\lambda,i}) \odot (\iota_{E_i 1_\lambda} * \iota_{1_\lambda}) \\ & = (\varepsilon_{\lambda+\hat{\alpha}_i,i} \odot \iota_{E_i 1_\lambda} \odot \iota_{E_i 1_\lambda}) * (\iota_{E_i 1_\lambda} \odot \eta_{\lambda,i} \odot \iota_{1_\lambda}) = (\varepsilon_{\lambda+\hat{\alpha}_i,i} \odot \iota_{E_i 1_\lambda}) * (\iota_{E_i 1_\lambda} \odot \eta_{\lambda,i}) \end{aligned}$$

$$\begin{array}{c}
\lambda \xrightarrow{\qquad E_i 1_\lambda \qquad} \lambda+\hat{\alpha}_i \\
\| \\
\lambda \xrightarrow{1_\lambda} \lambda \xrightarrow{E_i 1_\lambda} \lambda+\hat{\alpha}_i \\
\eta_{\lambda,i}\Downarrow \qquad\qquad \Downarrow \iota_{E_i 1_\lambda} \\
\lambda \xrightarrow{F_iE_i 1_\lambda} \lambda \xrightarrow{E_i 1_\lambda} \lambda+\hat{\alpha}_i \\
\| \\
\lambda \xrightarrow{\quad E_iF_iE_i 1_\lambda \quad} \lambda+\hat{\alpha}_i \\
\| \\
\lambda \xrightarrow{E_i 1_\lambda} \lambda+\hat{\alpha}_i \xrightarrow{\qquad E_iF_i 1_{\lambda+\hat{\alpha}_i} \qquad} \lambda+\hat{\alpha}_i \\
\iota_{E_i 1_\lambda}\Downarrow \qquad\qquad\qquad \Downarrow \varepsilon_{\lambda+\hat{\alpha}_i,i} \\
\lambda \xrightarrow{E_i 1_\lambda} \lambda+\hat{\alpha}_i \xrightarrow{\qquad 1_{\lambda+\hat{\alpha}_i} \qquad} \lambda+\hat{\alpha}_i \\
\| \\
\lambda \xrightarrow[\qquad E_i 1_\lambda \qquad]{} \lambda+\hat{\alpha}_i\,.
\end{array}$$

This follows from the fact that E_i and F_i are adjunction morphisms $\mathrm{Rep}(G)(E_iM, E_iM') \simeq \mathrm{Rep}(G)(M, F_iE_iM')$ via $\phi \mapsto F_i\phi \circ \eta_M$ with inverse $\varepsilon_{E_iM'} \circ E_i\psi \leftarrowtail \psi$. Thus, for $f \in \mathrm{Rep}(G)(M, M')$

$$E_if = \varepsilon_{E_iM'} \circ E_i(F_iE_if \circ \eta_M) = \varepsilon_{E_iM'} \circ E_iF_iE_if \circ E_i\eta_M,$$

and one has a commutative diagram

$$\begin{tikzcd}
E_iM \arrow[r, "E_if"] \arrow[d, "E_i\eta_M"] \arrow[dd, bend right=60, "\mathrm{id}_{E_iM}"'] & E_iM' \arrow[d, "E_i\eta_{M'}"'] \arrow[dd, bend left=60, "\mathrm{id}_{E_iM'}"] \\
E_iF_iE_iM \arrow[r, "E_iF_iE_if"] \arrow[d, "\varepsilon_{E_iM}"] & E_iF_iE_iM' \arrow[d, "\varepsilon_{E_iM'}"'] \\
E_iM \arrow[r, "E_if"'] & E_iM'.
\end{tikzcd}$$

To see the invertibility of (2)-(4), we note that the $E_i^\lambda : \mathrm{R}_{\iota_n(\lambda)}(G) \to \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_i)}(G)$ and $F_i^\lambda : \mathrm{R}_{\iota_n(\lambda)}(G) \to \mathrm{R}_{\iota_n(\lambda-\hat{\alpha}_i)}(G)$ define an $\mathfrak{sl}_2$-categorification [Ro, Def. 5.20. p. 58]: an $\mathfrak{sl}_2$-categorification on the 2-category of $\Bbbk$-linear abelian category $\mathrm{Rep}(G)$ of finite dimensional G-modules [Ro, p. 5] is the data of an adjoint pair (E_i, F_i) of exact functors $\mathrm{Rep}(G) \to \mathrm{Rep}(G)$ and 2-morphisms $\mathbb{X} \in \mathrm{Cat}(\mathrm{Rep}(G), \mathrm{Rep}(G))(E_i, E_i)$ and $\mathbb{T} \in \mathrm{Cat}(\mathrm{Rep}(G), \mathrm{Rep}(G))(E_i^2, E_i^2)$ such that under the isomorphism $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)] \to \wedge^n(\mathrm{nat}_p)$ from (3.7)

(i) the actions of $[E_i]$ and $[F_i]$ on $[\mathrm{Rep}(G)]$ give a locally finite representation of $\mathfrak{sl}_2$,

(ii) the classes of simple objects are weight vectors,

(iii) F_i is isomorphic to a left adjoint of E_i,

(iv) $\mathbb{X}$ has a single eigenvalue i,

(v) the action on E_i^m of $\mathbb{X}_j := E_i^{m-j}\mathbb{X}E_i^{j-1}$ for $j \in [1, m]$ and of $\mathbb{T}_{\underline{j}} := E_i^{m-i-1}\mathbb{T}E_i^{j-1}$ for $j \in [1, m[$ induce an action of the degenerate affine Hecke algebra $\overline{\mathrm{H}}_m$.

Then (3) and (4) (resp. (2)) follow from [Ro, Th. 5.22 and its proof] (resp. [Ro, Th. 5.25 and its proof]).

(4.8′) As we have observed in (3.9), the set $P(\otimes^n \mathrm{nat}_p)$ of $\otimes^n(\mathrm{nat}_p)$ coincides with $P(\wedge^n \mathrm{nat}_p) = \mathbb{Z}\delta + \{\sum_{j=1}^p n_j\hat{\varepsilon}_j | n_j \in \mathbb{N}, \sum_{j=1}^p n_j = n\}$, and hence we may denote the bijection $P(\otimes^n \mathrm{nat}_p) \to \Lambda/(\mathcal{W}_a \bullet)$ by ι_n from (3.8). Define $\mathbb{T} \in \mathrm{Cat}(\mathrm{Rep}(G_1T), \mathrm{Rep}(G_1T))(E^2, E^2)$ just as on $\mathrm{Rep}(G)$, and for each $\lambda \in P$ let

$$\mathrm{R}_{\iota_n(\lambda)}(G_1T) = \begin{cases} \mathrm{Rep}_{\iota_n(\lambda)}(G_1T) & \text{if } \lambda \in P(\otimes^n \mathrm{nat}_p) = P(\wedge^n \mathrm{nat}_p), \\ 0 & \text{else.} \end{cases}$$

As $E_i^\lambda : \mathrm{R}_{\iota_n(\lambda)}(G_1T) \to \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_i)}(G_1T)$ and $F_i^\lambda : \mathrm{R}_{\iota_n(\lambda)}(G_1T) \to \mathrm{R}_{\iota_n(\lambda-\hat{\alpha}_i)}(G_1T)$, $i \in \mathbb{F}_p$, form an $\mathfrak{sl}_2$-categorification by (3.10), exactly the same arguments for $\mathrm{Rep}(G)$ yields

Corollary: *The data defined on* $\mathrm{Rep}(G_1T)$ *just as on* $\mathrm{Rep}(G)$ *constitutes a 2-representation of* $\mathcal{U}(\widehat{\mathfrak{gl}_p})$.

(4.9) Recall $\varpi = \hat{\varepsilon}_1 + \cdots + \hat{\varepsilon}_n \in P(\wedge^n(\mathrm{nat}_p)))$ from (3.8). $\forall s \in \mathcal{S}_a$, set

$$\mathrm{T}^s = \begin{cases} E_{n-j}^{\varpi} & \text{if } s = s_{\alpha_j}, j \in [1, n[, \\ E_0^{\varpi+\hat{\alpha}_n+\cdots+\hat{\alpha}_{p-1}} E_{p-1}^{\varpi+\hat{\alpha}_n+\cdots+\hat{\alpha}_{p-2}} \dots E_{n+1}^{\varpi+\hat{\alpha}_n} E_n^{\varpi} & \text{if } s = s_{\alpha_0,1}, \end{cases}$$

$$\mathrm{T}_s = \begin{cases} F_{n-j}^{\varpi+\hat{\alpha}_{n-j}} & \text{if } s = s_{\alpha_j}, j \in [1, n[, \\ F_n^{\varpi+\hat{\alpha}_n} F_{n+1}^{\varpi+\hat{\alpha}_n+\hat{\alpha}_{n+1}} \dots F_{p-1}^{\varpi+\hat{\alpha}_n+\cdots+\hat{\alpha}_{p-1}} F_0^{\varpi+\hat{\alpha}_n+\cdots+\hat{\alpha}_{p-1}+\hat{\alpha}_0} & \text{if } s = s_{\alpha_0,1}, \end{cases}$$

and $\Theta_s = \mathrm{T}_s\mathrm{T}^s$. By (3.8) each Θ_s may be taken to be the s-wall crossing functor on $\mathrm{Rep}_{[n \det]}(G)$. We have obtained a strict monoidal functor

$$\mathcal{U}(\widehat{\mathfrak{gl}_p})(\varpi, \varpi) \to \mathrm{Cat}(\mathrm{Rep}_{[n \det]}(G), \mathrm{Rep}_{[n \det]}(G)) \tag{1}$$

such that $F_{n-j}E_{n-j}1_\varpi \mapsto \Theta_{s_j}$ $j \in [1, n[$, and $F_nF_{n+1} \dots F_{p-1}F_0E_0E_{p-1} \dots E_{n+1}E_n1_\varpi \mapsto \Theta_{s_{\alpha_0,1}}$. This is really a homomorphism of monoids with respect to $\circ$ (resp. the composition of the wall-crossing functors) on the 2-category $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ (resp. $\mathrm{Rep}_{[n \det]}(G)$); $\mathrm{Ob}(\mathcal{U}(\widehat{\mathfrak{gl}_p})(\varpi, \varpi))$ admits an addition by direct sum, but not a structure of abelian group.

As $\iota_n(\varpi) = n \det = \det^{\otimes n} \in A^+$, we may regard $\mathrm{R}_{\iota_n(\varpi)}(G) = \mathrm{Rep}_{[n \det]}(G)$ as the principal block $\mathrm{Rep}_0(G)$; $\mathrm{Rep}_0(G) \simeq \mathrm{R}_{\iota_n(\varpi)}(G)$ via $M \mapsto \det^{\otimes n} \otimes M$. Then (1) reads as a strict monoidal functor

$$\mathcal{U}(\widehat{\mathfrak{gl}_p})(\varpi, \varpi) \to \mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G)). \tag{2}$$

In order to obtain a strict monoidal functor $\mathcal{D}_{\mathrm{BS}} \to \mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G))$ such that $B_s\langle m\rangle \mapsto \Theta_s$ $\forall s \in \mathcal{S}_a$ $\forall m \in \mathbb{Z}$, it now suffices to construct a strict monoidal functor $\mathcal{D}_{\mathrm{BS}} \to \mathcal{U}(\widehat{\mathfrak{gl}_p})(\varpi, \varpi)$ such that $\forall j \in [1, n[$, $\forall m \in \mathbb{Z}$, $B_{s_{\alpha_j}}\langle m\rangle \mapsto F_{n-j}E_{n-j}1_\varpi$ and that $B_{s_{\alpha_0,1}}\langle m\rangle \mapsto F_nF_{n+1} \dots F_{p-1}F_0E_0E_{p-1} \dots E_{n+1}E_n1_\varpi$. Instead of constructing a strict monoidal functor $\mathcal{D}_{\mathrm{BS}} \to \mathcal{U}(\widehat{\mathfrak{gl}_p})(\varpi, \varpi)$, however, we make some further reductions.

(4.10) First let $P_+ = \{k\delta + \sum_{i=1}^p n_i\hat{\varepsilon}_i \in P | k \in \mathbb{Z}, n_i \in \mathbb{N}\ \forall i\} \supset \{k\delta + \sum_{i=1}^p n_i\hat{\varepsilon}_i | k \in \mathbb{Z}, n_i \in \mathbb{N}\ \forall i, \sum_{i=1}^p n_i = n\} = P(\wedge^n \mathrm{nat}_p)$. Let $\mathcal{U}_+(\widehat{\mathfrak{gl}_p})$ denote the 2-category having the same data as that of $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ but $\forall \lambda, \mu \in P$, $\forall X, Y \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \mu)$, $\{\mathcal{U}_+(\widehat{\mathfrak{gl}_p})(\lambda, \mu)\}(X, Y) = \{\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \mu)\}(X, Y)/\mathcal{I}_+(X, Y)$ with $\mathcal{I}_+(X, Y)$ denoting the $\Bbbk$-linear span of those $f : X \Rightarrow Y$ which factors through some $Z_2 \circ Z_1$ with $Z_1 \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \nu)$ $\exists \nu \in P \setminus P_+$

$$
\begin{tikzcd}
X \arrow[rr, Rightarrow, "f"] \arrow[dr, dotted] & & Y, \\
& Z_2 \circ Z_1 \arrow[ur, dotted] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\lambda \arrow[rr] \arrow[dr, dotted, "Z_1"'] & & \mu; \\
& \nu \arrow[ur, dotted, "Z_2"'] &
\end{tikzcd}
$$

$\mathcal{U}_+(\widehat{\mathfrak{gl}_p})(\lambda, \mu)$ is just an additive category, not having enough structure to define its quotient.

As $\mathrm{R}_{\iota_n(\nu)}(G) = 0$ unless $\nu \in P(\wedge^n \mathrm{nat}_p) \subset P_+$, the 2-representation of $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ on $(\mathrm{R}_{\iota_n(\lambda)}(G))_{\lambda \in P}$ in (4.3) induces a 2-representation of $\mathcal{U}_+(\widehat{\mathfrak{gl}_p})$.

(4.10′) As $P(\otimes^n \mathrm{nat}_p) = P(\wedge^n \mathrm{nat}_p)$, the 2-repsresentation of $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ on $(\mathrm{R}_{\iota_n(\lambda)}(G_1T))_{\lambda \in P}$ induces a 2-repsresentation on $\mathcal{U}_+(\widehat{\mathfrak{gl}_p})$

(4.11) We "restrict" next the 2-representation of $\mathcal{U}_+(\widehat{\mathfrak{gl}_p})$ to $\mathcal{U}(\widehat{\mathfrak{gl}_n})$. As $p > n$, one can imbed $\mathfrak{sl}_n(\mathbb{C})$ as a subalgebra of $\mathfrak{sl}_p(\mathbb{C})$ via

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$

As the trace form on $\mathfrak{sl}_p(\mathbb{C})$ restricts to the one on $\mathfrak{sl}_n(\mathbb{C})$, the imbedding extends to an imbedding of $\widehat{\mathfrak{sl}_n}$ into $\widehat{\mathfrak{sl}_p}$, and further to an imbedding of $\widehat{\mathfrak{gl}_n} = \widehat{\mathfrak{sl}_n} \oplus \mathbb{C}$ into $\widehat{\mathfrak{gl}_p} = \widehat{\mathfrak{sl}_p} \oplus \mathbb{C}$ with $(0,1) = \mathrm{diag}(\underbrace{1, \dots, 1}_{n}) \mapsto \mathrm{diag}(\underbrace{1, \dots, 1}_{p}) = (0,1)$. In particular, $\mathfrak{h}_{\widehat{\mathfrak{gl}_n}} = \mathfrak{h}_{\mathfrak{gl}_n(\mathbb{C})} \oplus \mathbb{C}K \oplus \mathbb{C}d$, $\mathfrak{h}_{\mathfrak{gl}_n(\mathbb{C})}$ denoting the CSA of $\mathfrak{gl}_n(\mathbb{C})$ consisting of the diagonals, is a direct summand of $\mathfrak{h}_{\mathfrak{gl}_p(\mathbb{C})} \oplus \mathbb{C}K \oplus \mathbb{C}d = \mathfrak{h}_{\widehat{\mathfrak{gl}_p}}$ as a $\mathbb{C}$-Lie algebra with $\mathrm{diag}(\underbrace{1, \dots, 1}_{n}) \mapsto \mathrm{diag}(\underbrace{1, \dots, 1}_{p})$, and hence one may regard

$$P_{\widehat{\mathfrak{gl}_n}} = \{\lambda \in (\mathfrak{h}_{\widehat{\mathfrak{gl}_n}})^* | \lambda(\hat{h}_i) \in \mathbb{Z}\ \forall i \in [0, n[\} \hookrightarrow \{\lambda \in (\mathfrak{h}_{\widehat{\mathfrak{gl}_p}})^* | \lambda(\hat{h}_i) \in \mathbb{Z}\ \forall i \in [0, p[\} = P.$$

If we let nat_n denote the natural module for $\widehat{\mathfrak{gl}_n}$, it may be imbedded as a direct summand of nat_p as $\widehat{\mathfrak{gl}_n}$-modules

$$
\begin{aligned}
\mathrm{nat}_p = (\coprod_{i=1}^p \mathbb{C}a_i) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] &= \{(\coprod_{i=1}^n \mathbb{C}a_i) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]\} \oplus \{(\coprod_{i=n+1}^p \mathbb{C}a_i) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]\} \\
&= \mathrm{nat}_n \oplus \{(\coprod_{i=n+1}^p \mathbb{C}a_i) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]\}
\end{aligned}
$$

with $\widehat{\mathfrak{gl}_n}$ acting on the 2nd summand by annihilating $\mathfrak{sl}_n(\mathbb{C})$. Let us denote the direct summand nat_n by $\mathrm{nat}_p^{[n]}$. Then the set of weights on $\mathrm{nat}_p^{[n]}$ is given by $P(\mathrm{nat}_p^{[n]}) = \{\hat{\varepsilon}_i + m\delta \in P | i \in$

$[1,n], m \in \mathbb{Z}\}$, and $\wedge^n \mathrm{nat}_n$ is a direct summand of

$$\wedge^n \mathrm{nat}_p \simeq \coprod_{j=0}^{n} (\wedge^j \mathrm{nat}_p^{[n]}) \otimes \wedge^{n-j}\{(\coprod_{i=n+1}^{p} \mathbb{C}a_i) \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}]\}$$

[服部, Prop. 21.3, p. 125] as a $\widehat{\mathfrak{gl}_n}$-module. Explicitly, one may identify $\wedge^n \mathrm{nat}_n$ with $\wedge^n(\mathrm{nat}_p^{[n]}) = \coprod_{\lambda \in P(\wedge^n \mathrm{nat}_n)} (\wedge^n \mathrm{nat}_p)_\lambda$ with $P(\wedge^n \mathrm{nat}_p^{[n]}) = \{\sum_{i=1}^n n_i \hat{\varepsilon}_i + m\delta \in P | n_i \in \mathbb{N}, \sum_{i=1}^n n_i = n, m \in \mathbb{Z}\}$; the other summand has weights involving some $\hat{\varepsilon}_j$, $j > n$.

Note, however, that the imbedding of $\widehat{\mathfrak{gl}_n}$ into $\widehat{\mathfrak{gl}_p}$ is not compatible with the Chevalley elements associated to the index 0, e.g., $\hat{e}_0 = te(1,n)$, $\hat{f}_0 = t^{-1}e(n,1)$ in $\widehat{\mathfrak{gl}_n}$ while $\hat{e}_0 = te(1,p)$, $\hat{f}_0 = t^{-1}e(p,1)$ in $\widehat{\mathfrak{gl}_p}$. Although $te(1,n)$ and $t^{-1}e(n,1)$ have complicated expressions in terms of Chevalley elements in $\widehat{\mathfrak{gl}_p}$, their actions on $\mathrm{nat}_p^{[n]}$ are given, resp., by

$$\hat{e}_0 \hat{e}_{p-1} \dots \hat{e}_{n+1} \hat{e}_n \quad \text{and} \quad \hat{f}_n \hat{f}_{n+1} \dots \hat{f}_{p-1} \hat{f}_0. \tag{1}$$

Recall from (3.7) the isomorphism $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)] \to \wedge^n \mathrm{nat}_p$, and set

$$\mathrm{Rep}^{[n]}(G) = \coprod_{\lambda \in P(\wedge^n \mathrm{nat}_p^{[n]})} \mathrm{Rep}_{\iota_n(\lambda)}(G).$$

One thus obtains an action of $\widehat{\mathfrak{gl}_n}$ on $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}^{[n]}(G)] \simeq \wedge^n(\mathrm{nat}_p^{[n]})$. To avoid confusion about the nodes 0 on $\widehat{\mathfrak{gl}_n}$ and on $\widehat{\mathfrak{gl}_p}$ we will write ∞ for the node 0 in $\widehat{\mathfrak{gl}_n}$ after [RW]; $\hat{e}_\infty$ and $\hat{f}_\infty$ act on $\wedge^n(\mathrm{nat}_p^{[n]})$ as the elements in (1), resp.

One can, moreover, upgrade the action to a 2-representation of $\mathcal{U}(\widehat{\mathfrak{gl}_n})$ on $[\mathrm{Rep}^{[n]}(G)]$ as follows: in the notation from (4.3), $\forall \lambda \in P_{\widehat{\mathfrak{gl}_n}}$,

(i) let

$$\mathcal{C}_\lambda = \mathrm{R}_{\iota_n(\lambda)}(G) = \begin{cases} \mathrm{Rep}_{\iota_n(\lambda)}(G) & \text{if } \lambda \in P(\wedge^n \mathrm{nat}_p^{[n]}), \\ 0 & \text{else,} \end{cases}$$

(ii) $\forall i \in [1, n[$, let

$$E_i^\lambda = E_i|_{\mathrm{R}_{\iota_n(\lambda)}(G)} : \mathrm{R}_{\iota_n(\lambda)}(G) \to \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_i)}(G),$$
$$F_i^\lambda = F_i|_{\mathrm{R}_{\iota_n(\lambda)}(G)} : \mathrm{R}_{\iota_n(\lambda)}(G) \to \mathrm{R}_{\iota_n(\lambda-\hat{\alpha}_i)}(G),$$

and, corresponding to $E_\infty 1_\lambda$ and $F_\infty 1_\lambda$ in $\mathcal{U}(\widehat{\mathfrak{gl}_n})$, let

$$E_\infty^\lambda = (E_0 E_{p-1} \dots E_{n+1} E_n)|_{\mathrm{R}_{\iota_n(\lambda)}(G)} : \mathrm{R}_{\iota_n(\lambda)}(G) \to \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_n+\hat{\alpha}_{n+1}+\dots+\hat{\alpha}_{p-1}+\hat{\alpha}_0)}(G),$$
$$F_\infty^\lambda = (F_n F_{n+1} \dots F_{p-1} F_0)|_{\mathrm{R}_{\iota_n(\lambda)}(G)} : \mathrm{R}_{\iota_n(\lambda)}(G) \to \mathrm{R}_{\iota_n(\lambda-\hat{\alpha}_0-\hat{\alpha}_{p-1}-\dots-\hat{\alpha}_{n+1}-\hat{\alpha}_n)}(G).$$

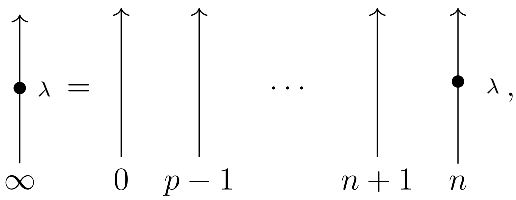
(iii) $\forall i, j \in [1, n[$, define

$$\begin{aligned}
x_i^\lambda &\in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_i)}(G))(E_i^\lambda, E_i^\lambda),\\
\tau_{(j,i)}^\lambda &\in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_i+\hat{\alpha}_j)}(G))(E_i^{\lambda+\hat{\alpha}_j} E_j^\lambda, E_j^{\lambda+\hat{\alpha}_i} E_i^\lambda),\\
\eta_i^\lambda &\in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda)}(G))(\mathrm{id}, F_i^{\lambda+\hat{\alpha}_i} E_i^\lambda),\\
\varepsilon_i^\lambda &\in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda)}(G))(E_i^{\lambda-\hat{\alpha}_i} F_i^\lambda, \mathrm{id})
\end{aligned}$$

to be $x_{\lambda,i}, \tau_{\lambda,(j,i)}, \eta_{\lambda,i}, \varepsilon_{\lambda,i}$, as in (4.3.iii), resp. Define

$$x_\infty^\lambda \in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_n+\hat{\alpha}_{n+1}+\cdots+\hat{\alpha}_{p-1}+\hat{\alpha}_0)}(G))(E_\infty^\lambda, E_\infty^\lambda)$$

to be



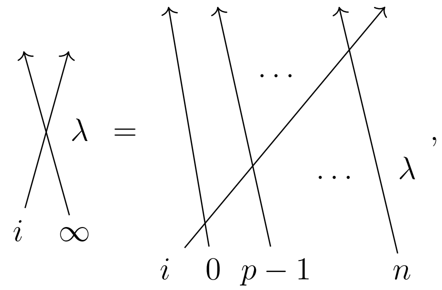
which reads

$$\iota_{E_0 E_{p-1} \dots E_{n+2} E_{n+1}|_{\mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_n)}(G)}} * x_n^\lambda$$
$$= \iota_{E_0^{\lambda+\hat{\alpha}_n+\hat{\alpha}_{n+1}+\cdots+\hat{\alpha}_{p-1}}} * \iota_{E_{p-1}^{\lambda+\hat{\alpha}_n+\hat{\alpha}_{n+1}+\cdots+\hat{\alpha}_{p-2}}} * \cdots * \iota_{E_{n+1}^{\lambda+\hat{\alpha}_n}} * x_n^\lambda.$$

Define

$$\tau_{(\infty,i)}^\lambda \in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_i+\hat{\alpha}_n+\hat{\alpha}_{n+1}+\cdots+\hat{\alpha}_{p-1}+\hat{\alpha}_0)}(G))(E_i^{\lambda+\hat{\alpha}_n+\cdots+\hat{\alpha}_{p-2}+\hat{\alpha}_{p-1}+\hat{\alpha}_0} E_\infty^\lambda, E_\infty^{\lambda+\hat{\alpha}_i} E_i^\lambda),$$

to be



which reads

$$
\begin{aligned}
&\big(\iota_{E_0E_{p-1}\dots E_{n+2}E_{n+1}|_{\mathrm{R}_{\iota n(\lambda+\hat\alpha_n+\hat\alpha_i)}(G)}} * \tau^{\lambda}_{(n,i)}\big) \odot \dots \\
&\quad\odot \big(\iota_{E_0E_{p-1}|_{\mathrm{R}_{\iota n(\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-2}+\hat\alpha_i)}(G)}} * \tau^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-3}}_{(p-2,i)} * \iota_{E_{p-3}\dots E_{n+1}E_n|_{\mathrm{R}_{\iota n(\lambda)}(G)}}\big) \\
&\quad\odot \big(\iota_{E_0|_{\mathrm{R}_{\iota n(\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-1}+\hat\alpha_i)}(G)}} * \tau^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-2}}_{(p-1,i)} * \iota_{E_{p-2}\dots E_{n+1}E_n|_{\mathrm{R}_{\iota n(\lambda)}(G)}}\big) \\
&\quad\odot \big(\tau^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-1}}_{(0,i)} * \iota_{E_{p-1}\dots E_{n+1}E_n|_{\mathrm{R}_{\iota n(\lambda)}(G)}}\big) \\
&= \big(\iota_{E_0^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-1}+\hat\alpha_i}} * \iota_{E_{p-1}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-2}+\hat\alpha_i}} * \iota_{E_{p-2}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-3}}} * \dots * \iota_{E_{n-1}^{\lambda+\hat\alpha_n+\hat\alpha_i}} \tau^{\lambda}_{(n,i)}\big) \\
&\quad\odot \dots \\
&\quad\odot \big(\iota_{E_0^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-1}+\hat\alpha_i}} * \iota_{E_{p-1}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-2}+\hat\alpha_i}} * \tau^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-2}}_{(i,p-2)} * \\
&\qquad\quad \iota_{E_{p-3}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-4}}} * \dots * \iota_{E_{n+1}^{\lambda+\hat\alpha_n}} * \iota_{E_n^{\lambda}}\big) \\
&\quad\odot \big(\iota_{E_0^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-1}+\hat\alpha_i}} * \tau^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-1}}_{(i,p-1)} * \iota_{E_{p-2}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-3}}} * \dots * \iota_{E_{n+1}^{\lambda+\hat\alpha_n}} * \iota_{E_n^{\lambda}}\big) \\
&\quad\odot \big(\tau^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-1}}_{(0,i)} * \iota_{E_{p-1}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-2}}} * \dots * \iota_{E_{n+1}^{\lambda+\hat\alpha_n}} * \iota_{E_n^{\lambda}}\big),
\end{aligned}
$$

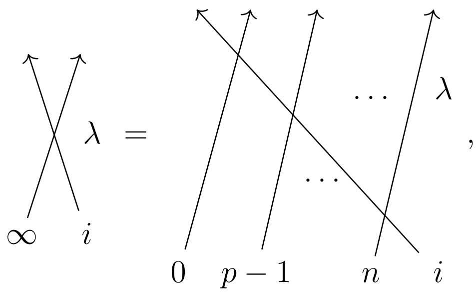
i.e., with $\gamma = \lambda + \hat\alpha_n + \hat\alpha_{n+1} + \dots + \hat\alpha_{p-1} + \hat\alpha_0 + \hat\alpha_i$, suppressing the restrictions and the superscripts,

$$
\begin{tikzcd}
\lambda \arrow[rrrr, "E_iE_0E_{p-1}\dots E_{n+1}E_n"] \arrow[d, phantom, ""] &&&& \gamma \\
\lambda \arrow[r, "E_{p-1}\dots E_{n+1}E_n"] \arrow[d, phantom, "\Downarrow \iota_{E_{p-1}\dots E_{n+1}E_n}"] & \lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-1} \arrow[rrr, "E_iE_0"] \arrow[rrr, phantom, "\parallel", shift left=4] \arrow[rrrd, phantom, "\Downarrow \tau_{(0,i)}"] &&& \gamma \\
\lambda \arrow[r, "E_{p-1}\dots E_{n+1}E_n"] & \lambda+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-1} \arrow[rrr, "E_0E_i"] \arrow[d, phantom, "\parallel"] &&& \gamma \\
\lambda \arrow[r, "E_{p-2}\dots E_{n+1}E_n"] \arrow[d, phantom, "\Downarrow \iota_{E_{p-2}\dots E_{n+1}E_n}"] & \lambda+\hat\alpha_n+\dots+\hat\alpha_{p-2} \arrow[rr, "E_iE_{p-1}"] \arrow[rrd, phantom, "\Downarrow \tau_{(p-1,i)}"] && \gamma-\hat\alpha_0 \arrow[r, "E_0"] \arrow[rd, phantom, "\Downarrow \iota_{E_0}"] & \gamma \\
\lambda \arrow[r, "E_{p-2}\dots E_{n+1}E_n"] & \lambda+\hat\alpha_n+\dots+\hat\alpha_{p-2} \arrow[rr, "E_{p-1}E_i"] \arrow[rrd, phantom, "\parallel"] && \gamma-\hat\alpha_0 \arrow[r, "E_0"] & \gamma \\
\lambda \arrow[r, "E_{p-3}\dots E_{n+1}E_n"] \arrow[d, phantom, "\Downarrow \iota_{E_{p-3}\dots E_{n+1}E_n}"] & \lambda+\hat\alpha_n+\dots+\hat\alpha_{p-3} \arrow[rr, "E_iE_{p-2}"] \arrow[rrd, phantom, "\Downarrow \tau_{(p-2,i)}"] && \gamma-\hat\alpha_0-\hat\alpha_{p-1} \arrow[r, "E_0E_{p-1}"] \arrow[rd, phantom, "\Downarrow \iota_{E_0}"] & \gamma \\
& \dots && \dots & \\
\lambda \arrow[r, "E_iE_n"] \arrow[rd, phantom, "\Downarrow \tau_{(n,i)}"] & \lambda+\hat\alpha_n+\hat\alpha_i \arrow[rrr, "E_0E_{p-1}\dots E_{n+1}"] \arrow[rrrd, phantom, "\Downarrow \iota_{E_0E_{p-1}\dots E_{n+1}}"] &&& \gamma \\
\lambda \arrow[r, "E_nE_i"'] & \lambda+\hat\alpha_n+\hat\alpha_i \arrow[rrr, "E_0E_{p-1}\dots E_{n+1}"'] \arrow[d, phantom, "\parallel"] &&& \gamma \\
\lambda \arrow[rrrr, "E_0E_{p-1}\dots E_{n+1}E_nE_i"'] &&&& \gamma.
\end{tikzcd}
$$

Define

$$
\tau^{\lambda}_{(i,\infty)} \in \mathrm{Cat}(\mathrm{R}_{\iota n(\lambda)}(G), \mathrm{R}_{\iota n(\lambda+\hat\alpha_i+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-1}+\hat\alpha_0)}(G))(E_\infty^{\lambda+\hat\alpha_i}E_i^{\lambda}, E_i^{\lambda+\hat\alpha_n+\dots+\hat\alpha_{p-2}+\hat\alpha_{p-1}+\hat\alpha_0}E_\infty^{\lambda})
$$

to be



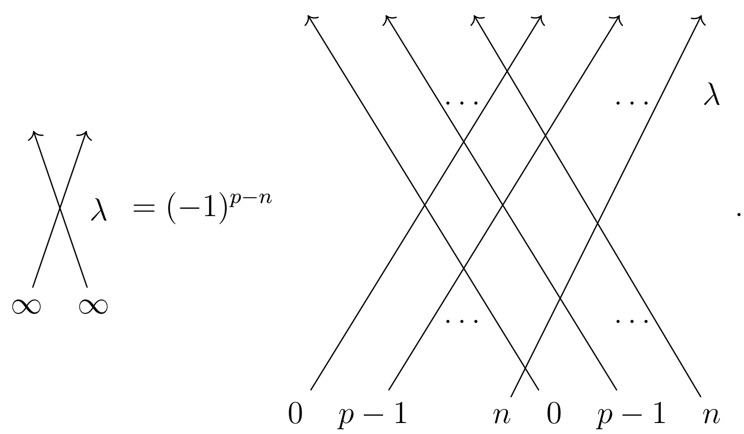
which reads

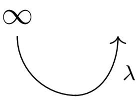
$$
\begin{aligned}
&\big(\tau_{(i,0)}{}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\cdots+\hat\alpha_{p-1}} * \iota_{E_{p-1}\dots E_{n+1}E_n|_{\mathrm{R}_{\iota n(\lambda)}(G)}}\big) \odot \dots \\
&\qquad \odot \big(\iota_{E_0E_{p-1}\dots E_{n+3}|_{\mathrm{R}_{\iota n(\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\hat\alpha_{n+2}+\hat\alpha_i)}(G)}} * \tau^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}}_{(i,n+2)} * \iota_{E_{n+1}E_n|_{\mathrm{R}_{\iota n(\lambda)}}}\big) \\
&\qquad \odot \big(\iota_{E_0E_{p-1}\dots E_{n+3}E_{n+2}|_{\mathrm{R}_{\iota n(\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\hat\alpha_i)}(G)}} * \tau^{\lambda+\hat\alpha_n}_{(i,n+1)} * \iota_{E_n|_{\mathrm{R}_{\iota n(\lambda)}}}\big) \\
&\qquad \odot \big(\iota_{E_0E_{p-1}\dots E_{n+2}E_{n+1}|_{\mathrm{R}_{\iota n(\lambda+\hat\alpha_n+\hat\alpha_i)}(G)}} * \tau^{\lambda}_{(i,n)}\big) \\
&\;= \big(\tau^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\cdots+\hat\alpha_{p-1}}_{(i,0)} * \iota_{E_{p-1}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\cdots+\hat\alpha_{p-2}}} * \iota_{E_{p-2}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\cdots+\hat\alpha_{p-3}}} * \cdots * \iota_{E_{n-1}^{\lambda+\hat\alpha_n+\hat\alpha_i}}\big) \\
&\qquad \odot \big(\iota_{E_0^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\cdots+\hat\alpha_{p-1}+\hat\alpha_i)}} * \tau^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\cdots+\hat\alpha_{p-2}}_{(i,p-1)} * \iota_{E_{p-2}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\cdots+\hat\alpha_{p-3}}} * \iota_{E_{p-3}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\cdots+\hat\alpha_{p-4}}} * \\
&\qquad\qquad \cdots * \iota_{E_{n+1}^{\lambda+\hat\alpha_n}} * \iota_{E_n^\lambda}\big) \\
&\qquad \odot \dots \\
&\\
&\qquad \odot \big(\iota_{E_0^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\cdots+\hat\alpha_{p-1}+\hat\alpha_i)}} * \iota_{E_{p-1}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\cdots+\hat\alpha_{p-2}}} * \cdots * \iota_{E_{n+2}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\hat\alpha_i}} * \tau^{\lambda+\hat\alpha_n}_{(i,n+1)} * \iota_{E_n^\lambda}\big) \\
&\qquad \odot \big(\iota_{E_0^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\cdots+\hat\alpha_{p-1}+\hat\alpha_i)}} * \iota_{E_{p-1}^{\lambda+\hat\alpha_n+\hat\alpha_{n+1}+\cdots+\hat\alpha_{p-2}+\hat\alpha_i}} * \cdots * \iota_{E_{n+1}^{\lambda+\hat\alpha_n+\hat\alpha_i}} * \tau^{\lambda}_{(i,n)}\big).
\end{aligned}
$$

Define

$$
\tau^\lambda_{(\infty,\infty)} \in \mathrm{Cat}(\mathrm{R}_{\iota n(\lambda)}(G), \mathrm{R}_{\iota n(\lambda+2\hat\alpha_n+2\hat\alpha_{n+1}+\cdots+2\hat\alpha_{p-1}+2\hat\alpha_0)}(G)) \\
\qquad (E_\infty^{\lambda+\hat\alpha_n+\cdots+\hat\alpha_{p-2}+\hat\alpha_{p-1}+\hat\alpha_0} E_\infty^\lambda, E_\infty^{\lambda+\hat\alpha_n+\cdots+\hat\alpha_{p-2}+\hat\alpha_{p-1}+\hat\alpha_0} E_\infty^\lambda)
$$

to be

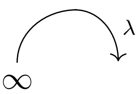


Define $\eta_\infty^\lambda \in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda)}(G))(\mathrm{id}, F_\infty^{\lambda+\hat{\alpha}_n+\cdots+\hat{\alpha}_{p-2}+\hat{\alpha}_{p-1}+\hat{\alpha}_0} E_\infty^\lambda)$, denoted , to be

$$
\begin{aligned}
\big(\iota_{F_nF_{n+1}\dots F_{p-1}|_{\mathrm{Rep}_{\iota_n(\lambda+\hat{\alpha}_n+\cdots+\hat{\alpha}_{p-1})}(G)}} * \eta_0^{\lambda+\hat{\alpha}_n+\hat{\alpha}_{n+1}+\cdots+\hat{\alpha}_{p-1}} * \iota_{E_{p-1}\dots E_{n+1}\dots E_n|_{\mathrm{Rep}_{\iota_n(\lambda)}(G)}}\big)& \\
\odot \cdots \odot \big(\iota_{F_nF_{n+1}|_{\mathrm{Rep}_{\iota_n(\lambda+\hat{\alpha}_n+\hat{\alpha}_{n+1})}(G)}} * \eta_{n+2}^{\lambda+\hat{\alpha}_n+\hat{\alpha}_{n+1}} * \iota_{E_{n+1}E_n|_{\mathrm{Rep}_{\iota_n(\lambda)}(G)}}\big)& \\
\odot \big(\iota_{F_n}|_{\mathrm{Rep}_{\iota_n(\lambda+\hat{\alpha}_n)}(G)} * \eta_{n+1}^{\lambda+\hat{\alpha}_n} * \iota_{E_n|_{\mathrm{Rep}_{\iota_n(\lambda)}(G)}}\big) \odot \eta_n^\lambda&
\end{aligned}
$$

with $\eta_0^{\lambda+\hat{\alpha}_n+\hat{\alpha}_{n+1}+\cdots+\hat{\alpha}_{p-1}}$ and $\eta_i^{\lambda+\hat{\alpha}_n+\hat{\alpha}_{n+1}+\cdots+\hat{\alpha}_{i-1}}$, $i \in [n, p[$, as in (4.3.iii): suppressing the superscripts

$$
\begin{tikzcd}
\lambda \arrow[rrr, "\mathrm{id}"] \arrow[rrr, phantom, shift right=4ex, "\Downarrow\eta_n"] & & & \lambda \\
\lambda \arrow[rrr, "F_nE_n"] \arrow[rrr, phantom, shift right=3ex, "\|"] & & & \lambda \\
\lambda \arrow[r, "E_n"] \arrow[r, phantom, shift right=4ex, "\Downarrow\iota_{E_n}"] & \lambda+\hat{\alpha}_n \arrow[r, "\mathrm{id}"] \arrow[r, phantom, shift right=4ex, "\Downarrow\eta_{n+1}"] & \lambda+\hat{\alpha}_n \arrow[r, "F_n"] \arrow[r, phantom, shift right=4ex, "\Downarrow\iota_{F_n}"] & \lambda \\
\lambda \arrow[r, "E_n"] & \lambda+\hat{\alpha}_n \arrow[r, "F_{n+1}E_{n+1}"] \arrow[r, phantom, shift right=3ex, "\|"] & \lambda+\hat{\alpha}_n \arrow[r, "F_n"] & \lambda \\
\lambda \arrow[r, "E_{n+1}E_n"] \arrow[r, phantom, shift right=4ex, "\Downarrow\iota_{E_{n+1}E_n}"] & \lambda+\hat{\alpha}_n+\hat{\alpha}_{n+1} \arrow[r, "\mathrm{id}"] \arrow[r, phantom, shift right=4ex, "\Downarrow\eta_{n+2}"] & \lambda+\hat{\alpha}_n+\hat{\alpha}_{n+1} \arrow[r, "F_nF_{n+1}"] \arrow[r, phantom, shift right=4ex, "\Downarrow\iota_{F_nF_{n+1}}"] & \lambda \\
 & \cdots & \cdots \quad \cdots & \\
\lambda \arrow[r, "E_{p-1}\dots E_{n+1}E_n"] \arrow[r, phantom, shift left=4ex, "\Downarrow\iota_{E_{p-1}\cdots E_{n+1}E_n}"] & \lambda+\hat{\alpha}_n+\hat{\alpha}_{n+1}+\cdots+\hat{\alpha}_{p-1} \arrow[r, "F_0E_0"] \arrow[r, phantom, shift left=4ex, "\Downarrow\eta_0"] \arrow[r, phantom, shift right=3ex, "\|"] & \lambda+\hat{\alpha}_n+\hat{\alpha}_{n+1}+\cdots+\hat{\alpha}_{p-1} \arrow[r, "F_nF_{n+1}\dots F_{p-1}"] \arrow[r, phantom, shift left=4ex, "\Downarrow\iota_{F_nF_{n+1}\cdots F_{p-1}}"] & \lambda, \\
\lambda \arrow[rrr, "F_nF_{n+1}\dots F_{p-1}F_0E_0E_{p-1}\dots E_{n+1}E_n"'] & & & \lambda.
\end{tikzcd}
$$

Define finally $\varepsilon_\infty^\lambda \in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda)}(G))(E_\infty^{\lambda-\hat{\alpha}_n-\cdots-\hat{\alpha}_{p-2}-\hat{\alpha}_{p-1}-\hat{\alpha}_0} F_\infty^\lambda, \mathrm{id})$, denoted , to be

$$
\begin{aligned}
&\varepsilon_0^\lambda \odot \big(\iota_{E_0|_{\iota_n(\lambda-\hat{\alpha}_0)(G)}}\big) * \iota_{F_0|_{\mathrm{Rep}_{\iota_n(\lambda)}(G)}}\big) \odot \dots \\
&\quad \odot \big(\iota_{E_0E_{p-1}\dots E_{n+2}|_{\mathrm{Rep}_{\iota_n(\lambda-\hat{\alpha}_0-\hat{\alpha}_{p-1}-\cdots-\hat{\alpha}_{n+2})}(G)}} * \varepsilon_{n+1}^{\lambda-\hat{\alpha}_0-\hat{\alpha}_{p-1}-\cdots-\hat{\alpha}_{n+2}} * \iota_{F_{n+2}\dots F_{p-1}F_0|_{\mathrm{Rep}_{\iota_n(\lambda)}(G)}}\big) \\
&\qquad \odot \big(\iota_{E_0E_{p-1}\dots E_{n+1}|_{\mathrm{Rep}_{\iota_n(\lambda-\hat{\alpha}_0-\hat{\alpha}_{p-1}-\cdots-\hat{\alpha}_{n+1})}(G)}} * \varepsilon_n^{\lambda-\hat{\alpha}_0-\hat{\alpha}_{p-1}-\cdots-\hat{\alpha}_{n+1}} * \iota_{F_{n+1}\dots F_{p-1}F_0|_{\mathrm{Rep}_{\iota_n(\lambda)}(G)}}\big)
\end{aligned}
$$

with ε_0 and ε_i, $i \in [n, p[$, as in (4.3.iii):

$$\begin{tikzcd}[column sep=large]
\lambda \arrow[rrr, "E_0E_{p-1}\dots E_{n+1}E_nF_nF_{n+1}\dots F_{p-1}F_0"] & {} \arrow[dr, phantom, "\|"] & & \lambda \\
\lambda \arrow[r, "F_{n+1}\dots F_{p-1}F_0"] \arrow[dr, phantom, "\Downarrow\iota_{F_{n+1}\cdots F_{p-1}F_0}"] & \lambda-\hat\alpha_0-\hat\alpha_{p-1}-\cdots-\hat\alpha_{n+1} \arrow[r, "E_nF_n"] \arrow[dr, phantom, "\Downarrow\varepsilon_n"] & \lambda-\hat\alpha_0-\hat\alpha_{p-1}-\cdots-\hat\alpha_{n+1} \arrow[r, "E_0E_{p-1}\dots E_{n+1}"] \arrow[dr, phantom, "\Downarrow\iota_{E_0E_{p-1}\cdots E_{n+1}}"] & \lambda \\
\lambda \arrow[r, "F_{n+1}\dots F_{p-1}F_0"] & \lambda-\hat\alpha_0-\hat\alpha_{p-1}-\cdots-\hat\alpha_{n+1} \arrow[r, "\mathrm{id}"] \arrow[dr, phantom, "\|"] & \lambda-\hat\alpha_0-\hat\alpha_{p-1}-\cdots-\hat\alpha_{n+1} \arrow[r, "E_0E_{p-1}\dots E_{n+1}"] & \lambda \\
\lambda \arrow[r, "F_{n+2}\dots F_{p-1}F_0"] \arrow[dr, phantom, "\Downarrow\iota_{F_{n+2}\cdots F_{p-1}F_0}"] & \lambda-\hat\alpha_0-\hat\alpha_{p-1}-\cdots-\hat\alpha_{n+2} \arrow[r, "E_{n+1}F_{n+1}"] \arrow[dr, phantom, "\Downarrow\varepsilon_{n+1}"] & \lambda-\hat\alpha_0-\hat\alpha_{p-1}-\cdots-\hat\alpha_{n+2} \arrow[r, "E_0E_{p-1}\dots E_{n+2}"] \arrow[dr, phantom, "\Downarrow\iota_{E_0E_{p-1}\cdots E_{n+2}}"] & \lambda \\
\lambda \arrow[r, "F_{n+2}\dots F_{p-1}F_0"] & \lambda-\hat\alpha_0-\hat\alpha_{p-1}-\cdots-\hat\alpha_{n+2} \arrow[r, "\mathrm{id}"] \arrow[dr, phantom, "\|"] & \lambda-\hat\alpha_0-\hat\alpha_{p-1}-\cdots-\hat\alpha_{n+2} \arrow[r, "E_0E_{p-1}\dots E_{n+2}"] & \lambda \\
\lambda \arrow[r, "F_{n+3}\dots F_{p-1}F_0"] \arrow[dr, phantom, "\Downarrow\iota_{F_{n+3}\cdots F_{p-1}F_0}"] & \lambda-\hat\alpha_0-\hat\alpha_{p-1}-\cdots-\hat\alpha_{n+3} \arrow[r, "E_{n+1}F_{n+2}"] \arrow[dr, phantom, "\Downarrow\varepsilon_{n+2}"] & \lambda-\hat\alpha_0-\hat\alpha_{p-1}-\cdots-\hat\alpha_{n+3} \arrow[r, "E_0E_{p-1}\dots E_{n+3}"] \arrow[dr, phantom, "\Downarrow\iota_{E_0E_{p-1}\cdots E_{n+3}}"] & \lambda \\
{} \arrow[dr, phantom, "\Downarrow\iota_{F_0}"] & \cdots \arrow[dr, phantom, "\Downarrow\varepsilon_{p-1}"] & \cdots \arrow[dr, phantom, "\Downarrow\iota_{E_0}"] & {} \\
\lambda \arrow[r, "F_0"] & \lambda-\hat\alpha_0 \arrow[r, "\mathrm{id}"] \arrow[dr, phantom, "\|"] & \lambda-\hat\alpha_0 \arrow[r, "E_0"] & \lambda \\
\lambda \arrow[rrr, "E_0F_0"] & {} \arrow[dr, phantom, "\Downarrow\varepsilon_0"] & & \lambda \\
\lambda \arrow[rrr, "\mathrm{id}"'] & & & \lambda.
\end{tikzcd}$$

To check that the so defined generating 2-morphisms satisfy the required relations, one can lift the 2-morphisms to those in $\mathcal{U}_+(\widehat{\mathfrak{gl}_p})$ and check a number of the relations there [RW, 7.3]. For the rest see [RW, pp. 101-102].

Theorem [RW, Th. 7.4.1]: The data above defines a 2-representation of $\mathcal{U}(\widehat{\mathfrak{gl}_n})$.

(4.11′) To check that (4.11) carries over to $\mathrm{Rep}(G_1T)$, one has $\otimes^n \mathrm{nat}_n$ a direct summnad of $\otimes^n \mathrm{nat}_p \simeq \coprod_{j=0}^n (\otimes^j \mathrm{nat}_p^{[n]}) \otimes \otimes^{n-j} \{(\coprod_{i=n+1}^p \mathbb{C}a_i) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]\}$ as a $\widehat{\mathfrak{gl}_n}$-module;

$$\otimes^n \mathrm{nat}_n \simeq \otimes^n (\mathrm{nat}_p^{[n]}) = \coprod_{\lambda \in P(\otimes^n \mathrm{nat}_n)} (\otimes^n \mathrm{nat}_p)_\lambda$$

with $P(\otimes^n \mathrm{nat}_n) = \{\sum_{i=1}^n n_i \hat\varepsilon_i + m\delta \in P | n_i \in \mathbb{N}, \sum_{i=1}^n n_i = n, m \in \mathbb{Z}\}$. As $P(\otimes^n \mathrm{nat}_n) = P(\wedge^n \mathrm{nat}_n)$, the arguments of (4.11) carry over to $\mathrm{Rep}(G_1T)$.

(4.12) Could we lift the 2-representation in (4.11) to a 2-functor $\mathcal{U}(\widehat{\mathfrak{gl}_n}) \to \mathcal{U}_+(\widehat{\mathfrak{gl}_p})$?

Definition [Bor, Def. 7.2.1, pp. 287-288]: Given two strict 2-categories $\mathcal{A}$ and $\mathcal{B}$, a 2-functor $\Phi : \mathcal{A} \to \mathcal{B}$ consists of the data

(i) for each object A of $\mathcal{A}$, an object ΦA of $\mathcal{B}$,

(ii) $\forall A, A' \in |\mathcal{A}|$, a functor $\Phi_{A,A'} : \mathcal{A}(A, A') \to \mathcal{B}(\Phi A, \Phi A')$ compatible with the compositions and the units: $\forall A, A', A'' \in |\mathcal{A}|$, $\Phi_{A,A''} \circ c_{A,A',A''} = c_{\Phi A, \Phi A', \Phi A''} \circ (\Phi_{A,A'} \times \Phi_{A',A''})$

$$\begin{tikzcd}
\mathcal{A}(A,A') \times \mathcal{A}(A',A'') \arrow[rr, "c_{A,A',A''}"] \arrow[d, "\Phi_{A,A'} \times \Phi_{A',A''}"'] & \circlearrowleft & \mathcal{A}(A,A'') \arrow[d, "\Phi_{A,A''}"] \\
\mathcal{B}(\Phi A, \Phi A') \times \mathcal{B}(\Phi A', \Phi A'') \arrow[rr, "c_{\Phi A, \Phi A', \Phi A''}"'] & & \mathcal{B}(\Phi A, \Phi A'')
\end{tikzcd}$$

and $\Phi_{A,A} \circ u_A = u_{\Phi A}$

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "u_A"] \arrow[dr, "u_{\Phi A}"'] & \mathcal{A}(A,A) \arrow[d, "\Phi_{A,A}"] \\
\circlearrowleft & \mathcal{B}(\Phi A, \Phi A).
\end{tikzcd}$$

In the case of the 2-representation we defined $\mathcal{C}_\lambda = 0$ unless $\lambda \in P(\wedge^n \mathrm{nat}_n)$ while we cannot associate 0 to $E_i 1_\lambda$ for $\lambda \in P_{\widehat{\mathfrak{gl}_n}} \setminus P(\wedge^n \mathrm{nat}_n)$. To compensate that, consider now a data $\Phi : \mathcal{U}(\widehat{\mathfrak{gl}_n}) \to \mathcal{U}_+(\widehat{\mathfrak{gl}_p})$ such that

(Φ1) $|\mathcal{U}(\widehat{\mathfrak{gl}_n})| = P_{\widehat{\mathfrak{gl}_n}} \hookrightarrow P = |\mathcal{U}(\widehat{\mathfrak{gl}_p})|$,

(Φ2) $\forall \lambda, \mu \in P_{\widehat{\mathfrak{gl}_n}}$, define $\Phi_{\lambda,\mu} : \mathcal{U}(\widehat{\mathfrak{gl}_n})(\lambda,\mu) \to \mathcal{U}_+(\widehat{\mathfrak{gl}_p})(\Phi\lambda, \Phi\mu) = \mathcal{U}_+(\widehat{\mathfrak{gl}_p})(\lambda,\mu)$ to be a strict monoidal functor such that $\forall i \in [1, n[\cup \{\infty\}$ $\forall \nu \in P_{\widehat{\mathfrak{gl}_n}}$,

$$E_i 1_\nu \mapsto \begin{cases} E_0 E_{p-1} \dots E_{n+1} E_n 1_\nu & \text{if } i = \infty, \\ E_i 1_\nu & \text{else,} \end{cases} \qquad F_i 1_\nu \mapsto \begin{cases} F_n F_{n+1} \dots F_{p-1} F_0 1_\nu & \text{if } i = \infty, \\ E_i 1_\nu & \text{else,} \end{cases}$$

and for the generating 2-morphisms such that $\forall \lambda \in P_{\widehat{\mathfrak{gl}_n}}$, $\forall i, j \in [1, n[$,

$$\mathcal{U}(\widehat{\mathfrak{gl}_n})(\lambda,\lambda)(E_i 1_\lambda, E_i 1_\lambda) \ni x \mapsto \begin{cases} x \in \mathcal{U}_+(\widehat{\mathfrak{gl}_p})(\lambda,\lambda)(E_i 1_\lambda, E_i 1_\lambda) & \text{if } \lambda \in P(\wedge^n \mathrm{nat}_n), \\ 0 & \text{else,} \end{cases}$$

$$\mathcal{U}(\widehat{\mathfrak{gl}_n})(\lambda, \lambda + \hat{\alpha}_i + \hat{\alpha}_j)(E_i E_j 1_\lambda, E_j E_i 1_\lambda) \ni \tau \mapsto$$
$$\begin{cases} \tau \in \mathcal{U}_+(\widehat{\mathfrak{gl}_p})(\lambda, \lambda + \hat{\alpha}_i + \hat{\alpha}_j)(E_i E_j 1_\lambda, E_j E_i 1_\lambda) & \text{if } \lambda, \lambda + \hat{\alpha}_i, \lambda + \hat{\alpha}_j, \lambda + \hat{\alpha}_i + \hat{\alpha}_j \in P(\wedge^n \mathrm{nat}_n), \\ 0 & \text{else,} \end{cases}$$

$$\mathcal{U}(\widehat{\mathfrak{gl}_n})(\lambda,\lambda)(1_\lambda, F_i E_i 1_\lambda) \ni \eta \mapsto$$
$$\begin{cases} \eta \in \mathcal{U}_+(\widehat{\mathfrak{gl}_p})(\lambda,\lambda)(1_\lambda, F_i E_i 1_\lambda) & \text{if } \lambda, \lambda + \hat{\alpha}_i \in P(\wedge^n \mathrm{nat}_n), \\ 0 & \text{else,} \end{cases}$$

$$\mathcal{U}(\widehat{\mathfrak{gl}_n})(\lambda,\lambda)(E_i F_i 1_\lambda, 1_\lambda) \ni \varepsilon \mapsto$$
$$\begin{cases} \varepsilon \in \mathcal{U}_+(\widehat{\mathfrak{gl}_p})(\lambda,\lambda)(E_i F_i 1_\lambda, 1_\lambda) & \text{if } \lambda, \lambda - \hat{\alpha}_i \in P(\wedge^n \mathrm{nat}_n), \\ 0 & \text{else,} \end{cases}$$

etc. Does $\Phi : \mathcal{U}(\widehat{\mathfrak{gl}_n}) \to \mathcal{U}_+(\widehat{\mathfrak{gl}_p})$ define a 2-functor?

(4.13) Just as we defined $\mathcal{U}_+(\widehat{\mathfrak{gl}_p})$, define a 2-category $\mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})$ to be the 2-category having the same data as that of $\mathcal{U}(\widehat{\mathfrak{gl}_n})$ but setting, $\forall \lambda, \mu \in P_{\widehat{\mathfrak{gl}_n}}$, $\forall X, Y \in \mathcal{U}(\widehat{\mathfrak{gl}_n})(\lambda, \mu)$, $\{\mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\lambda, \mu)\}(X, Y) = \{\mathcal{U}(\widehat{\mathfrak{gl}_n})(\lambda, \mu)\}(X, Y)/\mathcal{I}^{[n]}(X, Y)$ with $\mathcal{I}^{[n]}(X, Y)$ denoting the $\Bbbk$-linear span of those $f : X \Rightarrow Y$ which factors through some $Z_2 \circ Z_1$, $Z_1 \in \mathcal{U}(\widehat{\mathfrak{gl}_n})(\lambda, \nu)$, $Z_2 \in \mathcal{U}(\widehat{\mathfrak{gl}_n})(\nu, \mu)$, $\nu \in P_{\widehat{\mathfrak{gl}_n}} \setminus P(\wedge^n \mathrm{nat}_n)$. By construction the 2-representation of $\mathcal{U}(\widehat{\mathfrak{gl}_n})$ factors through $\mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})$ to induce a strict monoidal functor $\mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi, \varpi) \to \mathrm{Cat}(\mathrm{Rep}_{[n\,\det]}(G), \mathrm{Rep}_{[n\,\det]}(G))$ such that $F_{n-j}E_{n-j}1_\varpi \mapsto \Theta_{s_j}$ $\forall j \in [1, n[$, and $F_\infty E_\infty 1_\varpi \mapsto \Theta_{s_{\alpha_0,1}}$. We are now reduced to construct a strict monoidal functor $\mathcal{D}_{\mathrm{BS}} \to \mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi, \varpi)$.

(4.13′) As $P(\otimes^n \mathrm{nat}_p) = P(\wedge^n \mathrm{nat}_p)$ again, the 2-representation of $\mathcal{U}(\widehat{\mathfrak{gl}_n})$ on $(\mathrm{Rep}_{\iota_n(\lambda)}(G_1T)|\lambda \in P(\otimes^n \mathrm{nat}_p))$ factors through $\mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})$ to induce a strict monoidal functor $\mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi, \varpi) \to \mathrm{Cat}(\mathrm{Rep}_{[n\,\det]}(G_1T), \mathrm{Rep}_{[n\,\det]}(G_1T))$ such that $F_{n-j}E_{n-j}1_\varpi \mapsto \Theta_{s_j}$ $\forall j \in [1, n[$, and $F_\infty E_\infty 1_\varpi \mapsto \Theta_{s_{\alpha_0,1}}$. It follows that the functor $\mathcal{D}_{\mathrm{BS}} \to \mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi, \varpi)$ of (4.13) will suffice to yield a strict monoidal functor $\mathcal{D}_{\mathrm{BS}} \to \mathrm{Cat}(\mathrm{Rep}_{[n\,\det]}(G_1T), \mathrm{Rep}_{[n\,\det]}(G_1T))$.

5° The Elias-Williamson diagrammatic category

We now attempt to give a "reasonably" precise definition of the Bott-Samelson diagrammatic category $\mathcal{D}_{\mathrm{BS}}$ and of the Elias-Williamson category $\mathcal{D}$. The assumption $p > n$ is enforced here [RW, Rmk. 4.2.1]. We state the fundamental existence theorem of a strict monoidal functor from $\mathcal{D}_{\mathrm{BS}}$ to the category $\mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi, \varpi)$, a quotient of $\mathcal{U}(\widehat{\mathfrak{gl}_n})(\varpi, \varpi)$ the category of 1-endomorphisms of weight ϖ of the affine Lie algebra $\widehat{\mathfrak{gl}_n}$. We leave, however, the lengthy proof consuming [RW, 8] as a black box.

(5.1) Let $\underline{R} = \mathrm{S}_\Bbbk(\Bbbk \otimes_{\mathbb{Z}} \mathbb{Z}R^\vee) = \Bbbk \otimes_{\mathbb{Z}} \mathrm{S}_{\mathbb{Z}}(\mathbb{Z}R^\vee)$ endowed with gradation such that $\deg(R^\vee) = 2$. An expression is a sequence $(s_1, s_2, \dots, s_r)$ of simple reflections $s_j \in \mathcal{S}_a$, which we denote by $\underline{s_1 s_2 \dots s_r}$. If $w = s_1 s_2 \dots s_r \in \mathcal{W}_a$, we often abbreviate the sequence as $\underline{w}$. We also write $\ell(\underline{w}) = r$. A subexpression of $\underline{w}$ is an expression $\underline{x}$ obtained from a subsequence of $\underline{w}$, in which case we write $\underline{x} \subseteq \underline{w}$.

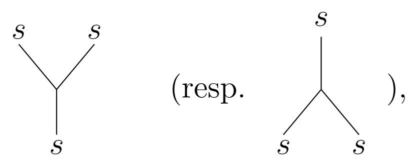
The category $\mathcal{D}_{\mathrm{BS}}$ is endowed with a shift of the grading autoequivalence $\langle 1 \rangle$, rather than a structure of graded category, consisting of objects, $B_{\underline{w}}\langle m \rangle$, $\underline{w}$ an expression of $w \in \mathcal{W}_a$, $m \in \mathbb{Z}$, such that $(B_{\underline{w}}\langle m \rangle)\langle 1 \rangle = B_{\underline{w}}\langle m+1 \rangle$. This is not even an additive category; the Karoubian envelope of its additive hull $\mathcal{D}$ appearing later, on the other hand, is a graded category [RW, 1.2, p. 3]. We will abbreviate $B_{\underline{w}}\langle 0 \rangle$ as $B_{\underline{w}}$. Under the product defined on the objects such that $(B_{\underline{w}}\langle m \rangle) \cdot (B_{\underline{v}}\langle m' \rangle) = B_{\underline{wv}}\langle m + m' \rangle$ with $\underline{wv}$ denoting the concatenation of $\underline{w}$ and $\underline{v}$, $\mathcal{D}_{\mathrm{BS}}$ comes equipped with a structure of monoidal category. Thus, $B_\emptyset$ is the unital object of $\mathcal{D}_{\mathrm{BS}}$. For $s \in \mathcal{S}_a$ by $\underline{s}$ we mean a sequence s, but we will abbreviate $B_{\underline{s}}\langle m \rangle$ as $B_s\langle m \rangle$.

We will use diagrams to denote morphisms in $\mathcal{D}_{\mathrm{BS}}$. An element of $\mathcal{D}_{\mathrm{BS}}(B_{\underline{v}}\langle m \rangle, B_{\underline{w}}\langle m' \rangle)$ is a $\Bbbk$-linear combination of certain equivalence classes of diagrams whose bottom has strands labelled by the simple reflections with multiplicities appearing in $\underline{v}$, and whose top has strands labeled by the simple reflections with multiplicities appearing in $\underline{w}$. Diagrams should be read from bottom to top. The monoidal product correspond to a horizontal concatenation, and the composition to a vertical concatenation. The diagrams, i.e., morphisms, are constructed by horizontal and vertical concatenations of images under autoequivalences $\langle m\rangle$, $m \in \mathbb{Z}$, of 4 different types of generators:

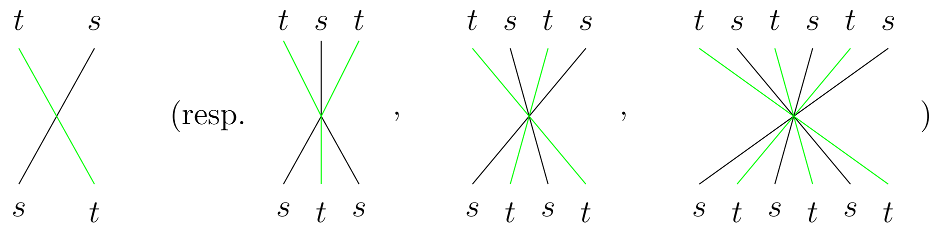
(G1) $\forall f \in \underline{R}$ homogeneous, $B_\emptyset \to B_\emptyset\langle \deg(f)\rangle$ represented diagrammatically as f with empty top and bottom,

(G2) $\forall s \in \mathcal{S}_a$, the upper dot $B_s \to B_\emptyset\langle 1\rangle$ (resp. the lower dot $B_\emptyset \to B_s\langle 1\rangle$) represented as

(G3) $\forall s \in \mathcal{S}_a$, the trivalent vertices $B_s \to B_{\underline{ss}}\langle -1\rangle$ (resp. $B_{\underline{ss}} \to B_s\langle -1\rangle$) represented as



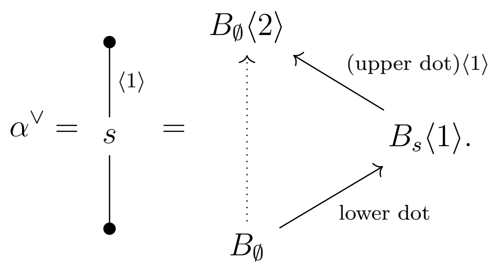
(G4) $\forall s,t \in \mathcal{S}_a$ with $s \neq t$ and $\mathrm{ord}(st) = m_{st}$ in $\mathcal{W}_a$, the $2m_{st}$-valent vertex $B_{\underbrace{\underline{st\ldots}}_{m_{st}}} \to B_{\underbrace{\underline{ts\ldots}}_{m_{st}}}$ represented as



if $m_{st} = 2$ (resp. 3, 4, 6).

Those generators are subject to a number of relations described in [EW, §5]. The relations define the "equivalence relations" on the morphisms. We recall only that isotopic diagrams are equivalent, and that, $\forall \alpha \in R^{\mathrm{s}}$, the morphism $\alpha^\vee \in \mathcal{D}_{\mathrm{BS}}(B_\emptyset, B_\emptyset\langle 2\rangle)$ in (G1) is the composition of morphisms in (G2) [EW, 5.1]:

(1)



As $\underline{R} = \Bbbk[\alpha^\vee | \alpha \in R^{\mathrm{s}}]$, the morphisms in (G2)-(G4) are, in fact, sufficient to generate all the morphisms in $\mathcal{D}_{\mathrm{BS}}$.

(5.2) There is also a monoidal equivalence $\tau : \mathcal{D}_{\mathrm{BS}} \to \mathcal{D}_{\mathrm{BS}}^{\mathrm{op}}$ sending each $B_{\underline{w}}\langle m\rangle$ to $B_{\underline{w}}\langle -m\rangle$ and reflecting diagrams along a horizontal axis [RW, 6.3].

$\forall X, Y \in \mathcal{D}_{\mathrm{BS}}$, set $\mathcal{D}_{\mathrm{BS}}^{\bullet}(X,Y) = \coprod_{m\in\mathbb{Z}} \mathcal{D}_{\mathrm{BS}}(X, Y\langle m\rangle)$, which is equipped with a structure of graded bimodule over $\underline{R}$ such that $\forall f \in \underline{R}$ homogeneous, $\forall \phi \in \mathcal{D}_{\mathrm{BS}}^{\bullet}(X,Y)$,

$$\begin{tikzcd} X = B_\emptyset \cdot X \arrow[r, "f\cdot X"] \arrow[d, "\phi"'] \arrow[dr, dotted, "f\cdot\phi"] & (B_\emptyset\langle \deg f\rangle)\cdot X = X\langle \deg f\rangle \arrow[d, "\phi\langle \deg f\rangle"] \\ Y = B_\phi \cdot Y \arrow[r, "f\cdot Y"'] & (B_\emptyset\langle \deg f\rangle)\cdot Y = Y\langle \deg f\rangle, \end{tikzcd}$$

$$\begin{tikzcd} X = X \cdot B_\emptyset \arrow[r, "X\cdot f"] \arrow[d, "\phi"'] \arrow[dr, dotted, "\phi\cdot f"] & X\cdot(B_\emptyset\langle \deg f\rangle) = X\langle \deg f\rangle \arrow[d, "\phi\langle \deg f\rangle"] \\ Y = Y \cdot B_\phi \arrow[r, "Y\cdot f"'] & Y\cdot(B_\emptyset\langle \deg f\rangle) = Y\langle \deg f\rangle. \end{tikzcd}$$

One has [EW, Cor. 6.13] that $\mathcal{D}_{\mathrm{BS}}^{\bullet}(X,Y)$ is free of finite rank as a left and as a right $\underline{R}$-module.

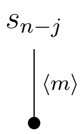
(5.3) Recall from (4.9) weight $\varpi = \hat{\varepsilon}_1 + \cdots + \hat{\varepsilon}_n \in P(\wedge^n \mathrm{nat}_p^{[n]})$. We now construct a strict monoidal functor $\mathcal{D}_{\mathrm{BS}} \to \mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi,\varpi)$ as follows: $\forall j \in [1, n[$, $\forall m \in \mathbb{Z}$, we assign

$$B_{s_{\alpha_{n-j}}}\langle m\rangle \mapsto F_jE_j1_\varpi = (F_j1_{\varpi+\hat{\alpha}_j})\circ(E_j1_\varpi),$$

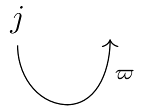
while

$$B_{s_{\alpha_0,1}}\langle m\rangle \mapsto F_\infty E_\infty 1_\varpi = (F_\infty 1_{\varpi+\hat{\alpha}_\infty})\circ(E_\infty 1_\varpi),$$

where $\hat{\alpha}_\infty = \delta + \hat{\varepsilon}_1 - \hat{\varepsilon}_n$ is a root for $\widehat{\mathfrak{gl}_n}$. As to the generating morphisms of $\mathcal{D}_{\mathrm{BS}}$, as for the objects we let $j \in [1, n[$ correspond to $s_{n-j} := s_{\alpha_{n-j}}$, and let ∞ correspond to $s_{\alpha_0,1}$, so we will let j vary over $[1, n]$ and read n as ∞ on the RHS. The assignment goes as follows: $\forall m \in \mathbb{Z}$,



$$= \quad \begin{array}{c} B_{s_{n-j}}\langle 1\rangle \\ \uparrow \langle m\rangle \\ B_\emptyset \end{array} \quad \mapsto \quad \begin{tikzcd} \varpi \arrow[r, "F_jE_j1_\varpi"] & \varpi \\ \varpi \arrow[r, "1_\varpi"'] \arrow[ur, phantom, "\Uparrow \eta_{\varpi,j}"] & \varpi \end{tikzcd} \quad =$$



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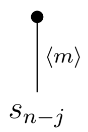
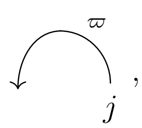
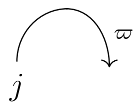


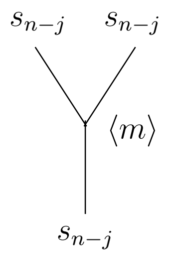
$$= \quad \begin{array}{c} B_\emptyset\langle 1\rangle \\ \uparrow \langle m\rangle \\ B_{s_{n-j}} \end{array} \quad \mapsto \quad \begin{tikzcd} \varpi \arrow[r, "1_\varpi"] & \varpi \\ \varpi \arrow[r, "F_jE_j1_\varpi"'] \arrow[ur, phantom, "\Uparrow \varepsilon'_{\varpi,j}"] & \varpi \end{tikzcd} \quad =$$



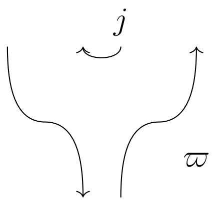
,

where $\varepsilon'_{\varpi,j} \in \mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi,\varpi)((F_j1_{\varpi+\hat{\alpha}_j})\circ(E_j1_\varpi), 1_\varpi)$ [Br, 1.10] is distinct from $\varepsilon_{\varpi,j} \in \mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})$

$(\varpi,\varpi)(E_j 1_{\varpi-\hat\alpha_j} F_j 1_\varpi, 1_\varpi)$ depicted as  in (4.1),

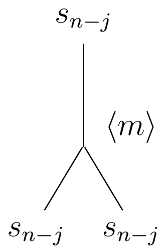


$$\langle m\rangle \quad = \quad \begin{matrix} B_{\underline{s_{n-j}s_{n-j}}}\langle -1\rangle \\ \uparrow{\scriptstyle\langle m\rangle} \\ B_{s_{n-j}} \end{matrix} \quad \mapsto \quad \begin{tikzcd} \varpi \arrow[r, "E_j1_\varpi"] & \varpi+\hat\alpha_j \arrow[r, "E_jF_j1_{\varpi+\hat\alpha_j}"] & \varpi+\hat\alpha_j \arrow[r, "F_j1_{\varpi+\hat\alpha_j}"] & \varpi \\ \varpi \arrow[r, "E_j1_\varpi"'] \arrow[ur, phantom, "\iota_{E_j1_\varpi}\Uparrow"] & \varpi+\hat\alpha_j \arrow[r, "1_{\varpi+\hat\alpha_j}"'] \arrow[ur, phantom, "\eta'_{\varpi+\hat\alpha_j,j}\Uparrow"] & \varpi+\hat\alpha_j \arrow[r, "F_j1_{\varpi+\hat\alpha_j}"'] \arrow[ur, phantom, "\Uparrow\iota_{F_j1_{\varpi+\hat\alpha_j}}"] & \varpi \end{tikzcd}$$

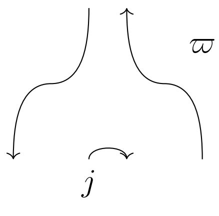
$=$ 

$= \iota_{F_j1_{\varpi+\hat\alpha_j}} * \eta'_{\varpi+\hat\alpha_j,j} * \iota_{E_j1_\varpi}$,

where $\eta'_{\varpi+\hat\alpha_j,j} \in \mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi+\hat\alpha_j,\varpi+\hat\alpha_j)(1_{\varpi+\hat\alpha_j}, E_j1_\varpi F_j 1_{\varpi+\hat\alpha_j})$ [Br, 1.10] is distinct from $\eta_{\varpi+\hat\alpha_j,j} \in \mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi+\hat\alpha_j,\varpi+\hat\alpha_j)(1_{\varpi+\hat\alpha_j}, F_j1_\varpi E_j 1_{\varpi+\hat\alpha_j})$,

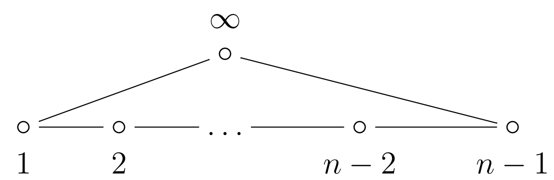


$$\langle m\rangle \quad = \quad \begin{matrix} B_{s_{n-j}}\langle -1\rangle \\ \uparrow{\scriptstyle\langle m\rangle} \\ B_{\underline{s_{n-j}s_{n-j}}} \end{matrix} \quad \mapsto \quad \begin{tikzcd} & \varpi \arrow[r, "F_jE_j1_\varpi"] \arrow[dr, phantom, "\|"] & \varpi & \\ \varpi \arrow[r, "E_j1_\varpi"] & \varpi+\hat\alpha_j \arrow[r, "1_{\varpi+\hat\alpha_j}"] & \varpi+\hat\alpha_j \arrow[r, "F_j1_{\varpi+\hat\alpha_j}"] & \varpi \\ \varpi \arrow[r, "E_j1_\varpi"'] \arrow[ur, phantom, "\iota_{E_j1_\varpi}\Uparrow"] & \varpi+\hat\alpha_j \arrow[r, "E_jF_j1_{\varpi+\hat\alpha_j}"'] \arrow[ur, phantom, "\varepsilon_{\varpi+\hat\alpha_j,j}\Uparrow"] & \varpi+\hat\alpha_j \arrow[r, "F_j1_{\varpi+\hat\alpha_j}"'] \arrow[ur, phantom, "\Uparrow\iota_{F_j1_{\varpi+\hat\alpha_j}}"] & \varpi \\ & \varpi \arrow[r, "F_jE_jF_jE_j1_\varpi"'] \arrow[ur, phantom, "\|"] & \varpi & \end{tikzcd}$$

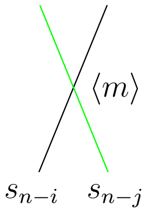
$=$ 

$= \iota_{F_j1_{\varpi+\hat\alpha_j}} * \varepsilon_{\varpi+\hat\alpha_j,j} * \iota_{E_j1_\varpi}$.

Replace now the quiver in (3.1) by



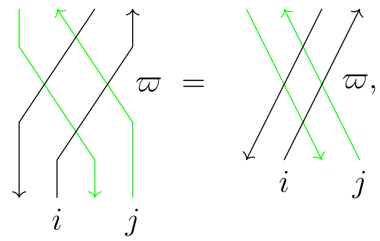
$\forall i, j \in [1, n]$ with $(n-i) \not\to (n-j)$ in the new quiver,



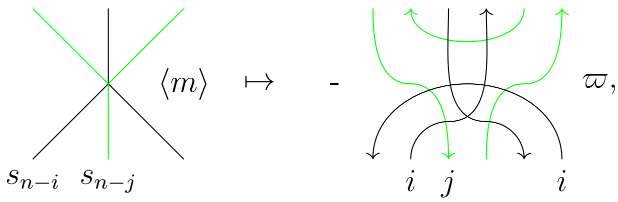
$$= \begin{array}{c} B_{s_{n-j}s_{n-i}} \\ \uparrow \langle m \rangle \\ B_{s_{n-i}s_{n-j}} \end{array} \qquad \mapsto$$

$$\begin{tikzcd}
 & \varpi \arrow[rr, "F_jE_jF_iE_i1_\varpi"] & \arrow[d, phantom, "\|"] & \varpi & \\
\varpi \arrow[r, "E_i1_\varpi"] \arrow[dr, phantom, "\Uparrow\iota"] & \varpi+\hat{\alpha}_i \arrow[r, "F_i1_{\varpi+\hat{\alpha}_i}"] & \varpi \arrow[r, "E_j1_\varpi"] \arrow[d, phantom, "\Uparrow\sigma'"] & \varpi+\hat{\alpha}_j \arrow[r, "F_j1_{\varpi+\hat{\alpha}_j}"] \arrow[dr, phantom, "\Uparrow\iota"] & \varpi \\
\varpi \arrow[r, "E_i1_\varpi"'] & \varpi+\hat{\alpha}_i \arrow[r, "E_j1_{\varpi+\hat{\alpha}_i}"'] \arrow[d, phantom, "\Uparrow\tau_{\varpi,(j,i)}"] & \varpi+\hat{\alpha}_i+\hat{\alpha}_j \arrow[r, "F_i1_{\varpi+\hat{\alpha}_i+\hat{\alpha}_j}"'] & \varpi+\hat{\alpha}_j \arrow[r, "F_j1_{\varpi+\hat{\alpha}_j}"'] \arrow[d, phantom, "\Uparrow\tau'"] & \varpi \\
\varpi \arrow[r, "E_j1_\varpi"] \arrow[dr, phantom, "\Uparrow\iota"] & \varpi+\hat{\alpha}_j \arrow[r, "E_i1_{\varpi+\hat{\alpha}_i}"] & \varpi+\hat{\alpha}_j+\hat{\alpha}_i \arrow[r, "F_j1_{\varpi+\hat{\alpha}_j+\hat{\alpha}_i}"] \arrow[d, phantom, "\Uparrow\sigma"] & \varpi+\hat{\alpha}_i \arrow[r, "F_i1_{\varpi+\hat{\alpha}_i}"] \arrow[dr, phantom, "\Uparrow\iota"] & \varpi \\
\varpi \arrow[r, "E_j1_\varpi"'] & \varpi+\hat{\alpha}_j \arrow[r, "F_j1_{\varpi+\hat{\alpha}_j}"'] & \varpi \arrow[r, "E_i1_\varpi"'] \arrow[d, phantom, "\|"] & \varpi+\hat{\alpha}_i \arrow[r, "F_i1_{\varpi+\hat{\alpha}_i}"'] & \varpi \\
 & \varpi \arrow[rr, "F_iE_iF_jE_j1_\varpi"'] & & \varpi, &
\end{tikzcd}$$

which is



where $\sigma \in \mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi+\hat{\alpha}_j, \varpi+\hat{\alpha}_i)(E_iF_j1_{\varpi+\hat{\alpha}_j}, F_jE_i1_{\varpi+\hat{\alpha}_j})$ (resp. $\tau' \in \mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi+\hat{\alpha}_j+\hat{\alpha}_i, \varpi)$ $(F_iF_j1_{\varpi+\hat{\alpha}_j+\hat{\alpha}_i}, F_jF_i1_{\varpi+\hat{\alpha}_j+\hat{\alpha}_i})$, $\sigma' \in \mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi+\hat{\alpha}_i, \varpi+\hat{\alpha}_j)(F_iE_j1_{\varpi+\hat{\alpha}_i}, E_jF_i1_{\varpi+\hat{\alpha}_i})$) is taken from [Br, 1.6] (resp. [Br, 1.10], [Br, 1.11]). Finally, $\forall i, j \in [1, n]$ with $(n-j) \to (n-1)$ in the quiver (3.1),

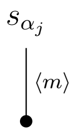


Theorem [RW, Th. 8.1.1]: *The data above defines a strict monoidal functor $\mathcal{D}_{\mathrm{BS}} \to \mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi, \varpi)$.*

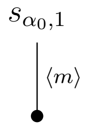
(5.4) Composed with the strict monoidal functor $\mathcal{U}^{[n]}(\widehat{\mathfrak{gl}_n})(\varpi, \varpi) \to \mathrm{Cat}(\mathrm{Rep}_{[n\,\det]}(G), \mathrm{Rep}_{[n\,\det]}(G))$ from (4.13) we have obtained a strict monoidal functor $\mathcal{D}_{\mathrm{BS}} \to \mathrm{Cat}(\mathrm{Rep}_{[n\,\det]}(G), \mathrm{Rep}_{[n\,\det]}(G))$ such that $\forall s \in \mathcal{S}_a$, $\forall m \in \mathbb{Z}$, $B_s\langle m\rangle \mapsto \Theta_s$; recall from (4.9) that

$$B_{s_{\alpha_j}}\langle m\rangle \mapsto F_{n-j}E_{n-j}1_\varpi \mapsto \Theta_{s_{\alpha_j}}\ \forall j \in [1, n[, \qquad B_{s_{\alpha_0,1}}\langle m\rangle \mapsto F_\infty E_\infty 1_\varpi \mapsto \Theta_{s_{\alpha_{0,1}}},$$

and



$$\mapsto \eta_{\varpi,n-j} \mapsto \eta_j^\varpi \in \mathrm{Cat}(\mathrm{Rep}_{[n\,\det]}(G), \mathrm{Rep}_{[n\,\det]}(G))(\mathrm{id}, \Theta_{s_{\alpha_j}})\ \forall j \in [1, n[,$$



$$\mapsto \eta_{\infty,\varpi} \mapsto \eta_\infty^\varpi$$

$$= (\iota_{F_0F_{p-1}\dots F_{n+1}1_{\varpi+\hat\alpha_0+\hat\alpha_{p-1}+\dots+\hat\alpha_{n+1}}} * \eta_n^{\varpi+\hat\alpha_0+\hat\alpha_{p-1}+\dots+\hat\alpha_{n+1}} * \iota_{E_{n+1}\dots E_{p-1}\dots E_0 1_\varpi}) \odot \dots \odot$$
$$(\iota_{F_0F_{p-1}1_{\varpi+\hat\alpha_0+\hat\alpha_{p-1}}} * \eta_{p-2}^{\varpi+\hat\alpha_0+\hat\alpha_{p-1}} * \iota_{E_{p-1}E_0 1_\varpi}) \odot (\iota_{F_0 1_{\varpi+\hat\alpha_0}} * \eta_{p-1}^{\varpi+\hat\alpha_0} * \iota_{E_0 1_\varpi}) \odot \eta_0^\varpi$$
$$\in \mathrm{Cat}(\mathrm{Rep}_{[n\,\det]}(G), \mathrm{Rep}_{[n\,\det]}(G))(\mathrm{id}, \Theta_{s_{\alpha_0,1}}).$$

Finally, there is an autoequivalence $\iota : \mathcal{D}_{\mathrm{BS}} \to \mathcal{D}_{\mathrm{BS}}$ such that $B_{\underline{s_1\dots s_r}}\langle m\rangle \mapsto B_{\underline{s_r\dots s_1}}\langle m\rangle$ $\forall$ sequences $\underline{s_1 \dots s_r}$ in $\mathcal{S}_a$, $\forall m \in \mathbb{Z}$, and on each morphism reflecting the corresponding diagrams along a vertical axis [RW, 4.2]. In particular, $\forall X, Y \in \mathrm{Ob}(\mathcal{D}_{\mathrm{BS}})$, $\iota(XY) = \iota(Y)\iota(X)$. Thus, combined with ι, we have obtained a strict monoidal functor $\mathcal{D}_{\mathrm{BS}} \to \mathrm{Cat}(\mathrm{Rep}_{[n\,\det]}(G), \mathrm{Rep}_{[n\,\det]}(G))^{\mathrm{op}}$ such that $\forall s \in \mathcal{S}_a$, $\forall m \in \mathbb{Z}$, $B_s\langle m\rangle \mapsto \Theta_s$. As $\mathrm{Rep}_{[n\,\det]}(G)$ is equivalent to the principal block $\mathrm{Rep}_0(G)$ by tensoring with $\det^{\otimes -n}$, we have now

Corollary [RW, Th. 1.5.1]: *There is a strict monoidal functor $\Psi : \mathcal{D}_{\mathrm{BS}} \to \mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G))^{\mathrm{op}}$ such that $\forall s \in \mathcal{S}_a$, $\forall m \in \mathbb{Z}$, $B_s\langle m\rangle \mapsto \Theta_s$.*

(5.4′) Together with (4.13′) we have also obtained

Corollary: *There is a strict monoidal functor $\Psi : \mathcal{D}_{\mathrm{BS}} \to \mathrm{Cat}(\mathrm{Rep}_0(G_1T), \mathrm{Rep}_0(G_1T))^{\mathrm{op}}$ such that $\forall s \in \mathcal{S}_a$, $\forall m \in \mathbb{Z}$, $B_s\langle m\rangle \mapsto \Theta_s$.*

(5.5) Recall that a Coxeter system $(\mathcal{X}, \mathcal{Y})$ is the free group $\mathcal{X}$ with a finite set $\mathcal{Y}$ of generators subject to the relations that each $y \in \mathcal{Y}$ is an involution and that $\forall y, z \in \mathcal{Y}$ distinct with $\mathrm{ord}(yz) = m_{yz}$, $\underbrace{yz\dots}_{m_{yz}} = \underbrace{zy\dots}_{m_{yz}}$; we allow m_{yz} to be ∞, in which case we impose no such relation. Given $x \in \mathcal{X}$, the minimal length of sequences of elements of $\mathcal{Y}$ to express x as a product is called the length of x and is denoted $\ell(x)$, in which case the expression $x = y_1 \dots y_{\ell(x)}$ is called a reduced expression of x. There is a PO on $(\mathcal{X}, \mathcal{Y})$, called the Chevalley-Bruhat order, such that $x \le x'$ iff x is obtained as the product of a subsequence of a reduced expression of x'. Our pairs $(\mathcal{W}_a, \mathcal{S}_a)$ and $(\mathcal{W}, \mathcal{S})$ form Coxeter systems.

Let now $\mathcal{H}$ (resp. $\mathcal{H}_f$) denote the 岩堀-Hecke algebra over the Laurent polynomial ring $\mathbb{Z}[v, v^{-1}]$ associated to the Coxeter system $(\mathcal{W}_a, \mathcal{S}_a)$ (resp. $(\mathcal{W}, \mathcal{S})$). Thus, $\mathcal{H}$ has generators $\{H_s | s \in \mathcal{S}_a\}$ subject to the quadratic relations $H_s^2 = 1 + (v^{-1} - v)H_s$ $\forall s \in \mathcal{S}_a$ and the braid relations $\underbrace{H_s H_t \dots}_{m_{st}} = \underbrace{H_t H_s \dots}_{m_{st}}$ $\forall s, t \in \mathcal{S}_a$ distinct with $m_{st} = \mathrm{ord}(st)$. It follows that each H_s, $s \in \mathcal{S}_a$, is invertible with $H_s^{-1} = H_s + (v - v^{-1})$. Setting $\forall x, y \in \mathcal{W}_a$ with $\ell(x) + \ell(y) = \ell(xy)$, $H_{xy} = H_x H_y$, one has that $\mathcal{H}$ (resp. $\mathcal{H}_f$) admits a standard $\mathbb{Z}[v, v^{-1}]$-linear basis $\{H_x | x \in \mathcal{W}_a\}$ (resp. $\{H_x | x \in \mathcal{W}\}$ with $H_e = 1$. For this and other reasons we often write 1 for e. Under the specialization $v \rightsquigarrow 1$ one has an isomorphism of rings

(1) $$\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H} \simeq \mathbb{Z}[\mathcal{W}_a].$$

There is $\bar{?} \in \mathrm{Rng}(\mathcal{H}, \mathcal{H})$ such that $\bar{v} = v^{-1}$ and that $\overline{H_x} = (H_{x^{-1}})^{-1}$ $\forall x \in \mathcal{W}_a$. On $\mathcal{H}$ there is also a Kazhdan-Lusztig basis $\{\underline{H}_x | x \in \mathcal{W}_a\}$ such that $\overline{\underline{H}_x} = \underline{H}_x$ $\forall x \in \mathcal{W}_a$ and $\underline{H}_x \in H_x + \sum_{y<x} v\mathbb{Z}[v] H_y$ [S97, claim 2.3, p. 84]. In particular, $\underline{H}_e = H_e = 1$, $\underline{H}_s = H_s + v$ $\forall s \in \mathcal{S}_a$, and $\mathcal{H}_f = \coprod_{w \in \mathcal{W}} \mathbb{Z}[v, v^{-1}] \underline{H}_w$. If $\underline{w} = \underline{s_1 \dots s_r}$ is an expression in $\mathcal{W}_a$, set $\underline{H}_{\underline{w}} = \underline{H}_{s_1} \dots \underline{H}_{s_r}$. In particular, $\underline{H}_\emptyset = \underline{H}_e = 1$ and $\underline{H}_{\underline{s}} = \underline{H}_s$ $\forall s \in \mathcal{S}_a$.

Recall from [S97, p. 86] a $\mathbb{Z}[v, v^{-1}]$-algebra homomorphism $\mathcal{H}_f \to \mathbb{Z}[v, v^{-1}]$ such that $s \mapsto -v$ $\forall s \in \mathcal{S}$, which defines a structure of right $\mathcal{H}_f$-module on $\mathbb{Z}[v, v^{-1}]$, called the "sign" representation and denoted sgn. We define the "anti-spherical" right $\mathcal{H}$-module as $\mathcal{M}^{\mathrm{asph}} = \mathrm{sgn} \otimes_{\mathcal{H}_f} \mathcal{H}$, which is denoted $\mathcal{N}$ (resp. $\mathcal{N}^0$) in [S97, p.86 (resp. p. 98)]. Recall from [S97, Th. 3.1] that $\mathcal{M}^{\mathrm{asph}}$ has a standard basis $\{N_x = 1 \otimes H_x | x \in {}^f\mathcal{W}\}$ and a Kazhdan-Lusztig basis $\{\underline{N}_x = 1 \otimes \underline{H}_x | x \in {}^f\mathcal{W}\}$, ${}^f\mathcal{W} = \{x \in \mathcal{W}_a | \ell(wx) \geq \ell(x)\ \forall w \in \mathcal{W}\}$.

Let $\phi \in \mathrm{Mod}\mathcal{H}(\mathcal{H}, \mathcal{M}^{\mathrm{asph}})$ via $H \mapsto 1 \otimes H$. Then [S97, pf of Prop. 3.4]

(2) $$\phi(\underline{H}_x) = \begin{cases} \underline{N}_x & \text{if } x \in {}^f\mathcal{W} \\ 0 & \text{else.} \end{cases}$$

Also [S97, p. 86] $\forall s \in \mathcal{S}_a$, $\forall x \in {}^f\mathcal{W}$,

(3) $$N_x \underline{H}_s = \begin{cases} N_{xs} + vN_x & \text{if } xs \in {}^f\mathcal{W} \text{ and } xs > x \\ N_{xs} + v^{-1}N_x & \text{if } xs \in {}^f\mathcal{W} \text{ and } xs < x \\ 0 & \text{else.} \end{cases}$$

Under the specialization $v \rightsquigarrow 1$ one has an isomorphism

(4) $$\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{asph}} \simeq \mathrm{sgn}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a] \simeq [\mathrm{Rep}_0(G)]$$

such that $1 \otimes N_x \mapsto 1 \otimes x \mapsto [\nabla(x \bullet 0)]$ $\forall x \in {}^f\mathcal{W}$. If $\underline{y} = \underline{s_1 \dots s_r}$ is an expression in $\mathcal{W}_a$, set also $\underline{N}_{\underline{y}} = N_1 \underline{H}_{s_1} \dots \underline{H}_{s_r} = 1 \otimes \underline{H}_{s_1} \dots \underline{H}_{s_r}$. By (3) and the translation principle (1.10), under (4) one has

(5) $$1 \otimes \underline{N}_{\underline{y}} \mapsto 1 \otimes (s_1 + 1) \dots (s_r + 1) \mapsto [\Theta_{s_r} \dots \Theta_{s_1} \nabla(0)].$$

(5.6) Let $\mathcal{D} = \mathrm{Kar}(\mathcal{D}_{\mathrm{BS}})$ denote the Karoubian envelope of the additive hull of $\mathcal{D}_{\mathrm{BS}}$ [Bor, Prop. 6.5.9, p. 274]. Thus an object of $\mathcal{D}$ is a direct summand of a finite direct sum of objects of $\mathcal{D}_{\mathrm{BS}}$.

The category $\mathcal{D}$ is a graded category inheriting the autoequivalence $\langle 1\rangle$, is Krull-Schmidt, and remains strict monoidal [RW, 1.2, 1.3]. By a Krull-Schmidt category we mean an additive category in which every object is isomorphic to a finite direct sum of indecomposable objects, and an object is indecomposable if and only if its endomorphism ring is local [EW, 6.6]. Recall from [EW, Th. 6.25] that $\forall w \in \mathcal{W}_a$, $\exists!$ indecomposable $B_w \in \mathrm{Ob}(\mathcal{D})$ such that B_w is a direct summand of each $B_{\underline{w}}$ for a reduced expression $\underline{w}$ of w but is not a direct summand of any $B_{\underline{v}}$ for an expression $\underline{v}$ with $\ell(\underline{v}) < \ell(w)$. Any indecomposable object of $\mathcal{D}$ is isomorphic to some $B_w\langle m\rangle$ for a unique $w \in \mathcal{W}_a$ and a unique $m \in \mathbb{Z}$. In particular, $B_1 = B_\emptyset$ and $B_s = B_{\underline{s}}$ for each $s \in \mathcal{S}_a$. The split Grothendieck group $[\mathcal{D}]$ of $\mathcal{D}$ admits a structure of $\mathbb{Z}[v, v^{-1}]$-module such that $v \cdot [X] = [X\langle 1\rangle]$. As such there is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras [EW]

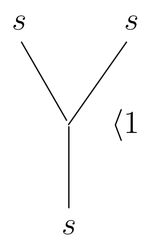
$$\mathcal{H} \to [\mathcal{D}] \quad \text{such that} \quad \underline{H}_s \mapsto [B_s]\ \forall s \in \mathcal{S}_a. \tag{1}$$

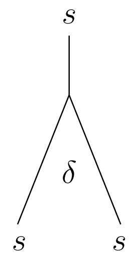
Then the right action of $\mathcal{D}_{\mathrm{BS}}$ on $\mathrm{Rep}_0(G)$ implies that the isomorphisms (5.5.4) are isomorphisms of right $\mathcal{H}$-modules.

$\forall x \in \mathcal{W}_a$, set ${}^p\underline{H}_x \in \mathcal{H}$ to be the pre-image of $[B_x]$ under (1). As the $[B_x]$ form a $\mathbb{Z}[v, v^{-1}]$-linear basis of $\mathcal{H}$, so do $({}^p\underline{H}_x | x \in \mathcal{W}_a)$ on $\mathcal{H}$, called the p-Kazhdan Lusztig basis of $\mathcal{H}$.

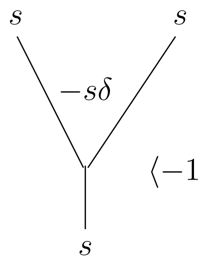
The auto-equivalence (resp. anti-auto-equivalence) ι (resp. τ) on $\mathcal{D}_{\mathrm{BS}}$ induces one on $\mathcal{D}$ denoted by the same letter. Thus, $\forall w \in \mathcal{W}_a$, $\iota(B_w) = B_{w^{-1}}$, $\tau(B_w) = B_w$.

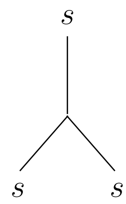
(5.7) Let $s \in \mathcal{S}_a$. Take $\delta \in \underline{R}$ with $\partial_s \delta = 1$, and let

$i_1 =$ 

$\langle 1\rangle \quad B_s\langle 1\rangle \to B_{\underline{ss}}$, $\quad p_1 =$ 

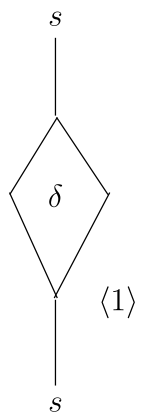
$B_{\underline{ss}} \xrightarrow{\delta} B_{\underline{ss}}\langle 2\rangle \to B_s\langle 1\rangle$,

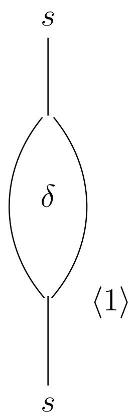
$i_2 =$ 

$\langle -1\rangle \quad B_{\underline{s}}\langle -1\rangle \to B_{\underline{ss}}\langle -2\rangle \xrightarrow{-s\delta} B_{\underline{ss}}$, $\quad p_2 =$ 

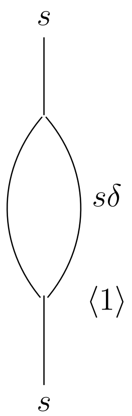
$B_{\underline{ss}} \to B_{\underline{s}}\langle -1\rangle$.

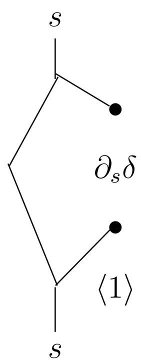
Then

$p_1 \circ i_1 =$ 

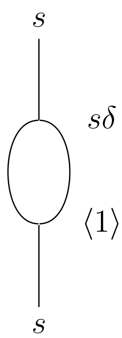
$=$ 

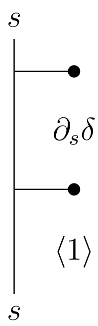
$$\begin{array}{c} B_{\underline{s}}\langle 1\rangle \\ \uparrow \\ B_{\underline{s}}\langle 1\rangle \end{array}$$

$=$ 

$+$ 

by the nil Hecke relation [EW, 5.2]

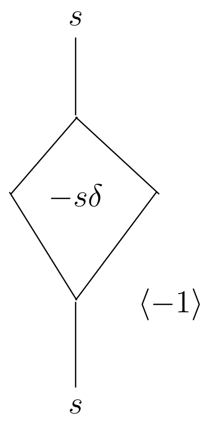
$=$ 

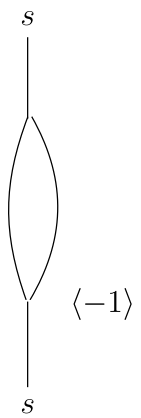
$+$ 

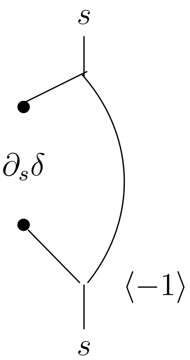
$=$ 

$\langle 1\rangle$ 1 by the needle relation [EW, 5.5] and the Frobenius unit [EW, 5.4]

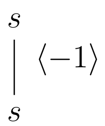
$= \mathrm{id}_{B_s\langle 1\rangle}$,

$p_2 \circ i_2 =$ 

$= \; -\delta$ 

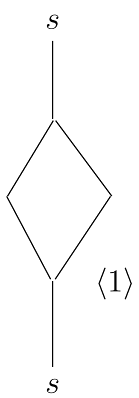
$+ \quad \partial_s \delta$ 

by the nil Hecke relation [EW, 5.2]

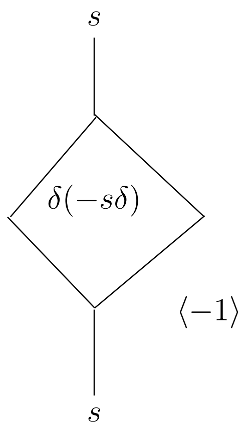
$= \; 1$ 

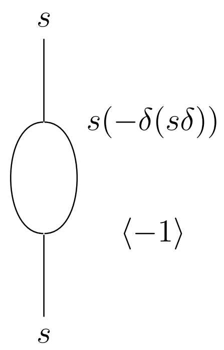
by the needle relation [EW, 5.5] and the Frobenius unit [EW, 5.4]

$= \mathrm{id}_{B_s\langle -1 \rangle}$,

$p_2 \circ i_1 =$ 

$= 0$ by the needle relation [EW, 5.5],

$p_1 \circ i_2 =$ 

$=$ 

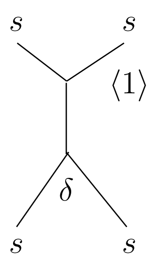
$+$ 

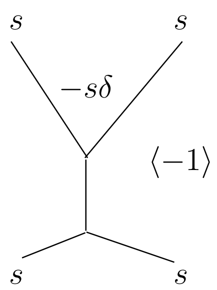
$- \partial_s(\delta(s\delta))$

by the nil Hecke relation [EW, 5.2]

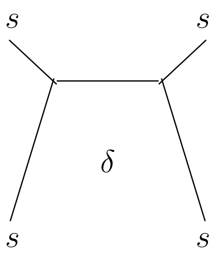
$= 0$ by the needle relation [EW, 5.5] as $\partial_s(\delta(s\delta)) = 0$,

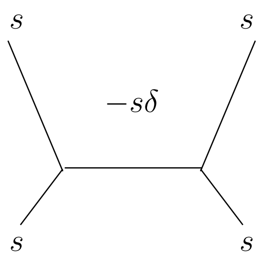
and

$i_1 \circ p_1 + i_2 \circ p_2 =$ 

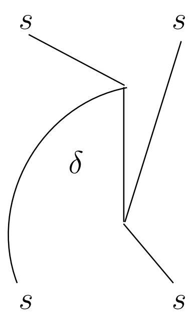
$+$ 

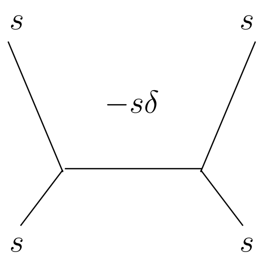
$\begin{array}{c} B_{\underline{ss}} \\ \uparrow \\ B_{\underline{ss}} \end{array}$

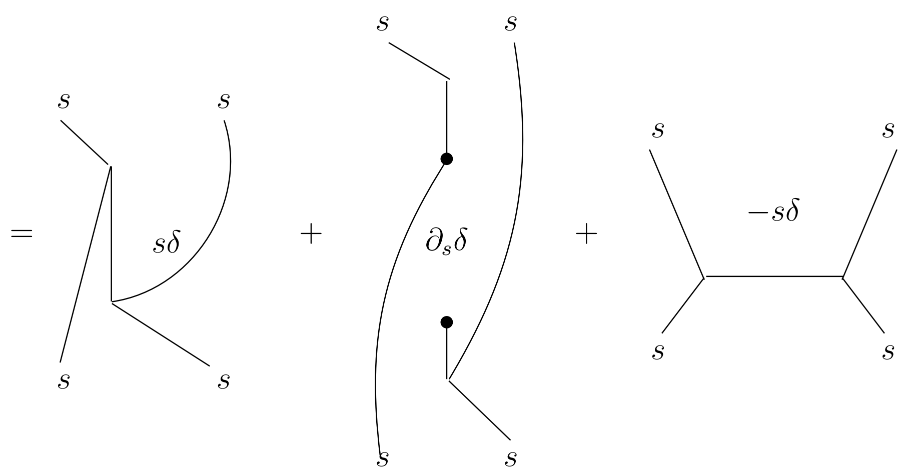
$=$ 

$+$ 

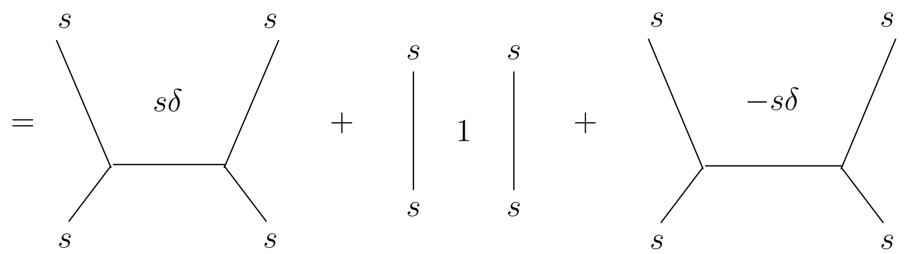
by the Frobenius associativity [EW, 5.3]

$=$ 

$+$ 



by the nil Hecke relation [EW, 5.2]



by the Frobenius associativity [EW, 5.3].

We have thus obtained

Lemma [RW, Lem. 4.3.1]: *In the additive hull* $\mathrm{Add}(\mathcal{D}_{\mathrm{BS}})$ *of* $\mathcal{D}_{\mathrm{BS}}$ *one has*

$$B_s \cdot B_s \simeq B_s\langle 1\rangle \oplus B_s\langle -1\rangle.$$

(5.8) **Lemma [RW, Lem. 4.2.3]:** *Given an expression* $\underline{s_1 \dots s_r}$ *in* $\mathcal{W}_a$*, if* $B_x\langle m\rangle$*,* $m \in \mathbb{Z}$*, is an indecomposable direct summand of* $B_{\underline{s_1 \dots s_r}}$ *in* $\mathcal{D}$*,* $s_1 x < x$ *in the Chevalley-Bruhat order.*

(5.9) Let $\mathcal{D}'_{\mathrm{BS}}$ be the set of objects $B_{\underline{w}}\langle m\rangle$ with expression $\underline{w}$ starting with some $s \in \mathcal{S}$ and $m \in \mathbb{Z}$, and set $\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}} = \mathcal{D}_{\mathrm{BS}}//\mathcal{D}'_{\mathrm{BS}}$ [RW, 4.4], [中岡, Prop. 3.2.51, p. 150]; as $\mathcal{D}_{\mathrm{BS}}$ is not additive, we define for $X, Y \in \mathrm{Ob}(\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}})$ the morphism set $\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}}(X, Y)$ to be

$$\mathcal{D}_{\mathrm{BS}}(X,Y)/\langle f \in \mathcal{D}_{\mathrm{BS}}(X,Y) \mid \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dotted] & & Y \\ & Z \arrow[ur, dotted] & \end{tikzcd} \ \exists Z \in \mathcal{D}'_{\mathrm{BS}}\rangle.$$

We will denote the image of $X \in \mathcal{D}_{\mathrm{BS}}$ in $\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}}$ under the quotient functor $\mathcal{D}_{\mathrm{BS}} \to \mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}}$ by $\bar{X}$. $\forall X \in \mathcal{D}_{\mathrm{BS}}$, one has that $\bar{X} = 0$ iff id_X factors through some $Y \in \mathcal{D}'_{\mathrm{BS}}$ [中岡, Cor, 3.2.46, p. 148]. The auto-equivalence $\langle 1\rangle$ on $\mathcal{D}_{\mathrm{BS}}$ induces one on $\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}}$ denoted by the same letter. Thus, $\overline{B_{\underline{x}}\langle m\rangle} = \bar{B}_{\underline{x}}\langle m\rangle$ $\forall \underline{x}, \forall m \in \mathbb{Z}$.

$\forall X, Y \in \mathrm{Ob}(\mathcal{D}_{\mathrm{BS}})$, put $(\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}})^{\bullet}(\bar{X}, \bar{Y}) = \coprod_{m \in \mathbb{Z}} \mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}}(\bar{X}, \bar{Y}\langle m\rangle)$. Consider the quotient map

$\mathcal{D}^\bullet_{\mathrm{BS}}(X,Y) \to (\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}})^\bullet(\bar{X},\bar{Y})$. $\forall \alpha \in R^s$, $\forall \phi \in \mathcal{D}_{\mathrm{BS}}(X, Y\langle m\rangle)$, one has from (5.1.1) a commutative diagram

$$\begin{tikzcd}
B_\emptyset \cdot X = X \arrow[r, "\alpha^\vee \phi"] \arrow[d] \arrow[dr, "\alpha^\vee X"] & Y\langle m+2\rangle = (Y\langle m\rangle)\langle 2\rangle \\
(B_s\langle 1\rangle)\cdot X \arrow[r] & (B_\emptyset\langle 2\rangle)\cdot X = X\langle 2\rangle. \arrow[u, "\phi\langle 2\rangle"']
\end{tikzcd}$$

As $(B_s\langle 1\rangle)\cdot X \in \mathcal{D}'_{\mathrm{BS}}$, $\alpha^\vee\phi = 0$ in $\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}}$. As $\underline{R} = \Bbbk[\alpha^\vee|\alpha \in R^s]$, if we regard $\Bbbk$ as the trivial $\underline{R}$-module, one obtains

$$\begin{tikzcd}
\mathcal{D}^\bullet_{\mathrm{BS}}(X,Y) \arrow[rr, two heads] \arrow[dr] & & (\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}})^\bullet(\bar{X},\bar{Y}). \\
& \Bbbk \otimes_{\underline{R}} \mathcal{D}^\bullet_{\mathrm{BS}}(X,Y) \arrow[ur, dotted] &
\end{tikzcd}$$

As $\mathcal{D}^\bullet_{\mathrm{BS}}(X,Y)$ is a free left $\underline{R}$-module of finite rank (5.2), $(\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}})^\bullet(\bar{X},\bar{Y})$ forms a finite dimensional $\Bbbk$-linear space.

Let $\mathcal{D}_{\mathcal{W}_a\setminus {}^f\mathcal{W}}$ be the additive full subcategory of $\mathcal{D}$ consisting of the direct sums of objects $B_w\langle m\rangle$, $w \in \mathcal{W}_a\setminus {}^f\mathcal{W}$, $m \in \mathbb{Z}$, and set $\mathcal{D}^{\mathrm{asph}} = \mathcal{D}//\mathcal{D}_{\mathcal{W}_a\setminus {}^f\mathcal{W}}$, which inherits a structure of graded category. $\forall w \in {}^f\mathcal{W}$, let $\bar{B}_w$ denote the image of B_w under the quotient functor $\mathcal{D} \to \mathcal{D}^{\mathrm{asph}}$. We will see presently in §6 that $\bar{B}_w$ remains nonzero in $\mathcal{D}^{\mathrm{asph}}$; we will first see that the right $\mathcal{D}$-action on $\mathrm{Rep}_0(G)$ factors through $\mathcal{D}^{\mathrm{asph}}$. If $\underline{w}$ is a reduced expression of $w \in {}^f\mathcal{W}$, $\nabla(0)B_{\underline{w}}$ has highest weight $w \bullet 0$. As $B_{\underline{w}}$ is a direct sum of B_w and some B_y's with $y < w$, we must have $\nabla(0)B_w \neq 0$, and hence $\bar{B}_w \neq 0$ in $\mathcal{D}^{\mathrm{asph}}$. Then, as a quotient of a local ring remains local [AF, 15.15, p. 170], the indecomposable objects of $\mathcal{D}^{\mathrm{asph}}$ are $\bar{B}_w\langle m\rangle$, $w \in {}^f\mathcal{W}$, $m \in \mathbb{Z}$. It follows from (5.8) that $\mathcal{D}^{\mathrm{asph}} = \mathrm{Kar}(\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}})$.

Strange as it may appear, if a reduced expression $\underline{w}$ of $w \in {}^f\mathcal{W}$ contains $s \in \mathcal{S}$, $\bar{B}_s = 0$ while $\bar{B}_w \neq 0$ as observed above, and hence $\bar{B}_{\underline{w}} \neq 0$. Nonetheless, (5.8) implies that $\mathcal{D}^{\mathrm{asph}}$ admits a structure of right $\mathcal{D}$-module. For let $\phi \in \mathcal{D}(X,Y)$ factor through some $Z \in \mathcal{D}_{\mathcal{W}_a\setminus {}^f\mathcal{W}}$. Let $B_x\langle m\rangle$ be a direct summand of Z, so x admits a reduced expression $\underline{s_1 \dots s_r}$ with $s_1 \in \mathcal{S}$. Given an expression $\underline{y}$ in $\mathcal{W}_a$, each direct summand $B_w\langle k\rangle$ of $B_x\langle m\rangle B_{\underline{y}}$ has $s_1 w < w$ by (5.8), and hence $w \notin {}^f\mathcal{W}$ and $B_w\langle k\rangle \in \mathcal{D}_{\mathcal{W}_a\setminus {}^f\mathcal{W}}$. As such, under the isomorphism of $\mathbb{Z}[v,v^{-1}]$-algebras $\mathcal{H} \to [\mathcal{D}]$ from (5.6) one has an isomorphism of right $\mathcal{H}$-modules

$$\mathcal{M}^{\mathrm{asph}} \to [\mathcal{D}^{\mathrm{asph}}]. \tag{1}$$

For each $w \in {}^f\mathcal{W}$ let ${}^p\underline{N}_w$ be the pre-image of $[\bar{B}_w] \in [\mathcal{D}^{\mathrm{asph}}]$: ${}^p\underline{N}_w = 1 \otimes {}^p\underline{H}_w$. Thus $({}^p\underline{N}_w|w \in^f \mathcal{W})$ forms a $\mathbb{Z}[v,v^{-1}]$-linear basis of $\mathcal{M}^{\mathrm{asph}}$, called the p-canonical basis. Writing ${}^p\underline{N}_w = \sum_{y \in {}^f\mathcal{W}} {}^pn_{y,w}N_y$, ${}^pn_{y,w} \in \mathbb{Z}[v,v^{-1}]$, we call ${}^pn_{y,w}$ an antisphereical p-Kazhdan-Lusztig polynomial.

(5.10) Fix now an expression $\underline{w} = \underline{s_1 \dots s_r}$. Each $e(\underline{w}) \in \{0,1\}^r$ defines a sub-expression $\underline{w}^{e(\underline{w})} = (s_1^{e(\underline{w})_1}, \dots, s_r^{e(\underline{w})_r})$ of $\underline{w}$ by deleting those terms with $e(\underline{w})_j = 0$, in which case we also let $w^{e(\underline{w})} = s_1^{e(\underline{w})_1} \dots s_r^{e(\underline{w})_r} \in \mathcal{W}_a$. The Bruhat stroll of $e(\underline{w})$ is the sequence $x_0 = e, x_1 =$

$s_1^{e(\underline{w})_1}, x_2 = s_1^{e(\underline{w})_1} s_2^{e(\underline{w})_2}, \dots, x_r = s_1^{e(\underline{w})_1} s_2^{e(\underline{w})_2} \dots s_r^{e(\underline{w})_r}$. $\forall j \in [1, r]$, we assign a symbol

$$\begin{cases} \text{U1} & \text{if } e(\underline{w})_j = 1 \text{ and } x_j = x_{j-1}s_j > x_{i-1}, \\ \text{D1} & \text{if } e(\underline{w})_j = 1 \text{ and } x_j = x_{j-1}s_j < x_{i-1}, \\ \text{U0} & \text{if } e(\underline{w})_j = 0 \text{ and } x_j = x_{j-1}s_j > x_{i-1}, \\ \text{D0} & \text{if } e(\underline{w})_j = 0 \text{ and } x_j = x_{j-1}s_j < x_{i-1}, \end{cases}$$

"U" (resp. "D") standing for Up (resp. Down). Let $d(e(\underline{w}))$ denote the number of U0's minus the number of D0's, called the defect of $e(\underline{w})$ [EW, 2.4]. For $\mathcal{W}' \subseteq \mathcal{W}_a$ we say $e(\underline{w})$ avoids $\mathcal{W}'$ iff $x_r \notin \mathcal{W}'$ and $x_{j-1}s_j \notin \mathcal{W}'$ $\forall j \in [1, r]$. We understand $e(\underline{w})$ avoids any $\mathcal{W}'$ in case $r = 0$.

Lemma [RW, Lem. 4.1.1]: *For each expression* $\underline{w}$ *one has in* $\mathcal{M}^{\mathrm{asph}}$

$$N_1 \underline{H}_{\underline{w}} = \sum_{e(\underline{w}) \text{ avoiding } \mathcal{W}_a \setminus {}^f\mathcal{W}} v^{d(e(\underline{w}))} N_{w^{e(\underline{w})}}.$$

(5.11) Let $w \in \mathcal{W}_a$. Define the rex graph Γ_w to have the vertices consisting of the reduced expressions of w and the edges connecting vertices iff they differ by one application of a braid relation $\underbrace{st\dots}_{m_{st}} = \underbrace{ts\dots}_{m_{st}}$ for $s, t \in \mathcal{S}_a$ distinct with $m_{st} = \mathrm{ord}(st)$ [RW, 4.3]. If $\underline{x}$ and $\underline{y}$ are 2 reduced expressions of w, a rex move $\underline{x} \rightsquigarrow \underline{y}$ is a directed path in Γ_w from the vertex $\underline{x}$ to the vertex $\underline{y}$. To such a path one can associate a morphism from $B_{\underline{x}}$ to $B_{\underline{y}}$ in $\mathcal{D}_{\mathrm{BS}}$ by composing the $2m_{st}$-valent morphisms (5.1.G4) associated to the braid relations encountered in the path.

Lemma [RW, Lem. 4.3.2]: *Let* $\underline{x} \rightsquigarrow \underline{y}$ *be a rex move in* Γ_w, *and let* $\underline{y} \rightsquigarrow \underline{x}$ *be the rex move in the reverse order. Let* $\gamma \in \mathcal{D}_{\mathrm{BS}}(B_{\underline{x}}, B_{\underline{x}})$ *associated to the concatenation* $\underline{x} \rightsquigarrow \underline{y} \rightsquigarrow \underline{x}$. *Then there is a finite set* J *and* $\phi_j \in \mathcal{D}_{\mathrm{BS}}(B_{\underline{x}}, B_{\underline{x}})$, $j \in J$, *factoring through some* $B_{\underline{z_j}}\langle k_j \rangle$

$$\begin{tikzcd} B_{\underline{x}} \arrow[rr, "\phi_j"] \arrow[dr, dotted] & & B_{\underline{x}} \\ & B_{\underline{z_j}}\langle k_j \rangle \arrow[ur, dotted] & \end{tikzcd}$$

with $\underline{z_j}$ *obtained from* $\underline{x}$ *by deleting at least 2 simple reflections and* $k_j \in \mathbb{Z}$ *such that* $\gamma = \mathrm{id}_{B_{\underline{x}}} + \sum_{j \in J} \phi_j$.

(5.12) Let $\underline{w} = s_1 \dots s_r$ be an expression. One has from [EW, Prop. 6.12] that $\mathcal{D}_{\mathrm{BS}}^{\bullet}(B_{\underline{w}}, B_{\emptyset})$ admits a basis of left $\underline{R}$-module consisting of the light leaves $L_{e(\underline{w})}$ $\forall e(\underline{w})$ expressing the unity of $\mathcal{W}_a$.

Proposition [RW, Prop. 4.5.1]: *Let* $\underline{w}$ *be an expression of an element in* $\mathcal{W}_a$. *One can choose the light leaves* $L_{e(\underline{w})}$ *with* $e(\underline{w})$ *expressing 1 and avoiding* $\mathcal{W}_a \setminus {}^f\mathcal{W}$ *to* $\Bbbk$*-linearly span* $(\mathcal{D}_{\mathrm{BS}}^{\mathrm{asph}})^{\bullet}(\bar{B}_{\underline{w}}, \bar{B}_{\emptyset})$.

6° Tilting characters

(6.1) One has from (5.4) a functor $\mathcal{D}_{\mathrm{BS}} \to \mathrm{Rep}_0(G)$ such that $B \mapsto \nabla(0)B$. If $\underline{x} = \underline{s_1 s_2 \dots s_r}$ is an expression of $x \in \mathcal{W}_a$,

$$B_{\underline{x}} \mapsto \nabla(0)B_{\underline{x}} = \nabla(0)B_{s_1}B_{s_2}\dots B_{s_r} = \Theta_{s_r}\dots\Theta_{s_2}\Theta_{s_1}\nabla(0),$$

the RHS of which we will denote by $\nabla(\underline{x})$. The functor naturally extends to another functor $\mathcal{D} \to \mathrm{Rep}_0(G)$, which we will denote by $\tilde{\Psi}$.

$\forall s \in \mathcal{S}$, $\tilde{\Psi}(B_s) = \nabla(0)B_s = \Theta_s\nabla(0) = 0$. $\forall x \in \mathcal{W}_a \setminus {}^f\mathcal{W}$, $\exists s \in \mathcal{S}$ and $y \in \mathcal{W}_a$ with $\ell(x) = \ell(y) + 1$ such that $x = sy$. If $\underline{y}$ is a reduced expression of y, B_x is a direct summand of $B_{\underline{sy}} = B_s B_{\underline{y}}$, and hence $\tilde{\Psi}(B_x)$ is a direct summand of $\tilde{\Psi}(B_{\underline{sy}}) = \tilde{\Psi}(B_s)B_{\underline{y}} = 0$. It follows that $\tilde{\Psi}$ factors through $\mathcal{D}^{\mathrm{asph}}$:

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "\tilde{\Psi}"] \arrow[d] & & \mathrm{Rep}_0(G), \\
\mathcal{D}//\mathcal{D}_{\mathcal{W}_a \setminus {}^f\mathcal{W}} = \mathcal{D}^{\mathrm{asph}} \arrow[urr, dotted] & &
\end{tikzcd}$$

which we denote by $\bar{\Psi}$. Composing with isomorphisms (5.5.4) one now obtains isomophisms of right $\mathcal{H}$-modules

(1) $$\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} [\mathcal{D}^{\mathrm{asph}}] \to \mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{asph}} \to \mathrm{sgn}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a] \to [\mathrm{Rep}_0(G)]$$

under which, $\forall w \in {}^f\mathcal{W}$, if $\underline{w} = \underline{s_1 \dots s_r}$,

(2) $$\begin{aligned} &1 \otimes [\bar{B}_w] \mapsto 1 \otimes {}^p\underline{N}_w, \\ &1 \otimes [B_{\underline{w}}] \mapsto 1 \otimes \underline{N}_{\underline{w}} \mapsto 1 \otimes (s_1+1)\dots(s_r+1) \mapsto [\Theta_{s_r}\dots\Theta_{s-1}\nabla(0)] = [\nabla(\underline{w})], \\ &\qquad 1 \otimes N_w \longmapsto [\nabla(w \bullet 0)]. \end{aligned}$$

The image of $1 \otimes {}^p\underline{N}_w$ turns out to be the indecomposable tilting module $T(w)$ of highest weight $w \bullet 0$. As ${}^p\underline{N}_w = \sum_{y \in {}^f\mathcal{W}} {}^p n_{y,w} N_y$ in $\mathcal{M}^{\mathrm{asph}}$ with p-Kazhdan-Lusztig polynomials ${}^p n_{y,w} \in \mathbb{Z}[v, v^{-1}]$, we will obtain

$$\mathrm{ch}\, T(w) = \sum_{y \in {}^f\mathcal{W}} {}^p n_{y,w}(1) \mathrm{ch}\, \nabla(y).$$

(6.2) We say that $M \in \mathrm{Rep}(G)$ admits a Δ- (resp. ∇-) filtration iff it possesses a filtration $M = M^0 > M^1 > \dots > M^r = 0$ in $\mathrm{Rep}(G)$ such that $\forall i \in [0, r[$, there is $\lambda_i \in \Lambda^+$ with $M^i/M^{i-1} \simeq \Delta(\lambda_i)$ (resp. $\nabla(\lambda_i)$), in which case we denote by $(M : \Delta(\lambda))$ (resp. $(M : \nabla(\lambda))$) the multiplicity of the appearance of $\Delta(\lambda)$ (resp. $\nabla(\lambda)$) in a Δ- (resp. ∇-) filtration; we will see in (6.8) that $(M : \Delta(\lambda)) = \dim \mathrm{Rep}(G)(M, \nabla(\lambda))$ while $(M : \nabla(\lambda)) = \dim \mathrm{Rep}(G)(\Delta(\lambda), M)$, and hence the number is independent of the choice of a filtration. We say that M is a tilting module iff it admits both a Δ- and a ∇-filtration. For each $\lambda \in \Lambda^+$ there is a unique, up to isomorphism, indecomposable tilting module of highest weight λ, which we denote by $T(\lambda)$. Any tilting module is a direct sum of $T(\lambda)$'s [J, E.3, 4]. By the linkage principle each $T(\lambda)$ belongs to single block $\mathrm{Rep}_{\mathcal{W}_a \bullet \lambda}(G)$.

Let $\mathrm{Tilt}(G)$ denote the full additive subcategory of $\mathrm{Rep}(G)$ consisting of tilting modules. We set $\mathrm{Tilt}_0(G) = \mathrm{Tilt}(G) \cap \mathrm{Rep}(G)$. As $\nabla(0) = T(0)$, as the translation functors send a tilting module to a tilting modele, and as $\mathrm{Tilt}_0(G)$ is Karoubian [J, E.1], $\bar{\Psi}$ factors through $\mathrm{Tilt}_0(G)$:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{asph}} \arrow[r, "\bar{\Psi}"] \arrow[dr, dotted] & \mathrm{Rep}_0(G) \\
& \mathrm{Tilt}_0(G). \arrow[u, hook]
\end{tikzcd}$$

(6.3) Let $\underline{w} = \underline{s_1 s_2 \dots s_r}$ be an expression of $w \in {}^f\mathcal{W}$, and write

$$T(\underline{w}) = T(0)B_{\underline{w}} = \Theta_{s_r} \dots \Theta_{s_2}\Theta_{s_1}T(0) = \Theta_{s_r} \dots \Theta_{s_2}\Theta_{s_1}\nabla(0).$$

Let us also abbreviate $T(w \bullet 0)$ as $T(w)$.

Let $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}$ be the degrading of $\mathcal{D}^{\mathrm{asph}}$: $\mathrm{Ob}(\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}) = \mathrm{Ob}(\mathcal{D}^{\mathrm{asph}})$ but $\forall X, Y \in \mathrm{Ob}(\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}})$, $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(X, Y) = (\mathcal{D}^{\mathrm{asph}})^\bullet(X, Y) = \coprod_{m \in \mathbb{Z}} \mathcal{D}^{\mathrm{asph}}(X, Y\langle m\rangle)$. In particular, $\forall m \in \mathbb{Z}$, $X \simeq X\langle m\rangle$ in $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}$; $\mathrm{id}_X \in \mathcal{D}^{\mathrm{asph}}(X, X) \leq \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(X, X\langle m\rangle)$ admits an inverse $\mathrm{id}_{X\langle m\rangle} \in \mathcal{D}^{\mathrm{asph}}(X\langle m\rangle, X\langle m\rangle) \leq \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(X\langle m\rangle, X)$. By construction $\bar{\Psi}$ induces a functor $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}} \to \mathrm{Tilt}_0(G)$, which we denote by $\bar{\Psi}_{\mathrm{deg}}$. We will show that Cor. 5.3 implies

Theorem [RW, Th. 1.3.1]; *The functor $\bar{\Psi}_{\mathrm{deg}} : \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}} \to \mathrm{Tilt}_0(G)$ is an equivalence of categories such that $\forall w \in {}^f\mathcal{W}$, $\bar{B}_w \mapsto T(w)$ and $\bar{B}_{\underline{w}} \mapsto T(\underline{w})$.*

(6.4) **Corollary [RW, Cor. 1.4.1]:** $\forall w \in {}^f\mathcal{W}$,

$$\mathrm{ch}\, T(w) = \sum_{y \in {}^f\mathcal{W}} {}^p n_{y,w}(1)\mathrm{ch}\,\nabla(y).$$

(6.5) To obtain only the character formula of $T(w)$, $w \in {}^f\mathcal{W}$, one has only to show that $\bar{\Psi}(B_w) = T(w)$.

To see the equivalence of $\bar{\Psi}_{\mathrm{deg}} : \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}} \to \mathrm{Tilt}_0(G)$, we first show that it is fully faithful. For that we have by (5.6) only to show for each pair of expressions $\underline{x}$ and $\underline{y}$ of $x, y \in {}^f\mathcal{W}$ that $\bar{\Psi}$ induces an isomorphism $(\mathcal{D}^{\mathrm{asph}})^\bullet(\bar{B}_{\underline{x}}, \bar{B}_{\underline{y}}) \simeq \mathrm{Rep}_0(T(\underline{x}), T(\underline{y}))$; if $\underline{w}$ is a reduced expression of $w \in {}^f\mathcal{W}$,

$$B_{\underline{w}} = B_w \oplus \coprod_{\substack{y<w \\ k\in\mathbb{Z}}} (B_y\langle k\rangle)^{\oplus m(y,k)} \quad \exists m(y,k) \in \mathbb{N},$$

from which the density of $\bar{\Psi}_{\mathrm{deg}}$ will also follow. For that we will make use of the structure of highest weight category on $\mathrm{Rep}_0(G)$.

We thus start with some generalities on highest weightcategories. Let $\mathcal{A}$ be a $\Bbbk$-linear abelian category whose objects all have finite length. Let Ξ denote a set parametrizing the isomorphism classes of simple objects of $\mathcal{A}$, and for $\lambda \in \Xi$ let $L(\lambda)$ denote the corresponding simple object in $\mathcal{A}$. Assume that Ξ is equipped with a PO $\preceq$. We say $\Omega \subseteq \Xi$ forms an ideal of Ξ iff

$\forall\lambda \in \Omega, \forall\mu \in \Xi$ with $\mu \preceq \lambda$, $\mu \in \Omega$, in which case we will write $\Omega \trianglelefteq \Xi$. We say $\Omega' \subseteq \Xi$ is a coideal of Ξ iff $\Xi \setminus \Omega'$ is an ideal.

$\forall\Omega \subseteq \Xi$, we let $\mathcal{A}_\Omega$ denote the Serre subcategory of $\mathcal{A}$ generated by the $L(\lambda)$, $\lambda \in \Omega$ [中岡, Def. 4.2.47, p. 260]; $\mathcal{A}_\Omega$ is the smallest full subcategory of $\mathcal{A}$ containing all $L(\lambda)$, $\lambda \in \Omega$, such that $\forall$ exact sequence $0 \to X \to Y \to Z \to 0$, $Y \in \mathcal{A}_\Omega$, iff $X, Z \in \mathcal{A}_\Omega$. We will abbreviate $\mathcal{A}_{\{\mu\in\Xi|\mu\preceq\lambda\}}$ (resp. $\mathcal{A}_{\{\mu\in\Xi|\mu\prec\lambda\}}$) as $\mathcal{A}_{\preceq\lambda}$ (resp. $\mathcal{A}_{\prec\lambda}$). Assume also that each $L(\lambda)$, $\lambda \in \Xi$, is equipped with nonzero morphisms $\Delta(\lambda) \to L(\lambda)$ and $L(\lambda) \to \nabla(\lambda)$ in $\mathcal{A}$ for some objects $\Delta(\lambda)$, $\nabla(\lambda)$. The following definition derives from [CPS], [BGS, Def. 3.2].

Definition [RW, Def. 2.1.1]: The category $\mathcal{A}$ is called a highest weight category iff $\forall\lambda \in \Xi$, the following holds:

(HW1) $\{\mu \in \Xi|\mu \preceq \lambda\}$ is finite,

(HW2) $\mathcal{A}(L(\lambda), L(\lambda)) = \Bbbk \mathrm{id}_{L(\lambda)}$,

(HW3) $\forall$ ideal Ω of Ξ such that λ is maximal in Ω, the structure morphism $\Delta(\lambda) \to L(\lambda)$ (resp. $L(\lambda) \to \nabla(\lambda)$) is a projective cover (resp. injective hull) in $\mathcal{A}_\Omega$,

(HW4) $\ker(\Delta(\lambda) \to L(\lambda)), \mathrm{coker}(L(\lambda) \to \nabla(\lambda)) \in \mathcal{A}_{\prec\lambda}$,

(HW5) $\forall\mu \in \Xi$, $\mathrm{Ext}^2_{\mathcal{A}}(\Delta(\lambda), \nabla(\mu)) = 0$,

in which case we call $(\Xi, \preceq)$ the weight poset of $\mathcal{A}$, and $\Delta(\lambda)$ (resp. $\nabla(\lambda)$) a standard (resp. costandard) object of $\mathcal{A}$. Here $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = \mathcal{D}(\mathcal{A})(X, Y[i])$ with $\mathcal{D}(\mathcal{A})$ denoting the derived category of $\mathcal{A}$, which may be described by the 米田-extensions [Weib, pp. 79-80], [dJ, 27]: on the set of exact sequences ξ_A in $\mathcal{A}$ of the form

$$0 \to Y \to A^{1-i} \to A^{2-i} \to \cdots \to A^0 \to X \to 0$$

one defines an equivalence relation such that ξ_A and ξ_B is equivalent iff there is another exact sequence ξ_C and a commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & A^{1-i} & \longrightarrow & A^{2-i} & \longrightarrow \dots \longrightarrow & A^0 & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & C^{1-i} & \longrightarrow & C^{2-i} & \longrightarrow \dots \longrightarrow & A^0 & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & B^{1-i} & \longrightarrow & B^{2-i} & \longrightarrow \dots \longrightarrow & B^0 & \longrightarrow & X & \longrightarrow & 0.
\end{array}$$

An equivalence class of such exact sequences is called a 米田-extension of X by Y of degree i. Given an exact sequence ξ_A, one has a qis s

$$\begin{array}{ccccccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & Y & \longrightarrow & A^{1-i} & \longrightarrow & A^{2-i} & \longrightarrow \dots \longrightarrow & A^1 & \longrightarrow & A^0 & \longrightarrow & 0 \longrightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & 0 \longrightarrow \dots
\end{array}$$

and a morphism f of complexes

$$\begin{array}{ccccccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & Y & \longrightarrow & A^{1-i} & \longrightarrow & A^{2-i} & \longrightarrow & \dots & \longrightarrow & A^0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & Y & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots,
\end{array}$$

which define an element $\frac{f}{s}$ of $\mathcal{D}(\mathcal{A})(X, Y[i])$. In turn, given $\frac{g}{t} \in \mathcal{D}(\mathcal{A})(X, Y[i])$, write $g : Z^\bullet \to X$ and $t : Z^\bullet \xrightarrow{\text{qis}} Y[i]$. Replacing $Z^\bullet$ by the truncation $\tau_{\leq 0}Z^\bullet : \dots Z^{-2} \to Z^{-1} \to \ker(\partial^0) \to 0 \to \dots$, we may assume that $Z^j = 0\ \forall j > 0$. Thus, one can write

$$\begin{array}{ccccccccccccc}
Z^{-i-1} & \xrightarrow{\partial^{-i-1}} & Z^{-i} & \xrightarrow{\partial^{-i}} & Z^{1-i} & \xrightarrow{\partial^{1-i}} & \dots & \longrightarrow & Z^0 & \longrightarrow & Y & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & & & & & & & \\
0 & \longrightarrow & X & \longrightarrow & 0 & & & & & & & &
\end{array}$$

with the top row exact. Then the sequence ξ_Z

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & X & \longrightarrow & (Z^{1-i} \oplus X)/Z^{-i} & \longrightarrow & Z^{2-i} & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & Z^0 & \longrightarrow & Y & \longrightarrow & 0 \\
 & & x & \longmapsto & [0, x] & & & & & & & & & & \\
 & & & & [z, x] & \longmapsto & \partial^{1-i}(z), & & & & & & & &
\end{array}$$

using Freyd-Mitchell imbedding theorem [Weib, p. 25], is exact; regarding $(Z^{1-i} \oplus X)/Z^{-i} = \{(\partial^{-i}(z), g(z)|z \in Z^{-i}\}$, if $[0, x] = 0$, there is $z \in Z^{-i}$ such that $\partial^{-i}(z) = 0$ and $g(z) = x$. Then $z \in \ker \partial^{-i} = \mathrm{im}\partial^{-i-1}$, and hence $x = g(z) = 0$. If $\partial^{1-i}(z) = 0$, $z \in \mathrm{im}\partial^{-i}$. Writing $z = \partial^{-i}(z')$ with $z' \in Z^{-i}$, one has $[\partial^{-i}(z'), x] = [0, x]$.

The assignments $[\xi_A] \mapsto \frac{f}{s}$ and $\frac{g}{t} \mapsto [\xi_Z]$ give a bijection between the 米田-extensions of degree i and $\mathcal{D}(\mathcal{A})(X, Y[i])$. With an addition on the 米田-extensions as defined in [Weib, p. 79], the bijection is an isomorphism of abelian groups. In particular, the zero extension

$$0 \to Y \xrightarrow{\text{id}} Y \to 0 \to \dots \to 0 \to X \xrightarrow{\text{id}} X \to 0$$

is assigned a mophism of complexes

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\text{id}} & Y & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0,
\end{array}$$

which is homotopic to 0.

Throughout the rest of §6, unless otherwise specified, $(\mathcal{A}, \Xi, \preceq)$ will denote a highest weight category.

(6.6) We verify that $(\mathrm{Rep}(G), \Lambda^+, \uparrow)$ forms a highest weight category, where $\uparrow$ is the strong linkage on Λ defined as follows. For $\lambda \in \Lambda$, $\alpha \in R$ and $m \in \mathbb{Z}$ we write $\lambda \uparrow s_{\alpha,m} \bullet \lambda = s_\alpha \bullet \lambda + pm\alpha$ iff $\lambda \leq s_{\alpha,m} \bullet \lambda$, and we let $\Uparrow$ denote the patial order $\uparrow$ generates; by abuse of notation we abbreviate $\uparrow\uparrow$ simply as $\uparrow$. We say λ is strongly linked to μ iff either $\lambda \uparrow \mu$ or $\mu \uparrow \lambda$. $\forall \lambda \in \Lambda^+$, each composition factor $L(\mu)$ of $\nabla(\lambda)$ has $\mu \uparrow \lambda$, called the strong linkage principle [J, II.6.13].

Put $\mathcal{A} = \mathrm{Rep}(G)$ and let $\hat{\mathcal{A}}$ denote the category of all rational G-modules, not necessarily finite dimensional. We actually show that both $\mathcal{A}$ and $(\hat{\mathcal{A}}, \Lambda^+, \leq)$ form highest weight categories.

To check (HW3) holding, assume $\lambda \in \Xi$ maximal in an ideal Ω of Ξ. Given a diagram

$$\begin{tikzcd} & \nabla(\lambda) \\ M \arrow[ur, "f"] \arrow[r, hook] & M' \end{tikzcd}$$

in $\hat{\mathcal{A}}_\Omega$ let $I(\lambda)$ be the injective hull of $\nabla(\lambda)$ in $\hat{\mathcal{A}}$ [J, I.3.9]. Thus, f extends to some $\hat{f} \in \hat{\mathcal{A}}(M', I(\lambda))$. As $I(\lambda)/\nabla(\lambda)$ admits a filtration whose subquotients are all of the form $\nabla(\nu)$, $\nu > \lambda$ [J, II.4.16, 6.20] and as $\mathrm{soc}\,\nabla(\lambda) = L(\lambda)$, by the maximality of λ in Ω we must have $\mathrm{im}\hat{f} \leq \nabla(\lambda)$.

To see that $\Delta(\lambda)$ is projective in $\hat{\mathcal{A}}_\Omega$, given

$$\begin{tikzcd} & \Delta(\lambda) \arrow[d, "f"] \\ M' \arrow[r, two heads, "g"'] & M \end{tikzcd}$$

in $\hat{\mathcal{A}}_\Omega$, we may assume $M, M' \in \mathcal{A}$. Taking the Chevalley dual [J, II.2.12], the assertion follows from the injectivity of ${}^\tau\Delta(\lambda) = \nabla(\lambda)$ in $\hat{\mathcal{A}}_\Omega$.

In $\hat{\mathcal{A}}$ the condition (HW5) holds [J, II.4.16], and hence also in $\mathcal{A}$ by [BGS, Lem. 3.2.3]:

$$\mathrm{Ext}^2_{\mathcal{A}}(\Delta(\lambda), \nabla(\mu)) \leq \mathrm{Ext}^2_{\hat{\mathcal{A}}}(\Delta(\lambda), \nabla(\mu)).$$

(6.7) Back to a general highest weight category $(\mathcal{A}, \Xi, \preceq)$, by (HW4) and (HW3) the structure morphism $L(\lambda) \to \nabla(\lambda)$ defines an injective hull in $\mathcal{A}_{\preceq\lambda}$, and hence is an essential mono [AF, pp. 72, 207]; $\forall M \leq \nabla(\lambda)$ with $M \cap L(\lambda) = 0$, $M = 0$. Then

$$\mathrm{soc}_{\mathcal{A}} \nabla(\lambda) = L(\lambda). \tag{1}$$

From the exact sequence $0 \to L(\lambda) \to \nabla(\lambda) \to \nabla(\lambda)/L(\lambda) \to 0$ one obtains

$$\begin{aligned} \mathcal{A}(L(\lambda), \nabla(\lambda)) &\simeq \mathcal{A}(L(\lambda), L(\lambda)) \quad \text{by (HW4)} \\ &\simeq \Bbbk \quad \text{by (HW2)}. \end{aligned} \tag{2}$$

In turn, from the exact sequence $0 \to \mathcal{A}(\nabla(\lambda)/L(\lambda), \nabla(\lambda)) \to \mathcal{A}(\nabla(\lambda), \nabla(\lambda)) \to \mathcal{A}(L(\lambda), \nabla(\lambda))$ one obtains by (HW4) that

$$\mathcal{A}(\nabla(\lambda), \nabla(\lambda)) \simeq \Bbbk. \tag{3}$$

Dually, the structure morphism $\Delta(\lambda) \to L(\lambda)$ is a superfluous epi [AF, pp. 72, 199]: $\forall M \leq \Delta(\lambda)$ with $\ker(\Delta(\lambda) \to L(\lambda)) + M = \Delta(\lambda)$, $M = \Delta(\lambda)$. Then

$$\mathrm{hd}_{\mathcal{A}}\Delta(\lambda) = L(\lambda), \tag{4}$$

$$\mathcal{A}(\Delta(\lambda), L(\lambda)) \simeq \Bbbk \simeq \mathcal{A}(\Delta(\lambda), \Delta(\lambda)). \tag{5}$$

Lemma: *Let $\lambda, \mu \in \Xi$.*

(i) If $\mathrm{Ext}^1_{\mathcal{A}}(L(\lambda), \nabla(\mu)) \neq 0$, $\lambda \succ \mu$.

(ii) If $\mathrm{Ext}^1_{\mathcal{A}}(\Delta(\lambda), L(\mu)) \neq 0$, $\lambda \prec \mu$.

Proof: (i) Just suppose $\lambda \not\succ \mu$. If $\Omega = \Xi_{\preceq\lambda} \cup \Xi_{\preceq\mu}$, μ is maximal in Ω, and hence $\nabla(\mu)$ is injective in $\mathcal{A}_\Omega$. Then

$$\begin{aligned} 0 &= \mathrm{Ext}^1_{\mathcal{A}_\Omega}(L(\lambda), \nabla(\mu)) \quad \text{as } L(\lambda) \in \mathcal{A}_\Omega \\ &\simeq \mathrm{Ext}^1_{\mathcal{A}}(L(\lambda), \nabla(\mu)) \quad \text{with respect to the 米田 extension [Weib, p. 79]}, \end{aligned}$$

absurd.

Likewise (ii).

(6.8) For an object M of $\mathcal{A}$ a filtration of M whose subquotients consist all of standard (resp. costandard) objects is called a Δ- (resp. ∇-) filtration of M.

Proposition [RW, (2.1.1)]: *$\forall \lambda, \mu \in \Xi$, $\forall r \in \mathbb{N}$,*

$$\mathrm{Ext}^r_{\mathcal{A}}(\Delta(\lambda), \nabla(\mu)) \simeq \delta_{r,0}\delta_{\lambda,\mu}\Bbbk.$$

In particular, any nonzero morphism $\Delta(\lambda) \to \nabla(\lambda)$ factors $L(\lambda)$, and is unique up to scalar. Also, for $X \in \mathcal{A}$ admitting a ∇- (resp. Δ-) filtration, the multiplicity of each $\nabla(\lambda)$ (resp. $\Delta(\lambda)$), $\lambda \in \Xi$, is equal to $\dim \mathcal{A}(\Delta(\lambda), X)$ (resp. $\dim \mathcal{A}(X, \nabla(\lambda))$), which we will denote by $(X : \nabla(\lambda))$ (resp. $(X : \Delta(\lambda))$).

Proof: Assume that $\mathcal{A}(\Delta(\lambda), \Delta(\mu)) \neq 0$. Then $\lambda \preceq \mu \preceq \lambda$ by (HW4) and (6.7.2, 3), and hence $\lambda = \mu$. From the exact sequence $0 \to L(\lambda) \to \nabla(\lambda) \to \nabla(\lambda)/L(\lambda) \to 0$ one obtains

$$\begin{aligned} \mathcal{A}(\Delta(\lambda), \nabla(\lambda)) &\simeq \mathcal{A}(\Delta(\lambda), L(\lambda)) \quad \text{by (HW4) and (6.7.4)} \\ &\simeq \Bbbk \quad \text{by (6.7.5)}. \end{aligned}$$

Just suppose $\mathrm{Ext}^1_{\mathcal{A}}(\Delta(\lambda), \nabla(\mu)) \neq 0$. Then there is $\nu \preceq \lambda$ such that $\mathrm{Ext}^1_{\mathcal{A}}(L(\nu), \nabla(\mu)) \neq 0$. Then $\lambda \succeq \nu \succ \mu$ by (6.7.i). As $\Delta(\lambda)$ is projective in $\mathcal{A}_{\preceq\lambda}$,

$$0 = \mathrm{Ext}^1_{\mathcal{A}_{\preceq\lambda}}(\Delta(\lambda), \nabla(\mu)) \simeq \mathrm{Ext}^1_{\mathcal{A}}(\Delta(\lambda), \nabla(\mu)),$$

absurd.

Just suppose $\mathrm{Ext}^3_{\mathcal{A}}(\Delta(\lambda), \nabla(\mu)) \neq 0$ with an exact sequence

$$0 \to \nabla(\mu) \to X_1 \to X_2 \to X_3 \to \Delta(\lambda) \to 0 \tag{1}$$

representing a nonzero extension. In $\mathrm{Ext}^3_{\mathcal{A}}(\Delta(\lambda), \nabla(\mu))$, 0 is represented by an exact sequence $0 \longrightarrow \nabla(\mu) \xrightarrow{\mathrm{id}} \nabla(\mu) \longrightarrow 0 \longrightarrow \Delta(\lambda) \xrightarrow{\mathrm{id}} \Delta(\lambda) \longrightarrow 0$ [Weib, p. 79]. Let Ω be a finite ideal of Ξ such that $\mathcal{A}_\Omega$ contains all $\nabla(\mu), X_1, X_2, X_3, \Delta(\lambda)$. Recall from [BGS, Lem. 3.2.3] that $\mathcal{A}_\Omega$ forms a highest weight category. As Ω is finite, $\nabla(\mu)$ posseses an injective hull $I(\mu)$ in $\mathcal{A}_\Omega$ such that $I(\mu)/\nabla(\mu)$ admits a finite ∇-filtration with subquotients of the form $\nabla(\nu)$, $\nu \succ \mu$ [BGS, pf of Cor. 3.2.2]. Then $\mathrm{Ext}^2_{\mathcal{A}_\Omega}(\Delta(\lambda), I(\mu)/\nabla(\mu)) \twoheadrightarrow \mathrm{Ext}^3_{\mathcal{A}_\Omega}(\Delta(\lambda), \nabla(\mu))$ with $\mathrm{Ext}^2_{\mathcal{A}_\Omega}(\Delta(\lambda), I(\mu)/\nabla(\mu)) = 0$ by (HW5) [BX, Th. 7.5.1], and hence $\mathrm{Ext}^3_{\mathcal{A}_\Omega}(\Delta(\lambda), \nabla(\mu)) = 0$ Then (1) vanishes [BX, Th. 7.5.1], absurd. Repeat the argument to get all $\mathrm{Ext}^r_{\mathcal{A}}(\Delta(\lambda), \nabla(\mu)) = 0$, $r \geq 2$.

(6.9) **Remark:** If Ω is a finite ideal of Λ^+, $\mathrm{Rep}(G)_\Omega$ admits enough injectives and projectives [BGS, 3.2]; $\forall \lambda \in \Omega$, an injective hull of $L(\lambda)$ in $\mathrm{Rep}(G)_\Omega$ is given by $\Gamma_\Omega(I(\lambda))$ with $I(\lambda)$ an injective hull of $L(\lambda)$ in the category of all rational G-modules [BGS, Th. 3.2.1(T)].

(6.10) Let $\mathcal{C}$ be an abelian category and $\mathcal{C}'$ a Serre subcategory of $\mathcal{C}$. The Serre quotient $\mathcal{C}/\mathcal{C}'$ [Ga, III.1] consists of the same objects as of $\mathcal{C}$, and for $X, Y \in \mathrm{Ob}(\mathcal{C}/\mathcal{C}')$

$$(\mathcal{C}/\mathcal{C}')(X, Y) = \varinjlim_{\substack{X' \leq X \text{ with } X/X' \in \mathcal{C}' \\ Y' \in \mathcal{C}'}} \mathcal{C}(X', Y/Y'),$$

where the (X', Y') are directed such that $(X_1, Y_1) \leq (X_2, Y_2)$ iff $X_2 \leq X_1$ and $Y_1 \leq Y_2$, in which case one has

$$\begin{tikzcd} X_1 \arrow[r] & Y/Y_1 \arrow[d] \\ X_2 \arrow[u, hook] \arrow[r, dotted] & Y/Y_2. \end{tikzcd}$$

Given arbitrary (X_i, Y_i) with X/X_i and $Y_i \in \mathcal{C}'$, $i = 1, 2$, one checks that $X/(X_1 \cap X_2)$ and $(Y_1 \oplus Y_2)/(Y_1 \cap Y_2) \in \mathcal{C}'$; using the Freyd-Mitchell imbedding theorem, $(Y_1 \oplus Y_2)/(Y_1 \cap Y_2) = Y_1 \times_{Y_1 \cap Y_2} Y_2 \simeq Y_1 + Y_2$, $X_1/(X_1 \cap X_2) \simeq (X_1 + X_2)/X_2 \leq X/X_2$, and hence $X_1/(X_1 \cap X_2) \in \mathcal{C}'$. Then the exact sequence $0 \to X_1/(X_1 \cap X_2) \to X/(X_1 \cap X_2) \to X/X_1 \to 0$ yields that $X/(X_1 \cap X_2) \in \mathcal{C}'$. If $f \in \mathcal{C}(X_1, Y/Y_1)$ and $g \in \mathcal{C}(Y_2, Z/Z_1)$, one composes f and g as follows:

$$\begin{tikzcd} X_1 \arrow[rr, "f"] & & Y/Y_1 & Y_2 \arrow[rr, "g"] \arrow[d, two heads] & & Z/Z_1 \arrow[d, two heads] \\ f^{-1}((Y_1 + Y_2)/Y_1) \arrow[u, hook] \arrow[rr, dotted] & & (Y_1 + Y_2)/Y_1 \arrow[u, hook] \arrow[r, "\sim"] & Y_2/(Y_1 \cap Y_2) \arrow[rr, dotted] & & Z/(Z_1 + g^{-1}(g(Y_1 \cap Y_2))). \end{tikzcd}$$

Thus,

(i) $\mathcal{C}/\mathcal{C}'$ is abelian and the quotient functor $^-: \mathcal{C} \to \mathcal{C}/\mathcal{C}'$ is exact [Ga, Prop. 1, p. 62],

(ii) $\forall f \in \mathcal{C}(X, Y)$, $\bar{f} = 0$ (resp. monic, epic) iff $\mathrm{im} f \in \mathcal{C}'$ (resp. $\ker f, \mathrm{coker} f \in \mathcal{C}'$) [Ga, Lem. 2, p.366]. In particular, id_X vanishes in $\mathcal{C}/\mathcal{C}'$ iff $X \in \mathcal{C}'$.

(iii) $\forall f \in \mathcal{C}(X, Y)$, $\bar{f}$ is invertible iff $\mathrm{im} f$ and $\mathrm{coker} f \in \mathcal{C}'$ [Ga, Lem. 4, p.367].

Let $S = \{f \in \mathrm{Mor}(\mathcal{C}) | \ker f \text{ and } \mathrm{coker} f \in \mathcal{C}'\}$ and let $\mathcal{C}_S$ be the localization of $\mathcal{C}$ with respect to the multiplicative system S [中岡, Prop. 4.2.28, p. 260, Prop. 2.4.26, p. 113]. By (ii) and (iii) the universality of $\mathcal{C}_S$ [中岡, Def. 2.4.3, p. 99] yields

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}/\mathcal{C}'. \\ \mathcal{C}_S \arrow[ur, dotted] & \end{tikzcd}$$

If $X \in \mathrm{Ob}(\mathcal{C}')$, the zero morph $X \to 0$ in $\mathcal{C}$ is invertible in $\mathcal{C}_S$ [中岡, Def. 2.4.3, p. 99], and hence the universality of $\mathcal{C}/\mathcal{C}'$ [Ga, Cor. 2, p. 368] yields a quasi inverse of $\mathcal{C}_S \to \mathcal{C}/\mathcal{C}'$ above

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}_S. \\ \mathcal{C}/\mathcal{C}' \arrow[ur, dotted] & \end{tikzcd}$$

One has also [中岡, Cor. 3.2.50]

(1)
$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{qt}"] \arrow[d] & \mathcal{C}/\mathcal{C}' \\ \mathcal{C}//\mathcal{C}'. \arrow[ur, dotted, "\text{faithful}"'] & \end{tikzcd}$$

For a coideal Ω of Ξ put $\mathcal{A}^\Omega = \mathcal{A}/\mathcal{A}_{\Xi\setminus\Omega}$.

Lemma [AR, Lem. 2.2]/[BGS, 3.2]/[RW, Lem. 2.1.3]: *(i) If Ω is an ideal of Ξ, $(\mathcal{A}_\Omega, \Omega, \preceq)$ forms a highest weight category with the standard (resp. costandard) objects $\Delta(\lambda)$ (resp. $\nabla(\lambda)$), $\lambda \in \Omega$.*

(ii) If Ω is a coideal of Ξ, $(\mathcal{A}^\Omega, \Omega, \preceq)$ forms a highest weight category with the standard (resp. costandard) objects $\bar{\Delta}(\lambda)$ (resp. $\bar{\nabla}(\lambda)$), $\lambda \in \Omega$.

(6.11) Let $(\mathcal{A}, \Xi, \preceq)$ be a highest weight category.

Corollary: *Let Ω be a coideal of Ξ. $\forall M \in \mathcal{A}$ admitting a Δ-filtration, $\forall M' \in \mathcal{A}$ admitting a ∇-filtration, one has*

$$\mathcal{A}(M, M') \twoheadrightarrow \mathcal{A}^\Omega(M, M').$$

Proof: By (6.8) and (6.10) and by the snake lemma [中岡, Lem. 4.2.21, p. 244] we may assume that $M = \Delta(\lambda)$ and $M' = \nabla(\mu)$ for some $\lambda, \mu \in \Omega$, in which case the assertion follows from (6.8) and (6.10) again.

(6.12) Let $X \in \mathcal{A}$ admitting a ∇-filtration. A canonical ∇-flag of X is the data $\Gamma_\Omega X \leq X$ for each ideal Ω of Ξ such that

(i) $\cup_\Omega \Gamma_\Omega X = X$,

(ii) if $\Omega' \subseteq \Omega$ is another ideal of Ξ, $\Gamma_{\Omega'} X \leq \Gamma_\Omega X$,

(iii) $\forall \Omega \trianglelefteq \Xi$, $\forall \lambda \in \Omega$ maximal, with $\Omega' = \Omega \setminus \{\lambda\}$, $\Gamma_\Omega X / \Gamma_{\Omega'} X \simeq \coprod \nabla(\lambda)$.

We set $\Gamma_\emptyset X = 0$.

Lemma [RW, Lem. 2.2.1]: $\forall X \in \mathcal{A}$ with a ∇-filtration, a canonical ∇-flag exists and uniquely. By the unicity we will call a canonical ∇-flag of X simply the ∇-flag.

Proof: $\forall \lambda, \mu \in \Xi$, $\mathrm{Ext}^1_{\mathcal{A}}(\nabla(\lambda), \nabla(\mu)) = 0$ unless $\mu \prec \lambda$ by (6.7), and hence the existence; if $X = X^0 > X^1 > \cdots > X^r = 0$ with $X^i / X^{i+1} \simeq \nabla(\lambda_i)$, $\lambda_i \in \Lambda^+$, one can arrange the filtration such that $i < j$ if $\lambda_i > \lambda_j$.

To see the unicity, it is enough to show that for minimal $\lambda \in \Xi$ with $(X : \nabla(\lambda)) \neq 0$ there is unique $X' \leq X$ with $X' = \coprod \nabla(\lambda)$ such that X/X' admits a ∇-filtration and $(X/X' : \nabla(\lambda)) = 0$. But $\forall \mu \in \Xi$, $\mathcal{A}(\nabla(\lambda), \nabla(\mu)) = 0$ unless $\mu \preceq \lambda$ as $\mathrm{soc}_{\mathcal{A}} \nabla(\mu) = L(\mu)$ by (6.7.1). Also, $\mathcal{A}(\nabla(\lambda), \nabla(\lambda)) = \Bbbk$ by (6.7.2). We must then have $X' = \sum_{f \in \mathcal{A}(\nabla(\lambda), X)} \mathrm{im} f$.

(6.13) We call $X \in \mathcal{A}$ tilting iff it admits both a ∇- and a Δ-filtrations. We denote by $\mathrm{Tilt}(\mathcal{A})$ the additive full subcategory of $\mathcal{A}$ consisting of the tilting objects. Thus, $\mathrm{Tilt}(\mathcal{A})$ is Krull-Schmidt and the isomorphism classes of indecomposables are parametrized by Ξ [AR, Prop. A.4]/[J, E.3, E.6]/[Ri, 7.5]; $\forall \lambda \in \Xi$, the corresponding indecomposable tilting $T(\lambda)$ is characterized up to isomorphism by the properties

(1) $$(T(\lambda) : \nabla(\lambda)) = 1 \quad \text{and} \quad \forall \mu \in \Xi \text{ with } (T(\lambda) : \nabla(\mu)) \neq 0, \mu \preceq \lambda.$$

Recall also from [loc. cit] that

(2) $$(T(\lambda) : \Delta(\lambda)) = 1 \quad \text{and} \quad \forall \mu \in \Xi \text{ with } (T(\lambda) : \Delta(\mu)) \neq 0, \mu \preceq \lambda.$$

Lemma [RW, Lem. 2.3.1]: *Let $\lambda \in \Xi$.*

(i) $\mathcal{A}(\Delta(\lambda), T(\lambda)) = \Bbbk$, and nonzero morphism $\Delta(\lambda) \to T(\lambda)$ is injective.

(ii) $\mathcal{A}(T(\lambda), \nabla(\lambda)) = \Bbbk$, and nonzero morphism $T(\lambda) \to \nabla(\lambda)$ is surjective.

(iii) $\forall \phi \in \mathcal{A}(\Delta(\lambda), T(\lambda)) \setminus 0$, $\forall \psi \in \mathcal{A}(T(\lambda), \nabla(\lambda)) \setminus 0$, $\psi \circ \phi \neq 0$.

(6.14) For $\lambda \in \Xi$ set $\mathcal{A}^{\succeq \lambda} = \mathcal{A}^{\{\mu \in \Xi | \mu \succeq \lambda\}} = \mathcal{A}/\mathcal{A}_{\{\mu \in \Xi | \mu \not\succeq \lambda\}}$. By (6.10.ii, iii) and by (HW4) one has

(1) $$\Delta(\lambda) \simeq \nabla(\lambda) \simeq L(\lambda) \simeq T(\lambda) \quad \text{in } \mathcal{A}^{\succeq \lambda}.$$

Definition [RW, Def. 2.3.2]: Let $X \in \mathcal{A}$ admitting a ∇-filtration. A section of the ∇-flag of X is a triple $(\Pi, e, (\phi^X_\pi | \pi \in \Pi))$ such that

(i) $e : \Pi \to \Xi$ is a map,

(ii) $\forall \pi \in \Pi$, $\phi^X_\pi \in \mathcal{A}(T(e(\pi)), X)$ such that $\forall \lambda \in \Xi$, $(\phi^X_\pi | \pi \in e^{-1}(\lambda))$ forms a $\Bbbk$-linear basis of $\mathcal{A}^{\succeq\lambda}(T(\lambda), X) \simeq \mathcal{A}^{\succeq\lambda}(\Delta(\lambda), X)$ under the quotient $\mathcal{A} \to \mathcal{A}^{\succeq\lambda}$. In particular, for $\lambda \in \Xi$ with $\mathcal{A}^{\succeq\lambda}(T(\lambda), X) = 0$, $e^{-1}(\lambda) = \emptyset$. Such exists by (6.11).

$\forall\lambda \in \Xi$, one has

$$\begin{aligned}
(2)\quad \dim \mathcal{A}^{\succeq\lambda}(T(\lambda), X) &= \dim \mathcal{A}^{\succeq\lambda}(\Delta(\lambda), X) \\
&= (X : \nabla(\lambda))_{\mathcal{A}^{\succeq\lambda}} \quad \text{the multiplicity of } \nabla(\lambda) \text{ in } X \text{ in } \mathcal{A}^{\succeq\lambda} \text{ by (6.10.ii)} \\
&= (X : \nabla(\lambda)) \quad \text{by (6.8.ii, iii) and (6.12),}
\end{aligned}$$

and hence

$$|\{\phi_\pi^X | \pi \in e^{-1}(\lambda)\}| = (X : \nabla(\lambda)), \quad |\Pi| = \sum_{\lambda\in\Xi} (X : \nabla(\lambda)).$$

(6.15) **Lemma [RW, Lem. 2.3.4]:** *Let $X \in \mathcal{A}$ with a ∇-filtration, and let $(\Pi, e, (\phi_\pi^X | \pi \in \Pi))$ be a section of the ∇-flag of X. Let $\Omega \trianglelefteq \Xi$ and put $\Pi_\Omega = e^{-1}(\Omega)$. $\forall\pi \in \Pi_\Omega$, $\phi_\pi^X \in \mathcal{A}(T(e(\pi)), X)$ factors through $\Gamma_\Omega X \hookrightarrow X$*

$$\begin{tikzcd}
T(e(\pi)) \arrow[r, "\phi_\pi^X"] \arrow[dr, dotted, "\phi_\pi^{\Gamma_\Omega X}"'] & X \\
& \Gamma_\Omega X. \arrow[u, hook]
\end{tikzcd}$$

If $e_\Omega = e|_{\Pi_\Omega}$, $(\Pi_\Omega, e_\Omega, (\phi_\pi^{\Gamma_\Omega X} | \pi \in \Pi_\Omega))$ forms a section of the ∇-flag of $\Gamma_\Omega X$.

Proof: Let $\lambda \in \Omega$. An exact sequence $0 \to \Gamma_\Omega X \to X \to X/\Gamma_\Omega X \to 0$ induces another short exact sequence $0 \to \mathcal{A}(T(\lambda), \Gamma_\Omega X) \to \mathcal{A}(T(\lambda), X) \to \mathcal{A}(T(\lambda), X/\Gamma_\Omega X) \to 0$. $\forall\mu \in \Xi$ with $(X/\Gamma_\Omega X : \nabla(\mu)) \neq 0$, $\mu \not\preceq \lambda$, and hence $\mathcal{A}(T(\lambda), \Gamma_\Omega X) \to \mathcal{A}(T(\lambda), X)$ is bijective. Thus, $\forall\pi \in e^{-1}(\lambda)$, ϕ_π^X factors through $\Gamma_\Omega X$. Also,

$$\begin{aligned}
\dim \mathcal{A}^{\succeq\lambda}(T(\lambda), X) &= (X : \nabla(\lambda)) \quad \text{by (6.14.2)} \\
&= (\Gamma_\Omega X : \nabla(\lambda)) \\
&= \dim \mathcal{A}^{\succeq\lambda}(T(\lambda), \Gamma_\Omega X) \quad \text{by (6.14.2) again.}
\end{aligned}$$

The assertion follows.

(6.16) Likewise

Lemma [RW, Lem. 2.3.5]: *Let $X \in \mathcal{A}$ with a ∇-filtration, and let $(\Pi, e, (\phi_\pi^X | \pi \in \Pi))$ be a section of the ∇-flag of X. Let $\Omega \triangleleft \Xi$ and put $\Pi^\Omega = \Pi \setminus \Pi_\Omega = \Pi \setminus e^{-1}(\Omega)$. $\forall\pi \in \Pi^\Omega$, define $\phi_\pi^{X/\Gamma_\Omega X}$ to be the composite of $\phi_\pi^X \in \mathcal{A}(T(e(\pi)), X)$ with the quotient $X \to \Gamma_\Omega X$*

$$\begin{tikzcd}
T(e(\pi)) \arrow[r, "\phi_\pi^X"] \arrow[dr, dotted, "\phi_\pi^{X/\Gamma_\Omega X}"'] & X \arrow[d] \\
& X/\Gamma_\Omega X.
\end{tikzcd}$$

If $e^\Omega = e|_{\Pi^\Omega}$, $(\Pi^\Omega, e^\Omega, (\phi_\pi^{X/\Gamma_\Omega X} | \pi \in \Pi^\Omega))$ forms a section of the ∇-flag of $X/\Gamma_\Omega X$.

(6.17) Back to $\mathrm{Rep}(G)$ under the standing hypothesis that $p > n$, for $\lambda, \nu \in \Lambda^+$ let us write $\nu \downarrow \lambda$ to mean $\lambda \uparrow \nu$. Thus, $\downarrow\lambda = \{\nu \in \Lambda^+ | \nu \downarrow \lambda\} = \{\nu \in \Lambda^+ | \lambda \uparrow \nu\}$.

Now, for each $s \in \mathcal{S}_a$ take $\mu_s \in \Lambda^+ \cap \overline{A^+}$ as in (3.8), and let $\mathrm{Rep}_s(G)$ be the block of μ_s. Let $\mathrm{T}^s : \mathrm{Rep}_0(G) \to \mathrm{Rep}_s(G)$ and $\mathrm{T}_s : \mathrm{Rep}_s(G) \to \mathrm{Rep}_0(G)$ be the adjoint pair of translation functors as in (4.9). If $\Lambda_0^+ = \Lambda^+ \cap (\mathcal{W}_a \bullet 0)$ (resp. $\Lambda_s^+ = \Lambda^+ \cap (\mathcal{W}_a \bullet \mu_s)$), $(\mathrm{Rep}_0(G), \Lambda_0^+, \uparrow|_{\Lambda_0^+})$ (resp. $(\mathrm{Rep}_s(G), \Lambda_s^+, \uparrow|_{\Lambda_s^+})$) forms a highest weight category; $\Lambda_0^+, \Lambda_s^+ \lhd \Lambda^+$, and $\mathrm{Rep}_0(G) = \mathrm{Rep}(G)_{\Lambda_0^+}$, $\mathrm{Rep}_s(G) = \mathrm{Rep}(G)_{\Lambda_s^+}$. If $\lambda \in \Lambda_0^+$, by $\downarrow\lambda$ we will mean a coideal $\{\nu \in \Lambda_0^+ | \ \nu \downarrow \lambda\} = \{\nu \in \Lambda_0^+ | \ \lambda \uparrow \nu\}$ of Λ_0^+. Likewise for $\mu \in \Lambda_s^+$. Writing $\lambda = w \bullet 0$, $w \in {}^f\mathcal{W}$, set $\lambda^s = ws \bullet 0$.

Assume $\lambda \uparrow \lambda^s$, and let $\mu \in \Lambda_s^+$ such that λ belongs to an alcove whose closure contains μ. Then [J, E.11]

$$\mathrm{T}_s T(\mu) \simeq T(\lambda^s). \tag{1}$$

We fix such an isomorphism once and for all. As $(\mathrm{T}^s T(\lambda) : \nabla(\mu)) = 1$ with μ maximal in $\{\nu \in \Lambda_s^+ | (\mathrm{T}^s T(\lambda) : \nabla(\nu)) \neq 0\}$, $T(\mu)$ is a direct summand of $\mathrm{T}^s T(\lambda)$ of multiplicity 1. Accordingly, we fix a split mono and a split epi

$$T(\mu) \rightleftarrows \mathrm{T}^s T(\lambda). \tag{2}$$

One has also [J, E.11]

$$\mathrm{T}^s T(\lambda^s) \simeq T(\mu) \oplus T(\mu). \tag{3}$$

(6.18) **Lemma [RW, Lem. 3.2.2]:** *Let $y \in {}^f\mathcal{W}$ and $s \in \mathcal{S}_a$ such that $ys > y$ and that $ys \in {}^f\mathcal{W}$. If $\lambda = y \bullet 0$, under the quotient $\mathrm{Rep}(G) \to \mathrm{Rep}(G)^{\downarrow\lambda}(G)$ one has an isomorphism*

$$\mathrm{Rep}_0(G)(\Delta(\lambda), \Theta_s\Delta(\lambda)) \to \mathrm{Rep}_0(G)^{\downarrow\lambda}(\Delta(\lambda), \Theta_s\Delta(\lambda)),$$

both of dimension 1.

Proof: Let $\mu \in \Lambda_s^+$ lying in the closure of the alcove containing λ. Then

$$\mathrm{Rep}_0(G)(\Delta(\lambda), \Theta_s\Delta(\lambda)) \simeq \mathrm{Rep}_s(G)(\mathrm{T}^s\Delta(\lambda), \mathrm{T}^s\Delta(\lambda)) \simeq \mathrm{Rep}_s(G)(\Delta(\mu), \Delta(\mu)) \simeq \Bbbk.$$

If $q : \Delta(\lambda) \to L(\lambda)$ is the quotient and $i : L(\lambda) \hookrightarrow \nabla(\lambda)$, one has commutative diagrams

$$\begin{array}{ccc}
\mathrm{Rep}_0(G)(\Delta(\lambda), \Theta_s\Delta(\lambda)) & \xrightarrow{\mathrm{Rep}_0(G)(\Delta(\lambda),\Theta_s q)} & \mathrm{Rep}_0(G)(\Delta(\lambda), \Theta_s L(\lambda)) \\
\sim\downarrow & & \downarrow\sim \\
\mathrm{Rep}_s(G)(\Delta(\mu), \Delta(\mu)) & \xrightarrow[\mathrm{Rep}_0(G)(\Delta(\mu),\mathrm{T}^s q)]{\sim} & \mathrm{Rep}_s(G)(\Delta(\mu), L(\mu))
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Rep}_0(G)(\Delta(\lambda), \Theta_s L(\lambda)) & \xrightarrow{\mathrm{Rep}_0(G)(\Delta(\lambda),\Theta_s i)} & \mathrm{Rep}_0(G)(\Delta(\lambda), \Theta_s\nabla(\lambda)) \\
\sim\downarrow & & \downarrow\sim \\
\mathrm{Rep}_s(G)(\Delta(\mu), L(\mu)) & \xrightarrow[\mathrm{Rep}_0(G)(\Delta(\mu),\mathrm{T}^s i)]{\sim} & \mathrm{Rep}_s(G)(\Delta(\mu), \nabla(\mu)).
\end{array}$$

Putting these together, the composite $\Delta(\lambda) \xrightarrow{q} L(\lambda) \xrightarrow{i} \nabla(\lambda)$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_0(G)(\Delta(\lambda), \Theta_s\Delta(\lambda)) & \xrightarrow[\sim]{\mathrm{Rep}_0(G)(\Delta(\lambda),\Theta_s(i\circ q))} & \mathrm{Rep}_0(G)(\Delta(\lambda), \Theta_s\nabla(\lambda)) \\ \downarrow & & \downarrow \\ \mathrm{Rep}_0(G)^{\downarrow\lambda}(\Delta(\lambda), \Theta_s\Delta(\lambda)) & \xrightarrow[\mathrm{Rep}_0(G)^{\downarrow\lambda}(\Delta(\lambda),\mathrm{T}^s(i\circ q))]{} & \mathrm{Rep}_s(G)^{\downarrow\lambda}(\Delta(\lambda), \Theta_s\nabla(\lambda)). \end{array} \tag{1}$$

If $L(\nu)$ is a composition factor of $\ker(q)$, there is an epi $\Delta(\nu) \twoheadrightarrow L(\nu)$. As $\nu < \lambda$, $\Theta_s\Delta(\nu)$ vanishes in $\mathrm{Rep}(G)^{\downarrow\lambda}$, and so therefore does $\Theta_s L(\nu)$ in $\mathrm{Rep}(G)^{\downarrow\lambda}$. Then $\Theta_s q$ is invertible in $\mathrm{Rep}(G)^{\downarrow\lambda}$, and so is $\Theta_s i$ likewise. It follows that the bottom horizontal map of (1) is invertible.

If $\lambda^s = ys \bullet 0$, as $\Theta_s\nabla(\lambda)$ has a ∇-filtration such that $0 \to \nabla(\lambda) \to \Theta_s\nabla(\lambda) \to \nabla(\lambda^s) \to 0$ is exact, and as $\mathrm{Rep}_0(G)^{\downarrow\lambda}$ is a highest weight category, one has a commutative diagram

$$\begin{array}{ccc} \Bbbk \simeq \mathrm{Rep}_0(G)(\Delta(\lambda), \nabla(\lambda)) & \xrightarrow{\sim} & \mathrm{Rep}_0(G)(\Delta(\lambda), \Theta_s\nabla(\lambda)) \\ \sim\downarrow & & \downarrow \\ \Bbbk \simeq \mathrm{Rep}_0(G)^{\downarrow\lambda}(\Delta(\lambda), \nabla(\lambda)) & \xrightarrow{\sim} & \mathrm{Rep}_s(G)^{\downarrow\lambda}(\Delta(\lambda), \Theta_s\nabla(\lambda)). \end{array}$$

Thus, the right vertical map in (1) is bijective, and hence also the left and the assertion follows.

(6.19) Recall also

Lemma: *Let $s \in \mathcal{S}_a$. $\forall\lambda \in \Lambda_0^+$ with $\lambda^s \notin \Lambda^+$, $\forall M \in \mathrm{Rep}_s(G)$ with a ∇-filtration,*

$$(\mathrm{T}_s M : \nabla(\lambda)) = \dim \mathrm{Rep}_0(G)(\Delta(\lambda), \mathrm{T}_s M) = \dim \mathrm{Rep}_s(G)(\mathrm{T}^s\Delta(\lambda), M) = 0.$$

(6.20) To compute $\mathrm{Rep}_0(G)(T(\underline{x}), T(\underline{y}))$ inductively, let $M \in \mathrm{Rep}_0(G)$ with a ∇-filtration. We now give a prescription to construct a section of the ∇-flag of $\Theta_s M = \Theta_s M$, $s \in \mathcal{S}_a$, from one on M.

Let $(\Pi, e, (\phi_\pi^M | \pi \in \Pi))$ be a section of the ∇-flag of M. Set $\Pi^s = \{\pi \in \Pi | e(\pi)^s \in \Lambda^+\}$. Define a map $e^s : \Pi^s \to \Lambda_s^+$ by defining $e^s(\pi) \in \Lambda_s^+$, $\pi \in \Pi^s$, to be the one lying in the closure of the alcove containing $e(\pi)$. As $|\Pi| = \sum_{\lambda\in\Lambda_0^+}(M : \nabla(\lambda))$ and as $\mathrm{T}^s\nabla(\lambda) = 0$ for $\lambda \in \Pi \setminus \Pi^s$,

$$|\Pi^s| = \sum_{\mu\in\Lambda_s^+} (\mathrm{T}^s M : \nabla(\mu)).$$

We now define $\phi_\pi^{\mathrm{T}^s M} \in \mathrm{Rep}_s(G)(T(e^s(\pi)), \mathrm{T}^s M)$ for $\pi \in \Pi^s$.

Case 1: $e(\pi) \downarrow e(\pi)^s$, i.e., $e(\pi)^s \uparrow e(\pi)$.
Recall from (6.17.1) a fixed isomorphism $\mathrm{T}_s T(e^s(\pi)) \simeq T(e(\pi))$. Then

$$\mathrm{Rep}_0(G)(T(e(\pi)), M) \simeq \mathrm{Rep}_0(G)(\mathrm{T}_s T(e^s(\pi)), M) \simeq \mathrm{Rep}_0(G)(T(e^s(\pi)), \mathrm{T}^s M),$$

under which we define $\phi_\pi^{\mathrm{T}^s M}$ to be the image of ϕ_π^M:

$$\begin{tikzcd} T(e^s(\pi)) \arrow[r, dotted, "\phi_\pi^{\mathrm{T}^s M}"] \arrow[d, "\mathrm{adj}"'] & \mathrm{T}^s M \\ \mathrm{T}^s\mathrm{T}_s T(e^s(\pi)) \arrow[r, "\sim"] & \mathrm{T}^s T(e(\pi)). \arrow[u, "\mathrm{T}^s(\phi_\pi^M)"'] \end{tikzcd}$$

By construction, defined under the isomorphisms, $\phi_\pi^{\mathrm{T}^s M} \neq 0$.

Case 2: $e(\pi) \uparrow e(\pi)^s$.
Then $T(e^s(\pi))$ is a direct summand of $\mathrm{T}^s T(e(\pi))$. Using a split mono fixed in (6.17.2), define

$$\begin{tikzcd} T(e^s(\pi)) \arrow[r, hook] \arrow[dr, dotted, "\phi_\pi^{\mathrm{T}^s M}"'] & \mathrm{T}^s T(e(\pi)) \arrow[d, "\mathrm{T}^s(\phi_\pi^M)"] \\ & \mathrm{T}^s M. \end{tikzcd}$$

To see that $\phi_\pi^{\mathrm{T}^s M} \neq 0$, put $\lambda = e(\pi)$ and $\mu = e^s(\pi)$, and take $\Omega = \Lambda_0^+ \cap (\uparrow \lambda)$. As $\mathrm{im}(\phi_\pi^M) = \nabla(\lambda)$ mod $\Gamma_{\Omega \setminus \lambda} M$, $[\mathrm{im}(\phi_\pi^M) : L(\lambda)] = 1$, and hence

$$[\mathrm{im}\mathrm{T}^s(\phi_\pi^M) : L(\mu)] = [\mathrm{T}^s \mathrm{im}(\phi_\pi^M) : \mathrm{T}^s L(\lambda)] = 1 = [\mathrm{T}^s T(\lambda) : L(\mu)] = [T(\mu) : L(\mu)].$$

One must therefore have $[\mathrm{im}\phi_\pi^{\mathrm{T}^s M} : L(\mu)] = [\mathrm{im}\mathrm{T}^s \phi_\pi^M : L(\mu)] = 1$.

Proposition [RW, Prop. 3.3.2]: $(\Pi^s, e^s, (\phi_\pi^{\mathrm{T}^s M} | \pi \in \Pi^s))$ *constructed above gives a section of the* ∇*-flag of* $\mathrm{T}^s M$.

Proof: We are to show that, $\forall \mu \in \Lambda_s^+$, the image of $(\phi_\pi^{\mathrm{T}^s M} | \pi \in (e^s)^{-1}(\mu))$ forms a basis of $\mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s M)$. In particular,

$$|(e^s)^{-1}(\mu)| = \dim \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s M) = (\mathrm{T}^s M : \nabla(\mu)).$$

Assume first that $M \simeq \nabla(\lambda)^{\oplus|\Pi|}$ for some $\lambda \in \Lambda_0^+$. $\forall \pi \in \Pi$, put $M_\pi = \mathrm{im}(\phi_\pi^M) \simeq \nabla(\lambda)$. Then $M = \coprod_{\pi \in \Pi} M_\pi$. and hence we may assume $M = \nabla(\lambda)$, $\Pi = \{\pi\}$, $e(\pi) = \lambda$ and $\phi_\pi^{\nabla(\lambda)} : T(\lambda) \to \nabla(\lambda)$ is the quotient. If $\lambda^s \notin \Lambda^+$, $\mathrm{T}^s \nabla(\lambda) = 0$, $\Pi^s = \emptyset$, and we are done. If $\lambda^s \in \Lambda^+$, $\Pi^s = \{\pi\}$. Put $\mu = e^s(\pi) \in \Lambda^+$. As $\dim \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s M) = \dim \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \nabla(\mu)) = 1$, the assertion follows from the fact that $\phi_\pi^{\mathrm{T}^s \nabla(\lambda)} \neq 0$.

In general, we may assume $0 \neq \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s M)$ for some $\mu \in \Lambda_s^+$; otherwise $\mathrm{T}^s M = 0$ and $\Pi^s = \emptyset$. Then there is a unique $\lambda \in \Lambda_0^+$ with μ lying in the closure of the alcove containing λ such that $\lambda \uparrow \lambda^s$, in which case $\forall \pi \in \Pi^s$, $e^s(\pi) = \mu$ iff $e(\pi) \in \{\lambda, \lambda^s\}$. Thus,

$$(e^s)^{-1}(\mu) = e^{-1}(\lambda) \sqcup e^{-1}(\lambda^s).$$

Let $\Omega = \Lambda_0^+ \cap (\uparrow \lambda)$, $\Omega' = \Omega \setminus \{\lambda\}$, and $\Omega'' = \Omega \cup \{\lambda^s\}$. Thus, $\Omega, \Omega', \Omega'' \trianglelefteq \Lambda_0^+$, $\Gamma_{\Omega'} M \hookrightarrow \Gamma_\Omega M \hookrightarrow \Gamma_{\Omega''} M \hookrightarrow M$ with $\Gamma_\Omega M / \Gamma_{\Omega'} M \simeq \nabla(\lambda)^{\oplus(M:\nabla(\lambda))} = \nabla(\lambda)^{\oplus|e^{-1}(\lambda)|}$ and $\Gamma_{\Omega''} M / \Gamma_\Omega M \simeq \nabla(\lambda^s)^{\oplus(M:\nabla(\lambda^s))} = \nabla(\lambda^s)^{\oplus|e^{-1}(\lambda^s)|}$. Then $\mathrm{T}^s(\Gamma_{\Omega'} M) \hookrightarrow \mathrm{T}^s(\Gamma_\Omega M) \hookrightarrow \mathrm{T}^s(\Gamma_{\Omega''} M) \hookrightarrow \mathrm{T}^s M$ with

(1) $\mathrm{T}^s(\Gamma_\Omega M)/\Gamma_{\Omega'} M) \simeq \mathrm{T}^s(\Gamma_\Omega M)/\mathrm{T}^s(\Gamma_{\Omega'} M) \simeq \nabla(\mu)^{\oplus|e^{-1}(\lambda)|}$ and

$$\mathrm{T}^s(\Gamma_{\Omega''} M)/\Gamma_\Omega M) \simeq \mathrm{T}^s(\Gamma_{\Omega''} M)/\mathrm{T}^s(\Gamma_\Omega M) \simeq \nabla(\mu)^{\oplus|e^{-1}(\lambda^s)|}.$$

A short exact sequence $0 \to \Gamma_{\Omega''}M \hookrightarrow M \to M/\Gamma_{\Omega''}M \to 0$ induces by (6.8), as T^s is exact, another short exact sequence

$$
\begin{aligned}
(2) \quad 0 \to \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s(\Gamma_{\Omega''}M)) \to \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s M) \\
\to \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s(M/\Gamma_{\Omega''}M)) \to 0.
\end{aligned}
$$

As $(\mathrm{T}^s(M/\Gamma_{\Omega''}M) : \nabla(\mu)) = 0$, one has

$$\mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s(\Gamma_{\Omega''}M)) \xrightarrow{\sim} \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s M).$$

By (6.15) all ϕ_π^M, $\pi \in e^{-1}(\lambda)$ (resp. $e^{-1}(\lambda^s)$), factor through $\Gamma_\Omega M$ (resp. $\Gamma_{\Omega''}M$):

$$
\begin{tikzcd}
T(\lambda) \arrow[r, "\phi_\pi^M"] \arrow[dr, dotted, "\phi_\pi^{\Gamma_\Omega M}"'] & M \\
& \Gamma_\Omega M \arrow[u, hook]
\end{tikzcd}
\quad (\text{resp.} \quad
\begin{tikzcd}
T(\lambda^s) \arrow[r, "\phi_\pi^M"] \arrow[dr, dotted, "\phi_\pi^{\Gamma_{\Omega''} M}"'] & M \\
& \Gamma_{\Omega''} M \arrow[u, hook]
\end{tikzcd}
\quad)
$$

with $(\phi_\pi^{\Gamma_\Omega M} | \pi \in e^{-1}(\lambda))$ (resp. $(\phi_\pi^{\Gamma_\Omega M} | \pi \in e^{-1}(\lambda^s))$) giving a $\Bbbk$-linear basis of $\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\lambda), \Gamma_\Omega M)$ (resp. $\mathrm{Rep}_0(G)^{\downarrow\lambda^s}(T(\lambda^s), \Gamma_{\Omega''}M)$). In particular, all ϕ_π^M, $\pi \in e^{-1}(\lambda) \sqcup e^{-1}(\lambda^s)$, factor through $\Gamma_{\Omega''}M$. By construction in Case 2 (resp. Case 1) one has a commutative diagram

$$
\begin{tikzcd}
& \mathrm{T}^s T(\lambda) \arrow[d, "\mathrm{T}^s(\phi_\pi^M)"'] \arrow[dd, bend left=60, "\mathrm{T}^s(\phi_\pi^{\Gamma_\Omega M})"] \\
T(\mu) \arrow[ur, hook] \arrow[r, "\phi_\pi^{\mathrm{T}^s M}"'] \arrow[dr, "\phi_\pi^{\mathrm{T}^s(\Gamma_\Omega M)}"'] & \mathrm{T}^s M \\
& \mathrm{T}^s(\Gamma_\Omega M) \arrow[u, hook]
\end{tikzcd}
\quad (\text{resp.} \quad
\begin{tikzcd}
T(\mu) \arrow[r, "\phi_\pi^{\mathrm{T}^s M}"] \arrow[d, "\mathrm{adj}"'] \arrow[rr, bend left, "\phi_\pi^{\mathrm{T}^s(\Gamma_{\Omega''} M)}"] & \mathrm{T}^s M & \mathrm{T}^s(\Gamma_{\Omega''}M) \arrow[l, hook'] \\
\mathrm{T}^s \mathrm{T}_s T(\mu) \arrow[r, "\sim"'] & \mathrm{T}^s T(\lambda^s) \arrow[u, "\mathrm{T}^s(\phi_\pi^M)"] \arrow[ur, "\mathrm{T}^s(\phi_\pi^{\Gamma_{\Omega''} M})"'] &
\end{tikzcd}
\quad);
$$

if we write $T(\mu) \overset{i}{\hookrightarrow} \mathrm{T}^s T(\lambda)$ and $\mathrm{T}^s(\Gamma_\Omega M) \overset{i'}{\hookrightarrow} \mathrm{T}^s M$,

$$i' \circ \phi_\pi^{\mathrm{T}^s(\Gamma_\Omega M)} = i' \circ \mathrm{T}^s(\phi_\pi^{\Gamma_\Omega M}) \circ i = \mathrm{T}^s(\phi_\pi^M) \circ i = \phi_\pi^{\mathrm{T}^s M}.$$

Thus, all $\phi_\pi^{\mathrm{T}^s M}$, $\pi \in e^{-1}(\mu)$, factor through $\mathrm{T}^s(\Gamma_{\Omega''}M)$. It suffices then by (3) to show that $(\phi_\pi^{\mathrm{T}^s(\Gamma_{\Omega''}M)} | \pi \in (e^s)^{-1}(\mu))$ forms a $\Bbbk$-linear basis of $\mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s(\Gamma_{\Omega''}M))$, i.e., we may now assume that $M = \Gamma_{\Omega''}M$.

Consider next a short exact sequence $0 \to \Gamma_{\Omega'}M \to M \to M/\Gamma_{\Omega'}M \to 0$. As in (2) one obtains a short exact sequence

$$
\begin{aligned}
0 \to \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s(\Gamma_{\Omega'}M)) \to \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s M) \\
\to \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s(M/\Gamma_{\Omega'}M)) \to 0.
\end{aligned}
$$

As $(\mathrm{T}^s(M/\Gamma_{\Omega'}M) : \nabla(\mu)) = 0$, one has

$$
\begin{aligned}
\mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s M) &\xrightarrow{\sim} \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s(M/\Gamma_{\Omega'}M)) \\
&\simeq \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s M/\mathrm{T}^s(\Gamma_{\Omega'}M)).
\end{aligned}
$$

Denoting the image of each $\phi_\pi^{\mathrm{T}^s M}$ by $\overline{\phi_\pi^{\mathrm{T}^s M}}$, it now suffices to show that $(\overline{\phi_\pi^{\mathrm{T}^s M}}|\pi \in (e^s)^{-1}(\mu))$ forms a $\Bbbk$-linear basis of $\mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s M/\mathrm{T}^s(\Gamma_{\Omega'} M))$. One has

$$(\overline{\phi_\pi^{\mathrm{T}^s M}}|\pi \in (e^s)^{-1}(\mu)) = (\overline{\phi_\pi^{\mathrm{T}^s(\Gamma_\Omega M)}}|\pi \in e^{-1}(\lambda)) \sqcup (\overline{\phi_\pi^{\mathrm{T}^s M}}|\pi \in e^{-1}(\lambda^s)),$$

the union on the RHS being disjoint from a short exact sequence

$$\begin{aligned} 0 \to \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s(\Gamma_\Omega M/\Gamma_{\Omega'} M)) \to \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s(M/\Gamma_{\Omega'} M)) \\ \to \mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s(M/\Gamma_\Omega M)) \to 0. \end{aligned}$$

By (6.16), $(\phi_\pi^{\Gamma_\Omega M/\Gamma_{\Omega'} M}|\pi \in e^{-1}(\lambda))$ (resp. $(\phi_\pi^{M/\Gamma_\Omega M}|\pi \in e^{-1}(\lambda^s))$) gives a $\Bbbk$-linear basis of $\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\lambda), \Gamma_\Omega M/\Gamma_{\Omega'} M)$ (resp. $\mathrm{Rep}_0(G)^{\downarrow\lambda^s}(T(\lambda^s), M/\Gamma_\Omega M)$). By construction in Case 2

$$\overline{\phi_\pi^{\mathrm{T}^s M}} = \begin{cases} \phi_\pi^{\mathrm{T}^s(\Gamma_\Omega M/\Gamma_{\Omega'} M)} & \text{if } \pi \in e^{-1}(\lambda), \\ \phi_\pi^{\mathrm{T}^s(M/\Gamma_\Omega M)} & \text{if } \pi \in e^{-1}(\lambda^s); \end{cases}$$

one has a commutative diagram

$$\begin{tikzcd} T(\mu) \arrow[r, hook] \arrow[rr, bend left, "\phi_\pi^{\mathrm{T}^s(\Gamma_\Omega M)}"] \arrow[dr, "\phi_\pi^{\mathrm{T}^s(\Gamma_\Omega M/\Gamma_{\Omega'} M)}"'] & \mathrm{T}^s T(\lambda) \arrow[r, "\mathrm{T}^s(\phi_\pi^{\Gamma_\Omega M})"] \arrow[d, "\mathrm{T}^s(\phi_\pi^{\Gamma_\Omega M/\Gamma_{\Omega'} M})"] & \mathrm{T}^s(\Gamma_\Omega M) \arrow[dl, bend left] \\ & \mathrm{T}^s(\Gamma_\Omega M/\Gamma_{\Omega'} M) & \end{tikzcd}$$

We are finally reduced to showing that $(\phi_\pi^{\mathrm{T}^s(\Gamma_\Omega M/\Gamma_{\Omega'} M)}|\pi \in e^{-1}(\lambda))$ (resp. $\phi_\pi^{\mathrm{T}^s(M/\Gamma_\Omega M)}|\pi \in e^{-1}(\lambda^s)$)) forms a basis of $\mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s(\Gamma_\Omega M/\Gamma_{\Omega'} M))$ (resp. $\mathrm{Rep}_s(G)^{\downarrow\mu}(T(\mu), \mathrm{T}^s(M/\Gamma_\Omega M))$). This has, however, already been done at the outset as $\Gamma_\Omega M/\Gamma_{\Omega'} M \simeq \nabla(\lambda)^{\oplus|e^{-1}(\lambda)|}$ (resp. $M/\Gamma_\Omega M \simeq \nabla(\lambda^s)^{\oplus|e^{-1}(\lambda^s)|}$).

(6.21) Consider next the case $M \in \mathrm{Rep}_s(G)$, $s \in \mathcal{S}_a$, with a ∇-filtration. Out of a section $(\Pi, e, (\phi_\pi^M|\pi \in \Pi))$ of the ∇-flag of M we will construct a section of the ∇-flag of $\mathrm{T}_s M$.

Put $\Pi' = \Pi \times \{0,1\}$ and define a map $e' : \Pi' \to \Lambda_0^+$ as follows: $\forall \pi \in \Pi$, $e'(\pi,0)$ and $e'(\pi,1)$ are such that $e'(\pi,0) \uparrow e'(\pi,1) = e'(\pi,0)^s$ and that $e(\pi)$ belongs to the closure of the alcove containing $e'(\pi,0)$. Recall from (6.17.1) the isomorphism $\mathrm{T}_s T(e(\pi)) \simeq T(e'(\pi,1))$, and define

$$\begin{tikzcd} T(e'(\pi,1)) \arrow[r, "\sim"] \arrow[dr, dotted, "\phi_{(\pi,1)}^{\mathrm{T}_s M}"'] & \mathrm{T}_s T(e(\pi)) \arrow[d, "\mathrm{T}_s(\phi_\pi^M)"] \\ & \mathrm{T}_s M. \end{tikzcd}$$

Recall also from (6.17.2) the projection $\mathrm{T}^s T(e'(\pi,0)) \twoheadrightarrow T(e(\pi))$, and define

$$\begin{tikzcd} T(e'(\pi,0)) \arrow[r, "\mathrm{adj}"] \arrow[d, dotted, "\phi_{(\pi,0)}^{\mathrm{T}_s M}"'] & \Theta_s T(e'(\pi,0)) \arrow[d, two heads] \\ \mathrm{T}_s M & \mathrm{T}_s T(e(\pi)). \arrow[l, "\mathrm{T}_s \phi_\pi^M"] \end{tikzcd}$$

As $\phi^{\mathrm{T}_s M}_{(\pi,0)}$ corresponds to the composite $\mathrm{T}^s T(e'(\pi,0)) \twoheadrightarrow T(e(\pi)) \xrightarrow{\phi^M_\pi} M$ under the isomorphism $\mathrm{Rep}(G)(T(e'(\pi,0)), \mathrm{T}_s M) \simeq \mathrm{Rep}(G)(\mathrm{T}^s T(e'(\pi,0)), M)$, it remains nonzero.

Proposition [RW, Prop. 3.4.2]: $(\Pi', e', (\phi^{\mathrm{T}_s M}_{\pi'} | \pi' \in \Pi'))$ *constructed above forms a section of the* ∇*-flag of* $\mathrm{T}_s M$.

Proof: Consider first the case $M = \nabla(\mu)^{\oplus|\Pi|}$ for some $\mu \in \Lambda_s^+$. As in (6.20) we may assume $M = \nabla(\mu)$. Thus we may assume that $\Pi = \{\pi\}, e(\pi) = \mu$, and that $\phi^{\nabla(\mu)}_\pi : T(\mu) \to \nabla(\mu)$ is the quotient. Put $\lambda = e'(\pi,0) \uparrow \lambda^s = e'(\pi,1)$. By definition

$$\begin{tikzcd} T(\lambda) \arrow[r, "\mathrm{adj}"] \arrow[d, dotted, "\phi^{\mathrm{T}_s\nabla(\mu)}_{(\pi,0)}"'] & \Theta_s T(\lambda) \arrow[d, two heads] \\ \mathrm{T}_s\nabla(\mu) & \mathrm{T}_s T(\mu), \arrow[l, two heads, "\mathrm{T}^s(\phi^{\nabla(\mu)}_\mu)"] \end{tikzcd} \qquad \begin{tikzcd} T(\lambda^s) \arrow[r, "\sim"] \arrow[dr, dotted, "\phi^{\mathrm{T}_s\nabla(\mu)}_{(\pi,1)}"'] & \mathrm{T}_s T(\mu) \arrow[d, two heads, "\mathrm{T}_s(\phi^{\nabla(\mu)}_\mu)"] \\ & \mathrm{T}_s\nabla(\mu). \end{tikzcd}$$

On the other hand, one has from (6.14.2)

$$\begin{aligned} \dim \mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\lambda), \mathrm{T}_s\nabla(\mu)) &= (\mathrm{T}_s\nabla(\mu) : \nabla(\lambda)) = 1 \\ &= (\mathrm{T}_s\nabla(\mu) : \nabla(\lambda^s)) = \dim \mathrm{Rep}_0(G)^{\downarrow\lambda^s}(T(\lambda^s), \mathrm{T}_s\nabla(\mu)). \end{aligned}$$

The assertion follows.

In general, let $\mu \in \mathrm{im}(e)$, $\pi \in e^{-1}(\mu)$, and put $\lambda = e'(\pi,0) \uparrow \lambda^s = e'(\pi,1)$. Let $\Omega = (\uparrow \mu)$ and $\Omega' = \Omega \setminus \{\mu\}$. Thus, $\Gamma_{\Omega'} M \hookrightarrow \Gamma_\Omega M \hookrightarrow M$ and $\Gamma_\Omega M / \Gamma_{\Omega'} M \simeq \nabla(\mu)^{\oplus|e^{-1}(\mu)|}$. As $\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\lambda), \mathrm{T}_s(M/\Gamma_\Omega M)) = 0 = \mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\lambda), \mathrm{T}_s(\Gamma_{\Omega'} M))$, one has

(1) $$\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\lambda), \mathrm{T}_s(\Gamma_\Omega M)) \simeq \mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\lambda), \mathrm{T}_s M),$$

(2) $$\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\lambda), \mathrm{T}_s(\Gamma_\Omega M)) \simeq \mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\lambda), \mathrm{T}_s(\Gamma_\Omega M/\Gamma_{\Omega'} M)).$$

Likewise,

(3) $$\mathrm{Rep}_0(G)^{\downarrow\lambda^s}(T(\lambda^s), \mathrm{T}_s(\Gamma_\Omega M)) \simeq \mathrm{Rep}_0(G)^{\downarrow\lambda^s}(T(\lambda), \mathrm{T}_s M),$$

(4) $$\mathrm{Rep}_0(G)^{\downarrow\lambda^s}(T(\lambda^s), \mathrm{T}_s(\Gamma_\Omega M)) \simeq \mathrm{Rep}_0(G)^{\downarrow\lambda^s}(T(\lambda^s), \mathrm{T}_s(\Gamma_\Omega M/\Gamma_{\Omega'} M)).$$

By (6.15) with

$$\begin{tikzcd} T(\mu) \arrow[r, "\phi^M_\pi"] \arrow[dr, dotted, "\phi^{\Gamma_\Omega M}_\pi"'] & M \\ & \Gamma_\Omega M \arrow[u, hook] \end{tikzcd}$$

$(e^{-1}(\mu), e|_{e^{-1}(\mu)}, (\phi^{\Gamma_\Omega M}_\pi | \pi \in e^{-1}(\mu)))$ forms a section of the ∇-flag of $\Gamma_\Omega M$. In turn, by (6.16) with

$$\begin{tikzcd} T(\mu) \arrow[r, "\phi^{\Gamma_\Omega M}_\pi"] \arrow[dr, dotted, "\phi^{\Gamma_\Omega M/\Gamma_{\Omega'} M}_\pi"'] & \Gamma_\Omega M \arrow[d, two heads] \\ & \Gamma_\Omega M/\Gamma_{\Omega'} M \end{tikzcd}$$

$(e^{-1}(\mu), e|_{e^{-1}(\mu)}, (\phi_\pi^{\Gamma_\Omega M/\Gamma_{\Omega'}M} | \pi \in e^{-1}(\mu)))$ forms a section of the ∇-flag of $\Gamma_\Omega M/\Gamma_{\Omega'}M$.

By above, corresponding to $(\phi_\pi^{\Gamma_\Omega M/\Gamma_{\Omega'}M} | \pi \in e^{-1}(\mu))$, $(\phi_{(\pi,0)}^{\mathrm{T}_s(\Gamma_\Omega M/\Gamma_{\Omega'}M)} \in \mathrm{Rep}_0(G)(T(\lambda), \mathrm{T}_s(\Gamma_\Omega M/\Gamma_{\Omega'}M)) | \pi \in e^{-1}(\mu))$ (resp. $(\phi_{(\pi,1)}^{\mathrm{T}_s(\Gamma_\Omega M/\Gamma_{\Omega'}M)} \in \mathrm{Rep}_0(G)(T(\lambda^s), \mathrm{T}_s(\Gamma_\Omega M/\Gamma_{\Omega'}M)) | \pi \in e^{-1}(\mu)))$ induces a $\Bbbk$-linear basis of $\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\lambda), \mathrm{T}_s(\Gamma_\Omega/\Gamma_{\Omega'}M))$ (resp. $\mathrm{Rep}_0(G)^{\downarrow\lambda^s}(T(\lambda^s), \mathrm{T}_s(\Gamma_\Omega/\Gamma_{\Omega'}M))$). Then, by (2) (resp. (4)), corresponding to $(\phi_\pi^{\Gamma_\Omega M} | \pi \in e^{-1}(\mu))$, $(\phi_{(\pi,0)}^{\mathrm{T}_s(\Gamma_\Omega M)} \in \mathrm{Rep}_0(G)(T(\lambda), \mathrm{T}_s(\Gamma_\Omega M)) | \pi \in e^{-1}(\mu))$ (resp. $(\phi_{(\pi,1)}^{\mathrm{T}_s(\Gamma_\Omega M)} \in \mathrm{Rep}_0(G)(T(\lambda^s), \mathrm{T}_s(\Gamma_\Omega M)) | \pi \in e^{-1}(\mu)))$ gives a $\Bbbk$-linear basis of $\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\lambda), \mathrm{T}_s(\Gamma_\Omega))$ (resp. $\mathrm{Rep}_0(G)^{\downarrow\lambda^s}(T(\lambda^s), \mathrm{T}_s(\Gamma_\Omega)))$. Finally, by (1) (resp. (3)), corresponding to $(\phi_\pi^M | \pi \in e^{-1}(\mu))$, $(\phi_{(\pi,0)}^{\mathrm{T}_s M} | \pi \in e^{-1}(\mu))$ (resp. $(\phi_{(\pi,1)}^{\mathrm{T}_s M} | \pi \in e^{-1}(\mu)))$ gives a $\Bbbk$-linear basis of $\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\lambda), \mathrm{T}_s M)$ (resp. $\mathrm{Rep}_0(G)^{\downarrow\lambda^s}(T(\lambda^s), \mathrm{T}_s M))$, as desired.

(6.22) We now consider the wall-crossing functor $\Theta_s = \Theta_s : \mathrm{Rep}_0(G) \to \mathrm{Rep}_0(G)$, $s \in \mathcal{S}_a$. If $\underline{w} = (s_1, \dots, s_m)$ is a reduced expression of $w \in {}^f\mathcal{W}$, $T(w \bullet 0)$ is a direct summand of multiplicity 1 in $T(\underline{w}) = \Theta_{s_m} \dots \Theta_{s_1} T(0)$.

Proposition [RW, Prop. 3.5.1]: *Let $s \in \mathcal{S}_a$, $\underline{x}$ a reduced expression of $x \in {}^f\mathcal{W}$ and $\underline{v}$ an arbitrary expression. Put $\lambda = x \bullet 0$ and $\lambda^s = xs \bullet 0$. Let us denote the quotients $\mathrm{Rep}_0(G) \to \mathrm{Rep}_0(G)^{\downarrow\lambda}$ and $\mathrm{Rep}_0(G) \to \mathrm{Rep}_0(G)^{\downarrow\lambda^s}$ by $\bar{?}$.*

(i) Assume $\lambda \uparrow \lambda^s$. Thus $\underline{xs}$ is a reduced expression of $xs \in {}^f\mathcal{W}$. Let I be a finite set, $f_i \in \mathrm{Rep}_0(G)(T(\underline{x}), T(\underline{v}))$, $i \in I$, such that $\sum_{i \in I} \Bbbk \bar{f}_i = \mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\underline{x}), T(\underline{v}))$; such exist by (6.11). Let J be a finite set, $g_j \in \mathrm{Rep}_0(G)(T(\underline{xs}), T(\underline{v}))$, $j \in J$, such that $\sum_{j \in J} \Bbbk \bar{g}_j = \mathrm{Rep}_0(G)^{\downarrow\lambda^s}(T(\underline{xs}), T(\underline{v}))$. Then there exist $f'_i \in \mathrm{Rep}_0(G)(T(\underline{x}), \Theta_s T(\underline{x}))$, $i \in I$, and $g'_j \in \mathrm{Rep}_0(G)(T(\underline{x}), \Theta_s T(\underline{xs}))$, $j \in J$, such that

$$\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\underline{x}), T(\underline{vs})) = \sum_{i \in I} \Bbbk \overline{\Theta_s(f_i) \circ f'_i} + \sum_{j \in J} \Bbbk \overline{\Theta_s(g_j) \circ g'_j}.$$

(ii) Assume that $\underline{x} = \underline{ys}$ for some reduced expression $\underline{y}$ of $y \in {}^f\mathcal{W}$. Thus $\lambda^s = y \bullet 0 \in \Lambda_0^+$ with $\lambda^s \uparrow \lambda$. Let I be a finite set, $f_i \in \mathrm{Rep}_0(G)(T(\underline{x}), T(\underline{v}))$, $i \in I$, such that $\sum_{i \in I} \Bbbk \bar{f}_i = \mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\underline{x}), T(\underline{v}))$. Let J be a finite set, $g_j \in \mathrm{Rep}_0(G)(T(\underline{y}), T(\underline{v}))$, $j \in J$, such that $\sum_{j \in J} \Bbbk \bar{g}_j = \mathrm{Rep}_0(G)^{\downarrow\lambda^s}(T(\underline{y}), T(\underline{v}))$. Then there exist $f'_i \in \mathrm{Rep}_0(G)(T(\underline{x}), \Theta_s T(\underline{x}))$, $i \in I$, and $g'_j \in \mathrm{Rep}_0(G)(T(\underline{x}), \Theta_s T(\underline{y}))$, $j \in J$, such that

$$\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\underline{x}), T(\underline{vs})) = \sum_{i \in I} \Bbbk \overline{\Theta_s(f_i) \circ f'_i} + \sum_{j \in J} \Bbbk \overline{\Theta_s(g_j) \circ g'_j}.$$

Proof: (i) One has $T(\underline{x}) \simeq T(x)$ in $\mathrm{Rep}_0(G)^{\downarrow\lambda}$ and $T(\underline{xs}) \simeq T(xs)$ in $\mathrm{Rep}_0(G)^{\downarrow\lambda^s}$. Fix split monos $\iota : T(\lambda) \hookrightarrow T(\underline{x})$ and $\iota^s : T(\lambda^s) \hookrightarrow T(\underline{xs})$. By shrinking I if necessary, we may assume that $f_i \circ \iota \in \mathrm{Rep}_0(G)(T(\lambda), T(\underline{v}))$, $i \in I$, constitute the part of a section, with domain $T(\lambda)$, of the ∇-flag of $T(\underline{v})$. Likewise for $g_j \circ \iota^s \in \mathrm{Rep}_0(G)(T(\lambda^s), T(\underline{v}))$.

Let $\mu \in \Lambda_s^+$ belonging to the closures of the alcove containing λ. If $\iota^\mu : T(\mu) \hookrightarrow \mathrm{T}^s T(\lambda)$ is the fixed mono, one has from (6.20) that

$$\begin{tikzcd} T(\mu) \arrow[r, dotted] \arrow[d, hook, "\iota^\mu"'] & \mathrm{T}^s T(\underline{v}), & i \in I, \\ \mathrm{T}^s T(\lambda) \arrow[r, "\mathrm{T}^s(\iota)"'] & \mathrm{T}^s T(\underline{x}) \arrow[u, "\mathrm{T}^s(f_i)"'] & \end{tikzcd}$$

together with

$$\begin{tikzcd} T(\mu) \arrow[rr, dotted] \arrow[d, "\mathrm{adj}"'] & & \mathrm{T}^s T(\underline{v}), & j \in J, \\ \mathrm{T}^s \mathrm{T}_s T(\mu) \arrow[r, dash, "\sim"'] & \mathrm{T}^s T(\lambda^s) \arrow[r, "\mathrm{T}^s(\iota^s)"'] & \mathrm{T}^s T(\underline{xs}) \arrow[u, "\mathrm{T}^s(g_j)"'] & \end{tikzcd}$$

form the part of a section, with domain $T(\mu)$, of the ∇-flag of $\mathrm{T}^s T(\underline{v})$. Then by (6.21)

$$\begin{tikzcd} T(\lambda) \arrow[rr, dotted] \arrow[d, "\mathrm{adj}"'] & & \Theta_s T(\underline{v}) = \Theta_s T(\underline{v}), & i \in I, \\ \Theta_s T(\lambda) = \Theta_s T(\lambda) \arrow[r, two heads] & \mathrm{T}_s T(\mu) \arrow[r, "\mathrm{T}_s(\mathrm{T}^s(\iota)\circ\iota^\mu)"'] & \Theta_s T(\underline{x}) = \Theta_s T(\underline{x}) \arrow[u, "\Theta_s(f_i)=\Theta_s(f_i)"'] & \end{tikzcd}$$

together with

$$\begin{tikzcd} T(\lambda) \arrow[rr, dotted] \arrow[d, "\mathrm{adj}"'] & & \Theta_s T(\underline{v}) = \Theta_s T(\underline{v}), & j \in J, \\ \Theta_s T(\lambda) = \Theta_s T(\lambda) \arrow[r, two heads] & \mathrm{T}_s T(\mu) \arrow[r, "\mathrm{T}_s(\mathrm{T}^s(\iota^s)\circ\mathrm{adj})"'] & \Theta_s T(\underline{xs}) = \Theta_s T(\underline{xs}) \arrow[u, "\Theta_s(g_j)=\Theta_s(g_j)"'] & \end{tikzcd}$$

form the part of a section, with domain $T(\lambda)$, of the ∇-flag of $\Theta_s T(\underline{v}) = T(\underline{vs})$. Thus, taking

$$\begin{tikzcd} T(\lambda) \arrow[r, "\mathrm{adj}"] & \Theta_s T(\lambda) \arrow[r, two heads] & \mathrm{T}_s T(\mu) \arrow[d, "\mathrm{T}_s(\mathrm{T}^s(\iota)\circ\iota^\mu)"] & \\ T(\underline{x}) \arrow[u, two heads] \arrow[rr, dotted, "f'_i"'] & & \Theta_s T(\underline{x}), & i \in I, \end{tikzcd}$$

and

$$\begin{tikzcd} T(\lambda) \arrow[r, "\mathrm{adj}"] & \Theta_s T(\lambda) \arrow[r, two heads] & \mathrm{T}_s T(\mu) \arrow[d, "\mathrm{T}_s(\mathrm{T}^s(\iota^s)\circ\mathrm{adj})"] & \\ T(\underline{x}) \arrow[u, two heads] \arrow[rr, dotted, "g'_j"'] & & \Theta_s T(\underline{xs}), & j \in J, \end{tikzcd}$$

will do. Likewise (ii).

(6.23) Recall from (5.11) a rex move between reduced expressions of an element of $\mathcal{W}_a$, a path from one to the other by consecutive applications of braid relations.

Lemma [RW, Lem. 5.2.2]: *Let $\underline{x}, \underline{y}$ be 2 reduced expressions of $w \in {}^f\mathcal{W}$. Let $\underline{x} \rightsquigarrow \underline{y}$ be a rex move and let $\phi_{\underline{x},\underline{y}} \in \mathcal{D}_{\mathrm{BS}}(B_{\underline{x}}, B_{\underline{y}})$ be the associated morphism. If $\lambda = w \bullet 0$, $\tilde{\Psi}(\phi_{\underline{x},\underline{y}}) \in \mathrm{Tilt}_0(G)^{\downarrow\lambda}(T(\underline{x}), T(\underline{y}))$ is invertible.*

Proof: Let $\underline{y} \rightsquigarrow \underline{x}$ be the rex move reversing $\underline{x} \rightsquigarrow \underline{y}$, and let $\phi_{\underline{y},\underline{x}} \in \mathcal{D}_{\mathrm{BS}}(B_{\underline{y}}, B_{\underline{x}})$ be the associated morphism. By (5.11) one can write $\phi_{\underline{y},\underline{x}} \circ \phi_{\underline{x},\underline{y}} = \mathrm{id}_{B_{\underline{x}}} + \sum_{j\in J} \phi_j$ for some finite set J such that each $\phi_j \in \mathcal{D}_{\mathrm{BS}}(B_{\underline{x}}, B_{\underline{x}})$ factors through some $B_{\underline{z}_j}\langle k_j\rangle$ with $\ell(\underline{z}_j) \leq \ell(w) - 2$ and $k_j \in \mathbb{Z}$

$$\begin{tikzcd} B_{\underline{x}} \arrow[rr, "\phi_j"] \arrow[dr, dotted] & & B_{\underline{x}}. \\ & B_{\underline{z}_j}\langle k_j\rangle \arrow[ur, dotted] & \end{tikzcd}$$

As $\tilde{\Psi}(B_{\underline{z}_j}\langle k_j\rangle) = T(\underline{z}_j) = 0$ in $\mathrm{Tilt}_0(G)^{\downarrow\lambda}$, $\tilde{\Psi}(\phi_{\underline{y},\underline{x}} \circ \phi_{\underline{x},\underline{y}}) = \mathrm{id}_{T(\underline{x})}$. Likewise $\tilde{\Psi}(\phi_{\underline{x},\underline{y}} \circ \phi_{\underline{y},\underline{x}}) = \mathrm{id}_{T(\underline{y})}$.

(6.24) $\forall s \in \mathcal{S}_a$, recall $\overset{s}{\downarrow\!\bullet} \in \mathcal{D}_{\mathrm{BS}}(B_\emptyset, B_s\langle 1\rangle)$. By construction (5.3), cf. (3.6), $\Psi(\overset{s}{\downarrow\!\bullet}) \in \mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G))^{\mathrm{op}}(\mathrm{id}_{\mathrm{Rep}_0(G)}, \Theta_s)$ is the unit associated to the adjunction $(\mathrm{T}^s, \mathrm{T}_s)$. Thus, $\forall M \in \mathrm{Rep}_0(G)$, under the isomorphism $\mathrm{Rep}_0(G)(M, \Theta_s M) \simeq \mathrm{Rep}_0(G)(\mathrm{T}^s M, \mathrm{T}^s M)$ one has $\Psi(\overset{s}{\downarrow\!\bullet})_M$ corresponding to $\mathrm{id}_{\mathrm{T}^s M}$. In particular, if $\mathrm{T}^s M \neq 0$, $\Psi(\overset{s}{\downarrow\!\bullet})_M \neq 0$, and hence

Lemma [RW, Lem. 5.2.3]: $\forall M \in \mathrm{Rep}_0(G)$ *with* $\Theta_s M \neq 0$,

$$\Psi(\overset{s}{\downarrow\!\bullet})_M \in \mathrm{Rep}_0(G)(M, \Theta_s M) \setminus 0.$$

(6.25) For 2 expressions $\underline{x}, \underline{y}$ of elements of $\mathcal{W}_a$ let $\alpha_{\underline{x},\underline{y}} : \mathcal{D}^{\bullet}_{\mathrm{BS}}(B_{\underline{x}}, B_{\underline{y}}) \to \mathrm{Rep}_0(G)(T(\underline{x}), T(\underline{y}))$ denote the $\Bbbk$-linear map induced by $\tilde{\Psi}$: $\forall \phi \in \mathcal{D}_{\mathrm{BS}}(B_{\underline{x}}, B_{\underline{y}}\langle m\rangle)$, $\Psi(\phi) \in \mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G))^{\mathrm{op}}$ $(\Psi(B_{\underline{x}}), \Psi(B_{\underline{y}}\langle m\rangle))$ is a natural transformation from functor $\Psi(B_{\underline{x}})$ to functor $\Psi(B_{\underline{y}})$, and we set $\alpha_{\underline{x},\underline{y}}(\phi) = \tilde{\Psi}(\phi) = \Psi(\phi)_{T(0)} \in \mathrm{Rep}_0(G)(T(\underline{x}), T(\underline{y}))$. In case $\underline{x}$ is a reduced expression for an element $x \in {}^f\mathcal{W}$ and if $\lambda = x \bullet 0$, define

$$\begin{tikzcd} \mathcal{D}^{\bullet}_{\mathrm{BS}}(B_{\underline{x}}, B_{\underline{y}}) \arrow[r, "\alpha_{\underline{x},\underline{y}}"] \arrow[dr, dotted, "\beta_{\underline{x},\underline{y}}"'] & \mathrm{Rep}_0(G)(T(\underline{x}), T(\underline{y})) \arrow[d] \\ & \mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\underline{x}), T(\underline{y})). \end{tikzcd}$$

We first show

Lemma [RW, Lem. 5.3.2]: *Assume that $\underline{y}$ is a reduced expression of $y \in {}^f\mathcal{W}$. Let $s \in \mathcal{S}_a$ with $ys > y$ such that $ys \in {}^f\mathcal{W}$. Then $\beta_{\underline{ys},\underline{ys}}, \beta_{\underline{ys},\underline{yss}}, \beta_{\underline{y},\underline{ys}}, \beta_{\underline{y},\underline{yss}}$ are all surjective.*

Proof: By (5.9) one has $B_{\underline{yss}} \simeq B_{\underline{ys}}\langle 1\rangle \oplus B_{\underline{ys}}\langle -1\rangle$ in $\mathcal{D}$. Then, letting $\mathcal{D}^m_{\mathrm{BS}}(B_{\underline{x}}, B_{\underline{ys}}) =$

$\mathcal{D}_{\mathrm{BS}}(B_{\underline{x}}, B_{\underline{ys}}\langle -m\rangle)\ \forall m \in \mathbb{Z}$, one has for $\underline{x} \in \{\underline{y}, \underline{ys}\}$ a commutative diagram

$$\begin{tikzcd}
\mathcal{D}^{\bullet}_{\mathrm{BS}}(B_{\underline{x}}, B_{\underline{yss}}) \arrow[r, "\sim"] \arrow[d, "\alpha_{\underline{x},\underline{yss}}"'] & \mathcal{D}^{\bullet -1}_{\mathrm{BS}}(B_{\underline{x}}, B_{\underline{ys}}) \oplus \mathcal{D}^{\bullet +1}_{\mathrm{BS}}(B_{\underline{x}}, B_{\underline{ys}}) \arrow[d, "\alpha^{\oplus 2}_{\underline{x},\underline{ys}}"] \\
\mathrm{Rep}_0(G)(T(\underline{x}), T(\underline{yss})) \arrow[r, "\sim"'] & \mathrm{Rep}_0(G)(T(\underline{x}), T(\underline{ys}))^{\oplus 2}.
\end{tikzcd}$$

Thus, one has only to show both $\beta_{\underline{ys},\underline{ys}}$ and $\beta_{\underline{y},\underline{ys}}$ are surjective.

Put $\lambda = y \bullet 0$ and $\lambda^s = ys \bullet 0$. As $\mathrm{Rep}_0(G)^{\downarrow \lambda^s}(T(\underline{ys}), T(\underline{ys})) \simeq \mathrm{Rep}_0(G)^{\downarrow \lambda^s}(L(\lambda^s), L(\lambda^s)) \simeq \Bbbk$, $\mathrm{Rep}_0(G)^{\downarrow \lambda^s}(T(\underline{ys}), T(\underline{ys})) = \Bbbk \mathrm{id}_{T(\underline{ys})}$. As $\beta_{\underline{ys},\underline{ys}}(\mathrm{id}) = \mathrm{id}$, $\beta_{\underline{ys},\underline{ys}}$ is surjective.

Fix $f \in \mathrm{Rep}_0(G)(\Delta(\lambda), T(\underline{y})) \setminus 0$, which is the composite of inclusions $\Delta(\lambda) \hookrightarrow T(\lambda) \hookrightarrow T(\underline{y})$ and is unique up to $\Bbbk^{\times}$. Put $\eta = \Psi(\begin{array}{c} s \\ | \\ \bullet \end{array}) \in \mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G))(\mathrm{id}, \Theta_s)$, which is the unit of an adjoint pair $(\mathrm{T}^s, \mathrm{T}_s)$. Thus one has a commutative diagram

(1)
$$\begin{tikzcd}
\Delta(\lambda) \arrow[r, "\eta_{\Delta(\lambda)}"] \arrow[d, "f"'] & \Theta_s \Delta(\lambda) \arrow[d, "\Theta_s(f)"] \\
T(\underline{y}) \arrow[r, "\eta_{T(\underline{y})}"'] & \Theta_s T(\underline{y}) = T(\underline{ys}).
\end{tikzcd}$$

Note that f is invertible in $\mathrm{Rep}_0(G)^{\downarrow \lambda}$. As $\mathrm{coker}(\Theta_s f) \simeq \Theta_s \mathrm{coker}(f) = 0$ in $\mathrm{Rep}_0(G)^{\downarrow \lambda}$, $\Theta_s f$ is also invertible in $\mathrm{Rep}_0(G)^{\downarrow \lambda}$. As $\eta_{\Delta(\lambda)} \neq 0$ in $\mathrm{Rep}_0(G)^{\downarrow \lambda}$ by (6.18), in $\mathrm{Rep}_0(G)^{\downarrow \lambda}$ one has from (1) that

$$0 \neq \eta_{T(\underline{y})} = \beta_{\underline{y},\underline{ys}}(\mathrm{id}_{B_{\underline{y}}} * \begin{array}{c} s \\ | \\ \bullet \end{array}) = \beta_{\underline{y},\underline{ys}}(\begin{array}{c} B_{\underline{y}} B_{\underline{s}}\langle 1\rangle = B_{\underline{ys}}\langle 1\rangle \\ \uparrow \\ B_{\underline{y}} \end{array}).$$

Finally, $\mathrm{Rep}_0(G)^{\downarrow \lambda}(T(\underline{y}), T(\underline{ys})) \simeq \mathrm{Rep}_0(G)^{\downarrow \lambda}(\Delta(\lambda), \Theta_s T(\underline{y})) \simeq \mathrm{Rep}_0(G)^{\downarrow \lambda}(\Delta(\lambda), \Theta_s \nabla(\lambda))$ is of dimension 1, and hence $\beta_{\underline{y},\underline{ys}}$ is surjective.

(6.26) Keep the notation of (6.25). Although we need it only in the case of $\underline{x} = \emptyset$ for the proof of Th. 6.3,

Proposition [RW, Prop. 5.3.1]: *$\beta_{\underline{x},\underline{y}}$ is surjective.*

Proof: We argue by induction on $\ell(\underline{y})$. Put $\lambda = x \bullet 0$.

Assume first $\ell(\underline{y}) = 0$. Thus, $T(\underline{y}) = T(\emptyset) = T(0)$. Then $\mathrm{Rep}_0(G)^{\downarrow \lambda}(T(\underline{x}), T(\emptyset)) = 0$ unless $\underline{x} = \emptyset$ in which case $\mathrm{Rep}_0(G)^{\downarrow 0}(T(\emptyset), T(\emptyset)) = \Bbbk \mathrm{id}_{T(\emptyset)}$. As $\mathcal{D}^{\bullet}_{\mathrm{BS}}(B_{\emptyset}, B_{\emptyset}) \ni \mathrm{id}_{B_{\emptyset}}$ and as $\beta_{\emptyset,\emptyset}(\mathrm{id}_{B_{\emptyset}}) = \mathrm{id}_{T(\emptyset)}$, the assertion holds.

Assume next $\ell(\underline{y}) > 0$. Write $\underline{y} = \underline{vs}$ with some $s \in \mathcal{S}_a$. Then $T(\underline{y}) = \Theta_s T(\underline{v})$, and we assume the assertion holding with $T(\underline{v})$. put $\lambda^s = xs \bullet 0$.

Case 1: $\lambda^s \notin \Lambda^+$.
As $(T(\underline{y}) : \nabla(\lambda)) = \dim \mathrm{Rep}(G)(\Delta(\lambda), T(\underline{y})) = \dim \mathrm{Rep}(G)(\Theta_s \Delta(\lambda), T(\underline{v})) = 0$ and as $\mathrm{Rep}_0(G)^{\downarrow\lambda}$ forms a highest weight category,

$$\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\underline{x}), T(\underline{y})) = \mathrm{Rep}_0(G)^{\downarrow\lambda}(\Delta(\lambda), T(\underline{y})) = 0,$$

and there is nothing to prove.

Case 2: $\lambda^s \in \Lambda^+$ and $\lambda^s \uparrow \lambda$, which does not happen if $\ell(\underline{x}) = 0$.
One has $xs < x$ and $xs \in {}^f\mathcal{W}$. If $\underline{u}$ is a reduced expression of xs, there is a rex move from $\underline{x}$ to $\underline{us}$, and hence we may assume $\underline{x} = \underline{us}$ by (6.23). As $\ell(\underline{v}) < \ell(\underline{y})$, there are by the induction hypothesis $\{f_i | i \in I\} \subseteq \mathrm{im}(\alpha_{\underline{x},\underline{v}}) \leq \mathrm{Rep}_0(G)(T(\underline{x}), T(\underline{v}))$ such that $\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\underline{x}), T(\underline{v})) = \sum_{i \in I} \Bbbk \bar{f}_i$ and $\{g_j | j \in J\} \subseteq \mathrm{im}(\alpha_{\underline{u},\underline{v}})$ such that $\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\underline{u}), T(\underline{v})) = \sum_{j \in J} \Bbbk \bar{g}_j$. By (6.22) one has $f'_i \in \mathrm{Rep}_0(T(\underline{x}), T(\underline{xs}))$, $i \in I$, and $g'_j \in \mathrm{Rep}_0(T(\underline{x}), T(\underline{us}))$, $j \in J$, such that

$$\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\underline{x}), T(\underline{y})) = \sum_{i \in I} \Bbbk \overline{\Theta_s(f_i) \circ f'_i} + \sum_{j \in J} \Bbbk \overline{\Theta_s(g_j) \circ g'_j}.$$

By (6.25) applied to $\underline{u}$ both $\beta_{\underline{x},\underline{x}}$ and $\beta_{\underline{x},\underline{xs}}$ are surjective, and hence we may assume $f'_i \in \mathrm{im}(\alpha_{\underline{x},\underline{xs}})$ and $g'_j \in \mathrm{im}(\alpha_{\underline{x},\underline{x}})$ $\forall i \in I$, $\forall j \in J$. Then $\Theta_s(f_i) \circ f'_i$ and $\Theta_s(g_j) \circ g'_j \in \mathrm{im}(\alpha_{\underline{x},\underline{y}})$ $\forall i, j$, and hence $\beta_{\underline{x},\underline{y}}$ is surjective.

Case 3: $\lambda \uparrow \lambda^s$.
One has $xs > x$ and $xs \in {}^f\mathcal{W}$. By induction there are $\{f_i | i \in I\} \subseteq \mathrm{im}(\alpha_{\underline{x},\underline{v}})$ and $\{g_j | j \in J\} \subseteq \mathrm{im}(\alpha_{\underline{xs},\underline{v}})$ such that $\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\underline{x}), T(\underline{v})) = \sum_{i \in I} \Bbbk \bar{f}_i$ and $\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\underline{xs}), T(\underline{v})) = \sum_{j \in J} \Bbbk \bar{g}_j$. By (6.22) again one has $f'_i \in \mathrm{Rep}_0(T(\underline{x}), T(\underline{xs}))$, $i \in I$, and $g'_j \in \mathrm{Rep}_0(T(\underline{x}), T(\underline{xss}))$, $j \in J$, such that

$$\mathrm{Rep}_0(G)^{\downarrow\lambda}(T(\underline{x}), T(\underline{y})) = \sum_{i \in I} \Bbbk \overline{\Theta_s(f_i) \circ f'_i} + \sum_{j \in J} \Bbbk \overline{\Theta_s(g_j) \circ g'_j}.$$

By (6.25) applied to $\underline{x}$ both $\beta_{\underline{x},\underline{xs}}$ and $\beta_{\underline{x},\underline{xss}}$ are surjective, and hence we may assume $f'_i \in \mathrm{im}(\alpha_{\underline{x},\underline{xs}})$ and $g'_j \in \mathrm{im}(\alpha_{\underline{x},\underline{xss}})$ $\forall i \in I$, $\forall j \in J$. Then $\Theta_s(f_i) \circ f'_i, \Theta_s(g_j) \circ g'_j \in \mathrm{im}(\alpha_{\underline{x},\underline{y}})$ $\forall i, j$, and hence $\beta_{\underline{x},\underline{y}}$ is surjective.

(6.27) Specialization $v \mapsto 1$ yields $\mathcal{H} \rightsquigarrow \mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H} \simeq \mathbb{Z}[\mathcal{W}_a]$ and $\mathcal{M}^{\mathrm{asph}} \rightsquigarrow \mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{M}^{\mathrm{asph}} = \mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathrm{sgn} \otimes_{\mathcal{H}_f} \mathcal{H} \simeq \mathrm{sgn}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a]$, the last of which we will abbreviate as M^{asph}. Thus, M^{asph} has a $\mathbb{Z}$-basis $N'_w = 1 \otimes w$, $w \in {}^f\mathcal{W}$. For an expression $\underline{w} = \underline{s_1 \dots s_r}$ of an element $w \in {}^f\mathcal{W}$ put $\underline{N'_{\underline{w}}} = 1 \otimes (1 + s_1) \dots (1 + s_r)$ in M^{asph}.

Lemma [RW, Lem. 5.4.1, 5.4.2]: *If $\underline{w}$ is an expression of $w \in {}^f\mathcal{W}$,*

$$\dim \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_\emptyset, \bar{B}_{\underline{w}}) \leq (T(\underline{w}) : \nabla(0)).$$

Proof: Recall first from (5.5.3) that $\forall s \in \mathcal{S}_a$, $\forall w \in {}^f\mathcal{W}$,

$$N'_w \cdot (1 + s) = \begin{cases} N'_w + N'_{ws} & \text{if } ws \in {}^f\mathcal{W}, \\ 0 & \text{else.} \end{cases}$$

Then by the translation principle (1.10), under the isomorphism of abelian groups $M^{\mathrm{asph}} \to [\mathrm{Rep}_0(G)]$ via $N'_w \mapsto [\nabla(w \bullet 0)]$ $\forall w \in {}^f\mathcal{W}$, one has for each $s \in \mathcal{S}_a$ a commutative diagram

$$\begin{tikzcd} M^{\mathrm{asph}} \arrow[r, "\sim"] \arrow[d, "1+s"'] & {[\mathrm{Rep}_0(G)]} \arrow[d, "\Theta_s"] \\ M^{\mathrm{asph}} \arrow[r, "\sim"'] & {[\mathrm{Rep}_0(G)].} \end{tikzcd}$$

As $\underline{N'_w} \mapsto [T(\underline{w})]$ by (6.1.2), $\underline{N'_w} \in (T(\underline{w}) : \nabla(0))N'_1 + \sum_{x \in {}^f\mathcal{W}\setminus 1} \mathbb{Z}N'_x$.

Using the anti-equivalence τ from (5.2) such that $\bar{B}_{\underline{x}}\langle m\rangle \mapsto \bar{B}_{\underline{x}}\langle -m\rangle$ $\forall \underline{x}$, $\forall m \in \mathbb{Z}$, one has $\dim(\mathcal{D}^{\mathrm{asph}})^\bullet(\bar{B}_\emptyset, \bar{B}_{\underline{w}}) = \dim(\mathcal{D}^{\mathrm{asph}})^\bullet(\bar{B}_{\underline{w}}, \bar{B}_\emptyset)$, which is equal to $\dim(\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}})^\bullet(\bar{B}_{\underline{w}}, \bar{B}_\emptyset)$ as $\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}}$ is a full subcategory of $\mathcal{D}^{\mathrm{asph}} = \mathrm{Kar}(\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}})$ by (5.9) [Bor, Prop. 6.5.9, p. 274]. In turn, $\dim(\mathcal{D}^{\mathrm{asph}}_{\mathrm{BS}})^\bullet(\bar{B}_{\underline{w}}, \bar{B}_\emptyset) \leq \sharp\{e(\underline{w})|e(\underline{w})$ is an expression of the unity avoiding $\mathcal{W}_a \setminus {}^f\mathcal{W}\}$ by (5.12). On the other hand, from (5.10) one has

$$N_1\underline{H}_{\underline{w}} = \sum_{e(\underline{w}) \text{ avoiding } \mathcal{W}_a\setminus {}^f\mathcal{W}} v^{d(e(\underline{w}))} N_{\underline{w}^{e(\underline{w})}},$$

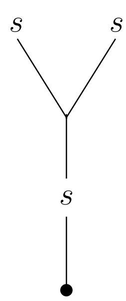
which under the specialization $v \rightsquigarrow 1$ yields

$$\begin{aligned} \underline{N'_w} &= \sum_{e(\underline{w}) \text{ avoiding } \mathcal{W}_a\setminus {}^f\mathcal{W}} N_{\underline{w}^{e(\underline{w})}} \\ &\in \sharp\{e(\underline{w})|e(\underline{w}) \text{ is an expression of the unity avoiding } \mathcal{W}_a \setminus {}^f\mathcal{W}\}N'_1 + \sum_{x \in {}^f\mathcal{W}\setminus 1} \mathbb{N}N'_x. \end{aligned}$$

(6.28) We are finally ready to prove Th. 6.3. We first show that $\bar{\Psi}$ is fully faithful. $\forall$ expressions $\underline{x}$ and $\underline{y}$, $\alpha_{\underline{x},\underline{y}} : \mathcal{D}^\bullet_{\mathrm{BS}}(B_{\underline{x}}, B_{\underline{y}}) \to \mathrm{Rep}_0(G)(T(\underline{x}), T(\underline{y}))$ induces $\bar{\alpha}_{\underline{x},\underline{y}} : \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_{\underline{x}}, \bar{B}_{\underline{y}}) = (\mathcal{D}^{\mathrm{asph}})^\bullet(\bar{B}_{\underline{x}}, \bar{B}_{\underline{y}}) \to \mathrm{Rep}_0(G)(T(\underline{x}), T(\underline{y}))$. By (5.6) and the additivity of $\bar{\Psi}$ we have only to show that each $\bar{\alpha}_{\underline{x},\underline{y}}$ is bijective. We argue by induction on $\ell(\underline{x})$.

If $\ell(\underline{x}) = 0$, $\underline{x} = \emptyset$. Then $\mathrm{Rep}_0(G)(T(\underline{x}), T(\underline{y})) = \mathrm{Rep}_0(G)^{\downarrow 0}(T(\emptyset), T(\underline{y}))$, and $\alpha_{\emptyset,\underline{y}} = \beta_{\emptyset,\underline{y}}$ is surjective by (6.26). On the other hand, $\dim \mathrm{Rep}_0(G)(T(\emptyset), T(\underline{y})) \geq \dim(\mathcal{D}^{\mathrm{asph}})^\bullet(\bar{B}_\emptyset, \bar{B}_{\underline{y}})$ by (6.27), and hence $\bar{\alpha}_{\emptyset,\underline{y}}$ is bijective.

Assume now that $\ell(\underline{x}) > 0$ and write $\underline{x} = \underline{ws}$ for some $s \in \mathcal{S}_a$. Recall from (5.3) that

(1) 

$$\begin{array}{ccc} B_{\underline{ss}} & & \Theta_s^2 \\ \uparrow & & \uparrow \\ B_{\underline{s}}\langle 1\rangle & \overset{\Psi}{\longmapsto} & \Theta_s \\ \uparrow & & \uparrow \\ B_\emptyset & & \mathrm{id}. \end{array}$$

As the LHS is the unit, say η'', associated to an adjunction $(?B_s, ?B_s)$ [EW], it induces a unit of adjunction $(?\bar{B}_s, ?\bar{B}_s)$ on $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}$, so therefore is $\Psi(\eta'')$ associated to an adjunction (Θ_s, Θ_s) [中岡, Cor. 2.2.9]. One has then a commutative daigram

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_{\underline{ws}}, \bar{B}_{\underline{y}}) \arrow[r, "(?B_{\underline{s}})\circ\eta''_{\bar{B}_{\underline{w}}}", "\sim"'] \arrow[d, "\bar{\alpha}_{\underline{ws},\underline{y}}"'] & \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_{\underline{w}}, \bar{B}_{\underline{ys}}) \arrow[d, "\bar{\alpha}_{\underline{w},\underline{ys}}"] \\
\mathrm{Rep}_0(T(\underline{ws}), T(\underline{y})) \arrow[r, "\sim", "\Theta_s(?)\circ\Psi(\eta''_{\bar{B}_{\underline{w}}})"'] & \mathrm{Rep}_0(T(\underline{w}), T(\underline{ys})).
\end{tikzcd}$$

As $\bar{\alpha}_{\underline{w},\underline{ys}}$ is bijective by induction, so is $\bar{\alpha}_{\underline{ws},\underline{y}} = \bar{\alpha}_{\underline{x},\underline{y}}$.

Finally, to see that $\bar{\Psi}_{\mathrm{deg}}(\bar{B}_w) = \tilde{\Psi}(B_w) = T(w \bullet 0)$ $\forall w \in {}^f\mathcal{W}$, we have only by (5.6) again to show that $\bar{\Psi}_{\mathrm{deg}}(\bar{B}_w)$ remains indecomposable. For that it suffices to show that $\mathrm{Rep}_0(\bar{\Psi}_{\mathrm{deg}}(\bar{B}_w), \bar{\Psi}_{\mathrm{deg}}(\bar{B}_w))$ is local [AF, p. 144], and hence to show that $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_w, \bar{B}_w)$ is local by what we have shown above; recall that a ring X is local iff $\forall x, y \in X$ with $x + y \in X^\times$, either $x \in X^\times$ or $y \in X^\times$. In particular, if X is local, the idempotents of X are just 0 and 1; if e is an idempotent of local X, $1 = e + (1 - e)$. If $e \in X^\times$, $1 - e = 0$ from $e(1 - e) = 0$; if $1 - e \in X^\times$, $e = 0$ likewise. We also have for local X, taking contrapositive of the definition, that $X \setminus X^\times$ is a unique maximal ideal of X.

As B_w is indecomposable in $\mathcal{D}^{\mathrm{asph}}$ and as $\mathcal{D}^{\mathrm{asph}}(\bar{B}_w, \bar{B}_w)$ is finite dimensional, $\mathcal{D}^{\mathrm{asph}}(\bar{B}_w, \bar{B}_w)$ is local; we have only to show that $\forall$ noninvertible $\phi \in \mathcal{D}^{\mathrm{asph}}(\bar{B}_w, \bar{B}_w)$, ϕ is nilpotent [AF, pf of Lem. 12.8]. As $\mathcal{D}^{\mathrm{asph}}(\bar{B}_w, \bar{B}_w)$ is finite dimensional, ϕ admits a minimal polynomial m_ϕ in $\Bbbk[x]$. If $m_\phi = (x - a_1)^{n_1} \dots (x - a_r)^{n_r}$ be a prime decomposition, put $m_{\phi,i} = \prod_{j \neq i}(x - a_j)^{n_j}$. Then one can write $1 = \sum_i f_i m_{\phi,i}$ for some $f_i \in \Bbbk[x]$. As $\mathrm{id}_{\bar{B}_w} = \sum_i \mathrm{ev}_\phi(f_i m_{\phi,i})$, m_ϕ must be a power of x.

As $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_w, \bar{B}_w)$ is finite dimensional, $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_w, \bar{B}_w) = (\mathcal{D}^{\mathrm{asph}})^\bullet(\bar{B}_w, \bar{B}_w)$ remains local [GG, Th. 3.1].

(6.29) Under the standing hypothesis $p > n$, in order to determine all the irreducible characters for G, it suffices by Steinberg's tensor product theorem (1.6) and by the translation functor to determine the irreducible characters of the G-modules of highest weight $x \bullet 0$ with $x \bullet 0 \in \Lambda_1$. Thus, let $\mathcal{W}_0 = \{x \in {}^f\mathcal{W} | \langle x \bullet 0 + \rho, \alpha^\vee \rangle < p(n-1)\ \forall \alpha \in R^+\}$. $\forall \lambda \in (p-1)\zeta + \Lambda^+$, write $\lambda = (p-1)\zeta + \lambda'_0 + p\lambda'_1$ with $\lambda'_0 \in \Lambda_1$ and $\lambda'_1 \in \Lambda^+$, and set $\check{\lambda} = (p-1)\zeta + w_0\lambda'_0 + p\lambda'_1$. One has then a bijection $(p-1)\zeta + \Lambda^+ \to \Lambda^+$ via $\lambda \mapsto \check{\lambda}$. Let $\Lambda^+ \to (p-1)\zeta + \Lambda^+$ be its inverse, denoted $\lambda \mapsto \hat{\lambda}$ [S97, p. 98]; if $\lambda = \lambda^0 + p\lambda^1$ with $\lambda^0 \in \Lambda_1$, $\hat{\lambda} = w_0 \bullet \lambda^0 + p(\lambda^1 + 2\zeta)$. $\forall y \in {}^f\mathcal{W}$, define $\hat{y} \in {}^f\mathcal{W}$ to be such that $\hat{y} \bullet 0 = \widehat{y \bullet 0}$.

Proposition [RW, Prop. 1.8.1]: *Assume $p \geq 2(n-1)$. $\forall x, y \in \mathcal{W}_0$,*

$$[\Delta(x \bullet 0) : L(y \bullet 0)] = (T(\hat{y} \bullet 0) : \nabla(x \bullet 0)).$$

Proof: Let $\Lambda^+_{<p(n-1)} = \{x \bullet 0 \in \Lambda^+ | x \in {}^f\mathcal{W}, \langle x \bullet 0, \alpha_0^\vee \rangle < p(n-1)\}$, $\alpha_0 = \alpha_1 + \dots + \alpha_{n-1} = \varepsilon_1 - \varepsilon_n$. Thus, $x \bullet 0 \in \Lambda^+_{<p(n-1)}$ $\forall x \in {}^f\mathcal{W}$. Let $\mathrm{Rep}(G)_{<p(n-1)}$ denote the Serre subcategory of $\mathrm{Rep}(G)$ generated by the $L(\lambda)$, $\lambda \in \Lambda^+_{<p(n-1)}$. As $\Lambda^+_{<p(n-1)}$ forms an ideal of $(\Lambda^+, \uparrow)$, $\mathrm{Rep}(G)_{<p(n-1)}$

forms a highest weight category by (6.10). Let $\mathcal{O}_{<p(n-1)} : \mathrm{Rep}(G) \to \mathrm{Rep}(G)_{<p(n-1)}$ denote the truncation functor sending M to the largest submodule of M whose composision factors all belong to $\mathrm{Rep}(G)_{<p(n-1)}$. As $\mathcal{O}_{<p(n-1)}$ is right adjoint to $\mathrm{Rep}(G)_{<p(n-1)} \hookrightarrow \mathrm{Rep}(G)$ [J, A.1.3], we have only to show that $\mathcal{O}_{<p(n-1)}T(\hat{y} \bullet 0)$ is the injective hull of $L(y \bullet 0)$ in $\mathrm{Rep}(G)_{<p(n-1)}$;

$$\begin{aligned}
[\Delta(x \bullet 0) : L(y \bullet 0)] &= \dim \mathrm{Rep}(G)_{<p(n-1)}(\Delta(x \bullet 0), \mathcal{O}_{<p(n-1)}T(\hat{y} \bullet 0)) \\
&= \mathrm{Rep}(G)(\Delta(x \bullet 0), T(\hat{y} \bullet 0)) \\
&= [T(\hat{y} \bullet 0) : \nabla(x \bullet 0)].
\end{aligned}$$

Put $\lambda = y \bullet 0$ and write $\lambda = \lambda^0 + p\lambda^1$ with $\lambda^0 \in \Lambda_1$ and $\lambda^1 \in \Lambda^+$. Then $\hat{y} \bullet 0 = \widehat{\lambda^0} + p\lambda^1$. As $y \in \mathcal{W}_0$,

$$\begin{aligned}
\langle p(\lambda^1 + \zeta), \alpha_0^\vee \rangle &\le \langle \lambda + \zeta + (p-1)\zeta, \alpha_0^\vee \rangle < p(n-1) + (p-1)(n-1) = (2p-1)(n-1) \\
&= p2(n-1) - (n-1) \le p^2 - (n-1) < p^2,
\end{aligned}$$

and hence $\langle \lambda^1 + \zeta, \alpha_0 \rangle < p$. Then $\Delta(\lambda^1) = \nabla(\lambda^1) = T(\lambda^1) = L(\lambda^1)$ by the linkage principle, and

$$\begin{aligned}
T(\hat{y} \bullet 0) &= T(\widehat{\lambda^0} + p\lambda^1) \simeq T(\widehat{\lambda^0}) \otimes T(\lambda^1)^{[1]} \quad \text{[J, E.9]} \\
&= T(\widehat{\lambda^0}) \otimes L(\lambda^1)^{[1]}.
\end{aligned} \tag{1}$$

Now, $T(\widehat{\lambda^0})$ is the injective hull of $L(\lambda^0)$ in the category of $\mathrm{Rep}(G_1)$ [J, E.9.1] and also in $\mathrm{Rep}_{<2p(n-1)}(G)$ defined anagolously to $\mathrm{Rep}_{<p(n-1)}(G)$ [J, II.11.11]. In particular, $\mathrm{soc}_G T(\widehat{\lambda^0}) = L(\lambda^0)$. Then

$$\begin{aligned}
\mathrm{soc}_G T(\hat{y} \bullet 0) &\simeq \mathrm{soc}_G(T(\widehat{\lambda^0}) \otimes L(\lambda^1)^{[1]}) \\
&\simeq \{\mathrm{soc}_G(T(\widehat{\lambda^0})\} \otimes L(\lambda^1)^{[1]}) \quad \text{[AK, Lem. 4.6]} \\
&\simeq L(\lambda^0) \otimes L(\lambda^1)^{[1]} \simeq L(\lambda),
\end{aligned}$$

and hence $\mathrm{soc}_{\mathrm{Rep}_{<p(n-1)}(G)} \mathcal{O}_{<p(n-1)} T(\hat{y} \bullet 0) = L(\lambda)$. $\forall \nu \in \Lambda^+_{<p(n-1)}$,

$$\begin{aligned}
(2)\quad \mathrm{Ext}^1_G(L(\nu), T(\hat{y} \bullet 0)) &\simeq \mathrm{Ext}^1_G(L(\nu), T(\widehat{\lambda^0}) \otimes L(\lambda^1)^{[1]}) \quad \text{by (1)} \\
&\simeq \mathrm{Ext}^1_G(L(\nu) \otimes L(-w_0\lambda^1)^{[1]}, T(\widehat{\lambda^0})) \simeq \mathrm{Ext}^1_G(L(\nu - pw_0\lambda^1), T(\widehat{\lambda^0})) \\
&\simeq \mathrm{Ext}^1_{\mathrm{Rep}_{<2p(n-1)}(G)}(L(\nu - pw_0\lambda^1), T(\widehat{\lambda^0})) \quad \text{using the 米田-extensions as} \\
&\qquad \langle \nu - pw_0\lambda^1 + \rho, \alpha_0^\vee \rangle < p(n-1) + \langle p\lambda^1, \alpha_0^\vee \rangle \le p(n-1) + \langle \lambda, \alpha_0^\vee \rangle < 2p(n-1) \\
&= 0.
\end{aligned}$$

Then $\forall M \hookrightarrow M'$ in $\mathrm{Rep}_{<p(n-1)}(G)$, one obtains a commutative exact diagram

$$\begin{tikzcd}
\mathrm{Rep}_{<p(n-1)}(G)(M', \mathcal{O}_{<p(n-1)}T(\hat{y} \bullet 0)) \arrow[r] \arrow[d, "\sim"', no head] & \mathrm{Rep}_{<p(n-1)}(G)(M, \mathcal{O}_{<p(n-1)}T(\hat{y} \bullet 0)) \arrow[d, "\sim", no head] & \\
\mathrm{Rep}(G)(M', T(\hat{y} \bullet 0)) \arrow[r] & \mathrm{Rep}(G)(M, T(\hat{y} \bullet 0)) \arrow[r] & \mathrm{Ext}^1_G(M'/M, T(\hat{y} \bullet 0))
\end{tikzcd}$$

with $\mathrm{Ext}^1_G(M'/M, T(\hat{y} \bullet 0)) = 0$ by (2). It follows that $\mathcal{O}_{<p(n-1)}T(\hat{y} \bullet 0)$ is injective in $\mathrm{Rep}_{<p(n-1)}(G)$, as desired.

(6.30) We now obtain under the hypothesis $p \geq 2(n-1)$ that $\forall x \in \mathcal{W}_0$,

$$[\Delta(x \bullet 0)] = \sum_{y \in \mathcal{W}_0} {}^p n_{x,\hat{y}}(1)[L(y \bullet 0)]$$

in $[\mathrm{Rep}(G)]$. Inverting the unipotent matrix $[\![{}^p n_{x,\hat{y}}(1)]\!]_{x,y \in \mathcal{W}_0}$ yields $\mathrm{ch}\, L(x \bullet 0)$ $\forall x \in \mathcal{W}_0$, from which one can derive all the irreducible characters of G.

Appendix A: The structure of the general linear groups as algebraic groups

This is meant also to be a preliminary to my lectures scheduled next semester on recent advances in the modular representation theory of algebraic groups.

Fix a field $\Bbbk$ and let $G = \mathrm{GL}_n(\Bbbk)$ denote the general linear group of invertible matrices over $\Bbbk$. We will describe some basic structure of G as an algebraic group. All the details can be found in Jantzen's (resp. Carter's) classic [J] (resp. [C]). We will often abbreviate $\mathrm{GL}_r(\Bbbk)$ as GL_r.

Precisely, given a category $\mathcal{C}$ let $\mathcal{C}(X, Y)$ denote the set of morphisms from the object X in $\mathcal{C}$ to the object Y in $\mathcal{C}$. Let **Commrng** denote the category of commutative rings and **Set** the category of sets. A scheme is a functor from **Commrng** to **Set**. If A is a commutative ring, let $\mathfrak{Sp}A$ be a scheme such that $(\mathfrak{Sp}A)(C) = \mathbf{Commrng}(A, C)$. For any scheme $\mathfrak{X}$ if $f \in \mathbf{Sch}(\mathfrak{Sp}A, \mathfrak{X})$, one has for any $\phi \in \mathbf{Commrng}(A, C)$ a commutative diagram

$$\begin{tikzcd}
\mathrm{id}_A \arrow[d, mapsto] & \mathbf{Commrng}(A,A) \arrow[d, "\mathbf{Commrng}(A,\phi)"'] & = & (\mathfrak{Sp}A)(A) \arrow[r, "f(A)"] \arrow[d, "(\mathfrak{Sp}A)(\phi)"'] & \mathfrak{X}(A) \arrow[d, "\mathfrak{X}(\phi)"] \\
\phi & \mathbf{Commrng}(A,C) & = & (\mathfrak{Sp}A)(C) \arrow[r, "f(C)"'] & \mathfrak{X}(C).
\end{tikzcd}$$

If $\mathfrak{X}_f = f(A)(\mathrm{id}_A)$, $f(C)(\phi) = \mathfrak{X}(\phi)(\mathfrak{X}_f)$, and hence f is uniquely determined by $\mathfrak{X}_f$. Conversely, given $x \in \mathfrak{X}(A)$, $\forall C \in \mathbf{Commrng}$, define $f_x(C) \in \mathbf{Set}((\mathfrak{Sp}A)(C), \mathfrak{X}(C))$ by $\phi \mapsto \mathfrak{X}(\phi)(x)$. Thus, f_x defines a morphism of schemes from $\mathfrak{Sp}A$ to $\mathfrak{X}$. We have obtained Yoneda's lemma:

$$\mathbf{Sch}(\mathfrak{Sp}A, \mathfrak{X}) \simeq \mathfrak{X}(A) \quad \text{via } f \mapsto \mathfrak{X}_f \text{ with inverse } f_x \hookleftarrow x.$$

In particular, if A' is another commutative ring,

$$\mathbf{Sch}(\mathfrak{Sp}A, \mathfrak{Sp}A') \simeq (\mathfrak{Sp}A')(A) = \mathbf{Commrng}(A', A).$$

Let $\mathbb{Z}[\xi_{ij}, \frac{1}{\det} | i, j \in [1, n]]$ be the polynomial ring in indeterminates $\xi_{ij}, i, j \in [1, n]$, localized at det, i.e., it is the subring of the rational function field $\mathbb{Q}(\xi_{ij} | i, j \in [1, n])$ in indeterminates ξ_{ij} generated by the ξ_{ij}'s and $\frac{1}{\det}$. Then GL_n is a functor $\mathbf{Commrng}(\mathbb{Z}[\xi_{ij}, \frac{1}{\det} | i, j \in [1, n]], ?) : \mathbf{Commrng} \to \mathbf{Set}$ from the category **Commrng** to the category **Set**. Thus $\mathrm{GL}_n(\Bbbk)$ is just the set of invertible matrices over $\Bbbk$ of degree n. We often denote the ring $\mathbb{Z}[\xi_{ij}, \frac{1}{\det} | i, j \in [1, n]]$ by $\mathbb{Z}[\mathrm{GL}_n]$. It is, moreover, equipped with an extra structure of Hopf algebra, which makes GL_n into a group functor from **Commrng** to the category **Grp** of groups.

(A.1) Let T be the subgroup of daiagonals of G, called a maximal torus of G. Thus, T is isomorphic to GL_1^n via $(a_1, \dots, a_n) \mapsto \mathrm{diag}(a_1, \dots, a_n)$. Set $\Lambda = \mathbf{Grp}_{\mathbb{Z}}(T, \mathrm{GL}_1)$. If $\varepsilon_i \in \Lambda$ such that $\mathrm{diag}(a_1, \dots, a_n) \mapsto a_i$, Λ is endowed with a structure of abelian group isomorphic to $\mathbb{Z}^{\oplus n}$ via $(r_1, \dots, r_n) \mapsto \sum_{i=1}^n r_i\varepsilon_i$, where $\sum_{i=1}^n r_i\varepsilon_i : t \mapsto \prod_i \varepsilon_i(t)^{r_i}$. In particular, $\mathbf{Grp}_{\mathbb{Z}}(\mathrm{GL}_1, \mathrm{GL}_1) \simeq \mathbb{Z}$ via $r \mapsto ?^r$. We are thus dealing only with a special kind of group homomorphisms, morphisms of algebraic groups. We call Λ the character group of T.

(A.2) For each $i, j \in [1, n]$ with $i \neq j$ define $x_{ij}(a) \in G, a \in \Bbbk$, to be the matrix such that $x_{ij}(a)_{kk} = 1\ \forall k$ and $x_{ij}(a)_{kl} = \delta_{ik}\delta_{jl}a\ \forall i \neq j$, and set $U(i, j) = \{x_{ij}(a) | a \in \Bbbk\}$ an elementary subgroup of G. Then $U(i, j)$ is isomorphic to the additive group $\mathbf{G}_a = \Bbbk$ via $a \mapsto x_{ij}(a)$. In particular, $x_{ij}(a)^{-1} = x_{ij}(-a)$.

$\forall t \in T$ and $b \in \Bbbk$, one has $tx_{ij}(b)t^{-1} = x_{ij}((\varepsilon_i - \varepsilon_j)(t)b)$, i.e.,

$$\mathrm{diag}(a_1, \dots, a_n)x_{ij}(b)\mathrm{diag}(a_1, \dots, a_n)^{-1} = x_{ij}(a_i a_j^{-1} b).$$

We let $R = \{\varepsilon_i - \varepsilon_j | i \neq j\}$ and call it the set of roots. If $\alpha = \varepsilon_i - \varepsilon_j \in R$, we will write U_α (resp. x_α) for $U(i, j)$ (resp. x_{ij}) and call U_α the root subgroup of G associated to α. If $R^+ = \{\varepsilon_i - \varepsilon_j | i < j\}$, $R = R^+ \sqcup (-R^+)$.

(A.3) Let $\alpha, \beta \in R$ with $\alpha + \beta \neq 0$. Let $\bar{x}_\alpha = x_\alpha(1) - I$ with I denoting the identity matrix, and define $\bar{x}_\beta$ (resp. $\bar{x}_{\alpha+\beta}$ if $\alpha + \beta \in R$) likewise. Define $N_{\alpha\beta} \in \{0, \pm 1\}$ to be

$$\bar{x}_\alpha \bar{x}_\beta - \bar{x}_\beta \bar{x}_\alpha = \begin{cases} N_{\alpha\beta}\bar{x}_{\alpha+\beta} & \text{if } \alpha + \beta \in R, \\ 0 & \text{else.} \end{cases}$$

Then $\forall a, b \in \Bbbk$

$$x_\beta(b)^{-1}x_\alpha(a)^{-1}x_\beta(b)x_\alpha(a) = \begin{cases} x_{\alpha+\beta}(N_{\alpha\beta}(-a)b) & \text{if } \alpha + \beta \in R, \\ I & \text{else,} \end{cases}$$

which is called Chevalley's commutator relation. It follows that $U = \prod_{\alpha \in R} U_{-\alpha}$ forms a subgroup of G consisting of the unimodular lower triangular matrices. Thus there is an isomorphism of schemes

$$\mathbb{A}^{|R^+|} \to U \quad \text{via} \quad (a_\alpha)_{\alpha \in R^+} \mapsto \prod_{\alpha \in R^+} x_{-\alpha}(a_\alpha).$$

Likewise for $U^+ = \prod_{\alpha \in R^+} U_\alpha$. Both U and U^+ are normalized by T, and we set $B = TU$, $B^+ = TU^+$, called opposite Borel subgroups of G.

(A.4) Dualizing Λ let $\Lambda^\vee = \mathbf{Grp}_{\mathbb{Z}}(\mathrm{GL}_1, T)$. If $\varepsilon_i^\vee \in \Lambda^\vee$ is defined by $c \mapsto \mathrm{diag}(1, \dots, 1, c, 1, \dots, 1)$ with c at the i-th place, $\Lambda^\vee$ is endowed with a structure of abelian group isomorphic to $\mathbb{Z}^{\oplus n}$ via $(r_1, \dots, r_n) \mapsto \sum_{i=1}^n r_i\varepsilon_i^\vee$, where $\sum_{i=1}^n r_i\varepsilon_i^\vee : c \mapsto \prod_i \varepsilon_i^\vee(c)^{r_i}$. Again we deal only with this kind of group homomorphisms.

$\forall \lambda \in \Lambda, \gamma \in \Lambda^\vee$, one has $\lambda \circ \gamma \in \mathbf{Grp}_{\mathbb{Z}}(\mathrm{GL}_1, \mathrm{GL}_1) \simeq \mathbb{Z}$ defining $\langle \lambda, \gamma \rangle \in \mathbb{Z}$ such that $(\lambda \circ \gamma)(c) = c^{\langle \lambda, \gamma \rangle}\ \forall c \in \mathrm{GL}_1$. One thus obtains a perfect pairing $\langle ?, ? \rangle : \Lambda \times \Lambda^\vee \to \mathbb{Z}$ via $(\lambda, \gamma) \mapsto \langle \lambda, \gamma \rangle$. We thus call $\Lambda^\vee$ the cocharacter group of T.

(A.5) Fix $\alpha \in R$. Let $\mathrm{SL}_2 = \mathrm{SL}_2(\Bbbk) = \{g \in \mathrm{GL}_2 | \det(g) = 1\}$ the special linear groupf of degree 2. There is a group homomorphism $\phi_\alpha : \mathrm{SL}_2 \to G$ such that $\forall a \in \Bbbk$,

$$\phi_\alpha \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = x_\alpha(a) \quad \text{and} \quad \phi_\alpha \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = x_{-\alpha}(a).$$

For $c \in \Bbbk^\times$ let $n_\alpha(c) = \phi_\alpha \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix}$ and $\alpha^\vee(c) = \phi_\alpha \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$. Then

$$n_\alpha(c) = x_\alpha(c)x_{-\alpha}(-c^{-1})x_\alpha(c) \in \mathrm{N}_G(T) \quad \text{and} \quad \alpha^\vee(c) = n_\alpha(c)n_\alpha(1)^{-1} \in T.$$

More explicitly, if we let $E(i,j) \in \mathrm{M}_n(\Bbbk)$ such that $E(i,j)_{kl} = \delta_{ik}\delta_{jl}$,

$$\phi_{\varepsilon_i - \varepsilon_j} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = aE(i,i) + bE(i,j) + cE(j,i) + dE(j,j) + \sum_{k \neq i,j} E(k,k).$$

We call $\alpha^\vee \in \Lambda^\vee$ the coroot of α, and set $R^\vee = \{\alpha^\vee | \alpha \in R\}$. One has $\langle \alpha, \alpha^\vee \rangle = 2$. In case $\alpha = \varepsilon_i - \varepsilon_j$, $\alpha^\vee = \varepsilon_i^\vee - \varepsilon_j^\vee$. We say that the quadruple $(\Lambda, R, \lambda^\vee, R^\vee)$ forms a root datum of G, which is used to classify the reductive algebraic groups.

(A.6) Let $W = \mathrm{N}_G(T)/T$. As W acts on T, so does it on Λ and $\Lambda^\vee$ via

$$(w\lambda)(t) = \lambda(w^{-1}tw) \quad \text{and} \quad (w\gamma)(c) = w\gamma(c)w^{-1} \quad \forall w \in W, \lambda \in \Lambda, t \in T, \gamma \in \Lambda^\vee.$$

Thus the pairing $\langle ?, ? \rangle$ is W-invariant: $\langle w\lambda, w\gamma \rangle = \langle \lambda, \gamma \rangle$. As Λ separates T, the action of W on Λ is faithful.

$\forall \alpha \in R$, set $s_\alpha = n_\alpha(1)$. Then

(1) $$W = \langle s_\alpha | \alpha \in R \rangle,$$
(2) $$s_\alpha \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \quad \forall \alpha \in R, \lambda \in \Lambda.$$

More specifically, if $\alpha = \varepsilon_i - \varepsilon_j$, $i \neq j$, and $\lambda = \varepsilon_k$, $k \in [1,n]$,

$$s_{\varepsilon_i - \varepsilon_j} \varepsilon_k = \varepsilon_k - \langle \varepsilon_k, \varepsilon_i^\vee - \varepsilon_j^\vee \rangle (\varepsilon_i - \varepsilon_j) = \begin{cases} \varepsilon_j & \text{if } k = i, \\ \varepsilon_i & \text{if } k = j, \\ \varepsilon_k & \text{else.} \end{cases}$$

It follows that the injective group homomorphism $W \to \mathfrak{S}_\Lambda$ induces an isomorphism $W \to \mathfrak{S}_n$ such that $\forall i \neq j$,

$$s_{\varepsilon_i - \varepsilon_j} \mapsto (i\ j).$$

Thus, if $w \mapsto \sigma$,

$$w\mathrm{diag}(a_1, \ldots, a_n)w^{-1} = \mathrm{diag}(a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(1)}), \quad w \sum_i \lambda_i \varepsilon_i = \sum_i \lambda_i \varepsilon_{\sigma(i)} = \sum_i \lambda_{\sigma^{-1}(i)} \varepsilon_i.$$

If $e_1, \ldots, e_n$ is the standard basis of $\Bbbk^{\oplus n}$ affording G, $we_i = e_{\sigma(i)}$ up to $\Bbbk^\times$. In particular,

(3) $$s_\alpha^2 = 1 \quad \forall \alpha \in R,$$
(4) $$WR = R, \quad WR^\vee = R^\vee,$$
(5) $$ws_\alpha w^{-1} = s_{w\alpha} \quad \forall w \in W, \forall \alpha \in R,$$
(6) $$W = \langle s_i | i \in [1, n[\rangle \quad \text{with } s_i = s_{\alpha_i} \text{ and } \alpha_i = \varepsilon_i - \varepsilon_{i+1}.$$

We call $\alpha_1, \dots, \alpha_{n-1}$ the simple roots, and put $R^s = \{\alpha_i | i \in [1, n[\}$. The matrix $[\![\langle \alpha_i, \alpha_j^\vee \rangle]\!]$ of degree $n-1$ is called the Cartan matrix.

Also, $\forall w \in W$, $\forall \alpha \in R$, $a \in \Bbbk$,

$$wx_\alpha(a)w^{-1} = x_{w\alpha}(\pm a). \tag{7}$$

We cannot control $\pm$ as w is defined up to T.

(A.7) We call $R^+ = R \cap \sum_{i=1}^{n-1} \mathbb{N}\alpha_i = \{\varepsilon_i - \varepsilon_j | 1 \leq i < j < n\}$ the positive system of R determined by the simple roots $\alpha_1, \dots, \alpha_{n-1}$. We define a partial order on Λ such that $\lambda \geq \mu$ iff $\lambda - \mu \in \sum_{i=1}^{n-1} \mathbb{N}\alpha_i = \sum_{\alpha \in R^+} \mathbb{N}\alpha$.

If $S = \{s_i | i \in [1, n[\}$, (W, S) forms a Coxeter system. Define a length function $\ell : W \to \mathbb{N}$ such that $\ell(w)$, $w \in W$, is equal to the smallest number m with $w = s_{i_1} \dots s_{i_m}$, $s_{i_j} \in S$, in which case we say such a sequence is a reduced expression of w. $\forall w \in W$, $\forall s_i \in S$,

$$\ell(ws_i) = \begin{cases} \ell(w) + 1 & \text{if } w\alpha_i > 0, \\ \ell(w) - 1 & \text{if } w\alpha_i < 0, \end{cases} \tag{1}$$

$$\ell(w) = |\{\alpha \in R^+ | w\alpha < 0\}|. \tag{2}$$

There is a unique $w_0 \in W$ such that $w_0 R^+ = -R^+$, which corresponds to $\begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$. Thus,

$$w_0^2 = 1, \tag{3}$$

$$\ell(w_0) = |R^+| = \binom{n}{2}. \tag{4}$$

Let $\rho = \sum_{i=1}^n (n-i)\varepsilon_i$. $\forall \alpha \in R^s$,

$$\langle \rho, \alpha^\vee \rangle = 1,$$

and hence $s_\alpha \rho = \rho - \alpha$. Then $w\rho - \rho \in \mathbb{Z}R$ $\forall w \in W$. A new action of W on Λ defined by

$$w \bullet \lambda = w(\lambda + \rho) - \rho$$

will be important in the representation theory of G.

(A.8) For each $w \in W$ let BwB denote $B\hat{w}B$ with a lift $\hat{w} \in \mathrm{N}_G(T)$ of w. One has a Bruhat decomposition

$$G = \sqcup_{w \in W} BwB.$$

The multiplication induces an isomorphism of schemes

$$\prod_{\alpha \in R^+} U_\alpha \times T \times \prod_{\alpha \in R^+} U_{-\alpha} \to U^+B,$$

and U^+B is open in G, called a big cell, the closure $\overline{U^+B}$ being the whole of G. More generally, let $R^+(w) = \{\alpha \in R^+ | w^{-1}\alpha < 0\}$. If $U(w) = \langle U_{-\alpha} | \alpha \in R^+(w) \rangle$, the multiplication induces an isomorphism $\prod_{\alpha\in R^+(w)} U_{-\alpha} \to U(w)$. One has an isomorphism of schemes

$$\mathbb{A}^{\ell(w)} \times B \simeq U(w) \times B \to BwB \quad \text{via} \quad ((a_\alpha)_{\alpha\in R^+(w)}, b) \mapsto (\prod_{\alpha\in R^+(w)} x_{-\alpha}(a_\alpha))wb,$$

where $\mathbb{A}^{\ell(w)} = \mathfrak{Sp}(\mathbb{Z}[\xi_1, \dots, \xi_{\ell(w)}])$ is called the affine $\ell(w)$-space.

There is a partial order on W, called the Chevalley-Bruhat order, such that $x \geq y$ iff $\overline{BxB} \supseteq \overline{ByB}$. Then

$$\overline{BwB} = \sqcup_{x\leq w} BxB \quad \text{with } BwB \text{ open in } \overline{BwB}.$$

Given $g \in G$, by elementary row operations there is $b_1 \in B$ such that the first column of $b_1 g$ is e_i for some $i \in [1, n]$. Then by elementary column operations there is $b_1' \in B$ such that the i-th row of $b_1 g b_1'$ is $(1, 0 \dots, 0)$. Repeating the procedure, by elementary row operations there is $b_2 \in B$ such that the second column of $b_2 b_1 g b_1'$ is e_j for some $j \in [1, n] \setminus \{i\}$. Then by elementary column operations there is $b_2' \in B$ such that the j-th row of $b_2 b_1 g b_1' b_2'$ is $(0, 1, 0 \dots, 0)$. Thus, eventually there are $b, b' \in B$ such that bgb' is equal to a permutation matrix w.

More precisely, for $g = [\![g_{kl}]\!] \in G$ and $i, j \in [1, n]$ let $c_{ij}(g) = [\![g_{kl}]\!]_{1\leq k\leq j, i\leq l\leq n}$. For all $r \in [1, \min\{n - i + 1, j\}]$ let

$$\mathfrak{d}^r_{ij}(g) = \begin{cases} \mathbb{k} & \text{if } \operatorname{rk} c_{ij}(g) \geq r, \\ 0 & \text{else.} \end{cases}$$

Then

$$BwB = \{g \in G | \mathfrak{d}^r_{ij}(g) = \mathfrak{d}^r_{ij}(\hat{w}) \ \forall i, j, r\}.$$

Also, $x \leq y$ iff x is the product of a subsequence of a reduced expression of y.

Appendix B: Representation theory of the general linear groups after Riche and Williamson 集中講義

The lecture is meant to give an introduction/survey of the first 2 parts of a recent monumental work by Riche and Williamson [RW]. We will consider the representation theory of $\mathrm{GL}_n(\mathbb{k})$ over an algebraically closed field $\mathbb{k}$ of positive characteristic p.

1° 月曜日

(月 1) Set $G = \mathrm{GL}_n(\mathbb{k})$. We will consider only algebraic representations of G, that is, group homomorphisms $\phi : G \to \mathrm{GL}(M)$ with M a finite dimensional $\mathbb{k}$-linear space such that, choosing a basis of M and identifying $\mathrm{GL}(M)$ with $\mathrm{GL}_r(\mathbb{k})$, $r = \dim M$, the functions $y_{\nu\mu} \circ \phi$ on G, $\nu, \mu \in [1, r]$, all belong to $\mathbb{k}[x_{ij}, \det^{-1} | i, j \in [1, n]]$, where $y_{\nu\mu}(g') = g'_{\nu\mu}$ is the (ν, μ)-th element of $g' \in \mathrm{GL}_r(\mathbb{k})$ and $x_{ij}(g) = g_{ij}$ is the (i, j)-th element of $g \in \mathrm{GL}_n(\mathbb{k})$ [J, I.2.7, 2.9]. Given a representation ϕ we also say that M affords a G-module, and write gm for $\phi(g)m$, $g \in G, m \in M$. Set $\mathbb{k}[G] = \mathbb{k}[x_{ij}, \det^{-1} | i, j \in [1, n]]$.

A basic problem of the representation theory of G is the determination of simple representations. A nonzero G-module M is called simple/irreducible iff M admits no proper subspace M' such that $gm \in M'\ \forall g \in G\ \forall m \in M'$.

(月 2) A classification of the simple G-modules is well-known. To describe it, let B denote a Borel subgroup of G consisting of the lower triangular matrices and T a maximal torus of B consisting of the diagonals. Let $\Lambda = \mathbf{Grp}_{\Bbbk}(T, \mathrm{GL}_1(\Bbbk))$, called the character group of T. Recall that Λ is a free abelian group of basis $\varepsilon_1, \dots, \varepsilon_n$ such that $\varepsilon_i : \mathrm{diag}(a_1, \dots, a_n) \mapsto a_i$. We write the group operation on Λ additively; for $m_1, \dots, m_n \in \mathbb{Z}$, $\sum_{i=1}^n m_i\varepsilon_i : \mathrm{diag}(a_1, \dots, a_n) \mapsto a_1^{m_1} \dots a_n^{m_n}$. Let $R = \{\varepsilon_i - \varepsilon_j | i, j \in [1, n], i \neq j\}$ be the set of roots, and put $R^+ = \{\varepsilon_i - \varepsilon_j | i, j \in [1, n], i < j\}$, the set of positive roots such that the roots of B are $-R^+$: $B = T \ltimes U$ with $U = \prod_{\alpha \in R^+} U_{-\alpha}$, $U_{-\alpha} = \{x_{-\alpha}(a) | a \in \Bbbk\}$ such that if $-\alpha = \varepsilon_i - \varepsilon_j$, $\forall \nu, \mu \in [1, n]$,

$$x_{-\alpha}(a)_{\nu\mu} = \begin{cases} 1 & \text{if } \nu = \mu, \\ a & \text{if } \nu = i \text{ and } \mu = j, \\ 0 & \text{else.} \end{cases}$$

If $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$, $i \in [1, n[$, $R^{\mathrm{s}} = \{\alpha_1, \dots, \alpha_{n-1}\}$ forms a set of all simple roots of R^+. For $\alpha = \varepsilon_i - \varepsilon_j \in R$ let $\alpha^\vee \in \Lambda^\vee$ denote the coroot of α such that

$$\langle \varepsilon_k, \alpha^\vee \rangle = \begin{cases} 1 & \text{if } k = i, \\ -1 & \text{if } k = j, \\ 0 & \text{else.} \end{cases}$$

Let $\Lambda^+ = \{\lambda \in \Lambda | \langle \lambda, \alpha^\vee \rangle \geq 0\, \forall \alpha \in R^+\}$, called the set of dominant weights of T. We introduce a partial order on Λ such that $\lambda \geq \mu$ iff $\lambda - \mu \in \sum_{\alpha \in R^+} \mathbb{N}\alpha$.

(月 3) Any T-module M is simultaneously diagonalizable:

$$M = \coprod_{\lambda \in \Lambda} M_\lambda \quad \text{with} \quad M_\lambda = \{m \in M | tm = \lambda(t)m\, \forall t \in T\}.$$

We call M_λ the λ-weight space of M, its dimension the multiplicity of λ in M, λ a weight of M iff $M_\lambda \neq 0$, and the coproduct the weight space decomposition of M. Let $\mathbb{Z}[\Lambda]$ be the group ring of Λ with a basis e^λ, $\lambda \in \Lambda$. We call

$$\mathrm{ch}\, M = \sum_{\lambda \in \Lambda} (\dim M_\lambda) e^\lambda \in \mathbb{Z}[\Lambda]$$

the (formal) character of M; if M is a G-module, for $g \in G$ g is conjugate to $g_s g_u$ in the Jordan cannical form with $g_s \in T$ and $g_u \in U$ such that $g_s g_u = g_u g_s$. Then the trace $\mathrm{Tr}(g)$ on M is given by

$$\begin{aligned} \mathrm{Tr}(g) &= \mathrm{Tr}(g_s g_u) = \mathrm{Tr}(g_s) \\ &= \sum_\lambda \lambda(t) \dim M_\lambda, \end{aligned}$$

which does not make much sense in positive characterstic.

(月 4) Assume for the moment that $\Bbbk$ is of characteristic 0. Here the representation theory of G is well-understood. Any G-module is semisimple, i.e., a direct sum of simple G-modules [J, II.5.6.6]. For $\lambda \in \Lambda$ regard λ as a 1-dimensional B-module via the projection $B = T \ltimes U \to T$, and let $\nabla(\lambda) = \{f \in \Bbbk[G] | f(gb) = \lambda(b)^{-1}f(g)\, \forall g \in G \forall b \in B\}$ with G-action defined by $g \cdot f = f(g^{-1}?)$. The Borel-Weil theorem asserts that $\nabla(\lambda) \neq 0$ iff $\lambda \in \Lambda^+$ [J, II.2.6]. Any simple G-module is isomorphic to a unique $\nabla(\lambda)$, $\lambda \in \Lambda^+$, and $\mathrm{ch}\,\nabla(\lambda)$ is given by Weyl's character formula. To describe the formula, we have to recall the Weyl group $\mathcal{W} = \mathrm{N}_G(T)/T$ of G and its action on Λ: $\forall w \in \mathcal{W}$, $\forall \mu \in \Lambda$, we define $w\mu \in \Lambda$ by setting $(w\mu)(t) = \mu(w^{-1}tw)$ $\forall t \in T$. More concretely, identify Λ with $\mathbb{Z}^{\oplus n}$ via $\sum_{i=1}^n \mu_i \varepsilon_i \mapsto (\mu_1, \dots, \mu_n)$. Then $\mathcal{W} \simeq \mathfrak{S}_n$ such that $w\varepsilon_i = \varepsilon_{wi}$, i.e., $w\mu = (\mu_{w^{-1}1}, \dots, \mu_{w^{-1}n})$. Let also $\zeta = (0, -1, \dots, -n+1) \in \Lambda$, and set $w \bullet \lambda = w(\lambda + \zeta) - \zeta$; we replace the usual choice of $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$, which may not live in Λ, e.g., in the case of $\mathrm{GL}_2(\Bbbk)$, by ζ. Then [J, II.5.10] for $\lambda \in \Lambda^+$

$$\mathrm{ch}\,\nabla(\lambda) = \frac{\sum_{w \in \mathcal{W}} \det(w) e^{w(\lambda+\zeta)}}{\sum_{w \in \mathcal{W}} \det(w) e^{w\zeta}} = \frac{\sum_{w \in \mathcal{W}} \det(w) e^{w \bullet \lambda}}{\sum_{w \in \mathcal{W}} \det(w) e^{w \bullet 0}}.$$

In particular, $\nabla(\lambda)$ has highest weight λ of multiplicity 1: any weight of $\nabla(\lambda)$ is $\leq \lambda$, and $\dim \nabla(\lambda)_\lambda = 1$.

(月 5) Back to our original setting, each $\nabla(\lambda)$ in (月 4) is defined over $\mathbb{Z}$ and gives us a standard module, denoted by the same letter, having the same character [J, II.8.8]; this is a highly nontrivial result requiring the universal coefficient theorem [J, I.4.18] on induction and Kempf's vanishing theorem [J, II.4] among other things. In particular, the ambient space V of our G is $\nabla(\varepsilon_1)$; if $v_1, \dots, v_n$ is the standard basis of V, each v_i is of weight ε_i. More generally, let $\mathrm{S}(V) = \Bbbk[v_1, \dots, v_n]$ denote the symmetric algebra of V, and $\mathrm{S}^m(V)$ its homogeneous part of degree m. Then $\mathrm{S}^m(V) \simeq \nabla(m\varepsilon_1)$ [J, II.2.16]. Note, however, that $\mathrm{S}^p(V)$ has a proper G-submodule $\sum_{i=1}^n \Bbbk v_i^p$, and hence $\nabla(\lambda)$ is no longer simple in general. Nonetheless, each $\nabla(\lambda)$ has a unique simple submodule, which we denote by $L(\lambda)$ [J, II.2.3]. It has highest weight λ, and any simple G-module is isomorphic to a unique $L(\mu)$, $\mu \in \Lambda^+$ [J, II.2.4]. Thus, our basic problem is to find all $\mathrm{ch}\, L(\mu)$.

For that, as any composition factor of $\nabla(\lambda)$ is of the form $L(\mu)$, $\mu \leq \lambda$, with $L(\lambda)$ appearing just once, the finite matrix $[\![[\nabla(\nu) : L(\mu)]]\!]$ of the composition factor multiplicities for $\nu, \mu \leq \lambda$ is unipotent, from which $\mathrm{ch}\, L(\lambda)$ can be obtained as a $\mathbb{Z}$-linear combinations of $\mathrm{ch}\,\nabla(\nu)$'s.

(月 6) To find the irreducible characters, some reductions are in order. First, let $\Lambda_1 = \{\lambda \in \Lambda^+ | \langle \lambda, \alpha^\vee \rangle < p \,\forall \alpha \in R^{\mathrm{s}}\}$. If $\varpi_i := \varepsilon_1 + \cdots + \varepsilon_i$, $i \in [1, n]$, $\Lambda = \coprod_{i=1}^n \mathbb{Z}\varpi_i$, $\varpi_n = \det$, and $\Lambda^+ = \mathbb{Z}\det + \sum_{i=1}^{n-1} \mathbb{N}\varpi_i$. Thus, $\Lambda_1 = \mathbb{Z}\det + \{\sum_{i=1}^{n-1} a_i \varpi_i | a_i \in [0, p[\}$. One can write any $\lambda \in \Lambda^+$ in the form $\lambda = \sum_{i=0}^r p^i \lambda^i$, $\lambda^i \in \Lambda_1$. Then

Steinberg's tensor product theorem [J, II.3.17]:

$$L(\lambda) \simeq L(\lambda^0) \otimes L(\lambda^1)^{[1]} \otimes \cdots \otimes L(\lambda^r)^{[r]},$$

where $L(\lambda^k)^{[k]}$ is $L(\lambda^k)$ with G acting through the k-th Frobenius $F^k : G \to G$ via $[\![g_{ij}]\!] \mapsto [\![g_{ij}^{p^k}]\!]$.

Thus, if $\mathrm{ch}\, L(\lambda^k) = \sum_\mu m_\mu e^\mu$, $\mathrm{ch}\, L(\lambda^k)^{[k]} = \sum_\mu m_\mu e^{p^k \mu}$, and our problem is reduced to finding $\mathrm{ch}\, L(\lambda)$ for $\lambda \in \Lambda_1$ or $\mathrm{ch}\, L(\sum_{i=1}^{n-1} \lambda_i \varpi_i)$ for $\lambda_i \in [0, p[$; $\forall m \in \mathbb{Z}$, $\nabla(m \det + \sum_{i=1}^{n-1} \lambda_i \varpi_i) \simeq$

$\det^{\otimes m} \otimes \nabla(\sum_{i=1}^{n-1} \lambda_i \varpi_i)$ by the tensor identity [J, I.3.6], and hence also $L(m \det + \sum_{i=1}^{n-1} \lambda_i \varpi_i) \simeq \det^{\otimes m} \otimes L(\sum_{i=1}^{n-1} \lambda_i \varpi_i)$.

(月 7) Let $\mathcal{W}_a = \mathcal{W} \ltimes \mathbb{Z}R$, called the affine Weyl group of $\mathcal{W}$, acting on Λ with $\mathbb{Z}R$ by translation. For $\alpha \in R$ let $s_\alpha \in \mathcal{W}$ such that $s_\alpha : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, $\lambda \in \Lambda$, and $s_{\alpha_0,1} : \lambda \mapsto \lambda - \langle \lambda, \alpha_0^\vee \rangle \alpha_0 + \alpha_0$ with $\alpha_0 = \alpha_1 + \cdots + \alpha_{n-1} = \varepsilon_1 - \varepsilon_n$. Under the identification $\mathcal{W} \simeq \mathfrak{S}_n$ one has $s_{\alpha_i} \mapsto (i, i+1)$, $i \in [1, n[$. If $\mathcal{S} = \{s_\alpha | \alpha \in R^{\mathrm{s}}\}$ and $\mathcal{S}_a = \mathcal{S} \cup \{s_{\alpha_0,1}\}$, $(\mathcal{W}_a, \mathcal{S}_a)$ forms a Coxeter system with a subsystem $(\mathcal{W}, \mathcal{S})$ [J, II.6.3]. Let $\ell : \mathcal{W}_a \to \mathbb{N}$ denote the length function on $\mathcal{W}_a$ with respect to $\mathcal{S}_a$, and let $\leq$ denote the Chevalley-Bruhat order on $\mathcal{W}_a$.

We let $\mathcal{W}_a$ act on Λ by setting

$$x \bullet \lambda = px(\frac{1}{p}(\lambda + \zeta)) - \zeta \quad \forall \lambda \in \Lambda \ \forall x \in \mathcal{W}_a.$$

Let $\mathrm{Rep}(G)$ denote the category of finite dimensional representations of G. By $\mathrm{Ext}^1_G(M, M')$ we will mean the 米田-extension of M by M' in $\mathrm{Rep}(G)$ [Weib, pp. 79-80], [dJ, 27]; $\mathrm{Rep}(G)$ admits no nonzero injectives nor projectives.

The linkage principle [J, II.6.17]: $\forall \lambda, \mu \in \Lambda^+$,

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \neq 0 \Rightarrow \lambda \in \mathcal{W}_a \bullet \mu.$$

By the linkage principle one has a decomposition

$$\mathrm{Rep}(G) = \coprod_{\Omega \in \Lambda / \mathcal{W}_a \bullet} \mathrm{Rep}_\Omega(G),$$

where $\mathrm{Rep}_\Omega(G)$ consists of G-modules whose composition factors are all of the form $L(\lambda)$, $\lambda \in \Omega \cap \Lambda^+$. In particular, for $\lambda \in \Omega$

$$\mathrm{ch}\, L(\lambda) \in \mathrm{ch}\, \nabla(\lambda) + \sum_{\substack{\mu \in \Omega \\ \mu < \lambda}} \mathbb{Z}\, \mathrm{ch} \nabla(\mu).$$

We will abbreviate $\mathrm{Rep}_{\mathcal{W}_a \bullet 0}(G)$ as $\mathrm{Rep}_0(G)$ and call it the principal block of G.

(月 8) We extend the $\mathcal{W}_a\bullet$-action on Λ to one on $\Lambda_\mathbb{R} := \Lambda \otimes_\mathbb{Z} \mathbb{R}$. For each $\alpha \in R^+$ and $m \in \mathbb{Z}$ let $H_{\alpha,m} = \{x \in \Lambda_\mathbb{R} | \langle x + \zeta, \alpha^\vee \rangle = mp\}$. We call a connected component of $\Lambda_\mathbb{R} \setminus \cup_{\alpha \in R^+, m \in \mathbb{Z}} H_{\alpha,m}$ an alcove of $\Lambda_\mathbb{R}$. Thus, $\mathcal{W}_a$ acts on the set of alcoves $\mathcal{A}$ in $\Lambda_\mathbb{R}$ simply transitively [J, II.6.2.4]. We call $A^+ = \{x \in \Lambda_\mathbb{R} | \langle x + \zeta, \alpha^\vee \rangle > 0 \ \forall \alpha \in R^+, \langle x + \zeta, \alpha_0^\vee \rangle < p\}$ the bottom dominant alcove of $\mathcal{A}$. Thus the action induces a bijection $\mathcal{W}_a \to \mathcal{A}$ via $w \mapsto w \bullet A^+$. The closure $\overline{A^+}$ is a fundamental domain for $\mathcal{W}_a$ on $\Lambda_\mathbb{R}$ [J, II.6.2.4], i.e., $\forall x \in \Lambda_\mathbb{R}$, $(\mathcal{W}_a \bullet x) \cap \overline{A^+}$ is a singleton. For $A = \{x \in \Lambda_\mathbb{R} | p(m_\alpha - 1) < \langle x + \zeta, \alpha^\vee \rangle < pm_\alpha \ \forall \alpha \in R^+\} \in \mathcal{A}$, $m_\alpha \in \mathbb{Z}$, a facet of A is some $\{x \in \overline{A} |\ p | \langle x + \zeta, \alpha^\vee \rangle \ \forall \alpha \in R_0\}$, $R_0 \subseteq R^+$, and a wall of A is a facet with $|R_0| = 1$. Also, we call $\hat{A} = \{x \in \Lambda_\mathbb{R} | p(m_\alpha - 1) < \langle x + \zeta, \alpha^\vee \rangle \leq pm_\alpha \ \forall \alpha \in R^+\}$ the upper closure of A. One has [J, II.6.2.8]

$$\Lambda \cap A \neq \emptyset \ \exists A \in \mathcal{A} \quad \text{iff} \quad 0 \in A^+ \quad \text{iff} \quad p \geq n,$$

in which case each wall of an alcove contains an element of Λ [J, II.6.3]. Assume from now on throughout the rest of this section that $p \geq n$.

For $\nu \in \Lambda$ let $\mathrm{pr}_\nu = \mathrm{pr}_{\mathcal{W}_a \bullet \nu} : \mathrm{Rep}(G) \to \mathrm{Rep}(G)$ denote the projection onto $\mathrm{Rep}_{\mathcal{W}_a \bullet \nu}(G)$. Now let $\lambda, \mu \in \Lambda \cap \overline{A^+}$. We choose a finite dimensional G-module $V(\lambda, \mu)$ of highest weight $\nu \in \Lambda^+ \cap \mathcal{W}(\mu - \lambda)$ such that $\dim V(\lambda, \mu)_\nu = 1$, e.g., $V(\lambda, \mu) = \nabla(\nu), L(\nu)$. Define the translation functor $T_\lambda^\mu : \mathrm{Rep}(G) \to \mathrm{Rep}(G)$ by setting $T_\lambda^\mu M = \mathrm{pr}_\mu(V(\lambda, \mu) \otimes \mathrm{pr}_\lambda M)$ $\forall M \in \mathrm{Rep}(G)$. A different choice of $V(\lambda, \mu)$ yields an isomorphic functor [J, II.7.6 Rmk. 1]. Each T_λ^μ is exact. As T_μ^λ may be defined with $V(\lambda, \mu)$ replaced by $V(\lambda, \mu)^*$, T_λ^μ and T_μ^λ are adjoint to each other [J, II.7.6]: $\forall M, M' \in \mathrm{Rep}(G)$,

$$\mathrm{Rep}(G)(T_\lambda^\mu M, M') \simeq \mathrm{Rep}(G)(M, T_\mu^\lambda M'). \tag{1}$$

The translation principle: *Let $\lambda, \mu \in \Lambda \cap \overline{A^+}$.*

(i) If λ and μ belong to the same facet, T_λ^μ and T_μ^λ induce a quasi-inverse to each other between $\mathrm{Rep}_{\mathcal{W}_a \bullet \lambda}(G)$ *and* $\mathrm{Rep}_{\mathcal{W}_a \bullet \mu}(G)$ *[J, II.7.9].*

(ii) If λ belongs to a facet F and if $\mu \in \overline{F}$, $\forall x \in \mathcal{W}_a$, $T_\lambda^\mu \nabla(x \bullet \lambda) \simeq \nabla(x \bullet \mu)$ [J, II.7.11].

(iii) If $\lambda \in A^+$ and if $\mu \in \overline{A^+}$ with $\mathrm{C}_{\mathcal{W}_a \bullet}(\mu) = \{1, s\}$ for some $s \in \mathcal{S}_a$, then $\forall x \in \mathcal{W}_a$ with $x \bullet \lambda \in \Lambda^+$ and $xs \bullet \lambda > x \bullet \lambda$, there is an exact sequence [J, II.7.12]

$$0 \to \nabla(x \bullet \lambda) \to T_\mu^\lambda \nabla(x \bullet \mu) \to \nabla(xs \bullet \lambda) \to 0$$

with $T_\mu^\lambda \nabla(x \bullet \mu) \simeq T_\mu^\lambda T_\lambda^\mu \nabla(x \bullet \lambda) \simeq T_\lambda^\mu \nabla(xs \bullet \lambda)$. We note also that the morphisms $\nabla(x \bullet \lambda) \to T_\mu^\lambda \nabla(x \bullet \mu)$ and $T_\mu^\lambda \nabla(x \bullet \mu) \to \nabla(xs \bullet \lambda)$ are unique up to $\Bbbk^\times$;

$$\mathrm{Rep}(G)(\nabla(x \bullet \lambda), T_\mu^\lambda \nabla(x \bullet \mu)) \simeq \mathrm{Rep}(G)(T_\lambda^\mu \nabla(x \bullet \lambda), \nabla(x \bullet \mu)) \simeq \mathrm{Rep}(G)(\nabla(x \bullet \mu), \nabla(x \bullet \mu)) \simeq \Bbbk.$$

(iv) If $\lambda \in A^+$ and if $\mu \in \overline{A^+}$, then $\forall x \in \mathcal{W}_a$ with $x \bullet \lambda \in \Lambda^+$ [J, II.7.13, 7.15],

$$T_\lambda^\mu L(x \bullet \lambda) \simeq \begin{cases} L(x \bullet \mu) & \text{if } x \bullet \mu \in \widehat{x \bullet A^+}, \\ 0 & \text{else.} \end{cases}$$

Thus, all the irreducible characters are obtained by the translation principle from those belonging to the principal block.

(月 9) Weyl's character formula was described using the Weyl group $\mathcal{W}$ of G. To describe the irreducible characters in $\mathrm{Rep}_0(G)$, we require $\mathcal{W}_a$. Let ${}^f\mathcal{W} = \{x \in \mathcal{W}_a | \ell(yx) \geq \ell(x)\ \forall y \in \mathcal{W}\}$. There is a bijection ${}^f\mathcal{W} \to (\mathcal{W}_a \bullet 0) \cap \Lambda^+$ via $w \mapsto w \bullet 0$, and $\mathbb{Z}[\mathcal{W}_a]$ is a free left $\mathbb{Z}[\mathcal{W}]$-module of basis w, $w \in {}^f\mathcal{W}$. Let $\mathrm{sgn}_\mathbb{Z} = \mathbb{Z}$ be the sign representation of $\mathcal{W}$, defining a right $\mathbb{Z}[\mathcal{W}]$-module such that $s \mapsto -1$ $\forall s \in \mathcal{S}$. If $[\mathrm{Rep}_0(G)]$ denotes the Grothendieck group of $\mathrm{Rep}_0(G)$, it has a $\mathbb{Z}$-linear basis $[\nabla(w \bullet 0)]$, $w \in {}^f\mathcal{W}$, by the linkage principle. There follows an isomorphism of $\mathbb{Z}$-modules

$$\mathrm{sgn}_\mathbb{Z} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a] \to [\mathrm{Rep}_0(G)] \quad \text{via} \quad 1 \otimes w \mapsto [\nabla(w \bullet 0)]. \tag{1}$$

We call $\mathrm{sgn}_\mathbb{Z} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a]$ the antispherical module of $\mathbb{Z}[\mathcal{W}_a]$ and denote by M^{asph}. Thus, $\mathrm{Rep}_0(G)$ gives a "categorification" of $\mathrm{sgn}_\mathbb{Z} \otimes_{\mathbb{Z}[\mathcal{W}]} \mathbb{Z}[\mathcal{W}_a]$. The bijection is, moreover, an isomorphism of right $\mathbb{Z}[\mathcal{W}_a]$-modules as follows. For each $s \in \mathcal{S}_a$ choose $\mu \in \Lambda \cap \overline{A^+}$ such that

$\mathrm{C}_{\mathcal{W}_a}(\mu) = \{1, s\}$, and let $\mathrm{T}^s = T_0^\mu$ be a translation functor into the s-wall of A^+ and $\mathrm{T}_s = T_\mu^0$ a translation functor out of the s-wall. We call $\Theta_s = \mathrm{T}_s\mathrm{T}^s$ an s-wall crossing functor. In M^{asph} one has from [S97, p. 86], $\forall s \in \mathcal{S}_a$, $\forall w \in {}^f\mathcal{W}$,

$$1 \otimes w(1+s) = \begin{cases} 1 \otimes ws + 1 \otimes w & \text{if } ws \in {}^f\mathcal{W} \\ 0 & \text{else.} \end{cases}$$

Then, letting $1+s$, $s \in \mathcal{S}_a$, act on $[\mathrm{Rep}_0(G)]$ by Θ_s, makes (1) into an isomorphism of right $\mathbb{Z}[\mathcal{W}_a]$-modules by the translation principle (月 8.iii): $\forall s \in \mathcal{S}_a$,

$$1 \otimes w(1+s) \mapsto [\nabla(w \bullet 0)]\Theta_s = [\Theta_s \nabla(w \bullet 0)].$$

Thus, $[\mathrm{Rep}_0(G)]$ admits a right $\mathcal{W}_a$-action.

For $x \in {}^f\mathcal{W}$ let now $x^s \in M^{\mathrm{asph}}$ such that $x^s \mapsto [L(x \bullet 0)]$. Thus, $\forall y \in {}^f\mathcal{W}$, if $x^s = \sum_{y \in {}^f\mathcal{W}} a_{y,x} y$, $a_{y,x} \in \mathbb{Z}$,

$$\mathrm{ch}\, L(x \bullet 0) = \sum_{y \in {}^f\mathcal{W}} a_{y,x} \mathrm{ch}\, \nabla(y \bullet 0).$$

(月 10) As M^{asph} does not possess enough structure to describe the x^s or $a_{y,x}$ internally, we quantize $\mathbb{Z}[\mathcal{W}_a]$ to 岩堀-Hecke algebra $\mathcal{H}_a$. It is a free $\mathbb{Z}[v, v^{-1}]$-module of basis H_x, $x \in \mathcal{W}_a$, subject to the relations $H_e = 1$, e denoting the unity of $\mathcal{W}_a$, $H_x H_y = H_{xy}$ if $\ell(x) + \ell(y) = \ell(xy)$, and $H_s^2 = 1 + (v^{-1} - v)H_s$ $\forall s \in \mathcal{S}_a$ [S97]. For this and other reasons we will often denote the unity e of $\mathcal{W}_a$ by 1. Under the specialization $v \rightsquigarrow 1$ one has an isomorphism of rings

$$\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H} \simeq \mathbb{Z}[\mathcal{W}_a]. \tag{1}$$

Under the isomorphism we will regard $\mathbb{Z}[\mathcal{W}_a]$ as a right $\mathcal{H}$-module, and hence also $[\mathrm{Rep}_0(G)]$ as a right $\mathcal{H}$-module.

As $(H_s)^{-1} = H_s + (v - v^{-1})$ $\forall s \in \mathcal{S}_a$, every H_x is a unit of $\mathcal{H}$. There is a unique ring endomorphism $\overline{?}$ of $\mathcal{H}$ such that $v \mapsto v^{-1}$ and $H_x \mapsto (H_{x^{-1}})^{-1}$ $\forall x \in \mathcal{W}_a$. Then $\forall x \in \mathcal{W}_a$, there is unique $\underline{H}_x \in \mathcal{H}$ with $\overline{\underline{H}_x} = \underline{H}_x$ and such that $\underline{H}_x \in H_x + \sum_{y \in \mathcal{W}_a} v\mathbb{Z}[v] H_y$, in which case $\underline{H}_x \in H_x + \sum_{y < x} v\mathbb{Z}[v] H_y$ [S97, Th. 2.1]. In particular, $\underline{H}_s = H_s + v$ $\forall s \in \mathcal{S}_a$. For $x, y \in \mathcal{W}_a$ define $h_{x,y} \in \mathbb{Z}[v]$ by the equality $\underline{H}_x = \sum_{y \in \mathcal{W}_a} h_{y,x} H_y$. The $h_{y,x}$ are the celebrated Kazhdan-Lusztig polynomials of $\mathcal{H}$. Let $w_0 = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$ denote the longest element of $\mathcal{W}$. Then Lusztig's conjecture, which is now a theorem for $p \gg 0$, reads [S97, Prop. 3.7], [F, 2.4], [RW, 1.9] that $\forall x \in \mathcal{W}_a$ with $x \bullet 0 \in \Lambda_1$,

$$\mathrm{ch}\, L(x \bullet 0) = \sum_{y \in \mathcal{W}_a} (-1)^{\ell(x) - \ell(y)} h_{w_0 y, w_0 x}(1) \mathrm{ch}\, \nabla(y \bullet 0). \tag{2}$$

A few years ago, however, Williamson [W] astonished the community of representation theory by exhibiting counterexamples to the formula for not so small p. The present work by Riche and Williamson [RW] is their effort to remedy the situation.

(月 11) We have seen a commutative diagram of right $\mathcal{H}$-modules: $\forall w \in {}^f\mathcal{W}$, $\forall s \in \mathcal{S}_a$,

$$
\begin{tikzcd}
\mathbb{Z}\otimes_{\mathbb{Z}[v,v^{-1}]}\mathcal{H} \arrow[d, dash, "\sim"'] \arrow[ddrr] & 1\otimes H_w \arrow[dddrr, mapsto] & 1\otimes H_w(H_s+v) = 1\otimes H_w\underline{H}_s \arrow[ddddrr, mapsto] & & \\
\mathbb{Z}[\mathcal{W}_a] \arrow[d] & & & & \\
M^{\mathrm{asph}} = \mathrm{sgn}_{\mathbb{Z}}\otimes_{\mathbb{Z}[\mathcal{W}]}\mathbb{Z}[\mathcal{W}_a] \arrow[rr, "\sim"] & & {[\mathrm{Rep}_0(G)]} & & \\
1\otimes w \arrow[rrr, mapsto] & & & {[\nabla(w\bullet 0)]} & \\
1\otimes w(s+1) \arrow[rrrr, mapsto] & & & & {[\Theta_s\nabla(w\bullet 0)] = [\nabla(ws\bullet 0)]+[\nabla(w\bullet 0)].}
\end{tikzcd}
\tag{1}
$$

Lusztig's conjecture predicted for $p \geq n$ that, writing $\underline{H}_x = \sum_{y\in\mathcal{W}_a} h_{y,x}H_y$ with the Kazhdan-Lusztig polynomials $h_{y,x}$ for $x \in {}^f\mathcal{W}$ such that $x \bullet 0 \in \Lambda_1$,

$$\sum_{y\in\mathcal{W}_a}(-1)^{\ell(x)-\ell(y)}h_{w_0y,w_0x}(1)\otimes H_y \mapsto [L(w\bullet 0)],$$

which turned out to be false for not so small p.

To remedy the the scheme, enter the tilting G-modules. For $\nu \in \Lambda$ let $\Delta(\nu) = \nabla(-w_0\nu)^*$. Thus, it is nonzero iff $\nu \in \Lambda^+$, in which case it is called the Weyl module of highest weight ν. $\forall \lambda, \nu \in \Lambda^+$, $\forall i \in \mathbb{N}$, one has [J, II.4.13]

$$\mathrm{Ext}^i_G(\Delta(\nu), \nabla(\lambda)) = \delta_{i,0}\delta_{\lambda,\nu}\Bbbk. \tag{2}$$

We say a G-module M admits a Δ- (resp. ∇-) filtration iff it possesses a filtration $M = M^0 > M^1 > \cdots > M^r = 0$ in $\mathrm{Rep}(G)$ such that $\forall i \in [0, r[$, there is $\lambda_i \in \Lambda^+$ with $M^i/M^{i+1} \simeq \Delta(\lambda_i)$ (resp. $\nabla(\lambda_i)$), in which case we denote by $(M : \Delta(\lambda))$ (resp. $(M : \nabla(\lambda))$) the multiplicity of the appearance of $\Delta(\lambda)$ (resp. $\nabla(\lambda)$) in a Δ- (resp. ∇-) filtration. A tilting module is a G-module that admits both a Δ- and a ∇-filtration. Thus, for tilting M, M' one has, $\forall i > 0$,

$$\mathrm{Ext}^i_G(M, M') = 0$$

and, $\forall \lambda \in \Lambda^+$,

$$(M : \Delta(\lambda)) = \dim \mathrm{Rep}(G)(M, \nabla(\lambda)), \quad (M : \nabla(\lambda)) = \dim \mathrm{Rep}(G)(\Delta(\lambda), M).$$

For each $\lambda \in \Lambda^+$ there is a unique, up to isomorphism, indecomposable tilting module $T(\lambda)$ of highest weight λ, and any tilting module is a direct some of those $T(\lambda)$'s [J, E.3, 4]. Writing $\lambda = \lambda^0 + p\lambda^1$ with $\lambda^0 \in \Lambda_1$, put $\hat{\lambda} = w_0 \bullet \lambda^0 + p(\lambda^1 + 2\zeta)$. $\forall y \in {}^f\mathcal{W}$, define $\hat{y} \in {}^f\mathcal{W}$ to be such that $\hat{y} \bullet 0 = \widehat{y \bullet 0}$. Let $\mathcal{W}_0 = \{x \in {}^f\mathcal{W} | \langle x \bullet 0 + \rho, \alpha^\vee\rangle < p(n-1)\ \forall \alpha \in R^+\}$.

Reciprocity [RW, Prop. 1.8.1]: *Assume $p \geq 2(n-1)$. $\forall x, y \in \mathcal{W}_0$,*

$$[\nabla(x\bullet 0) : L(y \bullet 0)] = (T(\hat{y}\bullet 0) : \nabla(x \bullet 0)).$$

Thus, in order to determine the irreducible characters for $p \geq 2(n-1)$, by Steinberg's tensor product theorem and by the translation principle, we may now transform the problem into finding the multiplicities $(T(x\bullet 0):\nabla(y\bullet 0))\ \forall x,y\in {}^f\mathcal{W}$. If ${}^px\in M^{\mathrm{asph}}$ with ${}^px\mapsto [T(x\bullet 0)]$ under the bottom horizontal bijection in (1), one has in M^{asph}

$$ {}^px=\sum_{y\in {}^f\mathcal{W}}(T(x\bullet 0):\nabla(y\bullet 0))(1\otimes y). $$

(月 12) To describe ${}^px$, $x\in {}^f\mathcal{W}$, Riche and Williamson lift it to an element of $\mathcal{H}$, but a little more elaborately. Let $\mathcal{H}_f$ be the 岩堀-Hecke algebra of the Coxeter subsystem $(\mathcal{W},\mathcal{S})$. Thus, $\mathcal{H}_f$ is a $\mathbb{Z}[v,v^{-1}]$-subalgebra of $\mathcal{H}$, having the standard basis H_w, $w\in\mathcal{W}$. Let $\mathrm{sgn}=\mathbb{Z}[v,v^{-1}]$ be a right $\mathcal{H}_f$-module such that $H_s\mapsto -v\ \forall s\in\mathcal{S}$. We set $\mathcal{M}^{\mathrm{asph}}=\mathrm{sgn}\otimes_{\mathcal{H}_f}\mathcal{H}$ and call it the antipherical right module of $\mathcal{H}$. Then $\mathcal{M}^{\mathrm{asph}}$ has a standard $\mathbb{Z}[v,v^{-1}]$-linear basis $1\otimes H_w$, $w\in {}^f\mathcal{W}$, and the Kazhdan-Lusztig $\mathbb{Z}[v,v^{-1}]$-linear basis $1\otimes \underline{H}_w$, $w\in {}^f\mathcal{W}$ [S97, line -2, p. 88]. Thus, $\mathcal{M}^{\mathrm{asph}}$ is a quantization of the antispherical $\mathbb{Z}[\mathcal{W}_a]$-module $M^{\mathrm{asph}}=\mathrm{sgn}_{\mathbb{Z}}\otimes_{\mathbb{Z}[\mathcal{W}]}\mathbb{Z}[\mathcal{W}_a]$: under the specialization $v\mapsto 1$

(1) $$ \mathbb{Z}\otimes_{\mathbb{Z}[v,v^{-1}]}\mathcal{M}^{\mathrm{asph}}\simeq M^{\mathrm{asph}}\simeq[\mathrm{Rep}_0(G)]. $$

Lifting $y\in {}^f\mathcal{W}$ to H_y, we are after a favorable lift ${}^p\underline{H}_x\in\mathcal{H}$ of ${}^px\in M^{\mathrm{asph}}$, $x\in {}^f\mathcal{W}$, such that under (1)

(2) $$ 1\otimes {}^p\underline{H}_x\mapsto {}^px\mapsto [T(x\bullet 0)]. $$

(月 13) Recall that (月 12.1) is an isomorphism of right $\mathcal{H}$-modules: we are to have

$$ [T(x\bullet 0)]=[\nabla(0)]\,{}^p\underline{H}_x=[T(0)]\,{}^p\underline{H}_x. $$

Thus, to realize ${}^p\underline{H}_x$, $x\in {}^f\mathcal{W}$, [RW] exploits a categorification of $\mathcal{H}$ by the diagrammatic Hecke category $\mathcal{D}$ over $\Bbbk$ introduced by Elias and Williamson [EW], and shows that $\mathcal{D}$ act on $\mathrm{Rep}_0(G)$ from the right. The category $\mathcal{D}$, which we will call the EW-category for short, is a strict monoidal category generated by objects $B_s\langle m\rangle$, $s\in\mathcal{S}_a$, $m\in\mathbb{Z}$, and its indecomposable objects are the $B_x\langle m\rangle$, $x\in\mathcal{W}_a$, $m\in\mathbb{Z}$. The split Grothendieck group $[\mathcal{D}]$ of $\mathcal{D}$ comes equipped with a structure of $\mathbb{Z}[v,v^{-1}]$-module such that $v\bullet[M]=[M\langle 1\rangle]$, and there is an isomorphism of $\mathbb{Z}[v,v^{-1}]$-algebras, thabks to [EW],

(1) $$ \mathcal{H}\simeq[\mathcal{D}]\quad\text{such that}\quad \underline{H}_s\mapsto[B_s]\ \forall s\in\mathcal{S}_a, $$

under which [RW] chooses ${}^p\underline{H}_x\mapsto[B_x]\ \forall x\in\mathcal{W}_a$.

To verify that the choice is correct, i.e., correspondence (月 12.2) holds, let $\mathrm{Cat}(\mathrm{Rep}_0(G),\mathrm{Rep}_0(G))$ denote the functor category on $\mathrm{Rep}_0(G)$, which is strict monoidal with respect to the composition.

Theorem [RW, Th. 8.1.1]: *For $p>n\geq 3$ there is a strict monoidal functor*

$$ \Psi:\mathcal{D}\to\mathrm{Cat}(\mathrm{Rep}_0(G),\mathrm{Rep}_0(G))^{\mathrm{op}}\quad\textit{such that}\quad B_s\langle m\rangle\mapsto\Theta_s\ \forall s\in\mathcal{S}_a\ \forall m\in\mathbb{Z}. $$

Thus, the right action of $\mathcal{W}_a$ on $[\mathrm{Rep}_0(G)]$ is now categorified to an action of $\mathcal{D}$ on $\mathrm{Rep}_0(G)$.

(月 14) The functor Ψ induces another functor $\mathcal{D}\to\mathrm{Rep}_0(G)$ such that

$$ B_s\langle m\rangle\mapsto T(0)B_s\langle m\rangle=\Psi(B_s\langle m\rangle)T(0)=\Theta_sT(0)\quad\forall s\in\mathcal{S}_a,\forall m\in\mathbb{Z}. $$

Theorem [RW, Th. 1.3.1]: *Under the same hypothesis of (月 13), $\forall w \in {}^f\mathcal{W}$,*

$$T(0)B_w = \Psi(B_w)T(0) \simeq T(w \bullet 0).$$

2° 火曜日

We will assume from now on throughout the rest of the lecture that $p > n$, unless otherwise specified, which comes partly from the requirement to have the Elias-Williamson categorification $\mathcal{D}$ of the Soergel bimodules to be well-behaved.

(火 1) To define the EW-category $\mathcal{D}$, we start with the diagrammatic Bott-Samelson Hecke category $\mathcal{D}_{\mathrm{BS}}$. For that we have first to define a strict monoidal category.

Definition [中岡, Def. 3.5.2, p. 211]/[Bor, Def. II.6.1.1, p. 292]/ [Mac, pp. 255-256]: A strict monoidal category is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $I \in \mathrm{Ob}(\mathcal{C})$, and a natural "associativity" identity $\alpha_{A,B,C} : (A \otimes B) \otimes C = A \otimes (B \otimes C)$, a natural "left unital" identity $\lambda_A : I \otimes A = A$, and a natural "right unital" identity $\rho_A : A \otimes I = A$.

Thus, the category of endo-functors $\mathrm{Cat}(\mathrm{Rep}(G), \mathrm{Rep}(G))$ from $\mathrm{Rep}(G)$ to itself is a strict monoidal category under the composition of functors.

Given two strict monoidal categories $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{C}', \otimes', I', \alpha', \lambda', \rho')$ a strict monoidal functor $(F, F_2, F_0) : \mathcal{C} \to \mathcal{C}'$ consists of the following data

(M1) $F : \mathcal{C} \to \mathcal{C}'$ is a functor,

(M2) $\forall A, B \in \mathrm{Ob}(\mathcal{C})$, bifunctorial identity $F_2(A, B) \in \mathcal{C}'(F(A) \otimes' F(B), F(A \otimes B))$,

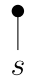
(M3) an identity $F_0 \in \mathcal{C}'(I', F(I))$.

(火 2) Let now $\underline{R} = \mathrm{S}_{\Bbbk}(\Bbbk \otimes_{\mathbb{Z}} \mathbb{Z}R^\vee) = \Bbbk \otimes_{\mathbb{Z}} \mathrm{S}_{\mathbb{Z}}(\mathbb{Z}R^\vee)$ endowed with gradation such that $\deg(R^\vee) = 2$. An expression of an element $x \in \mathcal{W}_a$ is a sequence $(s_1, s_2, \dots, s_r)$ of simple reflections $s_j \in \mathcal{S}_a$ such that $x = s_1 s_2 \dots s_r$. We often denote the sequence by $\underline{s_1 s_2 \dots s_r}$. We will even refer to an expression $\underline{x}$. The length of an expression $\underline{x}$ is denoted $\ell(\underline{x})$.

The objects of $\mathcal{D}_{\mathrm{BS}}$ are denoted $B_{\underline{x}}\langle m \rangle$, $x \in \mathcal{W}_a$, $m \in \mathbb{Z}$, parametrized by the expressions of elements of $\mathcal{W}_a$ and $\mathbb{Z}$. $\mathcal{D}_{\mathrm{BS}}$ is endowed with a shift of the grading autoequivalence $\langle 1 \rangle$ such that $(B_{\underline{x}}\langle m \rangle)\langle 1 \rangle = B_{\underline{x}}\langle m+1 \rangle$; this is not even an additive category, admitting no direct sums. We will abbreviate $B_{\underline{x}}\langle 0 \rangle$ as $B_{\underline{x}}$. Under the product defined on the objects such that $B_{\underline{x}}\langle m \rangle \cdot B_{\underline{y}}\langle m' \rangle = B_{\underline{xy}}\langle m + m' \rangle$ with $\underline{xy}$ denoting the concatenation of $\underline{x}$ and $\underline{y}$, $\mathcal{D}_{\mathrm{BS}}$ comes equipped with a structure of monoidal category. Thus, $B_\emptyset$ is the unital object of $\mathcal{D}_{\mathrm{BS}}$. For $s \in \mathcal{S}_a$ by $\underline{s}$ we mean a sequence s, but we will abbreviate $B_{\underline{s}}\langle m \rangle$ as $B_s\langle m \rangle$. The morphisms in $\mathcal{D}_{\mathrm{BS}}$ are defined using diagrams. An element of $\mathcal{D}_{\mathrm{BS}}(B_{\underline{v}}\langle m \rangle, B_{\underline{w}}\langle m' \rangle)$ is a $\Bbbk$-linear combination of certain equivalence classes of diagrams whose bottom has strands labelled by the simple reflections with multiplicities appearing in $\underline{v}$, and whose top has strands labeled by the simple reflections with multiplicities appearing in $\underline{w}$. Diagrams should be read from bottom to top. The monoidal product correspond to a horizontal concatenation, and the composition to a vertical concatenation. The diagrams, i.e., morphisms, are constructed by horizontal and vertical concatenations of images under autoequivalences $\langle m \rangle$, $m \in \mathbb{Z}$, of 4 different types of generators:

(G1) $\forall f \in \underline{R}$ homogeneous, $B_\emptyset \to B_\emptyset\langle \deg(f)\rangle$ represented diagrammatically as f with empty top and bottom,

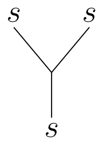
(G2) $\forall s \in \mathcal{S}_a$, the upper dot $B_s \to B_\emptyset\langle 1\rangle$ (resp. the lower dot $B_\emptyset \to B_s\langle 1\rangle$) represented as

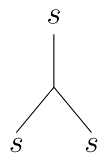


(resp.


),

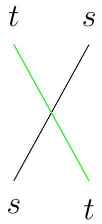
(G3) $\forall s \in \mathcal{S}_a$, the trivalent vertices $B_s \to B_{\underline{ss}}\langle -1\rangle$ (resp. $B_{\underline{ss}} \to B_s\langle -1\rangle$) represented as

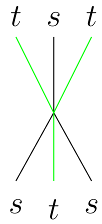


(resp.


),

(G4) $\forall s, t \in \mathcal{S}_a$ with $s \neq t$ and $\mathrm{ord}(st) = m_{st}$ in $\mathcal{W}_a$, the $2m_{st}$-valent vertex $B_{\underbrace{\underline{st\ldots}}_{m_{st}}} \to B_{\underbrace{\underline{ts\ldots}}_{m_{st}}}$ represented as



(resp.


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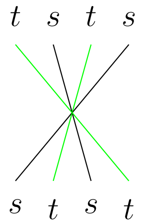

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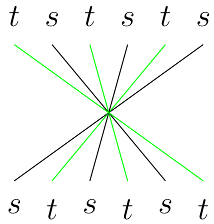
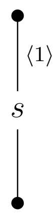


)

if $m_{st} = 2$ (resp. 3, 4, 6).

Those generators are subject to a number of relations described in [EW, §5]. The relations define the "equivalence relations" on the morphisms. We recall only that isotopic diagrams are equivalent, and that, $\forall \alpha \in R^{\mathrm{s}}$, the morphism $\alpha^\vee \in \mathcal{D}_{\mathrm{BS}}(B_\emptyset, B_\emptyset\langle 2\rangle)$ in (G1) is the composition of morphisms in (G2) [EW, 5.1]:

(1) $\alpha^\vee =$


$=$
$$\begin{tikzcd} B_\emptyset\langle 2\rangle & \\ & B_s\langle 1\rangle. \arrow[ul, "(\text{upper dot})\langle 1\rangle"'] \\ B_\emptyset \arrow[uu, dotted] \arrow[ur, "\text{lower dot}"'] & \end{tikzcd}$$

As $\underline{R} = \Bbbk[\alpha^\vee | \alpha \in R^{\mathrm{s}}]$, the morphisms in (G2)-(G4) are, in fact, sufficient to generate all the morphisms in $\mathcal{D}_{\mathrm{BS}}$.

There is also a monoidal equivalence $\tau : \mathcal{D}_{\mathrm{BS}} \to \mathcal{D}_{\mathrm{BS}}^{\mathrm{op}}$ sending each $B_{\underline{w}}\langle m\rangle$ to $B_{\underline{w}}\langle -m\rangle$ and reflecting diagrams along a horizontal axis [RW, 6.3].

(火 3) The EW category $\mathcal{D}$ is the Karoubian envelope $\mathrm{Kar}(\mathcal{D}_{\mathrm{BS}})$ of the additive hull of $\mathcal{D}_{\mathrm{BS}}$ [Bor, Prop. 6.5.9, p. 274]. Thus an object of $\mathcal{D}$ is a direct summand of a finite direct sum of objects of $\mathcal{D}_{\mathrm{BS}}$. The category $\mathcal{D}$ is a graded category inheriting the autoequivalence $\langle 1\rangle$, is Krull-Schmidt, and remains strict monoidal [RW, 1.2, 1.3]. By a Krull-Schmidt category we mean an additive category in which every object is isomorphic to a finite direct sum of indecomposable objects, and an object is indecomposable if and only if its endomorphism ring is local [EW, 6.6]. $\forall w \in \mathcal{W}_a$, $\exists!$ indecomposable $B_w \in \mathrm{Ob}(\mathcal{D})$ such that B_w is a direct summand of each $B_{\underline{w}}$ for a reduced expression $\underline{w}$ of w but is not a direct summand of any $B_{\underline{v}}$ for an expression $\underline{v}$ with $\ell(\underline{v}) < \ell(w)$. Any indecomposable object of $\mathcal{D}$ is isomorphic to some $B_w\langle m\rangle$ for a unique $w \in \mathcal{W}_a$ and a unique $m \in \mathbb{Z}$ [EW, Th. 6.25]. In particular, $B_1 = B_\emptyset$ and $B_s = B_{\underline{s}}$ for each $s \in \mathcal{S}_a$. Thus, $\mathcal{D}$ is generated by objects B_s, $s \in \mathcal{S}_a$. We will write B_x for $B_x\langle 0\rangle$.

Our first task is to define a strict monoidal functor $\mathcal{D}_{\mathrm{BS}} \to \mathrm{Cat}(\mathrm{Rep}(G), \mathrm{Rep}(G))^{\mathrm{op}}$ such that $B_s\langle m\rangle \mapsto \Theta_s$ $\forall s \in \mathcal{S}_a$ $\forall m \in \mathbb{Z}$. The difficulty lies in assignment of generating morphisms and verification of their relations in $\mathrm{Cat}(\mathrm{Rep}(G), \mathrm{Rep}(G))$. We have to find enough relations among the Θ_s's. For that we first make use of an action of the affine Lie algebra $\widehat{\mathfrak{gl}_p}$ over $\mathbb{C}$ on $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)]$, due to Chuang and Rouquier [ChR]. From now on throughout the rest of the lecture we will assume $n \geq 3$.

(火 4) We define the affine Lie algebra $\widehat{\mathfrak{gl}}_N$ associated to $\mathfrak{gl}_N(\mathbb{C})$ as follows. Consider first the Lie algebra $\widehat{\mathfrak{sl}}_N = \mathfrak{sl}_N(\mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}K \oplus \mathbb{C}d$ with $\mathfrak{sl}_N(\mathbb{C}[t, t^{-1}]) = \mathfrak{sl}_N(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ and the Lie bracket defined, for $x, y \in \mathfrak{sl}_N(\mathbb{C})$ and $k, m \in \mathbb{Z}$, by

$$\begin{gathered}[x \otimes t^k, y \otimes t^m] = [x, y] \otimes t^{k+m} + k\delta_{k+m,0}\mathrm{Tr}(xy)K, \\ [d, x \otimes t^m] = mx \otimes t^m, \quad [K, \widehat{\mathfrak{sl}}_N] = 0,\end{gathered}$$

which is the affine Lie algebra of type $\mathrm{A}_{N-1}^{(1)}$ in [谷崎, p. 164]. Then $\widehat{\mathfrak{gl}}_N = \widehat{\mathfrak{sl}}_N \oplus \mathbb{C}$ with $(0, 1) = \mathrm{diag}(1, \dots, 1)$ central in $\widehat{\mathfrak{gl}}_N$, so $\mathfrak{gl}_N(\mathbb{C}) = \mathfrak{sl}_N(\mathbb{C}) \oplus \mathbb{C} \leq \widehat{\mathfrak{gl}}_N$.

Let $e(i, j) \in \mathfrak{gl}_N(\mathbb{C})$, $i, j \in [1, N]$, denote a matrix unit such that $e(i, j)_{ab} = \delta_{a,i}\delta_{b,j}$ $\forall a, b \in [1, N]$. $\forall i \in [0, N[$, let

$$\hat{e}_i = \begin{cases} te(1, N) & \text{if } i = 0, \\ e(i+1, i) & \text{else,} \end{cases} \qquad \hat{f}_i = \begin{cases} t^{-1}e(N, 1) & \text{if } i = 0, \\ e(i, i+1) & \text{else,} \end{cases}$$

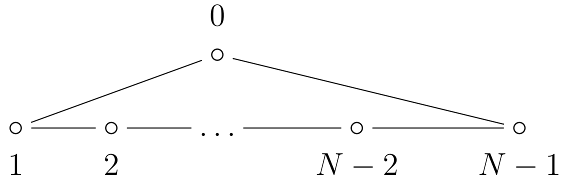
$$\hat{h}_i = [\hat{e}_i, \hat{f}_i] = \begin{cases} e(1, 1) - e(N, N) + K & \text{if } i = 0, \\ e(i+1, i+1) - e(i, i) & \text{else.} \end{cases}$$

The nonstandard indexing of $\hat{e}$ and $\hat{f}$ is chosen so that $\hat{e}_i$ (resp. $\hat{f}_i$) correspond to the endofunctor E_i (resp. F_i) of $\mathrm{Rep}_0(G)$ later in (火 9).

Set $\mathfrak{h} = \mathfrak{h}_f \oplus \mathbb{C}K \oplus \mathbb{C}d < \widehat{\mathfrak{gl}}_N$ with $\mathfrak{h}_f$ denoting the CSA of $\mathfrak{gl}_N(\mathbb{C})$ consisting of the diagonals. Define $(\hat{\varepsilon}_i, K^*, \delta | i \in [1, N])$ to be the dual basis of $(e(i, i), K, d | i \in [1, N])$ in $\mathfrak{h}^*$. Let $P = \{\lambda \in \mathfrak{h}^* | \lambda(\hat{h}_i) \in \mathbb{Z}\ \forall i \in [0, N[\}$. The simple roots of $\mathfrak{h}^*$ are defined by $\hat{\alpha}_0 = \delta - (\hat{\varepsilon}_N - \hat{\varepsilon}_1)$

and $\hat{\alpha}_i = \hat{\varepsilon}_{i+1} - \hat{\varepsilon}_i$, $i \in [1, N[$. Thus, $\forall i, j \in [0, N[$,

$$\hat{\alpha}_i(\hat{h}_j) = \begin{cases} 0 & \text{if } |i-j| \geq 2, \\ -1 & \text{if } |i-j| = 1 \text{ or } (i,j) \in \{(0, N-1), (N-1, 0)\}, \\ 2 & \text{if } i = j, \end{cases}$$
$$[\hat{h}_i, \hat{e}_j] = \hat{\alpha}_j(\hat{h}_i)\hat{e}_j, \quad [\hat{h}_i, \hat{f}_j] = -\hat{\alpha}_j(\hat{h}_i)\hat{f}_j.$$



(火 5) Let $A = \coprod_{i=1}^N \mathbb{C}a_i$ denote the natural module for $\mathfrak{gl}_N(\mathbb{C})$. Then $A \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ affords a module for $\mathfrak{sl}_N(\mathbb{C}[t, t^{-1}])$ such that $(x \otimes t^k) \cdot (a \otimes t^m) = (xa) \otimes t^{k+m}$ $\forall x \in \mathfrak{sl}_N(\mathbb{C})$, $\forall a \in A$ $\forall k, m \in \mathbb{Z}$. One may extend it to a representation of $\widehat{\mathfrak{gl}}_N$ by letting K act by 0, $\mathrm{diag}(1, \dots, 1)$ by the identity, and d by the formula $d \cdot (a \otimes t^m) = ma \otimes t^m$ $\forall a \in A$, $\forall m \in \mathbb{Z}$. We call the resulting $\widehat{\mathfrak{gl}}_N$-module the natural module and denote it by nat_N.

For $\lambda \in \mathbb{Z}$ write $\lambda = \lambda_0 + N\lambda_1$ with $\lambda_0 \in [1, N]$ and $\lambda_1 \in \mathbb{Z}$. Put $m_\lambda = a_{\lambda_0} \otimes t^{\lambda_1}$. Then $\mathrm{nat}_N = \coprod_{\lambda \in \mathbb{Z}} \mathbb{C}m_\lambda$: $\forall \mu \in \mathbb{Z}$, $a_1 \otimes t^\mu = m_{1+N\mu}, a_2 \otimes t^\mu = m_{2+N\mu}, \dots, a_N \otimes t^\mu = m_{N+N\mu}$, and $\hat{e}_0 a_N = te(1, N)a_N = ta_1 = a_1 \otimes t = m_{1+N}$. $\forall i \in [0, N[$,

$$\hat{e}_i m_\lambda = \begin{cases} m_{\lambda+1} & \text{if } i \equiv \lambda \bmod N, \\ 0 & \text{else,} \end{cases} \tag{1}$$

$$\hat{f}_i m_\lambda = \begin{cases} m_{\lambda-1} & \text{if } i \equiv \lambda - 1 \bmod N, \\ 0 & \text{else,} \end{cases} \tag{2}$$

and $\forall h \in \mathfrak{h}$,

$$h m_\lambda = (\hat{\varepsilon}_{\lambda_0} + \lambda_1 \delta)(h) m_\lambda. \tag{3}$$

In particular, all $\mathfrak{h}$-weight spaces of nat_N are 1-dimensional.

(火 6) Recall the natural module $V = \Bbbk^{\oplus n}$ for G with the standard basis $v_1, \dots, v_n$, and its dual V^* with the dual basis $v_1^*, \dots, v_n^*$. Thus, $V = L(\varepsilon_1) = \nabla(\varepsilon_1) = \Delta(\varepsilon_1) = T(\varepsilon_1)$ and $V^* = L(-w_0\varepsilon_1) = L(-\varepsilon_n) = \nabla(-\varepsilon_n) = \Delta(-\varepsilon_n) = T(-\varepsilon_n)$. Define 2 exact endofunctors E and F of $\mathrm{Rep}_0(G)$ by $E = V \otimes ?$ and $F = V^* \otimes ?$, resp. Define $\eta_\Bbbk \in \mathrm{Rep}(G)(\Bbbk, V^* \otimes V)$ such that $\eta_\Bbbk(1) = \sum_i v_i^* \otimes v_i$ and $\varepsilon_\Bbbk \in \mathrm{Rep}(G)(V \otimes V^*, \Bbbk)$ such that $v \otimes \mu \mapsto \mu(v)$; under a $\Bbbk$-linear isomorphism $V^* \otimes V \simeq \mathrm{Mod}_\Bbbk(V, V)$ via $f \otimes v \mapsto f(?)v$ with inverse $\sum_i v_i^* \otimes \phi(v_i) \leftarrow\!\shortmid \phi$, $\sum_i v_i^* \otimes v_i$ corresponds to id_V, and hence fixed by G. In turn, $\eta_\Bbbk$ defines a natural transformation $\eta : \mathrm{id}_{\mathrm{Rep}(G)} \Rightarrow FE$ via

$$\begin{tikzcd} M \arrow[r, dotted, "\eta_M"] \arrow[d, "\sim"'] & FE(M) \arrow[d, equal] \\ \Bbbk \otimes M \arrow[r, "\eta_\Bbbk \otimes M"'] & V^* \otimes V \otimes M, \end{tikzcd}$$

while $\varepsilon_{\Bbbk}$ defines a natural transformation $\varepsilon : EF \Rightarrow \mathrm{id}_{\mathrm{Rep}(G)}$ via

$$\begin{tikzcd} EF(M) \arrow[r, dotted, "\varepsilon_M"] \arrow[d, equal] & M \arrow[d, "\sim"] \\ V \otimes V^* \otimes M \arrow[r, "\varepsilon_{\Bbbk}\otimes M"'] & \Bbbk \otimes M \end{tikzcd}$$

to make η (resp. ε) into the unit (resp. counit) of an adjunction (E, F) [中岡, Cor. 2.2.9, pp. 65-66] such that

(1) $\quad \mathrm{Rep}(G)(M, FM') \xrightarrow{\sim} \mathrm{Rep}(G)(EM, M')$ via $\psi \mapsto \varepsilon_{M'} \circ E\psi$ with inverse $F\phi \circ \eta_M \leftarrow\!\shortmid \phi$.

Explicitly, $\forall m \in M$,

$$(F\phi \circ \eta_M)(m) = \sum_i v_i^* \otimes \phi(v_i \otimes m),$$

while, if we write $\psi(m) = \sum_i v_i^* \otimes \psi(m)_i$, $\forall v \in V$,

$$(\varepsilon_{M'} \circ E\psi)(v \otimes m) = \sum_i v_i^*(v)\psi(m)_i.$$

Now, let $\mathfrak{g} = \mathfrak{gl}_n(\Bbbk)$ equipped with the structure of G-module Ad: $g\bullet x = gxg^{-1}\ \forall g \in G\ \forall x \in \mathfrak{g}$; we identify $\mathfrak{g}$ with $\mathrm{Lie}(G) = \mathrm{Mod}_{\Bbbk}(\mathfrak{m}/\mathfrak{m}^2, \Bbbk)$, $\mathfrak{m} = (x_{ij}, x_{ii} - 1 | i, j \in [1, n], i \neq j) \triangleleft \Bbbk[G]$. $\forall M \in \mathrm{Rep}(G)$, the $\mathfrak{g}$-action on M given by differentiating the G-action $\Delta_M : M \to M \otimes \Bbbk[G]$

$$\begin{tikzcd} \mathfrak{g} \otimes M \arrow[r, dotted, "\mathfrak{a}"] \arrow[d, "\mathfrak{g}\otimes\Delta_M"'] & M & x(f)m \\ \mathfrak{g} \otimes M \otimes \Bbbk[G] \arrow[ur] & x \otimes m \otimes f \arrow[ur, mapsto] & \end{tikzcd}$$

is G-equivariant [J, I.7.18.1]. Let $\eta'_{\Bbbk} : \Bbbk \to V \otimes V^*$ via $1 \mapsto \sum_i v_i \otimes v_i^*$ to define an adjunction (F, E) as above. Using a natural isomorphism $\mathfrak{g} \simeq V^* \otimes V$ via $\mu(?)v \leftarrow\!\shortmid \mu \otimes v$, define for $M \in \mathrm{Rep}(G)$

$$\begin{tikzcd} V \otimes M \arrow[r, dotted, "\mathbb{X}_M"] \arrow[d, "\eta'_{\Bbbk}\otimes V\otimes M"'] & V \otimes M \\ V \otimes V^* \otimes V \otimes M \arrow[r, "\sim"'] & V \otimes \mathfrak{g} \otimes M, \arrow[u, "V\otimes\mathfrak{a}"'] \end{tikzcd}$$

which is functorial in M. Thus, one obtains an endomorphism $\mathbb{X} \in \mathrm{Cat}(\mathrm{Rep}(G), \mathrm{Rep}(G))(E, E)$ of E, i.e., a natural transformation from E to itself. In particular, each $\mathbb{X}_M$ is G-equivariant. In turn, $\mathbb{X}$ induces by adjunction (E, F) an endomorphism $\mathbb{X}'$ of F:

$$\begin{tikzcd} V^* \otimes M \arrow[r, dotted, "\mathbb{X}'_M"] \arrow[d, "\eta_{\Bbbk}\otimes V^*\otimes M"'] & V^* \otimes M \\ V^* \otimes V \otimes V^* \otimes M \arrow[r, "V^*\otimes\mathbb{X}_{V^*\otimes M}"'] & V^* \otimes V \otimes V^* \otimes M. \arrow[u, "V^*\otimes\varepsilon_{\Bbbk}\otimes M"'] \end{tikzcd}$$

Thus, $\forall M' \in \mathrm{Rep}(G)$,

(2)
$$\begin{tikzcd}
\mathrm{Rep}(G)(EM, M') & \mathrm{Rep}(G)(EM, M') \arrow[l, "\mathrm{Rep}(G)(\mathbb{X}_M, M')"'] \\
\mathrm{Rep}(G)(M, FM') \arrow[u, "\varepsilon_{M'} \circ E", "\sim"'] \arrow[ur, phantom, "\circlearrowright"] & \mathrm{Rep}(G)(M, FM'). \arrow[l, "\mathrm{Rep}(G)(M, \mathbb{X}'_{M'})"] \arrow[u, "\sim", "\varepsilon_{M'} \circ E"']
\end{tikzcd}$$

Let $\mathrm{Dist}(G)$ denote the algebra of distributions on G. As G is defined over $\mathbb{Z}$, $\mathrm{Dist}(G)$ has a $\mathbb{Z}$-form $\mathrm{Dist}(G_{\mathbb{Z}})$ which coincides with Kostant's $\mathbb{Z}$-form of the universal enveloping algebra $\mathbb{U}(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$. Put $\Omega = \sum_{i,j=1}^{n} e(i,j) \otimes e(j,i) \in \mathfrak{g} \otimes \mathfrak{g}$; $\mathrm{Tr}(e(i,j)e(k,l)) = \delta_{jk}\mathrm{Tr}(e(i,l)) = \delta_{jk}\delta_{il}$. For $x \in \mathfrak{g}$ put $\Delta(x) = x \otimes 1 + 1 \otimes x$. If M and M' are G-modules, recall that $\mathrm{Dist}(G)$ acts on the G-module $M \otimes M'$ via $x \mapsto \Delta(x)$, $x \in \mathfrak{g}$.

Lemma: *(i) $\forall v, v' \in V$, $\Omega \cdot (v \otimes v') = v' \otimes v$.*

(ii) $\forall x \in \mathfrak{g}$, $\Omega\Delta(x) = \Delta(x)\Omega$ in $\mathrm{Dist}(G) \otimes \mathrm{Dist}(G)$, and hence the action of Ω on $M \otimes M'$ for $M, M' \in \mathrm{Rep}(G)$ commutes with the action of $\mathrm{Dist}(G)$.

Proof: Exercise.

(火 7) We now describe $\mathbb{X}$ and $\mathbb{X}'$ using Ω. Recall from [HLA, 10.7, p. 76] that $\forall x \in \mathfrak{g}$ $\forall f \in V^*$ $\forall m \in M$,
$$x \cdot (f \otimes m) = (xf) \otimes m + f \otimes xm = -f(x?) \otimes m + f \otimes xm.$$
In particular, x acts on V^* via $-x^{\mathrm{t}}$ with respect to the dual basis:

(1)
$$e(i,j)v_k^* = -\delta_{ik}v_j^*.$$

Lemma [RW, 6.3]: *Let $M \in \mathrm{Rep}(G)$.*

(i) $\mathbb{X}_M : EM = V \otimes M \to V \otimes M = EM$ is given by the action of Ω.

(ii) $\mathbb{X}'_M : FM = V^* \otimes M \to V^* \otimes M = FM$ is given by the action of $-n\mathrm{id}_{V^* \otimes M} - \Omega$.

(iii) $(V \otimes \mathbb{X}_M) \circ \mathbb{X}_{V \otimes M} = \mathbb{X}_{V \otimes M} \circ (V \otimes \mathbb{X}_M)$.

(iv) $(V^{\otimes_2} \otimes \mathbb{X}_M) \circ \mathbb{X}_{V^{\otimes_2} \otimes M} = \mathbb{X}_{V^{\otimes_2} \otimes M} \circ (V^{\otimes_2} \otimes \mathbb{X}_M)$.

(v) $\mathbb{X}_{FM} \circ (V \otimes \mathbb{X}'_M) = (V \otimes \mathbb{X}'_M) \circ \mathbb{X}_{FM}$.

(vi) $\mathbb{X}'_{EM} \circ (V^* \otimes \mathbb{X}_M) = (V^* \otimes \mathbb{X}_M) \circ \mathbb{X}'_{EM}$.

Proof: Exercise.

(火 8) Recall from (火 6) the unit η and the counit ε of an adjoint pair (E, F), and also the unit η' and the counit ε' of an adjoint pair (F, E) induced by $\eta'_{\Bbbk} : \Bbbk \to V \otimes V^*$ via $1 \mapsto \sum_i v_i \otimes v_i^*$ and $\varepsilon'_{\Bbbk} : V^* \otimes V \to \Bbbk$ via $\xi \otimes v \mapsto \xi(v)$.

Lemma: *Let $M \in \mathrm{Rep}(G)$ and $r \in \mathbb{N}$.*

(i) $(\mathbb{X}'_{EM})^r \circ \eta_M = (V^* \otimes \mathbb{X}_M)^r \circ \eta_M, \quad \varepsilon_M \circ (\mathbb{X}_{FM})^r = \varepsilon_M \circ (V \otimes \mathbb{X}'_M)^r.$

(ii) $(\mathbb{X}_{FM})^r \circ \eta'_M = (V \otimes \mathbb{X}'_M)^r \circ \eta'_M, \quad \varepsilon'_M \circ (\mathbb{X}'_{EM})^r = \varepsilon'_M \circ (V^* \otimes \mathbb{X}_M)^r.$

Proof: Exercise.

(火 9) $\forall a \in \mathbb{k}$, let E_a (resp. F_a) denote the direct summand of E (resp. F) given by the generalized a-eigenspace of $\mathbb{X}$ (resp. $\mathbb{X}'$) acting on E (resp. F): $\forall M \in \mathrm{Rep}(G)$,

$$\begin{aligned} EM &= \coprod_{a \in \mathbb{k}} (E_a M) \quad \text{with} \quad E_a M = \cup_{r \in \mathbb{N}} \ker((\mathbb{X}_M - a\mathrm{id}_{EM})^r), \\ FM &= \coprod_{a \in \mathbb{k}} (F_a M) \quad \text{with} \quad F_a M = \cup_{r \in \mathbb{N}} \ker((\mathbb{X}'_M - a\mathrm{id}_{FM})^r). \end{aligned}$$

As $\mathbb{X}_M$ and $\mathbb{X}'_M$ are G-equivariant, each E_a (resp. F_a) is a direct summand of E (resp. F) as an endofunctor on $\mathrm{Rep}(G)$.

Lemma [RW, 6.3]: *Let $a \in \mathbb{k}$.*

(i) The unit η and the counit ε of the adjunction (E, F) induce a unit $\eta_a : \mathrm{id} \to F_a E_a$ and a counit $\varepsilon_a : E_a F_a \to \mathrm{id}$, resp., making (E_a, F_a) into an adjoint pair.

(ii) The unit η' and the counit ε' induce a unit $\eta'_a : \mathrm{id} \to E_a F_a$ and a counit $\varepsilon'_a : F_a E_a \to \mathrm{id}$ of an adjunction (F_a, E_a).

Proof: (i) We first show that η (resp. ε) factors through $\coprod_{a \in \mathbb{k}} \eta_a : \mathrm{id} \to \coprod_{a \in \mathbb{k}} F_a E_a$ (resp. $\coprod_{a \in \mathbb{k}} \varepsilon_a : \coprod_{a \in \mathbb{k}} E_a F_a \to \mathrm{id}$)

(1)
$$\begin{tikzcd} \mathrm{id} \arrow[r, "\eta"] \arrow[dr, dotted, "\coprod_{a\in\mathbb{k}} \eta_a"'] & FE \\ & \coprod_{a\in\mathbb{k}} F_a E_a \arrow[u, hook] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} EF \arrow[r, "\varepsilon"] \arrow[d] & \mathrm{id} \\ \coprod_{a\in\mathbb{k}} E_a F_a \arrow[ur, dotted, "\coprod_{a\in\mathbb{k}} \varepsilon_a"'] & \end{tikzcd}$$

Let $M \in \mathrm{Rep}(G)$, $m \in M$ and $d = \dim FEM$. Let $\eta(m)_{ab}$ be the $F_a E_b M$ component of $\eta_M(m)$. Then

$$\begin{aligned} 0 &= (\mathbb{X}'_{EM} - a\mathrm{id})^d \eta(m)_{ab} \quad \text{as } \eta(m)_{ab} \in F_a(E_b M) \\ &= ((V^* \otimes \mathbb{X}_M) - a\mathrm{id})^d \eta(m)_{ab} \quad \text{by (火 8.i)}. \end{aligned}$$

On the other hand, $0 = (V^* \otimes (\mathbb{X}_M - b\mathrm{id}))^d \eta(m)_{ab}$ as $\eta(m)_{ab} \in V^* \otimes (E_b M)$. It follows that $\eta(m)_{ab} = 0$ unless $a = b$, and hence $\mathrm{im}(\eta_M) \leq \coprod_{a \in \mathbb{k}} F_a E_a M$.

Let next $x \in E_a F_b M$ with $a \neq b$. Take polynomials $\phi, \psi \in \mathbb{k}[t]$ with $(t-a)^d \phi + (t-b)^d \psi = 1$.

Then

$$\begin{aligned}
\varepsilon_M(x) &= \varepsilon_M(\{\phi(\mathbb{X}_{FM})(\mathbb{X}_{FM} - a\mathrm{id})^d + \psi(\mathbb{X}_{FM})(\mathbb{X}_{FM} - b\mathrm{id})^d\}x) \\
&= \varepsilon_M(\psi(\mathbb{X}_{FM})(\mathbb{X}_{FM} - b\mathrm{id})^d x) \quad \text{as } x \in E_a(FM) \\
&= \varepsilon_M(\psi(\mathbb{X}_{FM})(V \otimes \mathbb{X}'_M - b\mathrm{id})^d x) \quad \text{by (火 8.i)} \\
&= \varepsilon_M(\psi(\mathbb{X}_{FM})(V \otimes (\mathbb{X}'_M - b\mathrm{id})^d)x) \\
&= 0 \quad \text{as } x \in E(F_bM),
\end{aligned}$$

and hence (1) holds.

Recall from (火 6.1) the adjunction $\mathrm{Rep}(G)(EM, M') \simeq \mathrm{Rep}(G)(M, FM')$ given by $f \mapsto (Ff) \circ \eta_M$ with inverse $g \mapsto \varepsilon_{M'} \circ Eg$. As each E_a (resp. F_a) is a direct summand of E (resp. F), one obtains commutative diagrams

$$\begin{tikzcd}
\mathrm{Rep}(G)(EM, M') \arrow[r, "F"] \arrow[d, equal] & \mathrm{Rep}(G)(FEM, FM') \arrow[r, "\mathrm{Rep}(G)(\eta_M, FM')"] \arrow[d, equal] & \mathrm{Rep}(G)(M, FM') \arrow[dd, equal] \\
\prod_a \mathrm{Rep}(G)(E_aM, M') \arrow[r, "\prod_a F"'] & \prod_a \mathrm{Rep}(G)(FE_aM, FM') \arrow[d, equal] & \\
& \prod_a \prod_b \mathrm{Rep}(G)(F_bE_aM, F_bM') \arrow[d] & \prod_a \mathrm{Rep}(G)(M, F_aM') \\
& \prod_a \mathrm{Rep}(G)(F_aE_aM, F_aM') \arrow[ur, "\prod_a \mathrm{Rep}(G)(\eta_{a,M}, F_aM')"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Rep}(G)(M, FM') \arrow[r, "E"] \arrow[d, equal] & \mathrm{Rep}(G)(EM, EFM') \arrow[r, "\mathrm{Rep}(G)(EM, \varepsilon_{M'})"] \arrow[d, equal] & \mathrm{Rep}(G)(EM, M') \arrow[dd, equal] \\
\prod_a \mathrm{Rep}(G)(M, F_aM') \arrow[r, dotted, "\prod_a E"'] & \prod_a \mathrm{Rep}(G)(EM, EF_aM') \arrow[d, equal] & \\
& \prod_a \prod_b \mathrm{Rep}(G)(E_bM, E_bF_aM') \arrow[d] & \prod_b \mathrm{Rep}(G)(E_bM, M'). \\
& \prod_b \mathrm{Rep}(G)(E_bM, E_bF_bM') \arrow[ur, "\prod_b \mathrm{Rep}(G)(E_bM, \varepsilon_{b,M'})"'] &
\end{tikzcd}$$

One thus obtains for each $a \in \Bbbk$ isomorphisms $\mathrm{Rep}(G)(E_aM, M') \simeq \mathrm{Rep}(G)(M, F_aM')$ via $f \mapsto F_a(f) \circ \eta_{a,M}$ and $\varepsilon_{a,M'} \circ E_a(g) \leftarrow\!\shortmid g$ inverse to each other.

(ii) As in (i) it suffices to show that the induced counit $\eta' : \mathrm{id} \to EF$ (resp. unit $\varepsilon' : FE \to \mathrm{id}$) factors through $\coprod_{a \in \Bbbk} E_aF_a$ (resp. $\coprod_{a \in \Bbbk} F_aE_a$)

$$\begin{tikzcd}
\mathrm{id} \arrow[r, "\eta'"] \arrow[dr, dotted] & EF \\
& \coprod_a E_aF_a \arrow[u]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
FE \arrow[r, "\varepsilon'"] \arrow[d] & \mathrm{id}. \\
\coprod_a F_aE_a \arrow[ur, dotted] &
\end{tikzcd}$$

Let $\eta'(m)_{ab}$ be the E_aF_bM-component of $\eta'_M(m)$. One has

$$0 = (\mathbb{X}_{FM} - a\mathrm{id})^d\eta'(m)_{ab} = ((V \otimes \mathbb{X}'_M) - a\mathrm{id})^d\eta'(m)_{ab} \quad \text{by (火 8.ii)}$$

while $0 = \{V \otimes (\mathbb{X}'_M - b\mathrm{id})\}^d\eta'_M(m)_{ab}$, and hence $\eta'_M(m) = 0$ unless $n + a = n + b$. Thus, $\mathrm{im}(\eta'_M) \leq \coprod_a E_aF_aM$.

Let finally $y \in F_aE_bM$ with $a \neq b$. Then, with $\phi, \psi \in \Bbbk[t]$ as above,

$$\begin{aligned}\varepsilon'_M(y) &= \varepsilon'_M(\{\phi(\mathbb{X}'_{EM})(\mathbb{X}'_{EM} - a\mathrm{id})^d + \psi(\mathbb{X}'_{EM})(\mathbb{X}'_{EM} - b\mathrm{id})^d\}y) = \varepsilon'_M(\psi(\mathbb{X}'_{EM})(\mathbb{X}'_{EM} - b\mathrm{id})^d y) \\ &= \varepsilon'_M(\psi(\mathbb{X}'_{EM})(V^* \otimes \mathbb{X}_M - b\mathrm{id})^d y) \quad \text{by (火 8.ii)} \\ &= 0, \quad \text{as desired.}\end{aligned}$$

3° 水曜日

To answer the question of the choice of ${}^p\underline{H}_w$ for $w \in {}^f\mathcal{W}$ we note that, as those correspond to the indecomposables B_w of $\mathcal{D}$, they extend to ${}^p\underline{H}_x$, $x \in \mathcal{W}_a$, to form a $\mathbb{Z}[v, v^{-1}]$-linear basis of $\mathcal{H}$, and coincide with the $\underline{H}_x$ for $p \gg 0$. Just like the latter have geometric counterpart the intersection cohomology over the affine flag variety, the ${}^p\underline{H}_x$ are related to the parity sheaves on the affine flag variety. Thus, the ${}^p\underline{H}_x$ are named p-KL polynomials.

(水 1) Recall now from (火 1) with $N = p$ the affine Lie algebra $\widehat{\mathfrak{gl}_p}$ over $\mathbb{C}$ and from (火 2) its natural representation nat_p.

Proposition [RW, 6.3]: *(i) $\forall a \in \Bbbk \setminus \mathbb{F}_p$, $E_a = 0 = F_a$, and hence $E = \coprod_{a \in \mathbb{F}_p} E_a$, $F = \coprod_{a \in \mathbb{F}_p} F_a$.*

(ii) Let $\phi : \mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)] \to \wedge^n(\mathrm{nat}_p)$ be a $\mathbb{C}$-linear isomorphism via

$$1 \otimes [\Delta(\lambda)] \mapsto m_{\lambda_1} \wedge m_{\lambda_2 - 1} \wedge \cdots \wedge m_{\lambda_n - n + 1} \quad \forall \lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+.$$

$\forall a \in \mathbb{F}_p$, regarding it as an element of $[0, p[$, one has a commutative diagram

$$\begin{tikzcd}
\wedge^n(\mathrm{nat}_p) \arrow[d, "\hat{e}_a"'] & \mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)] \arrow[l, "\phi"', "\sim"] \arrow[r, "\phi", "\sim"'] \arrow[d, shift right, "\mathbb{C}\otimes_{\mathbb{Z}}[E_a]"'] \arrow[d, shift left, "\mathbb{C}\otimes_{\mathbb{Z}}[F_a]"] & \wedge^n(\mathrm{nat}_p) \arrow[d, "\hat{f}_a"] \\
\wedge^n(\mathrm{nat}_p) & \mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)] \arrow[l, "\sim", "\phi"'] \arrow[r, "\sim", "\phi"'] & \wedge^n(\mathrm{nat}_p).
\end{tikzcd}$$

Thus, we may regard the exact functors E_a, F_a, $a \in [0, p[$, as part of an action of $\widehat{\mathfrak{gl}_p}$ on $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)]$ through ϕ.

(iii) The "block" decomposition $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)] = \coprod_{b \in \Lambda/\mathcal{W}_{a\bullet}} \mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}_b(G)]$ reads as the weight space decomposition of $\wedge^n(\mathrm{nat}_p)$ under ϕ; each $\phi(\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}_b(G)])$ provides a distinct weight space on $\wedge^n(\mathrm{nat}_p)$ of weight $\sum_{i=1}^n (\lambda_i - i + 1)_1\delta + \sum_{j=1}^p n_j\hat{\varepsilon}_j$ with $n_j = |\{k \in [1, n] | \lambda_k - k + 1 \equiv j \mod p\}|$ if $\lambda = (\lambda_1, \ldots, \lambda_n) \in b$; for $r \in \mathbb{Z}$ we write $r = r_0 + pr_1$ with $r_0 \in [1, p]$.

Proof: Details will be given in (水 3) with G replaced by G_1T.

(水 2) From (水 1.iii) we see that the set of weights of $\wedge^n(\mathrm{nat}_p)$ is

$$P(\wedge^n(\mathrm{nat}_p)) = \{k\delta + \sum_{i=1}^p n_i\hat{\varepsilon}_i | k \in \mathbb{Z}, n_i \in \mathbb{N}, \sum_{i=1}^p n_i = n\}.$$

We will denote the bijection $P(\wedge^n(\mathrm{nat}_p)) \to \Lambda/(\mathcal{W}_a\bullet)$ by ι_n. Note that $\Lambda/(\mathcal{W}_a\bullet)$ is infinite; $\Lambda = \mathbb{Z}\det \oplus \coprod_{i=1}^{n-1}\mathbb{Z}\varpi_i$ with $\mathcal{W}_a$ acting trivially on the $\mathbb{Z}\det$-component.

Let now $\varpi = \hat{\varepsilon}_1 + \cdots + \hat{\varepsilon}_n$. As $\phi([\Delta(n,\ldots,n)])$ has weight ϖ, $\iota_n(\varpi) = \mathcal{W}_a \bullet (n,\ldots,n) = \mathcal{W}_a \bullet n\det$ with $n\det \in A^+$. $\forall i \in [1,n[$, $\phi([\Delta(\underbrace{n,\ldots,n}_{n-i}, n+1, n,\ldots,n)])$ has weight $\varpi + \hat{\alpha}_i$, and hence $\iota_n(\varpi+\hat{\alpha}_i) = \mathcal{W}_a\bullet(\underbrace{n,\ldots,n}_{n-i}, n+1, n,\ldots,n) = \mathcal{W}_a\bullet(n\det+\varepsilon_{n-i+1})$. Put $\mu_{s_j} = n\det + \varepsilon_{j+1}$, $j \in [1,n[$. $\forall k \in [0,n[$,

$$\begin{aligned}\langle \mu_{s_j} + \zeta, \alpha_k^\vee\rangle &= \begin{cases} 1 + \langle \varepsilon_{j+1}, \alpha_k^\vee\rangle & \text{if } k \neq 0,\\ n-1+\langle \varepsilon_{j+1}, \varepsilon_1^\vee - \varepsilon_n^\vee\rangle & \text{if } k = 0\end{cases}\\ &= \begin{cases} 0 & \text{if } k = j,\\ 2 & \text{if } k = j+1,\\ n-1 & \text{if } k = 0 \text{ and } j \neq n-1,\\ n-2 & \text{if } k = 0 \text{ and } j = n-1,\\ 1 & \text{else,}\end{cases}\end{aligned}$$

and hence μ_{s_j} lies in the s_{α_j}-wall of A^+. For $\lambda \in \Lambda$, let us abbreviate $\mathcal{W}_a \bullet \lambda$ as $[\lambda]$, and write $i_{[\lambda]} : \mathrm{Rep}_{[\lambda]}(G) \hookrightarrow \mathrm{Rep}(G)$. Then

$$\begin{aligned} E_{n-j}|_{\mathrm{Rep}_{[n\det]}(G)} &= E_{n-j}|_{\mathrm{Rep}_{\iota_n(\varpi)}(G)} = \mathrm{pr}_{\iota_n(\varpi+\hat{\alpha}_{n-j})}(V\otimes ?) \quad \text{by (水 1)}\\ &\qquad \text{as the action of } \hat{e}_{n-j} \text{ increases the weight by } \hat{\alpha}_{n-j} \text{ (火 4)}\\ &= \mathrm{pr}_{[\mu_{s_j}]}(V \otimes \mathrm{pr}_{[n\det]}?) \circ i_{[n\det]} = \mathrm{pr}_{[\mu_{s_j}]}(\nabla(\varepsilon_1) \otimes \mathrm{pr}_{[n\det]}?) \circ i_{[n\det]}.\end{aligned}$$

We could abbreviate $\mathrm{pr}_{[\mu_{s_j}]}$ as $\mathrm{pr}_{\mu_{s_j}}$ after the convention in (月 8). As $\mu_{s_j} - n\det = \varepsilon_{j+1} \in \mathcal{W}\varepsilon_1$, $\mathrm{pr}_{[\mu_{s_j}]}(V \otimes \mathrm{pr}_{[n\det]}?)$ may be taken to be the translation functor $\mathrm{T}_{n\det}^{\mu_{s_j}}$ by (月 8), and hence

$$E_{n-j}|_{\mathrm{Rep}_{[n\det]}(G)} = \mathrm{T}_{n\det}^{\mu_{s_j}}|_{\mathrm{Rep}_{[n\det]}(G)}.$$

Likewise, as $n\det - \mu_{s_j} = -\varepsilon_{j+1} \in \mathcal{W}(-\varepsilon_n) = \mathcal{W}(-w_0\varepsilon_1)$ and as $V^* \simeq \nabla(-w_0\varepsilon_1)$, one may regard $F_{n-j}|_{\mathrm{Rep}_{[\mu_{s_j}]}(G)}$ as the translation functor $\mathrm{T}_{\mu_{s_j}}^{n\det}|_{\mathrm{Rep}_{[\mu_{s_j}]}(G)}$.

Consider next $\mu_{s_0} = (p+1, n, \ldots, n) \in \Lambda^+$. $\forall k \in [0,n[$,

$$\langle \mu_{s_0} + \zeta, \alpha_k^\vee\rangle = \begin{cases} p & \text{if } k = 0,\\ p-n+2 & \text{if } k = 1,\\ 1 & \text{else,}\end{cases}$$

and hence μ_{s_0} lies in the $s_{\alpha_0,1}$-wall of A^+. This proves part (i) of the following

Corollary [RW, Rmk. 6.4.7]: *(i) $\forall j \in [1,n[$, one may regard E_{n-j} (resp., F_{n-j}) as the translation functor $\mathrm{T}_{n\det}^{\mu_{s_j}}$ (resp. $\mathrm{T}^{n\det}_{\mu_{s_j}}$) restricted to $\mathrm{Rep}_{[n\det]}(G)$ (resp. $\mathrm{Rep}_{[\mu_{s_j}]}(G)$).*

(ii) One may take $E_0E_{p-1}\dots E_{n+1}E_n|_{\mathrm{Rep}_{[n\det]}(G)}$ (resp. $F_nF_{n+1}\dots F_{p-1}F_0|_{\mathrm{Rep}_{[\mu_{s_0}]}(G)}$) to be the translation functor $\mathrm{T}_{n\det}^{\mu_{s_0}}$ (resp. $\mathrm{T}^{n\det}_{\mu_{s_0}}$) restricted to $\mathrm{Rep}_{[n\det]}(G)$ (resp. $\mathrm{Rep}_{[\mu_{s_0}]}(G)$).

(水3) Analogous assertions hold for G_1T-modules with $\wedge^n$ replaced by $\otimes^n$ and $\Delta(\lambda)$, $\lambda \in \Lambda^+$, by $\hat\Delta(\lambda) = \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(B_1^+)} \lambda$, $\lambda \in \Lambda$. As the $[\hat\Delta(\lambda)]$, $\lambda \in \Lambda$, do not span the whole of $\mathrm{Rep}(G_1T)$ [J, II.9.9], we consider the additive full subcategory $\mathrm{Rep}'(G_1T)$ of $\mathrm{Rep}(G_1T)$ consisting of those admitting a filtration with subquotients $\hat\Delta(\lambda)$, $\lambda \in \Lambda$, and hence the Grothendieck group $[\mathrm{Rep}'(G_1T)]$ of $\mathrm{Rep}'(G_1T)$ has $\mathbb{Z}$-basis $[\hat\Delta(\lambda)]$, $\lambda \in \Lambda$; although $\mathrm{Rep}'(G_1T)$ does not form a Serre subcategory of $\mathrm{Rep}(G_1T)$ we may talk about its Grothendieck group [CR, 16.3].

Note that, as $\eta'_{\Bbbk}$ and $\mathfrak{a}$ are both G-equivariant, $\mathbb{X}_M$ is G_1T-equivariant $\forall M \in \mathrm{Rep}(G_1T)$, and hence all E_a, $a \in \Bbbk$, are G_1T-equivariant on $\mathrm{Rep}(G_1T)$. Likewise for the F_a's. One could also argue with (火6.ii).

Proposition: *(i) $\forall a \in \Bbbk \setminus \mathbb{F}_p$, $E_a = 0 = F_a$, and hence $E = \coprod_{a\in\mathbb{F}_p} E_a$, $F = \coprod_{a\in\mathbb{F}_p} F_a$.*

(ii) Let $\phi' : \mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}'(G_1T)] \to \otimes^n(\mathrm{nat}_p)$ be a $\mathbb{C}$-linear isomorphism via

$$[\hat\Delta(\lambda)] \mapsto m_{\lambda_1} \otimes m_{\lambda_2-1} \otimes \dots \otimes m_{\lambda_n-n+1} \quad \forall \lambda = (\lambda_1,\dots,\lambda_n) \in \Lambda.$$

$\forall a \in \mathbb{F}_p$, regarding it as an element of $[0,p[$, one has a commutative diagram

$$\begin{tikzcd}
\otimes^n(\mathrm{nat}_p) \arrow[d, "\hat e_a"'] & \mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}'(G_1T)] \arrow[l, "\phi'"', "\sim"] \arrow[r, "\phi'", "\sim"'] \arrow[d, shift right, "\mathbb{C}\otimes_{\mathbb{Z}}[E_a]"'] \arrow[d, shift left, "\mathbb{C}\otimes_{\mathbb{Z}}[F_a]"] & \otimes^n(\mathrm{nat}_p) \arrow[d, "\hat f_a"] \\
\otimes^n(\mathrm{nat}_p) & \mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}'(G_1T)] \arrow[l, "\sim", "\phi'"'] \arrow[r, "\sim", "\phi'"'] & \otimes^n(\mathrm{nat}_p).
\end{tikzcd}$$

Thus, we may regard the exact functors E_a, F_a, $a \in [0,p[$, as part of an action of $\widehat{\mathfrak{gl}_p}$ on $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}'(G_1T)]$ through ϕ'.

(iii) The "block" decomposition $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}'(G_1T)] = \coprod_{b\in\Lambda/\mathcal{W}_a\bullet} \mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}'_b(G_1T)]$ reads as the weight space decomposition of $\otimes^n(\mathrm{nat}_p)$ under ϕ'; each $\phi'(\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}'_b(G_1T)])$ provides a distinct weight space on $\otimes^n(\mathrm{nat}_p)$ of weight $\sum_{i=1}^n (\lambda_i - i + 1)_1\delta + \sum_{j=1}^p n_j\hat\varepsilon_j$ with $n_j = |\{k \in [1,n] | \lambda_k - k + 1 \equiv j \mod p\}|$ if $\lambda = (\lambda_1,\dots,\lambda_n) \in b$.

Proof: By the standing hypothesis $p > 3$. Let $\mathbb{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let $C = \sum_{i,j=1}^n e(i,j)e(j,i) \in \mathbb{U}(\mathfrak{g})$ be the Casimir element with respect to the trace form on V: $\mathrm{Tr}(e(j,i)e(k,l)) = \delta_{ik}\delta_{jl}$. Then

(1) $$C \text{ is central in } \mathbb{U}(\mathfrak{g}).$$

For let $x \in \mathfrak{g}$. Enumerate the $e(i,j)$ as $x_1,\dots,x_N$, $N = n^2$, and let $y_1,\dots,y_N$ be their dual basis with respect to the trace form. In $\mathbb{U}(\mathfrak{g})$

$$Cx = \sum_{i=1}^{N} x_i y_i x = \sum_{i=1}^{N} ([x_i y_i, x] + x x_i y_i) = xC + \sum_{i=1}^{N} [x_i y_i, x]$$

with $[x_i y_i, x] = [x_i, x] y_i + x_i [y_i, x]$. Write $[x_i, x] = \sum_{j=1}^{N} \xi_{ij} x_i$ and $[y_i, x] = \sum_{j=1}^{N} \xi'_{ij} y_i$ for some $\xi_{ij}, \xi'_{ij} \in \Bbbk$. Then $\xi_{ij} = \mathrm{Tr}([x_i, x] y_j) = \mathrm{Tr}(x_i [x, y_j]) = -\xi'_{ji}$, and hence $[x_i, x] y_i = \sum_{j=1}^{N} \xi_{ji} x_j y_i = -\sum_{j=1}^{N} \xi'_{ji} x_j y_i$ while $x_i [y_i, x] = \sum_{j=1}^{N} \xi'_{ij} x_i y_j$. It follows that

$$\sum_{i=1}^{N} [x_i y_i, x] = \sum_{i=1}^{N} ([x_i, x] y_i + x_i [y_i, x]) = \sum_{i=1}^{N} (-\sum_{j=1}^{N} \xi'_{ji} x_j y_i + \sum_{j=1}^{N} \xi'_{ij} x_i y_j) = 0,$$

and hence $Cx = xC$.

Let us denote by $\Delta : \mathbb{U}(\mathfrak{g}) \to \mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{g})$ the comultiplication on $\mathbb{U}(\mathfrak{g})$. Then in $\mathbb{U}(\mathfrak{g}) \otimes \mathbb{U}(\mathfrak{g})$ one has

$$\begin{aligned}\Delta(C) &= \sum_{i,j} (e(j,i) \otimes 1 + 1 \otimes e(j,i))(e(i,j) \otimes 1 + 1 \otimes e(i,j)) \\ &= \sum_{i,j} (e(j,i)e(i,j) \otimes 1 + e(i,j) \otimes e(j,i) + e(j,i) \otimes e(i,j) + 1 \otimes e(j,i)e(i,j)),\end{aligned}$$

and hence

$$\Omega = \frac{1}{2}\{\Delta(C) - C \otimes 1 - 1 \otimes C\}, \tag{2}$$

which also explains (3.3.ii) at least when $p \neq 2$. Write $C = 2\sum_{i<j} e(j,i)e(i,j) + \sum_{i=1}^{n} e(i,i)^2 + \sum_{i<j}(e(i,i) - e(j,j))$, using the fact that $e(i,j)e(j,i) = e(j,i)e(i,j) + e(i,i) - e(j,j)$.

Let $\lambda = (\lambda_1, \dots, \lambda_n) = \sum_{i=1}^{n} \lambda_i \varepsilon_i \in \Lambda$. As $\hat{\Delta}(\lambda) = \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(B_1^+)} \lambda$ and as $\mathbb{U}(\mathfrak{g}) \twoheadrightarrow \mathrm{Dist}(G_1)$, C acts on $\hat{\Delta}(\lambda)$ by the scalar

$$b_\lambda := \sum_{i=1}^{n} \lambda_i^2 + \sum_{i<j} (\lambda_i - \lambda_j); \tag{3}$$

if $i < j$, $e(i,j) \in \mathrm{Dist}(U_1^+)$ annihilates $1 \otimes 1$ while each $e(i,i)$ acts on $1 \otimes 1$ by scalar $\lambda(e(i,i)) = \lambda_i$.

One has

$$\begin{aligned}E\hat{\Delta}(\lambda) &= V \otimes \hat{\Delta}(\lambda) = V \otimes \mathrm{ind}_{B^+}^{G_1 B^+}(\lambda - 2(p-1)\rho) \quad \text{[J, II.9.2]} \\ &\simeq \mathrm{ind}_{B^+}^{G_1 B^+}(V \otimes (\lambda - 2(p-1)\rho)) \quad \text{by the tensor identity [J, I.3.6]},\end{aligned}$$

and hence $E\hat{\Delta}(\lambda)$ admits a filtration with the subquotients $\hat{\Delta}(\lambda + \varepsilon_i)$, $i \in [1, n]$. As C acts on $V \otimes \hat{\Delta}(\lambda)$ through the comultiplication and as $V = \Delta(\varepsilon_1)$, Ω acts by (2) and (3) on $\hat{\Delta}(\varepsilon_i + \lambda)$ by scalar

$$\frac{1}{2}(b_{\lambda+\varepsilon_i} - b_{\varepsilon_1} - b_\lambda) = \lambda_i - i + 1. \tag{4}$$

It follows from (火 7.i) that all the eigenvalues of $\mathbb{X}_{\hat{\Delta}(\lambda)}$ on $E\hat{\Delta}(\lambda)$ belong to $\mathbb{F}_p$. Thus, $\prod_{a\in\mathbb{F}_p}(\mathbb{X}_M - a)^{\dim M}$ annihilates any $M \in \mathrm{Rep}'(G_1T)$. Then $E_a = 0$ unless $a \in \mathbb{F}_p$, and hence $E = \coprod_{a\in\mathbb{F}_p} E_a$.

By (4) $\forall a \in \mathbb{F}_p$ $\forall \lambda \in \Lambda$,

$$[E_a][\hat{\Delta}(\lambda)] = \sum_{\substack{i\in[1,n]\\ \lambda_i-i+1\equiv a \bmod p}} [\hat{\Delta}(\lambda+\varepsilon_i)]. \tag{5}$$

For $\mu \in \Lambda$ write $\lambda \to_a \mu$ iff there is $i \in [1,n]$ with $\lambda_i - i + 1 \equiv a \mod p$ such that $\mu = \lambda + \varepsilon_i$. Then (5) reads

$$[E_a][\hat{\Delta}(\lambda)] = \sum_{\substack{\mu\in\Lambda\\ \lambda\to_a\mu}} [\hat{\Delta}(\mu)]. \tag{6}$$

Turning to F, as $F\hat{\Delta}(\lambda) = V^*\otimes\hat{\Delta}(\lambda) \simeq \mathrm{ind}_{B^+}^{G_1B^+}(V^*\otimes(\lambda - 2(p-1)\rho))$, the subquotients of $F\hat{\Delta}(\lambda)$ in its $\hat{\Delta}$-filtration are $\hat{\Delta}(\lambda-\varepsilon_i)$, $i \in [1,n]$. It follows that the eigenvalues of $\mathbb{X}_{\hat{\Delta}(\lambda)}$ on $F\hat{\Delta}(\lambda)$ are, as $V^* = \Delta(-\varepsilon_n)$, $-n - \frac{1}{2}(b_{\lambda-\varepsilon_i} - b_{-\varepsilon_n} - b_\lambda) = \lambda_i - i$ by (3.4). Then $F_a = 0$ unless $a \in \mathbb{F}_p$, and hence $F = \coprod_{a\in\mathbb{F}_p} F_a$. $\forall a \in \mathbb{F}_p$ $\forall \lambda \in \Lambda$,

$$[F_a][\hat{\Delta}(\lambda)] = \sum_{\substack{i\in[1,n]\\ \lambda_i-i\equiv a \bmod p}} [\hat{\Delta}(\lambda-\varepsilon_i)] = \sum_{\substack{\mu\in\Lambda\\ \mu\to_a\lambda}} [\hat{\Delta}(\mu)]. \tag{7}$$

Now,

$$(\phi'\circ[E_a])[\hat{\Delta}(\lambda)] = \phi'(\sum_{\substack{\mu\in\Lambda\\ \lambda\to_a\mu}}[\hat{\Delta}(\mu)]) = \sum_{\substack{\mu\in\Lambda\\ \lambda\to_a\mu}} m_{\mu_1}\otimes m_{\mu_2-1}\otimes\cdots\otimes m_{\mu_n-n+1}$$

while

$$\begin{aligned}(\hat{e}_a\circ\phi')[\hat{\Delta}(\lambda)] &= \hat{e}_a(m_{\lambda_1}\otimes m_{\lambda_2-1}\otimes\cdots\otimes m_{\lambda_n-n+1})\\ &= (\hat{e}_a m_{\lambda_1})\otimes m_{\lambda_2-1}\otimes\cdots\otimes m_{\lambda_n-n+1}\\ &\qquad + m_{\lambda_1}\otimes(\hat{e}_a m_{\lambda_2-1})\otimes m_{\lambda_3-2}\otimes\cdots\otimes m_{\lambda_n-n+1} + \dots\\ &\qquad + m_{\lambda_1}\otimes\cdots\otimes m_{\lambda_{n-1}-n+2}\otimes(\hat{e}_a m_{\lambda_n-n+1}).\end{aligned}$$

For $\mu \in \Lambda$ with $\lambda \to_a \mu$ there is $j \in [1,n]$ with $\lambda_j - j + 1 \equiv a \mod p$ such that $\forall k \in [1,n]$,

$$\mu_k = \begin{cases}\lambda_k + 1 & \text{if } k = j,\\ \lambda_k & \text{else.}\end{cases}$$

On the other hand, by (火 5.1)

$$\hat{e}_a m_{\lambda_i-i+1} = \begin{cases} m_{\lambda_i-i+2} & \text{if } \lambda_i - i + 1 \equiv a \mod p,\\ 0 & \text{else.}\end{cases}$$

Thus,

$$(\hat{e}_a \circ \phi')[\hat{\Delta}(\lambda)] = \sum_{\substack{i \\ \lambda_i - i+1 \equiv a \bmod p}} m_{\lambda_1} \otimes m_{\lambda_2-1} \otimes \cdots \otimes m_{\lambda_{i-1}-i+2} \otimes m_{\lambda_i - i+2} \otimes m_{\lambda_{i+1}-i} \otimes \cdots \otimes m_{\lambda_n - n+1}$$
$$= (\phi' \circ [E_a])[\hat{\Delta}(\lambda)].$$

Likewise, $\hat{f}_a \circ \phi' = \phi' \circ [F_a]$ $\forall a \in [0, p[$.

(iii) The weight of $m_{\nu_1} \otimes \cdots \otimes m_{\nu_n} \in \otimes^n(\mathrm{nat}_p)$ is, writing $\nu_i = \nu_{i0} + \nu_{i1}p$ with $\nu_{i0} \in [1,p]$,

$$(\hat{\varepsilon}_{\nu_{10}} + \nu_{11}\delta) + \cdots + (\hat{\varepsilon}_{\nu_{n0}} + \nu_{n1}\delta) = (\sum_{i=1}^n \nu_{i1})\delta + \sum_{i=1}^n \hat{\varepsilon}_{\nu_{i0}} = (\sum_{i=1}^n \nu_{i1})\delta + \sum_{j=1}^p n_j \hat{\varepsilon}_j$$

with $n_j = |\{i \in [1,n] | \nu_{i0} = j\}| = |\{i \in [1,n] | \nu_i \equiv j \mod p\}|$; in particular, $\sum_j n_j = n$ from the middle expression. It follows $\forall \lambda, \mu \in \Lambda$ that $\phi'([\hat{\Delta}(\lambda)]) = m_{\lambda_1} \otimes m_{\lambda_2-1} \otimes \cdots \otimes m_{\lambda_n-n+1}$ and $\phi'([\hat{\Delta}(\mu)]) = m_{\mu_1} \otimes m_{\mu_2-1} \otimes \cdots \otimes m_{\mu_n-n+1}$ have the same weight iff
$\sum_{i=1}^n (\lambda_i - i+1)_1 = \sum_{i=1}^n (\mu_i - i+1)_1$ and $\forall j \in [1,p]$, $|\{i \in [1,n] | \lambda_i - i + 1 \equiv j \mod p\}| = |\{i \in [1,n] | \mu_i - i+1 \equiv j \mod p\}|$
iff $\sum_{i=1}^n (\lambda+\zeta)_{i1} = \sum_{i=1}^n (\mu+\zeta)_{i1}$ and $\forall j \in [1,p]$, $|\{i \in [1,n] | (\lambda+\zeta)_i \equiv j \mod p\}| = |\{i \in [1,n] | (\mu+\zeta)_i \equiv j \mod p\}|$ as $\zeta = (0, -1, \dots, -n+1)$
iff $\exists \sigma \in \mathfrak{S}_n$ and $\nu_1, \dots, \nu_n \in \mathbb{Z}$ with $\nu_1 + \cdots + \nu_n = 0$: $(\lambda+\zeta) - \sigma(\mu+\zeta) = p(\nu_1, \dots, \nu_n)$
iff $\lambda + \zeta \in \mathcal{W}_a(\mu+\zeta)$ as $\{(\nu_1,\dots,\nu_n) \in \mathbb{Z}^{\oplus n} | \nu_1 + \cdots + \nu_n = 0\} = \mathbb{Z}R$
iff $\lambda \in \mathcal{W}_a \bullet \mu$, as desired.

(水 4) Let $a \in [0,p[$. We have seen above that $\mathbb{C} \otimes [\mathrm{Rep}'(G_1T)]$ admits a structure of $\mathfrak{sl}_2(\mathbb{C})$-module such that

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \mathbb{C} \otimes [E_a] \quad \text{and} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \mathbb{C} \otimes [F_a].$$

We show that the action extends to $\mathbb{C} \otimes [\mathrm{Rep}(G_1T)]$.

Corollary: *(i) There is a structure of* $\mathfrak{sl}_2(\mathbb{C})$*-module on* $\mathbb{C} \otimes [\mathrm{Rep}(G_1T)]$ *such that* $x \mapsto \mathbb{C} \otimes [E_a]$ *and* $y \mapsto \mathbb{C} \otimes [F_a]$. *As such, each* $1 \otimes [\hat{L}(\lambda)]$, $\lambda \in \Lambda$, *has weight* $\{\sum_{i=1}^n (\lambda_i - i+1)_1 \delta + \sum_{j=1}^p n_j \hat{\varepsilon}_j\}(\hat{h}_a)$ *with respect to* $[x,y]$. *Thus,* $\mathrm{Rep}(G_1T)$ *provides an* $\mathfrak{sl}_2$*-categorification of* $\mathbb{C} \otimes \mathbb{Z}[\mathrm{Rep}(G_1T)]$ *in the sense of [ChR]/[Ro].*

(ii) $\forall j \in [1,n[$, *one may regard* E_{n-j} *(resp.,* F_{n-j}*) as the translation functor* $\mathrm{T}_{n\det}^{\mu_{s_j}}$ *(resp.* $\mathrm{T}^{n\det}_{\mu_{s_j}}$*) restricted to* $\mathrm{Rep}_{[n\det]}(G_1T)$ *(resp.* $\mathrm{Rep}_{[\mu_{s_j}]}(G_1T)$ *). Also, one may take* $E_0E_{p-1}\dots E_{n+1}E_n|_{\mathrm{Rep}_{[n\det]}(G_1T)}$ *(resp.* $F_nF_{n+1}\dots F_{p-1}F_0|_{\mathrm{Rep}_{[\mu_{s_0}]}(G_1T)}$*) to be the translation functor* $\mathrm{T}_{n\det}^{\mu_{s_0}}$ *(resp.* $\mathrm{T}^{n\det}_{\mu_{s_0}}$*) restricted to* $\mathrm{Rep}_{[n\det]}(G_1T)$ *(resp.* $\mathrm{Rep}_{[\mu_{s_0}]}(G_1T)$*).*

Proof: (i) As E_a and F_a are exact on $\mathrm{Rep}(G_1T)$, they define

$$[E_a], [F_a] \in \mathrm{Mod}_{\mathbb{Z}}([\mathrm{Rep}(G_1T)], [\mathrm{Rep}(G_1T)]),$$

and hence also $\mathbb{C}\otimes_{\mathbb{Z}}[E_a], \mathbb{C}\otimes_{\mathbb{Z}}[F_a] \in \mathrm{Mod}_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}(G_1T)], \mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}(G_1T)])$. Let us abbreviate those as $[E_a]$ and $[F_a]$, resp. We thus get a $\mathbb{C}$-algebra homomorphism $\theta : T_{\mathbb{C}}(x,y) \to \mathrm{Mod}_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}(G_1T)], \mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}(G_1T)])$ such that $x \mapsto [E_a]$ and $y \mapsto [F_a]$, where $T_{\mathbb{C}}(x,y)$ denotes the tensor algebra of 2-dimensional $\mathbb{C}$-linear space $\mathbb{C}x \oplus \mathbb{C}y$. Put $z = x\otimes y - y \otimes x$. We show that

$$z\otimes x - x\otimes z - 2x, z\otimes y - y\otimes z + 2y \in \ker\theta,$$

and hence $\mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}(G_1T)]$ is equipped with a structure of $\mathbb{U}(\mathfrak{sl}_2(\mathbb{C}))$-module.

Now, we know from (水 3) that both $z\otimes x - x\otimes z - 2x$ and $z\otimes y - x\otimes z + 2y$ annihilate $\mathbb{C}$-linear subspace $\mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}'(G_1T)]$ of $\mathbb{C}\otimes_{\mathbb{Z}}[\mathrm{Rep}(G_1T)]$. We are to show that they both annihilate $[\hat{L}(\lambda)]$ $\forall\lambda\in\Lambda$. We have an exact sequence of G_1T-modules

$$0 \to M' \to M_r \to \cdots \to M_1 \to \hat{L}(\lambda) \to 0$$

such that all $M_i \in \mathrm{Rep}'(G)$ and that all of the composition factors $\hat{L}(\mu)$ of M' have $\mu \ll \lambda$. As $\hat{\Delta}(\mu) \twoheadrightarrow \hat{L}(\mu)$, the composition factors of $E_a\hat{L}(\mu)$ (resp. $F_a\hat{L}(\mu)$) are among those of $E_a\hat{\Delta}(\mu)$ (resp. $F_a\hat{\Delta}(\mu)$). For $X \in [\mathrm{Rep}(G_1T)]$ write $X = \sum_{\nu\in\Lambda} X_\nu[\hat{L}(\nu)]$ with $X_\nu \in \mathbb{Z}$ and set $\mathrm{supp}(X) = \{\hat{L}(\nu) | X_\nu \neq 0\}$. Thus,

$$\mathrm{supp}((zx - xz - 2x)[\hat{L}(\mu)]) \subseteq$$
$$\mathrm{supp}(xyx)[\hat{\Delta}(\mu)])\cup\mathrm{supp}(yxx)[\hat{\Delta}(\mu)])\cup\mathrm{supp}(xxy)[\hat{\Delta}(\mu)])\cup\mathrm{supp}(xyx)[\hat{\Delta}(\mu)])\cup\mathrm{supp}(x[\hat{\Delta}(\mu)]).$$

$\forall\nu\in\Lambda$, we have from (水 3.5)

$$\mathrm{supp}(x[\hat{\Delta}(\nu)]) = \bigcup_{\substack{i\in[1,n] \\ \nu_i - i + 1 \equiv a \mod p}} \mathrm{supp}([\hat{\Delta}(\nu + \varepsilon_i)]),$$
$$\mathrm{supp}(y[\hat{\Delta}(\nu)]) = \bigcup_{\substack{i\in[1,n] \\ \nu_i - i \equiv a \mod p}} \mathrm{supp}([\hat{\Delta}(\nu - \varepsilon_i)]).$$

It follows, as μ is far from λ, that

$$\mathrm{supp}((zx - xz - 2x)[\hat{L}(\mu)]) \cap \mathrm{supp}((zx - xz - 2x)[\hat{L}(\lambda)]) = \emptyset.$$

As $(zx - xz - 2x)[M_i] = 0$ $\forall i \in [1,r]$, we must then have $(zx - xz - 2x)[\hat{L}(\lambda)] = 0 = (zx - xz - 2x)[M'])$. Likewise, $(zy - yz + 2y)[\hat{L}(\lambda)] = 0$.

As all $[M_i]$'s have weight $\sum_{i=1}^n (\lambda_i - i + 1)_1\delta + \sum_{j=1}^p n_j\hat{\varepsilon}_j$, so does $[\hat{L}(\lambda)]$; again $\theta(z) - (\sum_{i=1}^n (\lambda_i - i + 1)_1\delta + \sum_{j=1}^p n_j\hat{\varepsilon}_j)(\hat{h}_a)$ annihilates $[\hat{L}(\lambda)]$.

(ii) The assertion holds on the $[n\det]$-block of $\mathrm{Rep}'(G_1T)$ by (水 2) and (水 3). Let $\lambda \in \mathcal{W}_a \bullet (n\det)$. As $\hat{L}(\lambda)$ is a quotient of $\hat{\Delta}(\lambda)$, $E_a\hat{L}(\lambda)$ is a quotient of $E_a\hat{\Delta}(\lambda)$, and hence $E_a\hat{L}(\lambda)$ belongs to the same block in the whole of $\mathrm{Rep}(G_1T)$ as $E_a\hat{\Delta}(\lambda)$ does. Likewise for $F_a\hat{L}(\lambda)$. The assertion holds by construction.

(水 5) **Remark:** As nat_p is locally finite with respect to the generators of $\widehat{\mathfrak{gl}_p}$, the same argument as in (水 4) yields that $\mathbb{C}\otimes[\mathrm{Rep}(G_1T)]$ admits a structure of $\widehat{\mathfrak{gl}_p}$-module; $\forall i \in [0,p[$, $\forall m \in \mathbb{Z}$, if $\hat{e}_i \bullet [\hat{\nabla}(\lambda)] = \sum_\mu [\hat{\nabla}(\mu)]$, $(\hat{e}_i \otimes t^m) \bullet [\hat{\nabla}(\lambda)] = \sum_\mu [\hat{\nabla}(\mu + pm\det)] = \sum_\mu [\hat{\nabla}(\mu) \otimes pm\det]$.

Accordingly, we define $(\hat{e}_i \otimes t^m) \bullet [\hat{L}(\lambda)] = \sum_\mu [\hat{L}(\mu) \otimes pm \det]$. Likewise for $\hat{f}_i \otimes t^m$. We let d act on $[\hat{L}(\lambda)]$, $\lambda \in \Lambda$, by the scalar $(\sum_{i=1}^n (\lambda_i - i + 1)_1 \delta + \sum_{j=1}^p n_j \hat{\varepsilon}_j)(d) = \sum_{i=1}^n (\lambda_i - i + 1)_1$. We let K annihilate the whole $[\mathrm{Rep}(G_1T)]$ and $(0, 1) = \mathrm{diag}(1, \ldots, 1)$ act as the identity on $[\mathrm{Rep}(G_1T)]$.

4° 木曜日

(木 1) We now wish to upgrade the $\widehat{\mathfrak{gl}_p}$-action on $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G)]$ to a categorical action of the Khovanov-Lauda-Rouquier, KLR for short, 2-category $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ on $\mathrm{Rep}(G)$ in such a way that $\mathbb{C} \otimes [E_a]$ and $\mathbb{C} \otimes [F_a]$, $a \in [0, p[$, are upgraded to form translation functors on $\mathrm{Rep}(G)$ as in (水 2). The 2-categorical action will provide ample 2-morphisms to realize an action of the Bott-Samelson diagrammatic category $\mathcal{D}_{\mathrm{BS}}$ on $\mathrm{Rep}(G)$. We will see that exactly the same argument gives an upgrading of $\widehat{\mathfrak{gl}_p}$-action on $\mathbb{C} \otimes_{\mathbb{Z}} [\mathrm{Rep}(G_1T)]$ in (水 5) to a $\mathcal{U}(\widehat{\mathfrak{gl}_p})$-action on $\mathrm{Rep}(G_1T)$.

We first take $N = p$ in § 火 to consider $\widehat{\mathfrak{gl}_p}$. We recall the definition of Rouquier's strict $\Bbbk$-linear additive 2-category $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ categorifying the enveloping algebra of $\widehat{\mathfrak{gl}_p}$ after Brundan [Br, Def. 1.1]. First, a $\Bbbk$-linear additive category is a category $\mathcal{C}$ with a zero object such that $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$, a direct sum $X \oplus Y$ exists with $\mathcal{C}(X, Y)$ forming a $\Bbbk$-linear space and that the compositions $\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$ are $\Bbbk$-linear [中岡, Def. 3.1.11]. Next,

Definition [中岡, Def. 3.5.22, p. 220]/[Bor, I.7]: A strict $\Bbbk$-linear additive 2-category $\mathcal{C}$ consists of the data

(i) a class $|\mathcal{C}|$, whose elements are called objects,

(ii) $\forall A, B \in |\mathcal{C}|$, a $\Bbbk$-linear additive category $\mathcal{C}(A, B)$, whose elements are called 1-morphisms and written as $f : A \to B$ with the morphisms in $\mathcal{C}(A, B)$ denoted as $\alpha : f \Rightarrow g$ and their compositions written

$$\begin{tikzcd} f \arrow[r, Rightarrow, "\alpha"] \arrow[dr, dotted, "\beta\odot\alpha"'] & g \arrow[d, Rightarrow, "\beta"] \\ & h, \end{tikzcd}$$

(iii) $\forall A, B, C \in |\mathcal{C}|$, a $\Bbbk$-bilinear bifunctor $c_{A,B,C} : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$ [中岡, Def. 3.1.11], written

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, phantom, "\alpha\Downarrow"] & B \arrow[r, "\ell"] \arrow[d, phantom, "\beta\Downarrow"] & C & & A \arrow[r, "\ell\circ f"] \arrow[d, phantom, "\Downarrow\beta*\alpha"] & C \\ A \arrow[r, "f'"'] & B \arrow[r, "\ell'"'] & C \arrow[r, mapsto, "c_{A,B,C}"] & {} & A \arrow[r, "\ell'\circ f'"'] & C, \end{tikzcd}$$

(iv) $\forall A \in |\mathcal{C}|$, there is a 1-morphism $1_A \in \mathcal{C}(A, A)$, subject to the axioms that $\forall A, B, C, D \in |\mathcal{C}|$,

$$\begin{tikzcd}
\mathcal{C}(A,B) \times \mathcal{C}(B,C) \times \mathcal{C}(C,D) \arrow[r, "\mathcal{C}(A,B)\times c_{B,C,D}"] \arrow[d, "c_{A,B,C}\times\mathcal{C}(C,D)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{C}(A,B) \times \mathcal{C}(B,D) \arrow[d, "c_{A,B,D}"] \\
\mathcal{C}(A,C) \times \mathcal{C}(C,D) \arrow[r, "c_{A,C,D}"'] & \mathcal{C}(A,D)
\end{tikzcd}$$

and $c_{A,A,B}(1_A, ?) = \mathrm{id}_{\mathcal{C}(A,B)} = c_{A,B,B}(?, 1_B)$. We will denote $\mathrm{id}_{1_A} \in \mathcal{C}(A,A)(1_A, 1_A)$ by ι_A.

Then, $\forall \alpha, \beta \in \mathrm{Mor}(\mathcal{C}(A,B))$, $\forall \mu, \nu \in \mathrm{Mor}(\mathcal{C}(B,C))$, the "interchange law" holds:

$$\begin{aligned}
(\nu * \beta) \odot (\mu * \alpha) &= c_{A,B,C}(\beta,\nu) \odot c_{A,B,C}(\alpha,\mu) \quad \text{by definition} \\
&= c_{A,B,C}((\beta,\nu) \odot (\alpha,\mu)) \quad \text{by the functoriality of } c_{A,B,C} \\
&= c_{A,B,C}(\beta \odot \alpha, \nu \odot \mu) \\
&= (\nu \odot \mu) * (\beta \odot \alpha);
\end{aligned}$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, phantom, "\alpha\Downarrow"] & B \arrow[r, "\ell"] \arrow[d, phantom, "\Downarrow\mu"] & C \\
A \arrow[r, "f'"'] \arrow[d, phantom, "\beta\Downarrow"] & B \arrow[r, "\ell'"'] \arrow[d, phantom, "\Downarrow\nu"] & C \\
A \arrow[r, "f''"'] & B \arrow[r, "\ell''"'] & C
\end{tikzcd}
\overset{c_{A,B,C}}{\longmapsto}
\begin{tikzcd}
A \arrow[r, "\ell\circ f"] \arrow[d, phantom, "\Downarrow\mu*\alpha"] & C \\
A \arrow[r, "\ell'\circ f'"'] \arrow[d, phantom, "\Downarrow\nu*\beta"] & C \\
A \arrow[r, "\ell''\circ f''"'] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, phantom, "\mu\odot\alpha\Downarrow"] & B \arrow[r, "\ell"] \arrow[d, phantom, "\Downarrow\nu\odot\beta"] & C \\
A \arrow[r, "f''"'] & B \arrow[r, "\ell''"'] & C.
\end{tikzcd}$$

(木 2) We now define

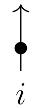
Definition [RW, 6.4.5]: A strict $\Bbbk$-linear additive 2-category $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ consists of the following data:

(i) $\forall i, j \in \mathbb{F}_p$ with $i \neq j$, $t_{ij} = \begin{cases} -1 & \text{if } j = i+1, \\ 1 & \text{else }, \end{cases}$

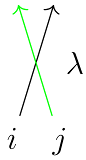
(ii) the objects of $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ are $P = \{\lambda \in \mathfrak{h}^* | \lambda(\hat{h}_i) \in \mathbb{Z}\ \forall i \in [0, p[\}$ from (火 4),

(iii) $\forall \lambda \in P$, $\forall i \in [0, p[$, generating 1-morphisms $E_i 1_\lambda \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda + \hat{\alpha}_i)$, $F_i 1_\lambda \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda - \hat{\alpha}_i)$,

(iv) $\forall \lambda \in P$, $\forall i, j \in [0, p[$, generating 2-morphisms

$x_{\lambda,i} =$  $\lambda \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda + \hat{\alpha}_i)(E_i 1_\lambda, E_i 1_\lambda)$

$$\begin{tikzcd}
\lambda \arrow[r, "E_i 1_\lambda"] \arrow[d, phantom, "\Downarrow x_{\lambda,i}"] & \lambda + \hat{\alpha}_i \\
\lambda \arrow[r, "E_i 1_\lambda"'] & \lambda + \hat{\alpha}_i,
\end{tikzcd}$$

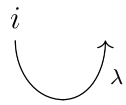
$$\tau_{\lambda,(j,i)} = $$  $$\lambda \ \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda+\hat{\alpha}_i+\hat{\alpha}_j)(E_iE_j1_\lambda, E_jE_i1_\lambda)$$

$$\begin{tikzcd} \lambda \arrow[r, "E_iE_j1_\lambda"] \arrow[d, phantom, "\Downarrow \tau_{\lambda,(j,i)}"] & \lambda+\hat{\alpha}_j+\hat{\alpha}_i \\ \lambda \arrow[r, "E_jE_i1_\lambda"'] & \lambda+\hat{\alpha}_i+\hat{\alpha}_j, \end{tikzcd}$$

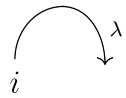
where $E_iE_j1_\lambda = (E_i1_{\lambda+\hat{\alpha}_j}) \circ (E_j1_\lambda) = c_{\lambda,\lambda+\hat{\alpha}_j,\lambda+\hat{\alpha}_j+\hat{\alpha}_i}(E_j1_\lambda, E_i1_{\lambda+\hat{\alpha}_j})$ and $E_jE_i1_\lambda = (E_j1_{\lambda+\hat{\alpha}_i}) \circ (E_i1_\lambda) = c_{\lambda,\lambda+\hat{\alpha}_i,\lambda+\hat{\alpha}_i+\hat{\alpha}_j}(E_i1_\lambda, E_j1_{\lambda+\hat{\alpha}_i})$:

$$\begin{tikzcd} \lambda \arrow[r, "E_j1_\lambda"] \arrow[dr, dotted, "E_iE_j1_\lambda"'] & \lambda+\hat{\alpha}_j \arrow[d, "E_i1_{\lambda+\hat{\alpha}_j}"] \\ & \lambda+\hat{\alpha}_j+\hat{\alpha}_i, \end{tikzcd} \qquad \begin{tikzcd} \lambda \arrow[r, "E_i1_\lambda"] \arrow[dr, dotted, "E_jE_i1_\lambda"'] & \lambda+\hat{\alpha}_i \arrow[d, "E_j1_{\lambda+\hat{\alpha}_i}"] \\ & \lambda+\hat{\alpha}_i+\hat{\alpha}_j, \end{tikzcd}$$

and

$$\eta_{\lambda,i} = $$  $$\in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\lambda)(1_\lambda, F_iE_i1_\lambda)$$

with 1_λ denoting the unital object of $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\lambda)$ from (木1.iv), and $F_iE_i1_\lambda = (F_i1_{\lambda+\hat{\alpha}_i})\circ(E_i1_\lambda)$, and finally

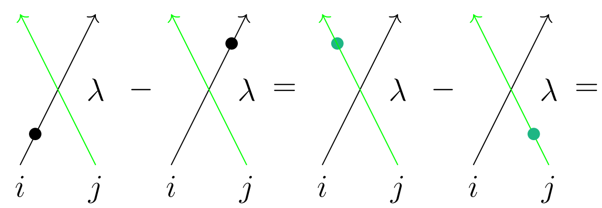
$$\varepsilon_{\lambda,i} = $$  $$\in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\lambda)(E_iF_i1_\lambda, 1_\lambda)$$

with $E_iF_i1_\lambda = (E_i1_{\lambda-\hat{\alpha}_i}) \circ (F_i1_\lambda)$. In the notation $\tau_{\lambda,(j,i)}$ we follow [RW, p. 90] to write (j,i) instead of (i,j) in accordance to the order of composition reading from the right.

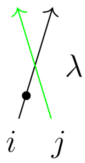
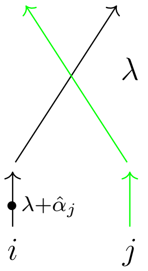
By (木1.iv) one has $\forall f \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\mu)$, $f\circ 1_\lambda = f$ and $1_\mu \circ f = f$. We will denote the identity 2-morphism of E_i1_λ in $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\lambda+\hat{\alpha}_i)(E_i1_\lambda, E_i1_\lambda)$ (resp. F_i1_λ in $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda,\lambda-\hat{\alpha}_i)(F_i1_\lambda, F_i1_\lambda)$) by $\underset{i}{\uparrow}\lambda$ (resp. $\overset{i}{\downarrow}\lambda$):

$$\iota_{E_i1_\lambda} = \mathrm{id}_{E_i1_\lambda} = \underset{i}{\uparrow}\lambda, \qquad \iota_{F_i1_\lambda} = \mathrm{id}_{F_i1_\lambda} = \overset{i}{\downarrow}\lambda.$$

Those 2-morphisms are subject to the relations in [Br, Def.1.1], e.g.,

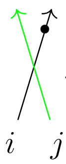
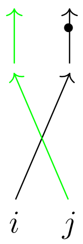
(1)  $$\begin{cases} \underset{i}{\uparrow}\,\underset{j}{\uparrow}\lambda & \text{if } i=j, \\ 0 & \text{else,} \end{cases}$$

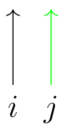
where

 $\lambda =$  $= \tau_{\lambda,(j,i)} \odot (x_{\lambda+\hat{\alpha}_j,i} * \iota_{E_j 1_\lambda}) \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda+\hat{\alpha}_j+\hat{\alpha}_i)(E_iE_j1_\lambda, E_jE_i1_\lambda),$

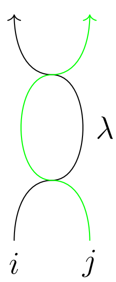
$$
\begin{tikzcd}
\lambda \arrow[r, "E_j1_\lambda"] \arrow[d, phantom, "\Downarrow \iota_{E_j1_\lambda}"] & \lambda+\hat{\alpha}_j \arrow[r, "E_i1_{\lambda+\hat{\alpha}_j}"] \arrow[d, phantom, "\Downarrow x_{\lambda+\hat{\alpha}_j,i}"] & \lambda+\hat{\alpha}_j+\hat{\alpha}_i \\
\lambda \arrow[r, "E_j1_\lambda"'] & \lambda+\hat{\alpha}_j \arrow[r, "E_i1_{\lambda+\hat{\alpha}_j}"'] & \lambda+\hat{\alpha}_j+\hat{\alpha}_i
\end{tikzcd}
\xmapsto{c_{\lambda,\lambda+\hat{\alpha}_j,\lambda+\hat{\alpha}_j+\hat{\alpha}_i}}
\begin{tikzcd}
\lambda \arrow[r, "E_iE_j1_\lambda"] \arrow[d, phantom, "\Downarrow x_{\lambda+\hat{\alpha}_j,i}*\iota_{E_j1_\lambda}"] & \lambda+\hat{\alpha}_j+\hat{\alpha}_i \\
\lambda \arrow[r, "E_iE_j1_\lambda"'] & \lambda+\hat{\alpha}_j+\hat{\alpha}_i,
\end{tikzcd}
$$

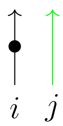
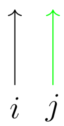
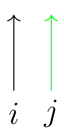
$$
\begin{tikzcd}
\lambda \arrow[r, "E_iE_j1_\lambda"] \arrow[d, phantom, "x_{\lambda+\hat{\alpha}_j,i}*\iota_{E_j1_\lambda}\Downarrow"] & \lambda+\hat{\alpha}_j+\hat{\alpha}_i \arrow[dd, bend left, dotted, Rightarrow, "\tau_{\lambda,(j,i)}\odot(x_{\lambda+\hat{\alpha}_j,i}*\iota_{E_j1_\lambda})"] \\
\lambda \arrow[r, "E_iE_j1_\lambda"'] \arrow[d, phantom, "\tau_{\lambda,(j,i)}\Downarrow"] & \lambda+\hat{\alpha}_j+\hat{\alpha}_i \\
\lambda \arrow[r, "E_jE_i1_\lambda"'] & \lambda+\hat{\alpha}_i+\hat{\alpha}_j,
\end{tikzcd}
$$

 $\lambda =$  $\lambda = (\iota_{E_j1_{\lambda+\hat{\alpha}_i}} * x_{\lambda,i}) \odot \tau_{\lambda,(j,i)} \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda+\hat{\alpha}_j+\hat{\alpha}_i)(E_iE_j1_\lambda, E_jE_i1_\lambda),$

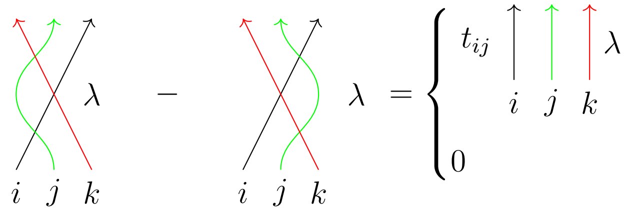
 $\lambda \quad = \quad \iota_{E_i1_{\lambda+\hat{\alpha}_j}} * \iota_{E_j1_\lambda} = \iota_{E_iE_j1_\lambda} \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda+\hat{\alpha}_j+\hat{\alpha}_i)(E_iE_j1_\lambda, E_jE_i1_\lambda),$

etc. We also impose, among others,

(2)  $\lambda \quad =$

- 0 — if $i = j$,
- t_{ij}  $\lambda \; + t_{ji}$  λ — if $i - j \equiv \pm 1 \mod p$,
-  λ — else,

the left hand side of which reads $\tau_{\lambda,(i,j)} \odot \tau_{\lambda,(j,i)}$, and

(3) 

$$= \begin{cases} t_{ij} \; \lambda & \text{if } i = j \text{ and } k - j \equiv \pm 1 \mod p, \\ 0 & \text{else,} \end{cases}$$

etc. On the LHS of (3) the first (resp. second) term reads $(\tau_{\lambda+\hat{\alpha}_i,(k,j)} * \iota_{E_i 1_\lambda}) \odot (\iota_{E_j 1_{\lambda+\hat{\alpha}_k+\hat{\alpha}_i}} * \tau_{\lambda,(k,i)}) \odot (\tau_{\lambda+\hat{\alpha}_k,(j,i)} * \iota_{E_k 1_\lambda})$ (resp. $(\tau_{\lambda,(j,i)} * \iota_{E_k 1_\lambda}) \odot (\tau_{\lambda+\hat{\alpha}_j,(k,i)} * \iota_{E_j 1_\lambda}) \odot (\iota_{E_i 1_{\lambda+\hat{\alpha}_k+\hat{\alpha}_j}} * \tau_{\lambda,(k,j)})$).

Recall from (木 1.ii) that each $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \mu)$ forms a $\Bbbk$-linear additive category, and hence $\forall X, Y \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \mu)$, $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \mu)(X, Y)$ carries a structure of $\Bbbk$-linear space. The 1-morphisms belonging to $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \mu)$ are direct sums of those

$$E_{i_m}^{a_m} F_{j_m}^{b_m} \dots E_{i_1}^{a_1} F_{j_1}^{b_1} 1_\lambda, \quad i_k, j_k \in [0, p[, \; a_k, b_k \in \mathbb{N} \text{ with } \mu = \lambda + \sum_{k=1}^{m} (a_k \hat{\alpha}_{i_k} - b_k \hat{\alpha}_{j_k})$$

[Ro12, 4.2.3]. In case $\mu = \lambda$, $\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda)$ forms a strict monoidal category with $\otimes$ in (火 1) given by the "composition" $\odot$ of 1-morphisms from (木 1) and $I \in \mathrm{Ob}(\mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda))$ given by 1_λ.

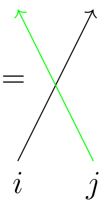
If $E_{i_m}^{a_m} F_{j_m}^{b_m} \dots E_{i_1}^{a_1} F_{j_1}^{b_1} 1_\lambda = E_{i_m}^{a_m} F_{j_m}^{b_m} \dots E_i^{a} E_i E_i^{a'} \dots E_{i_1}^{a_1} F_{j_1}^{b_1} 1_\lambda$ with $\nu = \lambda - b_1 \hat{\alpha}_{j_1} + a_1 \hat{\alpha}_{i_1} - \dots + a' \hat{\alpha}_i$,

$$x_{\nu,i} = $$  $$\lambda \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\nu, \nu + \hat{\alpha}_i)(E_i 1_\nu, E_i 1_\nu)$$

induces a 2-morphism $\iota_{E_{i_m}^{a_m} F_{j_m}^{b_m} \dots E_i^{a} 1_{\nu+\hat{\alpha}_i}} * x_{\nu,i} * \iota_{E_i^{a'} \dots E_{i_1}^{a_1} F_{j_1}^{b_1} 1_\lambda} \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda + \sum_{k=1}^m (a_k \hat{\alpha}_{i_k} - b_k \hat{\alpha}_{j_k}))$ $(E_{i_m}^{a_m} F_{j_m}^{b_m} \dots E_{i_1}^{a_1} F_{j_1}^{b_1} 1_\lambda, E_{i_m}^{a_m} F_{j_m}^{b_m} \dots E_{i_1}^{a_1} F_{j_1}^{b_1} 1_\lambda)$:

$$\begin{tikzcd} \lambda \arrow[r, "E_i^{a'} \dots E_{i_1}^{a_1} F_{j_1}^{b_1} 1_\lambda"] \arrow[d, phantom, "\Downarrow \iota_{E_i^{b} \dots E_{i_1}^{a_1} F_{j_1}^{b_1} 1_\lambda}"] & \nu \arrow[r, "E_i 1_\nu"] \arrow[d, phantom, "\Downarrow x_{\nu,i}"] & \nu + \hat{\alpha}_i \arrow[r, "E_{i_m}^{a_m} F_{j_m}^{b_m} \dots E_i^{a} 1_{\nu+\hat{\alpha}_i}"] \arrow[d, phantom, "\Downarrow \iota_{E_{i_m}^{a_m} F_{j_m}^{b_m} \dots E_i^{a} 1_{\nu+\hat{\alpha}_i}}"] & \lambda + \sum_{k=1}^m (a_k \hat{\alpha}_{i_k} - b_k \hat{\alpha}_{j_k}) \\ \lambda \arrow[r, "E_i^{a'} \dots E_{i_1}^{a_1} F_{j_1}^{b_1} 1_\lambda"'] & \nu \arrow[r, "E_i 1_\nu"'] & \nu + \hat{\alpha}_i \arrow[r, "E_{i_m}^{a_m} F_{j_m}^{b_m} \dots E_i^{a} 1_{\nu+\hat{\alpha}_i}"'] & \lambda + \sum_{k=1}^m (a_k \hat{\alpha}_{i_k} - b_k \hat{\alpha}_{j_k}). \end{tikzcd}$$

If $E_{i_m}^{a_m} F_{j_m}^{b_m} \dots E_{i_1}^{a_1} F_{j_1}^{b_1} 1_\lambda = E_{i_m}^{a_m} F_{j_m}^{b_m} \dots E_i^{a} E_i E_j E_j^{a'} \dots E_{i_1}^{a_1} F_{j_1}^{b_1} 1_\lambda$ with $\nu = \lambda - b_1 \hat{\alpha}_{j_1} + a_1 \hat{\alpha}_{i_1} - \dots + a' \hat{\alpha}_j$,

$$\tau_{\nu,(j,i)} = $$  $$\nu \quad \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\nu, \nu + \hat{\alpha}_i + \hat{\alpha}_j)(E_i E_j 1_\nu, E_j E_i 1_\nu),$$

induces a 2-morphism $\iota_{E_{i_m}^{a_m}F_{j_m}^{b_m}\ldots E_i^{a}1_{\nu+\hat{\alpha}_i+\hat{\alpha}_j}} * \tau_{\nu,(j,i)} * \iota_{E_j^{a'}\ldots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda} \in \mathcal{U}(\widehat{\mathfrak{gl}_p})(\lambda, \lambda + \sum_{k=1}^m (a_k\hat{\alpha}_{i_k} - b_k\hat{\alpha}_{j_k}))(E_{i_m}^{a_m}F_{j_m}^{b_m}\ldots E_i^{a}E_iE_jE_j^{a'}\ldots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda, E_{i_m}^{a_m}F_{j_m}^{b_m}\ldots E_i^{a}E_jE_iE_j^{a'}\ldots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda)$:

$$\begin{tikzcd}
\lambda \arrow[r, "E_j^{a'}\ldots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda"] \arrow[d, phantom, ""] & \nu \arrow[r, "E_iE_j1_\nu"] & \nu+\hat{\alpha}_i+\hat{\alpha}_j \arrow[r, "E_{i_m}^{a_m}F_{j_m}^{b_m}\ldots E_i^{a}1_{\nu+\hat{\alpha}_i+\hat{\alpha}_j}"] & \lambda+\sum_{k=1}^m(a_k\hat{\alpha}_{i_k}-b_k\hat{\alpha}_{j_k}) \\
\lambda \arrow[r, swap, "E_j^{a'}\ldots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda"] \arrow[u, phantom, ""] \arrow[r, phantom, shift left=6, "\Downarrow\iota_{E_j^{b}\ldots E_{i_1}^{a_1}F_{j_1}^{b_1}1_\lambda}"] & \nu \arrow[r, swap, "E_jE_i1_\nu"] \arrow[r, phantom, shift left=6, "\Downarrow\tau_{\nu,\nu+\hat{\alpha}_i+\hat{\alpha}_j}"] & \nu+\hat{\alpha}_i+\hat{\alpha}_j \arrow[r, swap, "E_{i_m}^{a_m}F_{j_m}^{b_m}\ldots E_i^{a}1_{\nu+\hat{\alpha}_i+\hat{\alpha}_j}"] \arrow[r, phantom, shift left=6, "\Downarrow\iota_{E_{i_m}^{a_m}F_{j_m}^{b_m}\ldots E_i^{a}1_{\nu+\hat{\alpha}_i+\hat{\alpha}_j}}"] & \lambda+\sum_{k=1}^m(a_k\hat{\alpha}_{i_k}-b_k\hat{\alpha}_{j_k}).
\end{tikzcd}$$

(木 3) **Definition [RW, 6.4.5]:** A 2-representation of $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ is a $\Bbbk$-linear functor from $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ to the 2-category of $\Bbbk$-linear additive categories, i.e., it consists of the following data:

(i) $\forall\lambda \in P$, a $\Bbbk$-linear additive category $\mathcal{C}_\lambda$,

(ii) $\forall\lambda \in P$, $\forall i \in [0, p[$, $\Bbbk$-linear functors $E_i1_\lambda \in \mathrm{Cat}(\mathcal{C}_\lambda, \mathcal{C}_{\lambda+\hat{\alpha}_i})$ and $F_i1_\lambda \in \mathrm{Cat}(\mathcal{C}_\lambda, \mathcal{C}_{\lambda-\hat{\alpha}_i})$,

(iii) $\forall\lambda \in P$, $\forall i, j \in [0, p[$,

$$\begin{aligned}
&x_{\lambda,i} \in \mathrm{Cat}(\mathcal{C}_\lambda, \mathcal{C}_{\lambda+\hat{\alpha}_i})(E_i1_\lambda, E_i1_\lambda),\\
&\tau_{\lambda,(j,i)} \in \mathrm{Cat}(\mathcal{C}_\lambda, \mathcal{C}_{\lambda+\hat{\alpha}_i+\hat{\alpha}_j})(E_iE_j1_\lambda, E_jE_i1_\lambda) \text{ with } E_iE_j1_\lambda = (E_i1_{\lambda+\hat{\alpha}_j}) \circ (E_j1_\lambda) \text{ and}\\
&\hphantom{\tau_{\lambda,(j,i)} \in \mathrm{Cat}(\mathcal{C}_\lambda, \mathcal{C}_{\lambda+\hat{\alpha}_i+\hat{\alpha}_j})(E_iE_j1_\lambda, E_jE_i1_\lambda) \text{ with }} E_jE_i1_\lambda = (E_j1_{\lambda+\hat{\alpha}_i}) \circ (E_i1_\lambda),\\
&\eta_{\lambda,i} \in \mathrm{Cat}(\mathcal{C}_\lambda, \mathcal{C}_\lambda)(\mathrm{id}_{\mathcal{C}_\lambda}, F_iE_i1_\lambda) \text{ with } F_iE_i1_\lambda = (F_i1_{\lambda+\hat{\alpha}_i}) \circ (E_i1_\lambda),\\
&\varepsilon_{\lambda,i} \in \mathrm{Cat}(\mathcal{C}_\lambda, \mathcal{C}_\lambda)(E_iF_i1_\lambda, \mathrm{id}_{\mathcal{C}_\lambda}) \text{ with } E_iF_i1_\lambda = (E_i1_{\lambda-\hat{\alpha}_i}) \circ (F_i1_\lambda),
\end{aligned}$$

subject to the same relations as $x_{\lambda,i}, \tau_{\lambda,(j,i)}, \eta_{\lambda,i}, \varepsilon_{\lambda,i}$ for $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ from (木 2).

(木 4) We now define a 2-representation of $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ on $\mathrm{Rep}(G)$, which is also due to [ChR]. Let $\mathbb{T} \in \mathrm{Cat}(\mathrm{Rep}(G), \mathrm{Rep}(G))(E^2, E^2)$ be a natural transformation defined by associating to each $M \in \mathrm{Rep}(G)$ a $\Bbbk$-linear map $\mathbb{T}_M : E^2M = V \otimes V \otimes M \to E^2M$ such that $v \otimes v' \otimes m \mapsto v' \otimes v \otimes m$ $\forall v, v' \in V$ $\forall m \in M$. Then

(1)
$$(V \otimes \mathbb{T}_M) \circ \mathbb{X}_{V^{\otimes 2}\otimes M} = \mathbb{X}_{V^{\otimes 2}\otimes M} \circ (V \otimes \mathbb{T}_M).$$

Using (火 6.i), one also checks

(2)
$$\mathbb{T}_M \circ (V \otimes \mathbb{X}_M) - \mathbb{X}_{V\otimes M} \circ \mathbb{T}_M = -\mathrm{id}_{E^2M}.$$

Recall from (水 2) the bijection $\iota_n : P(\wedge^n \mathrm{nat}_p) \to \Lambda/(\mathcal{W}_a\bullet)$. For $\lambda \in P$ let us write

$$\mathrm{R}_{\iota_n(\lambda)}(G) = \begin{cases} \mathrm{Rep}_{\iota_n(\lambda)}(G) & \text{if } \lambda \in P(\wedge^n \mathrm{nat}_p),\\ 0 & \text{else.}\end{cases}$$

Consider the following data:

(i) $\forall\lambda \in P$, let $\mathcal{C}_\lambda = \mathrm{R}_{\iota_n(\lambda)}(G)$.

(ii) $\forall \lambda \in P$, $\forall i \in [0, p[$, let $E_i 1_\lambda = E_i|_{\mathrm{R}_{\iota_n(\lambda)}(G)} : \mathrm{R}_{\iota_n(\lambda)}(G) \to \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_i)}(G)$ and $F_i 1_\lambda = F_i|_{\mathrm{R}_{\iota_n(\lambda)}(G)} : \mathrm{R}_{\iota_n(\lambda)}(G) \to \mathrm{R}_{\iota_n(\lambda-\hat{\alpha}_i)}(G)$ from (水 1). In particular, $E_i 1_\lambda = 0$ (resp. $F_i 1_\lambda = 0$) unless λ and $\lambda + \hat{\alpha}_i$ (resp. λ and $\lambda - \hat{\alpha}_i$) $\in P(\wedge^n \mathrm{nat}_p)$. Put for simplicity $E_i^\lambda = E_i|_{\mathrm{R}_{\iota_n(\lambda)}(G)}$ and $F_i^\lambda = F_i|_{\mathrm{R}_{\iota_n(\lambda)}(G)}$.

(iii) $\forall \lambda \in P$, $\forall i, j \in [0, p[$, define $x_{\lambda,i} \in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_i)}(G))(E_i^\lambda, E_i^\lambda)$ by associating to each $M \in \mathrm{R}_{\iota_n(\lambda)}(G)$ a $\Bbbk$-linear map $x_{M,i} = \mathbb{X}_M - i\mathrm{id}_{V \otimes M}$:

(3)
$$
\begin{tikzcd}
V \otimes M \arrow[r, "\mathbb{X}_M - i\mathrm{id}"] & V \otimes M \\
E_i^\lambda M \arrow[u, hook] \arrow[r, dotted] & E_i^\lambda M. \arrow[u, hook]
\end{tikzcd}
$$

Define $\tau_{\lambda,(j,i)} \in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda+\hat{\alpha}_i+\hat{\alpha}_j)}(G))(E_i^{\lambda+\hat{\alpha}_j} E_j^\lambda, E_j^{\lambda+\hat{\alpha}_i} E_i^\lambda)$ by associating to each $M \in \mathrm{R}_{\iota_n(\lambda)}(G)$ a $\Bbbk$-linear map $\tau_{M,(j,i)} : E_i^{\lambda+\hat{\alpha}_j} E_j^\lambda M \to E_j^{\lambda+\hat{\alpha}_i} E_i^\lambda M$ such that

(4)
$$
\tau_{M,(j,i)} = \begin{cases} \{\mathrm{id} + (V \otimes \mathbb{X}_M) - \mathbb{X}_{V\otimes M}\}^{-1}(\mathbb{T}_M - \mathrm{id}) & \text{if } j = i, \\ (V \otimes \mathbb{X}_M - \mathbb{X}_{V\otimes M})\mathbb{T}_M + \mathrm{id}_{V\otimes V\otimes M} & \text{if } j \equiv i - 1 \bmod p, \\ (V \otimes \mathbb{X}_M - \mathbb{X}_{V\otimes M})\{\mathrm{id} + (V \otimes \mathbb{X}_M) - \mathbb{X}_{V\otimes M}\}^{-1}(\mathbb{T}_M - \mathrm{id}) + \mathrm{id} & \text{else,} \end{cases}
$$

which is well-defined by [Ro, Th. 3.16]/[RW, Th. 6.4.2]; verification may formally be done using the degenerate affine Hecke algebra. In case $j = i$, $E_i^{\lambda+\hat{\alpha}_i} E_i^\lambda M$ is a generalized i-eigenspace of both $V \otimes \mathbb{X}_M$ and $\mathbb{X}_{V\otimes M}$. As $V \otimes \mathbb{X}_M$ and $\mathbb{X}_{V\otimes M}$ commute by (火 7.iii), $(V \otimes \mathbb{X}_M) - \mathbb{X}_{V\otimes M}$ is nilpotent on $E_i^{\lambda+\hat{\alpha}_i} E_i^\lambda M$, and hence $\mathrm{id} + (V \otimes \mathbb{X}_M) - \mathbb{X}_{V\otimes M}$ is invertible on $E_i^{\lambda+\hat{\alpha}_i} E_i^\lambda M$. Likewise in the 3rd case.

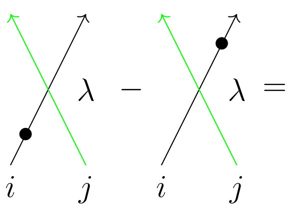
Define $\eta_{\lambda,i}$ to be the unit $\eta_i \in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda)}(G))(\mathrm{id}, F_i^{\lambda+\hat{\alpha}_i} E_i^\lambda)$ of the adjunction (E_i, F_i) on $\mathrm{R}_{\iota_n(\lambda)}(G)$ from (火 9). Define finally $\varepsilon_{\lambda,i}$ to be the counit $\varepsilon_i \in \mathrm{Cat}(\mathrm{R}_{\iota_n(\lambda)}(G), \mathrm{R}_{\iota_n(\lambda)}(G))$ $(E_i^{\lambda-\hat{\alpha}_i} F_i^\lambda, \mathrm{id})$ of the adjunction (E_i, F_i) on $\mathrm{R}_{\iota_n(\lambda)}(G)$ from (火 9) also.

Theorem [RW, Th. 6.4.6]: *The data above constitutes a 2-representation of* $\mathcal{U}(\widehat{\mathfrak{gl}_p})$.

5° 金曜日

(金 1) To see that Th. 木 4 holds, we must check that the 2-morphisms in (木 4.iii) satisfy the relations of those for $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ as given in (木 2).

Consider for example the relation from (木 2.1)



$$
= \begin{cases} \underset{i}{\uparrow}\,\underset{j}{\uparrow}\lambda & \text{if } i = j, \\ 0 & \text{else.} \end{cases}
$$

Accordingly, we must verify

$$\tau_{\lambda,(j,i)} \odot (x_{\lambda+\hat\alpha_j,i} * \iota_{E_j^\lambda}) - (\iota_{E_j^{\lambda+\hat\alpha_i}} * x_{\lambda,i}) \odot \tau_{\lambda,(j,i)} = \begin{cases} \mathrm{id} & \text{if } i = j \\ 0 & \text{else,} \end{cases} \tag{1}$$

i.e., in case $i = j$, for example, one must show on $E_i^{\lambda+\hat\alpha_i}E_i^\lambda M$ for $M \in \mathrm{R}_{\iota_n(\lambda)}(G)$ that

$$\begin{aligned} &\{\mathrm{id} + (V \otimes \mathbb{X}_M) - \mathbb{X}_{V\otimes M}\}^{-1}(\mathbb{T}_M - \mathrm{id}) \circ (\mathbb{X}_{E_iM} - i\mathrm{id}) - \\ &\qquad\qquad \{V \otimes (\mathbb{X}_M - i\mathrm{id})\} \circ \{\mathrm{id} + (V \otimes \mathbb{X}_M) - \mathbb{X}_{V\otimes M}\}^{-1}(\mathbb{T}_M - \mathrm{id}) = \mathrm{id}. \end{aligned}$$

For that the KLR-algebra $\mathrm{H}_3(\mathbb{F}_p)$ and the degenerate affine Hecke algebra $\bar{\mathrm{H}}_3$ of degree 3 come to rescue.

(金 2) To define the KLR-algebra, recall first $t_{ij} \in \{\pm 1\}$ from (木 2) for $i, j \in \mathbb{F}_p$ with $i \neq j$. Let $\mathfrak{S}_3$ act on $\mathbb{F}_p^3$ such that $\sigma\nu = (\nu_{\sigma^{-1}1}, \nu_{\sigma^{-1}2}, \nu_{\sigma^{-1}3})$ for $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{F}_p^3$. Put $\sigma_k = (k, k+1) \in \mathfrak{S}_3$, $k \in \{1, 2\}$. The algebra $\mathrm{H}_3(\mathbb{F}_p)$ is really a $\Bbbk$-linear additive category with objects $\mathbb{F}_p^3$ and morphisms generated by $x_{z,\nu} \in \mathrm{H}_3(\mathbb{F}_p)(\nu, \nu)$ and $\tau_{c,\nu} \in \mathrm{H}_3(\mathbb{F}_p)(\nu, \sigma_c\nu)$, $z \in [1, 3]$, $c \in [1, 2]$, $\nu \in \mathbb{F}_p^3$, subject to the relations

(KLR1) $\qquad x_{z,\nu}x_{z',\nu'} = x_{z',\nu}x_{z,\nu'}$,

(KLR2) $$\tau_{c,\sigma_c\nu}\tau_{c,\nu} = \begin{cases} 0 & \text{if } \nu_c = \nu_{c+1}, \\ t_{\nu_c,\nu_{c+1}}x_{c,\nu} + t_{\nu_{c+1},\nu_c}x_{c+1,\nu} & \text{if either } \nu_{c+1} \equiv \nu_c + 1 \text{ or } \nu_c \equiv \nu_{c+1} + 1, \\ \mathrm{id}_\nu & \text{else,} \end{cases}$$

(KLR3)
$$\tau_{c,\nu}x_{z,\nu} - x_{\sigma_c z,\sigma_c\nu}\tau_{c,\nu} = \begin{cases} -\mathrm{id}_\nu & \text{if } c = z \text{ and } \nu_c = \nu_{c+1}, \\ \mathrm{id}_\nu & \text{if } z = c+1 \text{ and } \nu_c = \nu_{c+1}, \\ 0 & \text{else.} \end{cases}$$

We do not care what $x_{z,\nu} : \nu \to \nu$ and $\tau_{c,\nu} : \nu \to \sigma\nu$ are as maps.

A representation of $\mathrm{H}_3(\mathbb{F}_p)$ consists of the data

(i) $\forall \nu \in \mathbb{F}_p^3$, a $\Bbbk$-linear space V_ν,

(ii) $\forall \nu \in \mathbb{F}_p^3$, $\forall z \in [1, 3]$, a $\Bbbk$-linear map $x_{z,\nu} : V_\nu \to V_\nu$,

(iii) $\forall \nu \in \mathbb{F}_p^3$, $\forall c \in [1, 2]$, a $\Bbbk$-linear map $\tau_{c,\nu} : V_\nu \to V_{\sigma_c\nu}$

satisfying the relations (KLR1-3).

(金 3) Recall next the degenerate affine Hecke algebra, daHa for short, $\bar{\mathrm{H}}_m$ of degree m; DAHA already stands for "double affine Hecke algebra". Thus, let $\Bbbk[X] = \Bbbk[X_1, \dots, X_m]$ be the polynomial $\Bbbk$-algebra in indeterminates $X_1, \dots, X_m$ with a natural $\mathfrak{S}_m$-action: $\sigma : X_i \mapsto X_{\sigma(i)}$. For transposition $\sigma_c = (c, c+1) \in \mathfrak{S}_m$, $c \in [1, m[$, let ∂_c denote the Demazure operator on $\Bbbk[X]$ defined by

$$f \mapsto \frac{f - \sigma_c f}{X_{c+1} - X_c},$$

which differs from the standard one by sign. The daHa $\bar{\mathrm{H}}_m$ is a $\Bbbk$-algebra with the ambient $\Bbbk$-linear space $\Bbbk\mathfrak{S}_m \otimes_\Bbbk \Bbbk[X]$ having $\Bbbk\mathfrak{S}_m$ and $\Bbbk[X]$ as $\Bbbk$-subalgebras such that, letting T_c denote $\sigma_c \in \mathfrak{S}_m$ in $\bar{\mathrm{H}}_m$,

$$fT_c = T_c\sigma_c(f) + \partial_c(f)T_c \quad \forall f \in \Bbbk[X], \forall c \in [1, m[. \tag{1}$$

If $r \leq m$, one has naturally $\bar{\mathrm{H}}_r \leq \bar{\mathrm{H}}_m$.

Lemma [RW, Lem. 6.4.5]: *There is a $\Bbbk$-algebra homomorphism*

$$\bar{\mathrm{H}}_m \to \mathrm{Cat}(\mathrm{Rep}(G), \mathrm{Rep}(G))(E^m, E^m)$$

such that $\forall M \in \mathrm{Rep}(G)$, $X_z \mapsto V^{\otimes_{m-z}} \otimes \mathbb{X}_{V^{\otimes_{z-1}} \otimes M}$, $z \in [1, m]$ *and* $T_c \mapsto V^{\otimes_{m-c-1}} \otimes \mathbb{T}_{V^{\otimes_{c-1}} \otimes M}$, $c \in [1, m[$.

Proof: One checks that the relations $T_c^2 = 1$ $\forall c \in [1, m[$, and the braid relations $T_cT_b = T_bT_c$ for b, c with $|b - c| \geq 2$, $T_cT_{c+1}T_c = T_{c+1}T_cT_{c+1}$ on $\mathrm{Cat}(\mathrm{Rep}(G), \mathrm{Rep}(G))(E^m, E^m)$. Also, the relations $X_zX_y = X_yX_z$, $z, y \in [1, m]$, hold on the RHS by generalizing (火 7). To check (1), we may assume $f \in \{X_1, \dots, X_m\}$ as $\forall g \in \Bbbk[X]$, $(fg)T_c = f(T_cg)$. Then the relations hold on the RHS by generalizing (木 4.1, 2).

(金 4) It follows for $M \in \mathrm{Rep}(G)$ that E^3M comes equipped with a structure of $\bar{\mathrm{H}}_3$-module. By (水 1)

$$E^3M = \coprod_{\nu \in \mathbb{F}_p^3} E^3_\nu M$$

with $E^3_\nu M = E_{\nu_3}E_{\nu_2}E_{\nu_1}M$ and $E_{\nu_i}(V^{\otimes_{i-1}} \otimes M)$ forming a generalized eigenspace of eigenvalue ν_i for $\mathbb{X}_{V^{\otimes_{i-1}} \otimes M}$, $i \in [1, 3]$. Thus, $E^3_\nu M$ affords a generalized eigenspace of eigenvalue ν_i for each X_i by (金 3). As such, it follows from a theorem of Brundan and Kleschev [BrK] and Rouquier [Ro], cf. [RW, Th. 6.4.2], that E^3M affords a representation of $\mathrm{H}_3(\mathbb{F}_p)$. Then (金 1.1) follows from (KLR3).

(金 5) **Remark:** As the set $P(\otimes^n \mathrm{nat}_p)$ of $\otimes^n(\mathrm{nat}_p)$ coincides with $P(\wedge^n \mathrm{nat}_p) = \mathbb{Z}\delta + \{\sum_{j=1}^p n_j\hat{\varepsilon}_j \mid n_j \in \mathbb{N}, \sum_{j=1}^p n_j = n\}$, we may denote the bijection $P(\otimes^n \mathrm{nat}_p) \to \Lambda/(\mathcal{W}_a\bullet)$ by ι_n from (水 2). Define $\mathbb{T} \in \mathrm{Cat}(\mathrm{Rep}(G_1T), \mathrm{Rep}(G_1T))(E^2, E^2)$ just as on $\mathrm{Rep}(G)$, and for each $\lambda \in P$ let

$$\mathrm{R}_{\iota_n(\lambda)}(G_1T) = \begin{cases} \mathrm{Rep}_{\iota_n(\lambda)}(G_1T) & \text{if } \lambda \in P(\otimes^n \mathrm{nat}_p) = P(\wedge^n \mathrm{nat}_p), \\ 0 & \text{else.} \end{cases}$$

Exactly the same arguments for $\mathrm{Rep}(G)$ yield a 2-representation of $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ on $\mathrm{Rep}(G_1T)$.

(金 6) Recall $\varpi = \hat{\varepsilon}_1 + \cdots + \hat{\varepsilon}_n \in P(\wedge^n(\mathrm{nat}_p)))$ from (水 2). $\forall s \in \mathcal{S}_a$, set

$$\mathrm{T}^s = \begin{cases} E^\varpi_{n-j} & \text{if } s = s_{\alpha_j}, \\ E_0^{\varpi + \hat{\alpha}_n + \cdots + \hat{\alpha}_{p-1}} E_{p-1}^{\varpi + \hat{\alpha}_n + \cdots + \hat{\alpha}_{p-2}} \dots E_{n+1}^{\varpi + \hat{\alpha}_n} E_n^\varpi & \text{if } s = s_{\alpha_0, 1}, \end{cases}$$

$$\mathrm{T}_s = \begin{cases} F_{n-j}^{\varpi + \hat{\alpha}_{n-j}} & \text{if } s = s_{\alpha_j}, \\ F_n^{\varpi + \hat{\alpha}_n} F_{n+1}^{\varpi + \hat{\alpha}_n + \hat{\alpha}_{n+1}} \dots F_{p-1}^{\varpi + \hat{\alpha}_n + \cdots + \hat{\alpha}_{p-1}} F_0^{\varpi + \hat{\alpha}_n + \cdots + \hat{\alpha}_{p-1} + \hat{\alpha}_0} & \text{if } s = s_{\alpha_j}, \end{cases}$$

and $\Theta_s = \mathrm{T}_s\mathrm{T}^s$. By (水 2) each Θ_s may be taken to be the s-wall crossing functor on $\mathrm{Rep}_{[n\det]}(G)$. We have obtained a strict monoidal functor

$$\mathcal{U}(\widehat{\mathfrak{gl}_p})(\varpi,\varpi) \to \mathrm{Cat}(\mathrm{Rep}_{[n\det]}(G), \mathrm{Rep}_{[n\det]}(G)) \tag{1}$$

such that $F_{n-j}E_{n-j}1_\varpi \mapsto \Theta_{s_j}$ $j \in [1,n[$, and $F_nF_{n+1}\dots F_{p-1}F_0E_0E_{p-1}\dots E_{n+1}E_n1_\varpi \mapsto \Theta_{s_{\alpha_0,1}}$.

As $\iota_n(\varpi) = n\det = \det^{\otimes n} \in A^+$, we may regard $\mathrm{R}_{\iota_n(\varpi)}(G) = \mathrm{Rep}_{[n\det]}(G)$ as the principal block $\mathrm{Rep}_0(G)$; $\mathrm{Rep}_0(G) \simeq \mathrm{R}_{\iota_n(\varpi)}(G)$ via $M \mapsto \det^{\otimes n} \otimes M$. Then (1) reads as a strict monoidal functor

$$\mathcal{U}(\widehat{\mathfrak{gl}_p})(\varpi,\varpi) \to \mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G)). \tag{2}$$

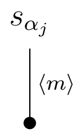
(金 7) In order to obtain a strict monoidal functor $\mathcal{D}_{\mathrm{BS}} \to \mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G))$ such that $B_s\langle m\rangle \mapsto \Theta_s$ $\forall s \in \mathcal{S}_a$ $\forall m \in \mathbb{Z}$, it now suffices to construct a strict monoidal functor $\mathcal{D}_{\mathrm{BS}} \to \mathcal{U}(\widehat{\mathfrak{gl}_p})(\varpi,\varpi)$ such that $\forall j \in [1,n[$, $\forall m \in \mathbb{Z}$, $B_{s_{\alpha_j}}\langle m\rangle \mapsto F_{n-j}E_{n-j}1_\varpi$ and that $B_{s_{\alpha_0,1}}\langle m\rangle \mapsto F_nF_{n+1}\dots F_{p-1}F_0E_0E_{p-1}\dots E_{n+1}E_n1_\varpi$. Such had been done by Mackaay, Stošić and Vas [MSV], Mackaay and Thiel [MT15], [MT17].

Instead of dealing directly with $F_nF_{n+1}\dots F_{p-1}F_0E_0E_{p-1}\dots E_{n+1}E_n1_\varpi$, however,[RW] considers "restriction" of the 2-representation of $\mathcal{U}(\widehat{\mathfrak{gl}_p})$ to $\mathcal{U}(\widehat{\mathfrak{gl}_n})$. We omit further details to state

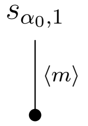
Theorem [RW, Th. 8.1.1]: *There is a strict monoidal functor*

$$\mathcal{D}_{\mathrm{BS}} \to \mathrm{Cat}(\mathrm{Rep}_{[n\det]}(G), \mathrm{Rep}_{[n\det]}(G))$$

such that $\forall s \in \mathcal{S}_a$, $\forall m \in \mathbb{Z}$, $B_s\langle m\rangle \mapsto \Theta_s$, *and* $\forall j \in [1,n[$,



$$\mapsto \eta^{\varpi}_{n-j} \in \mathrm{Cat}(\mathrm{Rep}_{[n\det]}(G), \mathrm{Rep}_{[n\det]}(G))(\mathrm{id}, \Theta_{s_{\alpha_j}}),$$



$$\begin{aligned}\mapsto &\big(\iota_{F_nF_{n+1}\dots F_{p-1}|_{\mathrm{Rep}_{\iota_n(\varpi+\hat\alpha_n+\dots+\hat\alpha_{p-1})}(G)}} * \eta_0^{\varpi+\hat\alpha_n+\hat\alpha_{n+1}+\dots+\hat\alpha_{p-1}} * \iota_{E_{p-1}\dots E_{n+1}\dots E_n|_{\mathrm{Rep}_{\iota_n(\varpi)}(G)}}\big)\\ &\odot\dots\odot \big(\iota_{F_nF_{n+1}|_{\mathrm{Rep}_{\iota_n(\varpi+\hat\alpha_n+\hat\alpha_{n+1})}(G)}} * \eta_{n+2}^{\varpi+\hat\alpha_n+\hat\alpha_{n+1}} * \iota_{E_{n+1}E_n|_{\mathrm{Rep}_{\iota_n(\varpi)}(G)}}\big)\\ &\odot\big(\iota_{F_n}|_{\mathrm{Rep}_{\iota_n(\varpi+\hat\alpha_n)}(G)} * \eta_{n+1}^{\varpi+\hat\alpha_n} * \iota_{E_n|_{\mathrm{Rep}_{\iota_n(\varpi)}(G)}}\big) \odot \eta_n^{\varpi} \in \mathrm{Cat}(\mathrm{Rep}_{[n\det]}(G), \mathrm{Rep}_{[n\det]}(G))(\mathrm{id}, \Theta_{s_{\alpha_0,1}}).\end{aligned}$$

(金 8) Finally, there is an autoequivalence $\iota : \mathcal{D}_{\mathrm{BS}} \to \mathcal{D}_{\mathrm{BS}}$ such that $B_{\underline{s_1\dots s_r}}\langle m\rangle \mapsto B_{\underline{s_r\dots s_1}}\langle m\rangle$ $\forall$ sequences $\underline{s_1\dots s_r}$ in $\mathcal{S}_a$, $\forall m \in \mathbb{Z}$, and on each morphism reflecting the corresponding diagrams along a vertical axis [RW, 4.2]. In particular, $\forall X, Y \in \mathrm{Ob}(\mathcal{D}_{\mathrm{BS}})$, $\iota(XY) = \iota(Y)\iota(X)$. Thus, combined with ι, we have obtained a strict monoidal functor $\mathcal{D}_{\mathrm{BS}} \to \mathrm{Cat}(\mathrm{Rep}_{[n\det]}(G), \mathrm{Rep}_{[n\det]}(G))^{\mathrm{op}}$

such that $\forall s \in \mathcal{S}_a$, $\forall m \in \mathbb{Z}$, $B_s\langle m\rangle \mapsto \Theta_s$. As $\mathrm{Rep}_{[n\det]}(G)$ is equivalent to the principal block $\mathrm{Rep}_0(G)$ by tensoring with $\det^{\otimes -n}$, we have now

Corollary [RW, Th. 1.5.1]: *There is a strict monoidal functor*

$$\Psi : \mathcal{D}_{\mathrm{BS}} \to \mathrm{Cat}(\mathrm{Rep}_0(G), \mathrm{Rep}_0(G))^{\mathrm{op}}$$

such that $\forall s \in \mathcal{S}_a$, $\forall m \in \mathbb{Z}$, $B_s\langle m\rangle \mapsto \Theta_s$.

(金 9) The functor Ψ induces another functor $\tilde{\Psi} : \mathcal{D}_{\mathrm{BS}} \to \mathrm{Rep}_0(G)$ such that $B \mapsto \nabla(0)B$. If $\underline{x} = \underline{s_1 s_2 \dots s_r}$ is an expression of $x \in \mathcal{W}_a$, one has

$$B_{\underline{x}} \mapsto \nabla(0)B_{\underline{x}} = \nabla(0)B_{s_1}B_{s_2}\dots B_{s_r} = \Theta_{s_r}\dots\Theta_{s_2}\Theta_{s_1}\nabla(0).$$

(金 10) Recall now from (火 3) the EW-category $\mathcal{D} = \mathrm{Kar}(\mathcal{D}_{\mathrm{BS}})$. The functor $\tilde{\Psi}$ naturally extends to a functor $\mathcal{D} \to \mathrm{Rep}_0(G)$, which we denote by the same letter. Our final objective is to show

Theorem [RW, Th. 1.3.1]: $\forall w \in {}^f\mathcal{W}$,

$$\tilde{\Psi}(B_w) = \nabla(0)B_w = T(w \bullet 0).$$

As $\nabla(0) = T(0)$, $\tilde{\Psi}(B_w)$ is tilting, and hence we have only to show that it is indecomposable. For that we will show that $\mathrm{Rep}(G)(T(0)B_w, T(0)B_w)$ is local. Let $\mathrm{Tilt}_0(G) = \mathrm{Tilt}(G) \cap \mathrm{Rep}(G)$. As $\nabla(0) = T(0)$, as the translation functors send a tilting module to a tilting module, and as $\mathrm{Tilt}_0(G)$ is Karoubian [J, E.1], $\tilde{\Psi}$ factors through $\mathrm{Tilt}_0(G)$:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\tilde{\Psi}"] \arrow[dr, dotted] & \mathrm{Rep}_0(G) \\ & \mathrm{Tilt}_0(G). \arrow[u, hook] \end{tikzcd}$$

(金 11) **Lemma [RW, Lem. 4.2.3]:** *Given an expression* $\underline{s_1 \dots s_r}$ *in* $\mathcal{W}_a$, *if* $B_x\langle m\rangle$, $m \in \mathbb{Z}$, *is an indecomposable direct summand of* $B_{\underline{s_1 \dots s_r}}$ *in* $\mathcal{D}$, $s_1 x < x$ *in the Chevalley-Bruhat order.*

This may appear strange. Recall, however, an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras $\mathcal{H} \xrightarrow{\sim} [\mathcal{D}]$ such that $\underline{H}_s \mapsto B_s$ $\forall s \in \mathcal{S}_a$. Let $s, t \in \mathcal{S}_a$ with $s \neq t$. One has

$$\begin{aligned} \underline{H}_s\underline{H}_t &= (H_s + v)(H_t + v) = H_{st} + v(H_s + H_t) + v^2 \\ &= \underline{H}_{st} \quad \text{by the characterization of KL-basis elements [S97, Th. 2.1]}, \\ \underline{H}_s^2 &= (H_s + v)^2 = H_s^2 + 2vH_s + v^2 = 1 + (v^{-1} - v)H_s + 2vH_s + v^2 \\ &= 1 + v^2 + (v^{-1} + v)H_s = (v^{-1} + v)(v + H_s), \end{aligned}$$

and hence

$$\underline{H}_s^2\underline{H}_t = (v^{-1} + v)(v + H_s)(H_t + v) = (v^{-1} + v)\{H_{st} + v(H_s + H_t) + v^2\} = (v^{-1} + v)\underline{H}_{st}.$$

As ${}^p\underline{H}_s = \underline{H}_s$, ${}^p\underline{H}_t = \underline{H}_t$, and as ${}^p\underline{H}_{st} = \underline{H}_{st}$,

$$[B_{\underline{sst}}] = (v^{-1}+v)[B_{st}] = [B_{st}\langle -1\rangle] + [B_{st}\langle 1\rangle],$$

and the lemma indeed holds in this case.

(金 12) Let $\mathcal{D}_{\mathcal{W}_a\setminus{}^f\mathcal{W}}$ be the additive full subcategory of $\mathcal{D}$ consisting of the direct sums of objects $B_w\langle m\rangle$, $w \in \mathcal{W}_a \setminus {}^f\mathcal{W}$, $m \in \mathbb{Z}$, and let $\mathcal{D}^{\mathrm{asph}} = \mathcal{D}//\mathcal{D}_{\mathcal{W}_a\setminus{}^f\mathcal{W}}$ be the quotient of $\mathcal{D}$ by $\mathcal{D}_{\mathcal{W}_a\setminus{}^f\mathcal{W}}$ [中岡, Prop. 3.2.51, p. 150]: $\forall X, Y \in \mathcal{D}$, let $\mathcal{I}(X,Y) = \{f \in \mathcal{D}(X,Y) |\ f$ factors through some $Z \in \mathcal{D}_{\mathcal{W}_a\setminus{}^f\mathcal{W}}\}$. Then $\mathcal{D}^{\mathrm{asph}}$ is the category with objects $\mathrm{Ob}(\mathcal{D})$ and $\forall X, Y \in \mathcal{D}$, $\mathcal{D}^{\mathrm{asph}}(X,Y) = \mathcal{D}(X,Y)/\mathcal{I}(X,Y)$. $\forall s \in \mathcal{S}$, $\tilde{\Psi}(B_s) = \nabla(0)B_s = \Theta_s\nabla(0) = 0$. $\forall x \in \mathcal{W}_a \setminus {}^f\mathcal{W}$, $\exists s \in \mathcal{S}$ and $y \in \mathcal{W}_a$ with $\ell(x) = \ell(y)+1$ such that $x = sy$. If $\underline{y}$ is a reduced expression of y, B_x is a direct summand of $B_{\underline{sy}} = B_sB_{\underline{y}}$, and hence $\tilde{\Psi}(B_x)$ is a direct summand of $\tilde{\Psi}(B_{\underline{sy}}) = \tilde{\Psi}(B_s)B_{\underline{y}} = 0$. It follows that $\tilde{\Psi}$ factors through $\mathcal{D}^{\mathrm{asph}}$:

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "\tilde{\Psi}"] \arrow[d] & & \mathrm{Rep}_0(G), \\
\mathcal{D}//\mathcal{D}_{\mathcal{W}_a\setminus{}^f\mathcal{W}} = \mathcal{D}^{\mathrm{asph}} \arrow[urr, dotted] & &
\end{tikzcd}$$

which we denote by $\bar{\Psi}$. If $\underline{w}$ is a reduced expression of $w \in {}^f\mathcal{W}$, $\nabla(0)B_{\underline{w}}$ has highest weight $w \bullet 0$. As $B_{\underline{w}}$ is a direct sum of B_w and some B_y's with $y < w$, we must have $\nabla(0)B_w \neq 0$, and hence $\bar{B}_w \neq 0$ in $\mathcal{D}^{\mathrm{asph}}$. Then, as a quotient of a local ring remains local [AF, 15.15, p. 170], the indecomposable objects of $\mathcal{D}^{\mathrm{asph}}$ are $\bar{B}_w\langle m\rangle$, $w \in {}^f\mathcal{W}$, $m \in \mathbb{Z}$. Thus, $\mathcal{D}^{\mathrm{asph}}$ is a graded category inheriting shift functor $\langle 1\rangle$, and the indecomposables of $\mathcal{D}^{\mathrm{asph}}$ are the images $\bar{B}_w\langle m\rangle$ of $B_w\langle m\rangle$, $w \in {}^f\mathcal{W}$, $m \in \mathbb{Z}$. Also, (金 11) implies that $\mathcal{D}^{\mathrm{asph}}$ admits a structure of right $\mathcal{D}$-module. For let $\phi \in \mathcal{D}(X,Y)$ factor through some $Z \in \mathcal{D}_{\mathcal{W}_a\setminus{}^f\mathcal{W}}$. Let $B_x\langle m\rangle$ be a direct summand of Z, so x admits a reduced expression $\underline{s_1 \dots s_r}$ with $s_1 \in \mathcal{S}$. Given an expression $\underline{y}$ in $\mathcal{W}_a$, each direct summand $B_w\langle k\rangle$ of $B_x\langle m\rangle B_{\underline{y}}$ has $s_1w < w$ by (金 11) again, and hence $w \notin {}^f\mathcal{W}$ and $B_w\langle k\rangle \in \mathcal{D}_{\mathcal{W}_a\setminus{}^f\mathcal{W}}$.

Let $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}$ be the degrading of $\mathcal{D}^{\mathrm{asph}}$: $\mathrm{Ob}(\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}) = \mathrm{Ob}(\mathcal{D}^{\mathrm{asph}})$ but $\forall X, Y \in \mathrm{Ob}(\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}})$, $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(X,Y) = (\mathcal{D}^{\mathrm{asph}})^\bullet(X,Y) = \coprod_{m\in\mathbb{Z}} \mathcal{D}^{\mathrm{asph}}(X, Y\langle m\rangle)$. In particular, $\forall m \in \mathbb{Z}$, $X \simeq X\langle m\rangle$ in $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}$; $\mathrm{id}_X \in \mathcal{D}^{\mathrm{asph}}(X,X) \leq \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(X, X\langle m\rangle)$ admits an inverse $\mathrm{id}_{X\langle m\rangle} \in \mathcal{D}^{\mathrm{asph}}(X\langle m\rangle, X\langle m\rangle) \leq \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(X\langle m\rangle, X)$. By construction $\bar{\Psi}$ induces a functor $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}} \to \mathrm{Tilt}_0(G)$, which we denote by $\bar{\Psi}_{\mathrm{deg}}$. $\forall w \in {}^f\mathcal{W}$, $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_w, \bar{B}_w) = (\mathcal{D}^{\mathrm{asph}})^\bullet(\bar{B}_w, \bar{B}_w)$ remains local [GG, Th. 3.1]. Our objective (金 10) will thus follow from

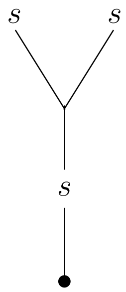
Theorem [RW, Th. 1.3.1]; *The functor* $\bar{\Psi}_{\mathrm{deg}} : \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}} \to \mathrm{Tilt}_0(G)$ *is an equivalence of categories.*

(金 13) For an expression $\underline{x} = \underline{s_1s_2 \dots s_r}$ of $x \in \mathcal{W}_a$, put $T(\underline{x}) = T(0)B_{\underline{x}} = \Theta_{s_r} \dots \Theta_{s_2}\Theta_{s_1}T(0)$. To establish the categorical equivalence, it suffices by induction and (火 3) to show that $\bar{\Psi}$ induces an isomorphism $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_{\underline{x}}, \bar{B}_{\underline{y}}) \xrightarrow{\sim} \mathrm{Rep}_0(T(\underline{x}), T(\underline{y}))$ $\forall \underline{x}, \underline{y}$. Let $\alpha_{\underline{x},\underline{y}} : \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_{\underline{x}}, \bar{B}_{\underline{y}}) \to \mathrm{Rep}_0(T(\underline{x}), T(\underline{y}))$ denote the $\Bbbk$-linear map induced by $\bar{\Psi}$. The surjectivity of $\alpha_{\underline{x},\underline{y}}$ requires introduction of highest weight categories and the Serre quotient of a highest weight category by a Serre subcategory. We will only show that

$$\dim \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_{\underline{x}}, \bar{B}_{\underline{y}}) \leq \dim \mathrm{Rep}_0(T(\underline{x}), T(\underline{y})).$$

If $\underline{x} = \underline{ws}$ for some $s \in \mathcal{S}_a$. Recall from (金 7) that

(1)



$$\begin{array}{ccc} B_{\underline{ss}} & & \Theta_s^2 \\ \uparrow & & \uparrow \\ B_{\underline{s}}\langle 1\rangle & \overset{\Psi}{\longmapsto} & \Theta_s \\ \uparrow & & \uparrow \\ B_{\emptyset} & & \mathrm{id}. \end{array}$$

As the LHS is the unit, say η^s, associated to an adjunction $(?B_s, ?B_s)$ [EW], it induces a unit of adjunction $(?\bar{B}_s, ?\bar{B}_s)$ on $\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}$, so therefore is $\Psi(\eta^s)$ associated to an adjunction (Θ_s, Θ_s) [中岡, Cor. 2.2.9]. One has then a commutative daigram

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_{\underline{ws}}, \bar{B}_{\underline{y}}) \arrow[r, "(?B_{\underline{s}})\circ\eta^s_{\bar{B}_{\underline{w}}}", "\sim"'] \arrow[d, "\alpha_{\underline{ws},\underline{y}}"'] & \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_{\underline{w}}, \bar{B}_{\underline{ys}}) \arrow[d, "\alpha_{\underline{w},\underline{ys}}"] \\
\mathrm{Rep}_0(T(\underline{ws}), T(\underline{y})) \arrow[r, "\sim", "\Theta_s(?)\circ\Psi(\eta^s_{\bar{B}_{\underline{w}}})"'] & \mathrm{Rep}_0(T(\underline{w}), T(\underline{ys})).
\end{tikzcd}$$

Thus the bijectivity is reduced to that of $\alpha_{\underline{w},\underline{ys}}$, and hence to the case $\underline{x} = \emptyset$.

(金 14) For any expression $\underline{x}$ of an element of $\mathcal{W}_a$ one has

$$\dim \mathrm{Rep}_0(T(\emptyset), T(\underline{x})) = \dim \mathrm{Rep}_0(\Delta(0), T(\underline{x})) = (T(\underline{x}) : \nabla(0)).$$

Lemma [RW, Lem. 5.4.1, 5.4.2]: *If $\underline{w}$ is an expression of $w \in {}^f\mathcal{W}$,*

$$\dim \mathcal{D}^{\mathrm{asph}}_{\mathrm{deg}}(\bar{B}_{\emptyset}, \bar{B}_{\underline{w}}) \leq (T(\underline{w}) : \nabla(0)).$$

(金 15) To see Lem. 金 14, fix an expression $\underline{w} = \underline{s_1 \dots s_r}$. Each $e(\underline{w}) \in \{0,1\}^r$ defines a subexpression $\underline{w}^{e(\underline{w})} = (s_1^{e(\underline{w})_1}, \dots, s_r^{e(\underline{w})_r})$ of $\underline{w}$ by deleting those terms with $e(\underline{w})_j = 0$, in which case we also let $w^{e(\underline{w})} = s_1^{e(\underline{w})_1} \dots s_r^{e(\underline{w})_r} \in \mathcal{W}_a$. The Bruhat stroll of $e(\underline{w})$ is the sequence $x_0 = e, x_1 = s_1^{e(\underline{w})_1}, x_2 = s_1^{e(\underline{w})_1} s_2^{e(\underline{w})_2}, \dots, x_r = s_1^{e(\underline{w})_1} s_2^{e(\underline{w})_2} \dots s_r^{e(\underline{w})_r}$. $\forall j \in [1, r]$, we assign a symbol

$$\begin{cases} \text{U1} & \text{if } e(\underline{w})_j = 1 \text{ and } x_j = x_{j-1}s_j > x_{i-1}, \\ \text{D1} & \text{if } e(\underline{w})_j = 1 \text{ and } x_j = x_{j-1}s_j < x_{i-1}, \\ \text{U0} & \text{if } e(\underline{w})_j = 0 \text{ and } x_j = x_{j-1}s_j > x_{i-1}, \\ \text{D0} & \text{if } e(\underline{w})_j = 0 \text{ and } x_j = x_{j-1}s_j < x_{i-1}, \end{cases}$$

"U" (resp. "D") standing for Up (resp. Down). Let $d(e(\underline{w}))$ denote the number of U0's minus the number of D0's, called the defect of $e(\underline{w})$ [EW, 2.4]. For $\mathcal{W}' \subseteq \mathcal{W}_a$ we say $e(\underline{w})$ avoids $\mathcal{W}'$

iff $x_r \notin \mathcal{W}'$ and $x_{j-1}s_j \notin \mathcal{W}'$ $\forall j \in [1,r]$. We understand $e(\underline{w})$ avoids any $\mathcal{W}'$ in case $r=0$. For each $x \in {}^f\mathcal{W}$ put $N_x = 1 \otimes H_x$ in $\mathcal{M}^{\text{asph}}$, and for each expression $\underline{w} = \underline{s_1 \dots s_r}$ of $w \in \mathcal{W}_a$ put $\underline{H}_{\underline{w}} = \underline{H}_{s_1} \dots \underline{H}_{s_r}$.

Lemma [RW, Lem. 4.1.1]: *For each expression $\underline{w}$ one has in $\mathcal{M}^{\text{asph}}$*

$$N_1 \underline{H}_{\underline{w}} = \sum_{e(\underline{w}) \text{ avoiding } \mathcal{W}_a \setminus {}^f\mathcal{W}} v^{d(e(\underline{w}))} N_{w^{e(\underline{w})}}.$$

(金 16) Let $\underline{w} = \underline{s_1 \dots s_r}$ be an expression. One has from [EW, Prop. 6.12] that $\mathcal{D}^\bullet_{\text{BS}}(B_{\underline{w}}, B_\emptyset)$ admits a basis of left $\underline{R}$-module consisting of the light leaves $L_{e(\underline{w})}$ $\forall e(\underline{w})$ expressing the unity of $\mathcal{W}_a$.

Proposition [RW, Prop. 4.5.1]: *Let $\underline{w}$ be an expression of an element in $\mathcal{W}_a$. One can choose the light leaves $L_{e(\underline{w})}$ with $e(\underline{w})$ expressing 1 and avoiding $\mathcal{W}_a \setminus {}^f\mathcal{W}$ to $\Bbbk$-linearly span $(\mathcal{D}^{\text{asph}}_{\text{BS}})^\bullet(\bar{B}_{\underline{w}}, \bar{B}_\emptyset)$.*

(金 17) We are now ready to show Lem. 金 11. Recall from (月 10) an isomorphism of right $\mathcal{H}$-modules $M^{\text{asph}} = \text{sgn}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\mathcal{W}_f]} \mathbb{Z}[\mathcal{W}_a] \simeq [\text{Rep}_0(G)]$. If we put $N'_w = 1 \otimes w$, $w \in {}^f\mathcal{W}$, $w \in {}^f\mathcal{W}$ forms a $\mathbb{Z}$-linear basis of M^{asph}, and for each $s \in \mathcal{S}_a$ one has a commutative diagram

$$\begin{array}{ccc} N'_w & \longmapsto & \nabla(w \bullet 0) \\ M^{\text{asph}} & \xrightarrow{\sim} & [\text{Rep}_0(G)] \\ {\scriptstyle \underline{H}_s}\big\downarrow & & \big\downarrow{\scriptstyle \Theta_s} \\ M^{\text{asph}} & \xrightarrow[\sim]{} & [\text{Rep}_0(G)]. \end{array}$$

For an expression $\underline{w} = \underline{s_1 \dots s_r}$ of an element $w \in {}^f\mathcal{W}$ put $\underline{N'_{\underline{w}}} = 1 \otimes (1+s_1)\dots(1+s_r)$ in M^{asph}. As $\underline{N'_{\underline{w}}} \mapsto [T(\underline{w})]$, $\underline{N'_{\underline{w}}} \in (T(\underline{w}) : \nabla(0))N'_1 + \sum_{x \in {}^f\mathcal{W} \setminus 1} \mathbb{Z}N'_x$.

Using the anti-equivalence τ from (火 2) such that $\bar{B}_{\underline{x}}\langle m \rangle \mapsto \bar{B}_{\underline{x}}\langle -m \rangle$ $\forall \underline{x}$, $\forall m \in \mathbb{Z}$, one has $\dim(\mathcal{D}^{\text{asph}})^\bullet(\bar{B}_\emptyset, \bar{B}_{\underline{w}}) = \dim(\mathcal{D}^{\text{asph}})^\bullet(\bar{B}_{\underline{w}}, \bar{B}_\emptyset)$, which is equal to $\dim(\mathcal{D}^{\text{asph}}_{\text{BS}})^\bullet(\bar{B}_{\underline{w}}, \bar{B}_\emptyset)$ as $\mathcal{D}^{\text{asph}}_{\text{BS}}$ is a full subcategory of $\mathcal{D}^{\text{asph}} = \text{Kar}(\mathcal{D}^{\text{asph}}_{\text{BS}})$ by (金 11) [Bor, Prop. 6.5.9, p. 274]. In turn, $\dim(\mathcal{D}^{\text{asph}}_{\text{BS}})^\bullet(\bar{B}_{\underline{w}}, \bar{B}_\emptyset) \leq \sharp\{e(\underline{w}) | e(\underline{w})$ is an expression of the unity avoiding $\mathcal{W}_a \setminus {}^f\mathcal{W}\}$ by (金 16). On the other hand, from (金 15) one has

$$N_1 \underline{H}_{\underline{w}} = \sum_{e(\underline{w}) \text{ avoiding } \mathcal{W}_a \setminus {}^f\mathcal{W}} v^{d(e(\underline{w}))} N_{w^{e(\underline{w})}},$$

which under the specialization $v \rightsquigarrow 1$ yields

$$\begin{aligned} \underline{N'_{\underline{w}}} &= \sum_{e(\underline{w}) \text{ avoiding } \mathcal{W}_a \setminus {}^f\mathcal{W}} N_{w^{e(\underline{w})}} \\ &\in \sharp\{e(\underline{w}) | e(\underline{w}) \text{ is an expression of the unity avoiding } \mathcal{W}_a \setminus {}^f\mathcal{W}\}N'_1 + \sum_{x \in {}^f\mathcal{W} \setminus 1} \mathbb{N}N'_x. \end{aligned}$$

One thus obtains

$$\begin{aligned}\dim(\mathcal{D}_{\mathrm{BS}}^{\mathrm{asph}})^\bullet(\bar{B}_{\underline{w}}, \bar{B}_\emptyset) &\leq \sharp\{e(\underline{w})|e(\underline{w}) \text{ is an expression of the unity avoiding } \mathcal{W}_a \setminus {}^f\mathcal{W}\} \\ &= (T(\underline{w}) : \nabla(0)).\end{aligned}$$

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