RESTRICTION OF EISENSTEIN SERIES AND STARK-HEEGNER POINTS

MING-LUN HSIEH AND SHUNSUKE YAMANA

Abstract. In a recent work of Darmon, Pozzi and Vonk, the authors consider a particular p-adic family of Hilbert Eisenstein series $E_k(1,\phi)$ associated with an odd character ϕ of the narrow ideal class group of a real quadratic field F and compute the first derivative of a certain one-variable twisted triple product p-adic L-series attached to $E_k(1,\phi)$ and an elliptic newform f of weight 2 on $\Gamma_0(p)$. In this paper, we generalize their construction to include the cyclotomic variable and thus obtain a two-variable twisted triple product p-adic L-series. Moreover, when f is associated with an elliptic curve E over $\mathbf{Q}$, we prove that the first derivative of this p-adic L-series along the weight direction is a product of the p-adic logarithm of a Stark-Heegner point of E over F introduced by Darmon and the cyclotomic p-adic L-function for E.

1. Introduction

In the work [DPV19], to each odd character ϕ of the narrow ideal class group of a real quadratic field F, the authors associate a one-variable p-adic family $E_k^{(p)}(1,\phi)$ of Hilbert Eisenstein series on $\Gamma_0(p)$ over a real quadratic field F and give the explicit spectral decomposition of the ordinary projection of the diagonal restriction $G_{2k}(\phi)$ of $E_k^{(p)}(1,\phi)$ around $k=1$. The coefficient $\lambda_f(k)$ of each normalized Hecke eigenform f of weight two on $\Gamma_0(p)$ in the spectral decomposition can be viewed as a certain (one-variable) twisted triple product p-adic L-function associated with $E_k^{(p)}(1,\phi)$ and f, and it is proved in [DPV19, Theorem C(2)] that the first derivative of $\lambda_f(k)$ at $k=1$ can be expressed in terms of the product of special values of the L-function for f and the p -adic logarithms of Stark-Heegner points or elliptic units over F introduced in [Dar01] and [DD06].

The purpose of this paper is to provide some partial generalizations of this work to the two-variable setting by introducing the cyclotomic variable. To begin with, we let F be a real quadratic field and let $\mathfrak{d}$ be the different of

---

Date: February 27, 2020.

2010 *Mathematics Subject Classification.* 11F67, 11F33.

Hsieh was partially supported by a MOST grant 108-2628-M-001-009-MY4. Yamana is partially supported by JSPS Grant-in-Aid for Scientific Research (C) 18K03210 and (B) 19H01778. This work is partially supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics).

$F/\mathbf{Q}$. Let $x \mapsto \overline{x}$ denote the non-trivial automorphism of F and let $\mathrm{N} : F \to \mathbf{Q}$, $\mathrm{N}(x) = x\overline{x}$ be the norm map. Let Δ_F be the discriminant of $F/\mathbf{Q}$. Let $\mathrm{Cl}^+(\mathcal{O}_F)$ be the narrow ideal class group of F. Let $\phi : \mathrm{Cl}^+(\mathcal{O}_F) \to \overline{\mathbf{Q}}^\times$ be an odd narrow ideal class character, i.e. $\phi((\delta)) = -1$ for any $\delta \in \mathcal{O}_F$ with $\overline{\delta} = -\delta$. Let $L(s,\phi)$ be the Hecke L-function attached to ϕ. Fix an odd rational prime p unramified in F. For $x \in \mathbf{Z}_p^\times$, let $\boldsymbol{\omega}(x)$ be the Teichmüller lift of $x \pmod p$ and let $\langle x \rangle := x\boldsymbol{\omega}^{-1}(x) \in 1 + p\mathbf{Z}_p$. Let $\mathscr{X} := \{x \in \mathbf{C}_p \mid |x|_p \leq 1\}$ be the p-adic closed unit disk and let $A(\mathscr{X})$ be the ring of rigid analytic functions on $\mathscr{X}$. Fix an embedding $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ througout. For each ideal $\mathfrak{m} \lhd \mathcal{O}_F$ corpime to p, define $\boldsymbol{\sigma}_\phi(\mathfrak{m}) \in A(\mathscr{X} \times \mathscr{X})$ by

$$\boldsymbol{\sigma}_\phi(\mathfrak{m})(k,s) = \sum_{\mathfrak{a} \lhd \mathcal{O}_F,\, \mathfrak{a} | \mathfrak{m}} \phi(\mathfrak{a}) \left\langle \mathrm{N}(\mathfrak{a}) \right\rangle^{\frac{k-s}{2}} \left\langle \mathrm{N}(\mathfrak{m}\mathfrak{a}^{-1}) \right\rangle^{\frac{s-2}{2}}.$$

Let $\mathscr{X}^{\mathrm{cl}} := \left\{ k \in \mathbf{Z}^{\geq 2} \mid k \equiv 2 \,(\mathrm{mod}\ 2(p-1)) \right\}$ be the set of classical points in $\mathscr{X}$. Let $h = \#\mathrm{Cl}^+(\mathcal{O}_F)$. Fix a set $\{\mathfrak{t}_\lambda\}_{\lambda = 1,\ldots,h}$ of representatives of the narrow ideal class group $\mathrm{Cl}^+(\mathcal{O}_F)$ with $(\mathfrak{t}_\lambda, p\mathcal{O}_F) = 1$. For each classical point $k \in \mathscr{X}^{\mathrm{cl}}$, the classical Hilbert-Eisenstein series $E_{\frac{k}{2}}(1,\phi)$ on $\mathrm{SL}_2(\mathcal{O}_F)$ of parallel weight $\frac{k}{2}$ is determined by the normalized Fourier coefficients

$$c(\mathfrak{m}, E_{\frac{k}{2}}(1,\phi)) = \boldsymbol{\sigma}_\phi(\mathfrak{m})(k,s), \quad c_\lambda(0, E_{\frac{k}{2}}(1,\phi)) = 4^{-1} L(1 - k/2, \phi).$$

Let I_F be the set of integral ideals of F. Let $\mathfrak{n} \in I_F$ and p be coprime. Let $\mathcal{M}^{(2)}(\mathfrak{n})$ be the space of two-variable p-adic families of Hilbert *semi-cusp* forms[1] of tame level $\mathfrak{n}$, which consists of functions

$$f \colon I_F \to \mathscr{A}(\mathscr{X} \times \mathscr{X}), \quad \mathfrak{m} \mapsto c(\mathfrak{m}, f)$$

such that the specialization $f(k,s) = \{c(\mathfrak{m}, f)(k,s)\}$ is the set of normalized Fourier coefficients of a p-adic Hilbert semi-cusp forms of parallel weight k on $\Gamma_0(p\mathfrak{n})$ for (k,s) in a p-adically dense subset $U \subset \mathbf{Z}_p \times \mathbf{Z}_p$. Define $\widehat{E}_\phi^{\{p\}} : I_F \to A(\mathscr{X} \times \mathscr{X})$ by the data

$$c(\mathfrak{m}, \widehat{E}_\phi^{\{p\}}) = \boldsymbol{\sigma}_\phi(\mathfrak{m}) \text{ if } (\mathfrak{m}, p\mathcal{O}_F) = 1,$$
$$c(\mathfrak{m}, \widehat{E}_\phi^{\{p\}}) = 0 \text{ otherwise.}$$

By definition, for $(k,s) \in \mathscr{X}^{\mathrm{cl}} \times \mathscr{X}^{\mathrm{cl}}$ with $k \geq 2s$, we have

$$\widehat{E}_\phi^{\{p\}}(k,s) = \langle \Delta_F \rangle^{\frac{s-2}{2}} \cdot \theta^{\frac{s-2}{2}} E^{\{p\}}_{\frac{k+4-2s}{2}}(1,\phi),$$

where $E_k^{\{p\}}(1,\phi)$ is the *p-depletion* of $E_k(1,\phi)$ and θ is Serre's differential operator $\theta(\sum_\beta a_\beta q^\beta) = \sum_\beta \mathrm{N}(\beta) a_\beta q^\beta$. Therefore, $\widehat{E}_\phi^{\{p\}}(k,s)$ is a p-adic Hilbert modular form of parallel weight k for all $(k,s) \in \mathbf{Z}_p^2$, and

[1] Recall that a Hilbert semi-cusp form is a Hilbert modular form having no constant in the Fourier expansion around the cusps at the infinity.

$\widehat{E}^{(p)}_\phi \in \mathcal{M}^{(2)}(\mathcal{O}_F)$. For each prime ideal $\mathfrak{q}$, define $\mathbf{U}_\mathfrak{q} \colon \mathcal{M}^{(2)}(\mathfrak{n}) \to \mathcal{M}^{(2)}(\mathfrak{n}\mathfrak{q})$ by $c(\mathfrak{m}, \mathbf{U}_\mathfrak{q} f) := c(\mathfrak{m}\mathfrak{q}, f)$. Let N be a positive integer such that $p \nmid N$ and

$$\text{(Splt)} \qquad N\mathcal{O}_F = \mathfrak{N}\overline{\mathfrak{N}}, \quad (\mathfrak{N}, \overline{\mathfrak{N}}) = 1.$$

Define $\boldsymbol{E}_\phi \in \mathcal{M}^{(2)}(\mathfrak{N})$ by

$$\boldsymbol{E}_\phi := \prod_{\mathfrak{q}|\mathfrak{N}} (1 - \phi(\mathfrak{q})^{-1} \langle \mathrm{N}(\mathfrak{q}) \rangle^{\frac{2s-2-k}{2}} \mathbf{U}_\mathfrak{q}) \cdot \widehat{E}^{\{p\}}_\phi$$

and the *diagonal restriction* $\boldsymbol{G}_\phi \in A(\mathscr{X} \times \mathscr{X})[\![q]\!]$ of $\boldsymbol{E}_\phi$ by

$$\boldsymbol{G}_\phi := \sum_{n>0} \Big(\sum_{\beta \in \mathfrak{d}^{-1}_+, \mathrm{Tr}(\beta) = n} c(\beta\mathfrak{d}, \boldsymbol{E}_\phi) \Big) q^n,$$

where $\mathfrak{d}^{-1}_+$ is the additive semigroup of totally positive elements in $\mathfrak{d}^{-1}$.

By definition $\boldsymbol{G}_\phi(k,s)$ is the q-expansion of a p-adic elliptic modular on $\Gamma_0(pN)$ of weight k obtained from the diagonal restriction of $\widehat{E}^{\{p\}}_\phi(k,s)$ for $(k,s) \in \mathscr{X}^{\mathrm{cl}} \times \mathscr{X}^{\mathrm{cl}}$ with $k \geq 2s$. Let $\mathscr{U}$ be an appropriate neighborhood around $2 \in \mathscr{X}$. Let $\mathbf{S}^{\mathrm{ord}}(N)$ be the space of ordinary $A(\mathscr{U})$-adic elliptic cusp forms on $\Gamma_0(Np)$, consisting of q-expansion $\boldsymbol{f} = \sum_{n>0} c(n, \boldsymbol{f}) q^n \in A(\mathscr{U})[\![q]\!]$ such that the weight k specialization $\boldsymbol{f}_k$ is a p-ordinary cusp forms of weight k on $\Gamma_0(pN)$ for $k \in \mathscr{X}^{\mathrm{cl}}$. By Hida theory, we know $\mathbf{S}^{\mathrm{ord}}(N)$ is a free $A(\mathscr{U})$-module of finite rank. It can be shown that the image $e\boldsymbol{G}_\phi$ under Hida's ordinary projector actually belongs to $\mathbf{S}^{\mathrm{ord}}(N) \widehat{\otimes}_{A(\mathscr{U})} A(\mathscr{U} \times \mathscr{X})$, where $A(\mathscr{X})$ is regarded as a subring of $A(\mathscr{U} \times \mathscr{X})$ via the pull-back of the first projection $\mathscr{U} \times \mathscr{X} \to \mathscr{U}$. We can thus decompose

$$e\boldsymbol{G}_\phi = \sum_{\boldsymbol{f}} \mathcal{L}_{\boldsymbol{E}_\phi, \boldsymbol{f}} \cdot \boldsymbol{f} + (\text{old forms}), \quad \mathcal{L}_{\boldsymbol{E}_\phi, \boldsymbol{f}} \in A(\mathscr{U} \times \mathscr{X}),$$

where $\boldsymbol{f}$ runs over the set of primitive Hida families of tame conductor N. We shall call $\mathcal{L}_{\boldsymbol{E}_\phi, \boldsymbol{f}} \in A(\mathscr{U} \times \mathscr{X})$ the twisted triple product p-adic L-function attached to the p-adic Hilbert Eisenstein series $\boldsymbol{E}_\phi$ and a primitive Hida family $\boldsymbol{f}$. We provide the following derivative formula for $\mathcal{L}_{\boldsymbol{E}_\phi, \boldsymbol{f}}$, which partially generalizes [DPV19, Theorem C(2)] to elliptic newforms of split tame conductor.

Theorem (Corollary 7.3)**.** *Let E be an elliptic curve over $\mathbf{Q}$ of conductor pN with N satisfying* (Splt)*. Let $\boldsymbol{f} \in A(\mathscr{U})[\![q]\!]$ be a primitive Hida family of tame level N such that the weight two specialization $\boldsymbol{f}_2$ is the elliptic newform associated with E. Suppose that p is inert in F. Then $\mathcal{L}_{\boldsymbol{E}_\phi, \boldsymbol{f}}(2, s) = 0$ and*

$$\frac{d}{dk} \big(\mathcal{L}_{\boldsymbol{E}_\phi, \boldsymbol{f}}(k, s+1) \big)|_{k=2} = \frac{1}{2}(1 + \phi(\mathfrak{N})^{-1} w_N) \cdot \log_E P_\phi \cdot L_p(E, s) \\ \times \frac{c_f}{m_E^2 2^{\alpha(E)}} \langle \Delta_F \rangle^{\frac{s-1}{2}},$$

where

- $\log_E P_\phi$ *is the p-adic loagrithm of the twisted Stark-Heegner point* $P_\phi \in E(F_p) \otimes \mathbf{Q}(\phi)$ *introduced in* [Dar01, (182)],
- $L_p(E,s)$ *is the Mazur-Tate-Teitelbaum cyclotomic p-adic L-function for E,*
- $c_f \in \mathbf{Z}^{>0}$ *is the congruence number for f,* $m_E \in \mathbf{Q}^\times$ *is the Mainn constant for E and* $2^{\alpha(E)} = [\mathrm{H}_1(E(\mathbf{C}),\mathbf{Z}) : \mathrm{H}_1(E(\mathbf{C}),\mathbf{Z})^+ \oplus \mathrm{H}_1(E(\mathbf{C}),\mathbf{Z})]$.

Remark 1.1.

- The definition of Stark-Heegner points P_ϕ for odd ϕ in [Dar01] depends on a choice of the purely imaginary period Ω_E^-. In the above theorem, we require $(\sqrt{-1})^{-1}\Omega_E^-$ to be positive.
- Our main motivation for this two-variable generalization is that we have the non-vanishing of the p-adic L-function $L_p(E,s)$ thanks to Rohrlich's theorem [Roh84], so $\log_E P_\phi$ can be computed from the twisted triple product p-adic L-function even when the central value $L(E,1)$ vanishes.
- The Eisenstein contribution in the spectral decomposition in Part (2) of [DPV19, Theorem C] is connected with the p-adic logatithms of elliptic units over F, while in our two-variable setting, $e\boldsymbol{G}_\phi$ is a p-adic family of cusp forms, so we do not get any information for elliptic units.

We briefly outline the proof. Let $\mathcal{L}_p(\boldsymbol{f}/F,\phi,k)$ be the (odd) square-root p-adic L-function associated with the primitive Hida family $\boldsymbol{f}$ and the character ϕ constructed in [BD09, Definition 3.4] with $w_\infty = -1$ and let $L_p(\boldsymbol{f},k,s)$ be the Mazur-Kitagawa two-variable p-adic L-function so that $L_p(\boldsymbol{f},2,s)$ is the cyclotomic p-adic L-function for $\boldsymbol{f}_2$. In Theorem 7.1, we prove the following factorization formula of $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}$:

$$C^*(k)\cdot \mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(k,s+1) = \mathcal{L}_p(\boldsymbol{f}/F,\phi,k)\cdot L_p(\boldsymbol{f},k,s), \tag{1.1}$$

where $C^*(k)$ is a meromorphic function on $\mathscr{X}$ holomorphic at all classical points $k \in \mathscr{X}^{\mathrm{cl}}$ with $C^*(2) = 1$. By the very construction, the square root p-adic L-function $\mathcal{L}_p(\boldsymbol{f}/F,\phi,k)$ interpolates the toric period integrals $B^\phi_{\boldsymbol{f}_k}$. Thus we get $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(2,s) = \mathcal{L}_p(\boldsymbol{f}/F,\phi,2) = 0$ by a classical theorem of Saito and Tunnell. Moreover, from the formula [BD09, Corollary 2.6], it is not difficult to deduce that the first derivative of $\mathcal{L}_p(\boldsymbol{f}/F,\phi,k)$ at $k=2$ is $2^{-1}(1+w_N\phi(\mathfrak{N})^{-1})\log_E P_\phi$, and hence we obtain Theorem from (1.1). The factorization formula (1.1) is established by the explicit interpolation formulae on both sides. In particular, the interpolation formula for $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(k,s)$ (Proposition 5.7) is the most technical part of this paper. Roughly speaking, for $(k,s) \in \mathscr{X}^{\mathrm{cl}} \times \mathscr{X}^{\mathrm{cl}}$ with $k \geq 2s$, Hida's p-adic Rankin-Selberg method shows that $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(k,s)$ is interpolated by the inner product between the diagonal restriction of a nearly holomorphic Hilbert Eisenstein series $\boldsymbol{E}_\phi(k,s)$ and $\boldsymbol{f}_k$. Therefore, a result of Keaton and Pitale [KP19, Proposition 2.3] tells us that $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(k,s)$ is a product of (i) the Waldspurger toric period integral $B^{\phi}_{\boldsymbol{f}_k}$ of $\boldsymbol{f}_k$ over F twisted by ϕ, (ii) the special value $L(\boldsymbol{f}_k,s)$ of the L-function for $\boldsymbol{f}_k$ and (iii) local zeta integrals $Z_{\mathcal{D}}(s,B_{W_v})$ for every place of $\mathbf{Q}$ in (4.5). Now items (i) and (ii) are basically interpolated by $\mathcal{L}(\boldsymbol{f}/K,\phi,k)$ and $L_p(\boldsymbol{f},k,s)$, so our task is to evaluate explicitly these local zeta integrals, which occupy the main body of Section 4. By the explicit interpolation formulae of these p-adic L-functions, we find immediately that the ratio C^* between $\mathcal{L}_p(\boldsymbol{f}/F,\phi,k)\cdot L_p(\boldsymbol{f},k,s)$ and $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(k,s+1)$ is independent of s, and hence C^* is a meromorphic function in k only. Finally, by a standard argument using Rohrlich's result on the non-vanishing of the cyclotomic p-adic L-functions for elliptic modular forms, we can conclude that $C^*(k)$ is holomorphic at all $k\in\mathscr{X}^{\mathrm{cl}}$ and $C^*(2)$ is essentially the congruence number.

This paper is organized as follows. After preparing the basic notation for modular forms and automorphic forms in Section 2, we give the construction of Hilbert Eisenstein series and compute the Fourier coefficients in Section 3. In Section 4, we compute the inner product between the diagonal restriction of Hilbert Eisenstein series and a p-stablized newform. The main local calculations are carried out in Proposition 4.3 for the split case, Proposition 4.4 for the non-split, and Proposition 4.5 for the p-adic case. In Section 5, we use p-adic Rankin-Selberg method to construct the p-adic L-function $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}$ and obtain the interpolation formula in Proposition 5.7 by combining the local calculations in Section 4. In order to make the comparison between p-adic L-functions easier, the interpolation formulae shall be presented in terms of automorphic L-functions in this paper. In Section 6, we review the theory of Λ-adic modular symbols in [Kit94] and the construction of the square root p-adic L-function $\mathcal{L}_p(\boldsymbol{f}/F,\phi,k)$. Our treatment for modular symbols is *semi-adelic*, which allows simple descriptions of Heck actions and are amenable to the calculations from the automorphic side. The connection with Greenberg-Stevens' approach [GS93] is explained in Remark 6.6. In Proposition 6.9, we give the complete interpolation formula for $\mathcal{L}_p(\boldsymbol{f}/F,\phi,k)$, including the evaluation at finite order characters of p-power conductors. Finally, we deduce the factorization formula and the derivative formula for $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}$ in Section 7.

2. Classical modular forms and automorphic forms

In this section, we recall basic definitions and standard facts about classical elliptic modular forms and automorphic forms on $\mathrm{GL}_2(\mathbf{A})$, following the notation in [Hsi20, §2] which we reproduce here for the reader's convenience. The main purpose of this section is to set up the notation and introduce some Hecke operators on the space of automorphic forms which will be frequently used in the construction of p-adic L-functions.

2.1. **Notation.** We denote by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{A}$, $\mathbf{R}_+$ the ring of rational integers, the field of rational, real, complex numbers, the ring of adeles of $\mathbf{Q}$ and the group of strictly positive real numbers. Let $\mu_n(F)$ denote the group of nth roots of unity in a field F. For a rational prime ℓ we denote by $\mathbf{Z}_\ell$, $\mathbf{Q}_\ell$ and $\mathrm{ord}_\ell : \mathbf{Q}_\ell \to \mathbf{Z}$ the ring of ℓ-adic integers, the field of ℓ-adic numbers and the additive valuation normalized so that $\mathrm{ord}_\ell(\ell) = 1$. Put $\widehat{\mathbf{Z}} = \prod_\ell \mathbf{Z}_\ell$. Define the idele $\varpi_\ell = (\varpi_{\ell,v}) \in \mathbf{A}^\times$ by $\varpi_{\ell,\ell} = \ell$ and $\varpi_{\ell,v} = 1$ if $v \neq \ell$.

Let F be a number field. We denote its integer ring by $\mathcal{O}_F$. We write $\mathrm{T}_{F/\mathbf{Q}}$ and $\mathrm{N}_{F/\mathbf{Q}}$ for the trace and norm from F to $\mathbf{Q}$. For each place v of F we denote by F_v be the completion of F with respect to v. Let $\mathbf{A}_F = \mathbf{A} \otimes_{\mathbf{Q}} F$ be the adele ring of F. Given $t \in \mathbf{A}_F^\times$, we write $t_v \in F_v^\times$ for its v-component. We shall regard F_v (resp. $F_v^\times$) as a subgroup of $\mathbf{A}_F$ (resp. $\mathbf{A}_F^\times$) in a natural way. Let $\boldsymbol{\alpha}_{F_v} = |\ |_{F_v}$ be the normalized absolute value on F_v. If $v = \mathfrak{q}$ is finite, then $|\varpi_\mathfrak{q}|_{F_\mathfrak{q}} = q_\mathfrak{q}^{-1}$, where $\varpi_\mathfrak{q}$ is a generator of the prime ideal of the integral ring $\mathcal{O}_\mathfrak{q}$ of $F_\mathfrak{q}$ and $q_\mathfrak{q}$ denotes the cardinality of the residue field of $\mathcal{O}_\mathfrak{q}$. Define the complete Dedekind zeta function by $\zeta_F(s) = \prod_v \zeta_{F_v}(s)$, where $\zeta_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$, and if $v = \mathfrak{q}$ is finite, then $\zeta_{F_\mathfrak{q}}(s) = (1 - q_\mathfrak{q}^{-s})^{-1}$. When $F = \mathbf{Q}$, we will write $\boldsymbol{\alpha}_v = |\ |_v$ and $\zeta_v(s) = \zeta_{\mathbf{Q}_v}(s)$. Let $\boldsymbol{\psi} : \mathbf{A}/\mathbf{Q} \to \mathbf{C}^\times$ be the additive character whose archimedean component is $\boldsymbol{\psi}_\infty(x) = e^{2\pi\sqrt{-1}x}$ and whose local component at ℓ is denoted by $\boldsymbol{\psi}_\ell : \mathbf{Q}_\ell \to \mathbf{C}^\times$. We define the additive character $\boldsymbol{\psi}_F = \prod_v \boldsymbol{\psi}_{F_v} : \mathbf{A}_F/F \to \mathbf{C}^\times$ by setting $\boldsymbol{\psi}_F := \boldsymbol{\psi} \circ \mathrm{Tr}_{F/\mathbf{Q}}$. Let $\mathcal{S}(\mathbf{A}_F^m) = \otimes_v' \mathcal{S}(F_v^m)$ denote the space of Schwartz functions on $\mathbf{A}_F^m$.

For any set X we denote by $\mathbb{I}_X$ the characteristic function of X. If R is a commutative ring and $G = \mathrm{GL}_2(R)$, we define homomorphisms $\mathbf{t} : R^\times \to G$ and $\mathbf{n} : R \to G$ by

$$\mathbf{t}(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{n}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

We denote by ρ the right translation of G on the space of $\mathbf{C}$-valued functions on G, i.e., $\rho(g)f(g') = f(g'g)$, and by $\mathbf{1} : G \to \mathbf{C}$ the constant function $\mathbf{1}(g) = 1$. For a function $f : G \to \mathbf{C}$ and a character $\omega : R^\times \to \mathbf{C}^\times$, let $f \otimes \omega : G \to \mathbf{C}$ denote the function $f \otimes \omega(g) = f(g)\omega(\det g)$. The subgroup $B(R)$ (resp. $N(R)$) of $\mathrm{GL}_2(R)$ consists of upper triangular (resp. upper triangular unipotent) matrices.

2.2. **Characters.** If F is a number field and $\chi : F^\times \backslash \mathbf{A}_F^\times \to \overline{\mathbf{Q}}^\times$ be a Hecke character of $\mathbf{A}_F^\times$, we denote by $\chi_v : F_v^\times \to \mathbf{C}^\times$ the local component of χ at a place v of F. When ω is a Hecke character of $\mathbf{A}^\times$, we denote by $\omega_F := \omega \circ \mathrm{N}_{F/\mathbf{Q}} : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ *the base change* of ω.

If v is non-archimedean and $\lambda : F_v^\times \to \mathbf{C}^\times$ is a character, let $c(\lambda)$ be the exponent of the conductor of λ.

2.3. **Automorphic forms on** $\mathrm{GL}_2(\mathbf{A})$**.** Fix a positive integer N. Define open compact subgroups of $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ by

$$U_0(N)=\left\{g\in \mathrm{GL}_2(\widehat{\mathbf{Z}})\;\middle|\; g\equiv\begin{pmatrix}* & *\\ 0 & *\end{pmatrix} \pmod{N\widehat{\mathbf{Z}}}\right\},$$
$$U_1(N)=\left\{g\in U_0(N)\;\middle|\; g\equiv\begin{pmatrix}* & *\\ 0 & 1\end{pmatrix} \pmod{N\widehat{\mathbf{Z}}}\right\}.$$

Let $\omega:\mathbf{Q}^\times\backslash\mathbf{A}^\times\to\mathbf{C}^\times$ be a finite order Hecke character of level N. We extend ω to a character of $U_0(N)$ defined by $\omega\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)=\prod_{\ell|N}\omega_\ell(d_\ell)$ for $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in U_0(N)$. For any integer k the space $\mathcal{A}_k(N,\omega)$ of automorphic forms on $\mathrm{GL}_2(\mathbf{A})$ of weight k, level N and character ω consists of automorphic forms $\varphi:\mathrm{GL}_2(\mathbf{A})\to\mathbf{C}$ such that

$$\varphi(z\gamma g\kappa_\theta u_{\mathrm{f}})=\omega(z)\varphi(g)e^{\sqrt{-1}k\theta}\omega(u_{\mathrm{f}}),\qquad \kappa_\theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

for $z\in\mathbf{A}^\times$, $\gamma\in\mathrm{GL}_2(\mathbf{Q})$, $\theta\in\mathbf{R}$ and $u_{\mathrm{f}}\in U_0(N)$. Let $\mathcal{A}^0_k(N,\omega)$ be the space of cusp forms in $\mathcal{A}_k(N,\omega)$.

Next we introduce important local Hecke operators on automorphic forms. At the archimedean place, let $V_\pm:\mathcal{A}_k(N,\omega)\to\mathcal{A}_{k\pm2}(N,\omega)$ be the normalized weight raising/lowering operator in [JL70, page 165] given by

(2.1)
$$V_\pm=\frac{1}{(-8\pi)}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes 1\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\otimes\sqrt{-1}\right)\in\mathrm{Lie}(\mathrm{GL}_2(\mathbf{R}))\otimes_{\mathbf{R}}\mathbf{C}.$$

Define the operator $\mathbf{U}_\ell$ acting on $\varphi\in\mathcal{A}_k(N,\omega)$ by

$$\mathbf{U}_\ell\varphi=\sum_{x\in\mathbf{Z}_\ell/\ell\mathbf{Z}_\ell}\rho\left(\begin{pmatrix}\varpi_\ell & x\\ 0 & 1\end{pmatrix}\right)\varphi,$$

and the level-raising operator $V_\ell:\mathcal{A}_k(N,\omega)\to\mathcal{A}_k(N\ell,\omega)$ at a finite prime ℓ by

$$V_\ell\varphi(g):=\rho(\mathbf{t}(\varpi_\ell^{-1}))\varphi.$$

Note that $\mathbf{U}_\ell V_\ell\varphi=\ell\varphi$ and that if $\ell\mid N$, then $\mathbf{U}_\ell\in\mathrm{End}_{\mathbf{C}}\mathcal{A}_k(N,\omega)$. For each prime $\ell\nmid N$, let $T_\ell\in\mathrm{End}_{\mathbf{C}}\mathcal{A}_k(N,\omega)$ be the usual Hecke operator defined by

$$T_\ell=\mathbf{U}_\ell+\omega_\ell(\ell)V_\ell.$$

Define the $\mathrm{GL}_2(\mathbf{A})$-equivariant pairing $\langle\,,\,\rangle:\mathcal{A}^0_{-k}(N,\omega)\otimes\mathcal{A}_k(N,\omega^{-1})\to\mathbf{C}$ by

$$\langle\varphi,\varphi'\rangle=\int_{\mathbf{A}^\times\mathrm{GL}_2(\mathbf{Q})\backslash\mathrm{GL}_2(\mathbf{A})}\varphi(g)\varphi'(g)\,\mathrm{d}^\tau g,\tag{2.2}$$

where $\mathrm{d}^\tau g$ is the Tamagawa measure of $\mathrm{PGL}_2(\mathbf{A})$. Note that $\langle T_\ell\varphi,\varphi'\rangle=\langle\varphi,T_\ell\varphi'\rangle$ for $\ell\nmid N$.

2.4. **Classical modular forms.** We recall a *semi-adelic* description of classical modular forms. Let $C^\infty(\mathfrak{H})$ be the space of $\mathbf{C}$-valued smooth functions on the half complex plane $\mathfrak{H} := \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$. The group $\mathrm{GL}_2(\mathbf{R})^+ := \{g \in \mathrm{GL}_2(\mathbf{R}) \mid \det g > 0\}$ acts on $\mathfrak{H}$ and the automorphy factor is given by

$$\gamma(z) = \frac{az+b}{cz+d}, \qquad\qquad J(\gamma, z) = cz + d$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{R})^+$ and $z \in \mathfrak{H}$.

Let k be any integer. The Maass-Shimura differential operators δ_k and ε on $C^\infty(\mathfrak{H})$ are defined by

$$\delta_k = \frac{1}{2\pi\sqrt{-1}}\left(\frac{\partial}{\partial z} + \frac{k}{2\sqrt{-1}y}\right), \qquad\qquad \varepsilon = -\frac{1}{2\pi\sqrt{-1}} y^2 \frac{\partial}{\partial \bar{z}}$$

(cf. [Hid93, (1a, 1b) page 310]), where $y = \mathrm{Im}(z)$ is the imaginary part of z. Let χ be a Dirichlet character of level N. For a non-negative integer m let $\mathcal{N}_k^{[m]}(N, \chi)$ denote the space of nearly holomorphic modular forms of weight k, level N and character χ. In other words $\mathcal{N}_k^{[m]}(N, \chi)$ consists of smooth slowly increasing functions $f : \mathfrak{H} \times \mathrm{GL}_2(\widehat{\mathbf{Q}}) \to \mathbf{C}$ such that

- $f(\gamma z, \gamma g_\mathrm{f} u) = (\det \gamma)^{-1} J(\gamma, z)^k f(z, g_\mathrm{f}) \chi^{-1}(u)$ for any $\gamma \in \mathrm{GL}_2(\mathbf{Q})^+$ and $u \in U_0(N)$;
- $\varepsilon^{m+1} f(z, g_\mathrm{f}) = 0$

(cf. [Hid93, page 314]). Let $\mathcal{N}_k(N, \chi) = \bigcup_{m=0}^\infty \mathcal{N}_k^{[m]}(N, \chi)$ (cf. [Hid93, (1a), page 310]). By definition $\mathcal{N}_k^{[0]}(N, \chi)$ coincides with the space $\mathcal{M}_k(N, \chi)$ of classical holomorphic modular forms of weight k, level N and character χ. Denote by $\mathcal{S}_k(N, \chi)$ the space of cusp forms in $\mathcal{M}_k(N, \chi)$. Let $\delta_k^m = \delta_{k+2m-2} \cdots \delta_{k+2}\delta_k$. If $f \in \mathcal{N}_k(N, \chi)$, then $\delta_k^m f \in \mathcal{N}_{k+2m}(N, \chi)$ ([Hid93, page 312]). Given a positive integer d, we define

$$V_d f(z, g_\mathrm{f}) = f(dz, g_\mathrm{f}); \qquad \mathbf{U}_d f(z, g_\mathrm{f}) = \sum_{j=0}^{d-1} f\left(z, g_\mathrm{f} \begin{pmatrix} d & j \\ 0 & 1 \end{pmatrix}\right).$$

The classical Hecke operators T_ℓ for primes $\ell \nmid N$ are given by

$$T_\ell f = \mathbf{U}_\ell f + \chi_\ell(\ell^{-1}) \ell^{k-2} V_\ell f.$$

We say that $f \in \mathcal{N}_k(N, \chi)$ is a *Hecke eigenform* if f is an eigenfunction of all the Hecke operators T_ℓ for $\ell \nmid N$ and the operators $\mathbf{U}_\ell$ for $\ell \mid N$.

2.5. To every nearly holomorphic modular form $f \in \mathcal{N}_k(N, \chi)$ we associate a unique automorphic form $\Phi(f) \in \mathcal{A}_k(N, \chi^{-1})$ defined by the formula

$$\Phi(f)(g) := f(g_\infty(\sqrt{-1}), g_\mathrm{f}) J(g_\infty, \sqrt{-1})^{-k} (\det g_\infty) \, |\det g|_{\mathbf{A}}^{\frac{k}{2}-1} \tag{2.3}$$

for $g = g_\infty g_{\mathrm{f}} \in \mathrm{GL}_2(\mathbf{R})\,\mathrm{GL}_2(\widehat{\mathbf{Q}})$ (cf. [Cas73, §3]). Conversely, we can recover the form f from $\Phi(f)$ by

$$f(x+\sqrt{-1}y,\, g_{\mathrm{f}}) = y^{-k/2}\,\Phi(f)\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix} g_{\mathrm{f}}\right)|\det g_{\mathrm{f}}|_{\mathbf{A}}^{1-\frac{k}{2}}. \tag{2.4}$$

We call $\Phi(f)$ the *adelic lift* of f.

The weight raising/lowering operators are the adelic avatar of the differential operators δ_k^m and ε on the space of automorphic forms. A direct computation shows that the map Φ from the space of modular forms to the space of automorphic forms is *equivariant* for the Hecke action in the sense that

$$\Phi(\delta_k^m f) = V_+^m\,\Phi(f), \qquad\qquad \Phi(\varepsilon f) = V_-\,\Phi(f), \tag{2.5}$$

and for a finite prime ℓ

$$\Phi(T_\ell f) = \ell^{k/2-1} T_\ell\,\Phi(f), \qquad \Phi(\mathbf{U}_\ell f) = \ell^{k/2-1}\mathbf{U}_\ell\,\Phi(f). \tag{2.6}$$

In particular, f is holomorphic if and only if $V_-\Phi(f) = 0$.

2.6. Preliminaries on irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_v)$.

2.6.1. *Measures.* We shall normalize the Haar measures on F_v and $F_v^\times$ as follows. Let $\mathrm{d}x_v$ be the self-dual Haar measures of F_v with respect to $\boldsymbol{\psi}_{F_v}$. Put $\mathrm{d}^\times x_v = \zeta_{F_v}(1)\frac{\mathrm{d}x_v}{|x_v|_{F_v}}$. If $F = \mathbf{Q}$, then $\mathrm{d}a_\infty$ denote the usual Lebesgue measure on $\mathbf{R}$ and $\mathrm{d}a_\ell$ be the Haar measure on $\mathbf{Q}_\ell$ with $\mathrm{vol}(\mathbf{Z}_\ell, \mathrm{d}a_\ell) = 1$. The Tamagawa measure of $\mathbf{A}_F$ is $\mathrm{d}x = \prod_v \mathrm{d}x_v$ while the Tamagawa measure of $\mathbf{A}_F^\times$ is defined by $\mathrm{d}^\times x = c_F^{-1}\prod_v \mathrm{d}x_v^\times$, where c_F denotes the residue of $\zeta_F(s)$ at $s = 1$.

Define the compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v$ of $\mathrm{GL}_2(\mathbf{A})$ by $\mathbf{K}_\infty = \mathrm{O}(2,\mathbf{R})$ and $\mathbf{K}_\ell = \mathrm{GL}_2(\mathbf{Z}_\ell)$. Let $\mathrm{d}u_v$ be the Haar measure on $\mathbf{K}_v$ so that $\mathrm{vol}(\mathbf{K}_v, \mathrm{d}u_v) = 1$. Let $\mathrm{d}g_v$ be the Haar measure on $\mathrm{PGL}_2(\mathbf{Q}_v)$ given by $\mathrm{d}^\tau g_v = |a_v|_v^{-1}\,\mathrm{d}x_v \mathrm{d}^\times a_v \mathrm{d}u_v$ for $g_v = \begin{pmatrix} a_v & x_v \\ 0 & 1\end{pmatrix} u_v$ with $a_v \in \mathbf{Q}_v^\times$, $x_v \in \mathbf{Q}_v$ and $u_v \in \mathbf{K}_v$. The Tamagawa measure on $\mathrm{PGL}_2(\mathbf{A})$ is given by $\mathrm{d}^\tau g = \zeta_{\mathbf{Q}}(2)^{-1}\prod_v \mathrm{d}^\tau g_v$.

2.6.2. *Representations of* $\mathrm{GL}_2(\mathbf{Q}_v)$. Denote by $\varrho \boxplus \upsilon$ the irreducible principal series representation of $\mathrm{GL}_2(\mathbf{Q}_v)$ attached to two characters $\varrho, \upsilon : \mathbf{Q}_v^\times \to \mathbf{C}^\times$ such that $\varrho\upsilon^{-1} \neq \boldsymbol{\alpha}_v^{\pm}$. If $v = \infty$ is the archimedean place and $k \geq 1$ is an integer, denote by $\mathcal{D}_0(k)$ the discrete series of lowest weight k if $k \geq 2$ or the limit of discrete series if $k = 1$ with central character sgn^k (the k-the power of the sign character $\mathrm{sgn}(x) = \frac{x}{|x|_\infty}$ of $\mathbf{R}^\times$).

2.6.3. *Whittaker models and the normalized Whittaker newforms.* Every irreducible admissible infinite dimensional representation π of $\mathrm{GL}_2(\mathbf{Q}_v)$ admits a Whttaker model $\mathcal{W}(\pi) = \mathcal{W}(\pi, \boldsymbol{\psi}_v)$ with respect to $\boldsymbol{\psi}_v$. Recall that $\mathcal{W}(\pi)$ is a subspace of smooth functions $W : \mathrm{GL}_2(\mathbf{Q}_v) \to \mathbf{C}$ such that

- $W(\mathbf{n}(x)g) = \boldsymbol{\psi}_v(x)W(g)$ for all $x \in \mathbf{Q}_v$,
- if $v = \infty$ is archimedean, then there exists an integer M such that

$$W(\mathbf{t}(a)) = O(|a|_\infty^M) \text{ as } |a|_\infty \to \infty.$$

The group $\mathrm{GL}_2(\mathbf{Q}_v)$ (or the Hecke algebra of $\mathrm{GL}_2(\mathbf{Q}_v)$) acts on $\mathcal{W}(\pi)$ via the right translation ρ. We introduce the (normalized) *local Whittaker newform* W_π in $\mathcal{W}(\pi)$ in the following way: if $v = \infty$ and $\pi = \mathcal{D}_0(k)$, then $W_\pi \in \mathcal{W}(\pi)$ is defined by

$$W_\pi\left(z\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\kappa_\theta\right) = \mathbb{I}_{\mathbf{R}_+}(y)\cdot\frac{y^{k/2}}{e^{2\pi y}}\cdot\mathrm{sgn}(z)^k\boldsymbol{\psi}_\infty(x)e^{\sqrt{-1}k\theta} \tag{2.7}$$

for $y, z \in \mathbf{R}^\times$ and $x, \theta \in \mathbf{R}$; if v is finite, then W_π is the unique function in $\mathcal{W}(\pi)^{\mathrm{new}}$ such that $W_\pi(\mathbf{1}_2) = 1$. The explicit formula for $W_\pi(\mathbf{t}(a))$ is well-known (See [Sch02, page 21] or [Sah16, Section 2.2] for example).

2.6.4. *L-factors and ε-factors.* Given $a \in \mathbf{Q}_v^\times$, we define an additive character $\boldsymbol{\psi}_v^a$ on $\mathbf{Q}_v$ by $\boldsymbol{\psi}_v^a(x) = \boldsymbol{\psi}_v(ax)$ for $x \in \mathbf{Q}_v$. We associate to a character $\varrho : \mathbf{Q}_v^\times \to \mathbf{C}^\times$ the L-factor $L(s,\varrho)$ and the ε-factor $\varepsilon(s,\varrho,\boldsymbol{\psi}_v^a)$ (cf. [Sch02, Section 1.1]). The gamma factor

$$\gamma(s,\varrho,\boldsymbol{\psi}_v^a) = \varepsilon(s,\varrho,\boldsymbol{\psi}_v^a)\frac{L(1-s,\varrho^{-1})}{L(s,\varrho)}$$

is obtained as the proportionality constant of the functional equation

$$\gamma(s,\varrho,\boldsymbol{\psi}_v^a)\int_{\mathbf{Q}_v^\times}\varphi(a)\varrho(a)|a|_v^s\,\mathrm{d}^\times a = \int_{\mathbf{Q}_v^\times}\widehat{\varphi}(a)\varrho(a)^{-1}|a|_v^{1-s}\,\mathrm{d}^\times a \tag{2.8}$$

for $\varphi \in \mathcal{S}(\mathbf{Q}_v)$, where

$$\widehat{\varphi}(y) = \int_{\mathbf{Q}_v}\varphi(x_v)\boldsymbol{\psi}_v(yx_v)\,\mathrm{d}x_v$$

is the Fourier transform with respect to $\boldsymbol{\psi}_v$. When $a = 1$, we write

$$\varepsilon(s,\varrho) = \varepsilon(s,\varrho,\boldsymbol{\psi}_v), \qquad \gamma(s,\varrho) = \gamma(s,\varrho,\boldsymbol{\psi}_v).$$

When $v = \ell$ is a finite prime, we denote the exponent of the conductor of ϱ by $c(\varrho)$. Recall that

$$\varepsilon(s,\varrho,\boldsymbol{\psi}_v^a) = \varrho(a)|a|_v^{-1}\varepsilon(0,\varrho)\ell^{-c(\varrho)s}. \tag{2.9}$$

Let π be an irreducible admissible representation of $\mathrm{GL}_2(\mathbf{Q}_v)$ with central character ω. Denote by $L(s,\pi)$ and $\varepsilon(s,\pi) = \varepsilon(s,\pi,\boldsymbol{\psi}_v)$ its L-factor and

ε-factor relative to $\boldsymbol{\psi}_v$ defined in [JL70, Theorem 2.18]. We write $\pi^\vee$ for the contragredient representation of π. The gamma factor

$$\gamma(s,\pi)=\varepsilon(s,\pi)\frac{L(1-s,\pi^\vee)}{L(s,\pi)}$$

is obtained as the proportionality constant of the functional equation (2.10)

$$\gamma\left(s+\frac{1}{2},\pi\right)\int_{\mathbf{Q}_v^\times}W(\mathbf{t}(a)g)\,|a|_v^s\,\mathrm{d}^\times a=\int_{\mathbf{Q}_v^\times}W(\mathbf{t}(a)J_1^{-1}g)\omega(a)^{-1}\,|a|_v^{-s}\,\mathrm{d}^\times a$$

for every $W\in\mathcal{W}(\pi)$.

2.7. **p-stabilized newforms.** Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$. The Whittaker function of $\varphi\in\pi$ with respect to the additive character $\boldsymbol{\psi}$ is given by

$$W_\varphi(g)=\int_{\mathbf{A}/\mathbf{Q}}\varphi(\mathbf{n}(x)g)\boldsymbol{\psi}(-x)\,\mathrm{d}x \tag{2.11}$$

for $g\in\mathrm{GL}_2(\mathbf{A})$, where $\mathrm{d}x$ is the Haar measure with $\mathrm{vol}(\mathbf{A}/\mathbf{Q},\mathrm{d}x)=1$. We have the Fourier expansion:

$$\varphi(g)=\sum_{\beta\in\mathbf{Q}^\times}W_\varphi(\mathbf{t}(\beta)g)$$

(cf. [Bum98, Theorem 3.5.5]). Let $f=\sum_n\mathbf{a}(n,f)q^n\in\mathcal{S}_k(N,\chi)$ be a normalized Hecke eigenform whose adelic lift $\varPhi(f)$ generates $\pi=\otimes'_v\pi_v$ of $\mathrm{GL}_2(\mathbf{A})$, having central character χ^{-1}. If f is a newform, then the conductor of π is N, the adelic lift $\varPhi(f)$ is the normalized new vector in π and the Mellin transform

$$\int_{\mathbf{A}^\times/\mathbf{Q}^\times}\varPhi(f)(\mathbf{t}(y))|y|_{\mathbf{A}}^s\,\mathrm{d}^\times y=L\left(s+\frac{1}{2},\pi\right)$$

is the automorphic L-function of π. Here $|y|_{\mathbf{A}}=\prod_v|y_v|_v$ and $\mathrm{d}^\times y$ is the product measure $\prod_v\mathrm{d}^\times y_v$.

Definition 2.1 (p-stabilized newform)**.** Let p be a prime and fix an isomorphism $\iota_p:\mathbf{C}\simeq\overline{\mathbf{Q}}_p$. We say that a normalized Hecke eigenform $f=\sum_{n=1}^\infty\mathbf{a}(n,f)q^n\in\mathcal{S}_k(Np,\chi)$ is a (ordinary) *p-stabilized newform* (with respoect to ι_p) if f is a new outside p and the eigenvalue of $\mathbf{U}_p$, i.e. the p-th Fourier coefficient $\iota_p(\mathbf{a}(p,f))$, is a p-adic unit. The prime-to-p part of the conductor of f is called *the tame conductor* of f.

The Whittaker function of $\varPhi(f)$ is a product of local Whittaker functions in $\mathcal{W}(\pi_v,\boldsymbol{\psi}_v)$ by the multiplicity one for new and ordinary vectors. To be precise, we have

$$W_{\varPhi(f)}(g)=W_{\pi_p}^{\mathrm{ord}}(g_p)\prod_{v\neq p}W_{\pi_v}(g_v)$$

for $g=(g_v)\in \mathrm{GL}_2(\mathbf{A})$. Here W_{π_v} is the normalized Whittaker newform of π_v and $W^{\mathrm{ord}}_{\pi_p}$ is the ordinary Whittaker function characterized by

$$W^{\mathrm{ord}}_{\pi_p}(\mathbf{t}(a))=\varrho_f(a)\,|a|^{\frac{1}{2}}\cdot \mathbb{I}_{\mathbf{Z}_p}(a) \text{ for } a\in \mathbf{Q}_p^\times, \tag{2.12}$$

where $\varrho_f:\mathbf{Q}_p^\times\to\mathbf{C}^\times$ is the unramified character with $\varrho_f(p)=\mathbf{a}(p,f)\cdot p^{(1-k)/2}$ (See [Hsi20, Corollary 2.3, Remark 2.5]).

3. The construction of Hilbert-Eisenstein series

3.1. **Eisenstein series.** We recall the construction of Eisenstein series described in [Jac72, §19]. Let F be a real quadratic field with integer ring $\mathcal{O}_F$. We denote the set of real places of F by $\Sigma_{\mathbf{R}}=\{\sigma_1,\sigma_2\}$, the different of F by $\mathfrak{d}$, the discriminant of F by Δ_F and the unique non-trivial automorphism of F by $x\mapsto\bar{x}$. For each finite prime $\mathfrak{q}$ of F we write $\mathcal{O}_\mathfrak{q}$ for the integer ring of $F_\mathfrak{q}$.

Let (μ,ν) be a pair of unitary Hecke characters of $\mathbf{A}_F^\times$. For each place v we write $\mathcal{B}(\mu_v,\nu_v,s)$ for the space of smooth functions $f_v:\mathrm{GL}_2(F_v)\to\mathbf{C}$ which satisfy

$$f_v\left(\begin{pmatrix}a&b\\0&d\end{pmatrix}g\right)=\mu_v(a)\nu_v(d)\left|\frac{a}{d}\right|_{F_v}^{s+\frac{1}{2}}f_v(g)$$

for $a,d\in F_v^\times$ and $b\in F_v$. Recall that $\mathcal{S}(F_v^2)$ denotes the space of Schwartz functions on F_v^2. We associate to $\Phi_v\in\mathcal{S}(F_v^2)$ the Godement section $f_{\mu_v,\nu_v,\Phi_v,s}\in\mathcal{B}(\mu_v,\nu_v,s)$ by

$$f_{\mu_v,\nu_v,\Phi_v,s}(g_v)=\mu_v(\det g_v)\,|\det g_v|_{F_v}^{s+\frac{1}{2}}\int_{F_v^\times}\Phi_v((0,t_v)g_v)(\mu_v\nu_v^{-1})(t_v)\,|t_v|_{F_v}^{2s+1}\,\mathrm{d}^\times t_v. \tag{3.1}$$

Let $\Phi=\otimes_v\Phi_v\in\mathcal{S}(\mathbf{A}_F^2)$. Define a function $f_{\mu,\nu,\Phi,s}:\mathrm{GL}_2(\mathbf{A}_F)\to\mathbf{C}$ by $f_{\mu,\nu,\Phi,s}(g)=\prod_v f_{\mu_v,\nu_v,\Phi_v,s}(g_v)$. The series

$$E_{\mathbf{A}}(g,f_{\mu,\nu,\Phi,s})=\sum_{\gamma\in B(F)\backslash\mathrm{GL}_2(F)}f_{\mu,\nu,\Phi,s}(\gamma g)$$

converges absolutely for $\mathrm{Re}(s)\gg 0$ and has meromorphic continuation to $s\in\mathbf{C}$. It admits the Fourier expansion

$$E_{\mathbf{A}}(g,f_{\mu,\nu,\Phi,s})=f_{\mu,\nu,\Phi,s}(g)+f_{\nu,\mu,\widehat{\Phi},-s}(g)+\sum_{\beta\in F^\times}W(\mathbf{t}(\beta)g,f_{\mu,\nu,\Phi,s}), \tag{3.2}$$

where $\widehat{\Phi}:=\otimes_v\widehat{\Phi}_v$ is the symplectic Fourier transform defined by

$$\widehat{\Phi}_v(x,y)=\iint_{F_v^2}\Phi_v(z,u)\boldsymbol{\psi}_{F_v}(zy-ux)\,\mathrm{d}z\mathrm{d}u.$$

We tentatively write $f_{v,s}=f_{\mu_v,\nu_v,\Phi_v,s}$. There exists an open compact subgroup $\mathcal{U}$ of F_v such that for any open compact subgroup $\mathcal{U}'$ containing $\mathcal{U}$

$$\int_{\mathcal{U}}f_{v,s}(J_1\mathbf{n}(x_v)g_v)\boldsymbol{\psi}_{F_v}(-x_v)\,\mathrm{d}x_v=\int_{\mathcal{U}'}f_{v,s}(J_1\mathbf{n}(x_v)g_v)\boldsymbol{\psi}_{F_v}(-x_v)\,\mathrm{d}x_v,$$

where $J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We define the regularized integral by

$$\begin{aligned} W(g_v, f_{\mu_v,\nu_v,\Phi_v,s}) &= \int_{F_v}^{\mathrm{st}} f_{v,s}(J_1\mathbf{n}(x_v)g_v)\boldsymbol{\psi}_{F_v}(-x_v)\,\mathrm{d}x_v \\ &:= \int_{\mathcal{U}} f_{v,s}(J_1\mathbf{n}(x_v)g_v)\boldsymbol{\psi}_{F_v}(-x_v)\,\mathrm{d}x_v. \end{aligned}$$

Then $W(g, f_{\mu,\nu,\Phi,s}) = \prod_v W(g_v, f_{v,s})$ for $g = (g_v) \in \mathrm{GL}_2(\mathbf{A}_F)$.

3.2. **The Eisenstein series $E_k(\mu, \nu)$.** Let N and C be positive integers such that $N\Delta_F$ and C are coprime. We assume that

(Spl) every prime factor of NC splits in F.

Then there are ideals $\mathfrak{N}$ and $\mathfrak{c}$ of $\mathcal{O}_F$ such that

$$N\mathcal{O}_F = \mathfrak{N}\overline{\mathfrak{N}}, \quad (\mathfrak{N}, \overline{\mathfrak{N}}) = 1 \qquad C\mathcal{O}_F = \mathfrak{c}\bar{\mathfrak{c}}, \quad (\mathfrak{c}, \bar{\mathfrak{c}}) = 1. \tag{3.3}$$

Fix a positive integer k. Assume that $\nu_{\sigma_i}\mu_{\sigma_i} = \mathrm{sgn}^k$ for $i = 1, 2$. We recall a construction of certain classical Eisenstein series $E_k(\mu, \nu)$ of parallel weight k, level $\Gamma_1(NC)$ and central character $\mu\nu$ following [Jac72]. We impose the following hypothese for (μ, ν):

Hypothesis 3.1.

- μ is unramified outside p,
- the prime-to-p part of the conductor of ν has a decomposition $\mathfrak{c}\mathfrak{c}'$ with $\bar{\mathfrak{c}} \subset \mathfrak{c}'$.

Definition 3.2. Let $k \geq 2$ be an integer. The quintuple

$$\mathcal{D} := (\mu, \nu, k, \mathfrak{N}, \mathfrak{c})$$

is called an Eisenstein datum of weight k. The Fourier transform of $\phi \in \mathcal{S}(F_v)$ is defined by

$$\widehat{\phi}(x) := \int_{F_v} \phi(y)\boldsymbol{\psi}_{F_v}(yx)\,\mathrm{d}y,$$

where the Haar measure $\mathrm{d}y$ is so chosen that $\widehat{\widehat{\phi}}(x) = \phi(-x)$. When $\mathfrak{q}$ is a finite prime, we associate to a character $\chi : F_{\mathfrak{q}}^\times \to \mathbf{C}$ a function $\phi_\chi \in \mathcal{S}(F_{\mathfrak{q}})$ by $\phi_\chi(x) = \mathbb{I}_{\mathcal{O}_{\mathfrak{q}}^\times}(x)\chi(x)$. We associate to $\mathcal{D}$ the Bruhat-Schwartz function

$$\Phi_{\mathcal{D}} = \bigotimes_v \Phi_{\mathcal{D},v} \in \mathcal{S}(\mathbf{A}_F^2)$$

defined as follows:

- $\Phi_{\mathcal{D},v}(x, y) = 2^{-k}(x + \sqrt{-1}y)^k e^{-\pi(x^2+y^2)}$ if $v \in \Sigma_{\mathbf{R}}$,
- $\Phi_{\mathcal{D},v}(x, y) = \phi_{\mu_v^{-1}}(x)\widehat{\phi}_{\nu_v^{-1}}(y)$ if $v \mid p$,
- $\Phi_{\mathcal{D},v}(x, y) = \mathbb{I}_{\mathfrak{N}\mathfrak{c}\mathcal{O}_v}(x)\phi_{\nu_v}(y)$ if $v \mid \mathfrak{N}\mathfrak{c}$,
- $\Phi_{\mathcal{D},v}(x, y) = \mathbb{I}_{\mathcal{O}_v}(x)\widehat{\phi}_{\nu_v^{-1}}(y)$ if $v \mid \bar{\mathfrak{c}}$,

- $\Phi_{\mathcal{D},v}(x,y) = \mathbb{I}_{\mathfrak{d}^{-1}\mathcal{O}_v}(x)\mathbb{I}_{\mathfrak{d}^{-1}\mathcal{O}_v}(y)\cdot|\Delta_F|_v^{\frac{1}{2}}$ if $v\nmid p\mathfrak{N}\mathfrak{c}$.

We define the associated Godement section by $f_{\mathcal{D},s} = f_{\mu,\nu,\Phi_{\mathcal{D}},s}$ and $f_{\mathcal{D},s,v} = f_{\mu_v,\nu_v,\Phi_{\mathcal{D},v},s}$.

Remark 3.3. If $v\in\Sigma_{\mathbf{R}}$, then $f_{\mathcal{D},s,v}$ is the unique function in $\mathcal{B}(\mu_v,\nu_v,s)$ such that

$$f_{\mathcal{D},s,v}(\kappa_\theta) = e^{\sqrt{-1}k\theta}\cdot 2^{-k}(\sqrt{-1})^k\pi^{-(s+\frac{k+1}{2})}\Gamma\left(s+\frac{k+1}{2}\right)$$

(see the proof of Lemma 3.6). If $v=\mathfrak{q}$ is a finite place, then for any integer M, let $\mathcal{U}_1(M)$ be the open-compact subgroup of $\mathrm{GL}_2(\mathcal{O}_\mathfrak{q})$ given by

$$\mathcal{U}_1(M) = \mathrm{GL}_2(\mathcal{O}_\mathfrak{q})\cap\begin{pmatrix}\mathcal{O}_\mathfrak{q} & \mathcal{O}_\mathfrak{q}\\ M\mathcal{O}_\mathfrak{q} & 1+M\mathcal{O}_\mathfrak{q}\end{pmatrix},$$

and $f_{\mathcal{D},s,\mathfrak{q}}\in\mathcal{B}(\mu_\mathfrak{q},\nu_\mathfrak{q},s)$ is invariant by $\mathcal{U}_1(p^rNC)$ under the right translation for some sufficiently large r.

Definition 3.4. Define the classical Eisenstein series $E_k^{\pm}(\mu,\nu):\mathfrak{H}^{\Sigma_{\mathbf{R}}}\to\mathbf{C}$ by

$$E_k^{\pm}(\mu,\nu)(x+y\sqrt{-1}) := y^{-\frac{k}{2}}E_{\mathbf{A}}\left(\begin{pmatrix}y & x\\ 0 & 1\end{pmatrix}, f_{\mathcal{D},s}\right)\Big|_{s=\pm\frac{k-1}{2}}\quad (x\in\mathbf{R}^2,\ y\in\mathbf{R}_+^2).$$

Then $E_k^{\pm}(\mu,\nu)$ is a Hilbert modular form of parellel weight k, level p^rNC and character $\mu^{-1}\nu^{-1}$. By definition

$$\Phi(E_k^{\pm}(\mu,\nu)|\mathfrak{H})(g) = E_{\mathbf{A}}((g,g),f_{\mathcal{D},s})|_{s=\pm\frac{k-1}{2}}$$

for $g\in\mathrm{GL}_2(\mathbf{A})$, where Φ is the adelic lift defined in (2.4).

Proposition 3.5. *For every non-negative integer t, we have*

$$\Phi(\delta_k^t E_k^{\pm}(\mu,\nu)) = E_{\mathbf{A}}(f_{\mathcal{D}_t,s})|_{s=\pm\frac{k-1}{2}},$$

where $\mathcal{D}_t=(\mu,\nu,k+2t,\mathfrak{N},\mathfrak{c})$ is an Eisenstein datum of weight $k+2t$.

Proof. Recall the differential operator V_+ defined in (2.1). Proposition 3.5 follows from (2.5) in view of the relation $V_+^t f_{\mathcal{D},s,\infty} = f_{\mathcal{D}_t,s,\infty}$ (see [JL70, Lemma 5.6 (iii)]). □

3.3. **Fourier coefficients of Eisenstein series.**

Lemma 3.6. *For $a\in\mathbf{R}^\times$, we have*

$$W(\mathbf{t}(a),f_{\mathcal{D},s,\infty})|_{s=\frac{k-1}{2}} = W(\mathbf{t}(a),f_{\mathcal{D},s,\infty})|_{s=\frac{1-k}{2}} = a^{\frac{k}{2}}e^{-2\pi a}\cdot\mathbb{I}_{\mathbf{R}_+}(a).$$

Proof. By definition, $W(\mathbf{t}(a), f_{\mathcal{D},s,\infty})$ equals

$$2^{-k}\mu\boldsymbol{\alpha}^{s+\frac{1}{2}}(a)\int_{\mathbf{R}}\int_{\mathbf{R}^\times} t^k(a+\sqrt{-1}x)^k e^{-\pi t^2(x^2+a^2)}\,\mathrm{sgn}(t)^k\,|t|^{2s+1}\,\boldsymbol{\psi}_\infty(-x)\,\mathrm{d}^\times t\mathrm{d}x$$

$$=\mu\boldsymbol{\alpha}^{s+\frac{1}{2}}(a)\cdot(-2\sqrt{-1})^{-k}\cdot\Gamma\left(s+\frac{k+1}{2}\right)\pi^{-(s+\frac{k+1}{2})}$$

$$\times\int_{\mathbf{R}}(x+\sqrt{-1}a)^{-(s+\frac{k+1}{2})}(x-\sqrt{-1}a)^{-(s-\frac{k-1}{2})}\boldsymbol{\psi}_\infty(-x)\,\mathrm{d}x.$$

By Cauchy's integral formula we find that

$$W(\mathbf{t}(a), f_{\mathcal{D},s,\infty})|_{s=\frac{k-1}{2}} =\mu\boldsymbol{\alpha}^{\frac{k}{2}}(a)\cdot(-2\pi\sqrt{-1})^{-k}\cdot\Gamma(k)\int_{\mathbf{R}}\frac{e^{-2\pi\sqrt{-1}x}}{(x+\sqrt{-1}a)^k}\,\mathrm{d}x$$

$$=\mu(a)\cdot a^{\frac{k}{2}}e^{-2\pi a}\cdot\mathbb{I}_{\mathbf{R}_+}(a),$$

and that

$$W(\mathbf{t}(a), f_{\mathcal{D},s,\infty})|_{s=\frac{1-k}{2}} =\mu\boldsymbol{\alpha}^{1-\frac{k}{2}}(a)(-2\sqrt{-1})^{-k}\pi^{-1}\int_{\mathbf{R}}\frac{(x-\sqrt{-1}a)^{k-1}e^{-2\pi\sqrt{-1}x}}{x+\sqrt{-1}a}\,\mathrm{d}x$$

$$=\mu(a)\cdot a^{\frac{k}{2}}e^{-2\pi a}\cdot\mathbb{I}_{\mathbf{R}_+}(a).$$

Since μ is a quadratic character, the lemma follows. □

Let $q_\mathfrak{q}=|\varpi_\mathfrak{q}|^{-1}=\sharp(\mathcal{O}_F/\mathfrak{q})$ denote the cardinality of the residue field.

Lemma 3.7. *Let $v=\mathfrak{q}$ be a prime ideal of $\mathcal{O}_F$. Let $a\in F_\mathfrak{q}^\times$. Put*

$$\chi_\mathfrak{q}=\mu_\mathfrak{q}^{-1}\nu_\mathfrak{q},\quad \gamma_\mathfrak{q}=\chi_\mathfrak{q}(\varpi_\mathfrak{q}),\quad q_\mathfrak{q}=|\varpi_\mathfrak{q}|^{-1}=\sharp(\mathcal{O}_F/\mathfrak{q}),\quad m=\mathrm{ord}_\mathfrak{q}(a).$$

Then $W(\mathbf{t}(a), f_{\mathcal{D},s,\mathfrak{q}})$ equals

$(\mathfrak{q}\nmid p\mathfrak{N}c)$ $$\mu_\mathfrak{q}(a)|a|^{s+\frac{1}{2}}\sum_{j=0}^{m+\mathrm{ord}_\mathfrak{q}(\mathfrak{d})}(\gamma_\mathfrak{q}q_\mathfrak{q}^{2s})^j,$$

$(\mathfrak{q}\mid\mathfrak{N})$ $$\mu_\mathfrak{q}(a)|a|^{s+\frac{1}{2}}\left(\sum_{j=0}^{m-\mathrm{ord}_\mathfrak{q}(\mathfrak{N})}(\gamma_\mathfrak{q}q_\mathfrak{q}^{2s})^j-q_\mathfrak{q}^{-1}\sum_{j=-1}^{m-\mathrm{ord}_\mathfrak{q}(\mathfrak{N})}(\gamma_\mathfrak{q}q_\mathfrak{q}^{2s})^j\right),$$

$(\mathfrak{q}\mid\mathfrak{c})$ $$\mu_\mathfrak{q}(a)|a|^{s+\frac{1}{2}}\cdot\varepsilon(-2s,\chi_\mathfrak{q})^{-1}\cdot\mathbb{I}_{\mathcal{O}_\mathfrak{q}}(a),$$

$(\mathfrak{q}\mid\bar{\mathfrak{c}})$ $$\mu_\mathfrak{q}(a)|a|^{s+\frac{1}{2}}\mathbb{I}_{\mathcal{O}_\mathfrak{q}}(a),$$

$(\mathfrak{q}=\mathfrak{p}\mid p)$ $$\mathbb{I}_{\mathcal{O}_\mathfrak{p}^\times}(a).$$

Proof. Fix a local uniformizer $\varpi_\mathfrak{q}\in\mathcal{O}_\mathfrak{q}$ of the prime ideal $\mathfrak{q}$. Note that if $\Phi=\Phi_1\otimes\Phi_2\in\mathcal{S}(F_\mathfrak{q}^2)$, then

(3.4)

$$f_{\mathcal{D},s,\mathfrak{q}}\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}a&x\\0&1\end{pmatrix}\right)=\mu_\mathfrak{q}(a)|a|^{s+\frac{1}{2}}\int_{F_\mathfrak{q}^\times}\Phi_1(ta)\Phi_2(tx)(\mu_\mathfrak{q}\nu_\mathfrak{q}^{-1})(t)|t|^{2s+1}\,\mathrm{d}^\times t$$

and hence

$$W(\mathbf{t}(a), f_{\mathcal{D},s,\mathfrak{q}}) = \mu_{\mathfrak{q}}(a)|a|^{s+\frac{1}{2}} \int_{F_{\mathfrak{q}}^{\times}} \Phi_1(ta)\widehat{\Phi}_2(-t^{-1})(\mu_{\mathfrak{q}}\nu_{\mathfrak{q}}^{-1})(t)|t|^{2s}\, \mathrm{d}^{\times}t.$$

If $\mathfrak{q} \nmid p\mathfrak{N}\mathfrak{c}$, then $\Phi_{\mathcal{D},\mathfrak{q}} = \mathbb{I}_{\mathfrak{d}^{-1}\mathcal{O}_{\mathfrak{q}}} \otimes \mathbb{I}_{\mathfrak{d}^{-1}\mathcal{O}_{\mathfrak{q}}}$, and hence

$$\begin{aligned} W(\mathbf{t}(a), f_{\mathcal{D},s,\mathfrak{q}}) =&\mu_{\mathfrak{q}}(a)|a|^{s+\frac{1}{2}} \int_{F_{\mathfrak{q}}^{\times}} \mathbb{I}_{\mathfrak{d}^{-1}\mathcal{O}_{\mathfrak{q}}}(t^{-1}a)\mathbb{I}_{\mathcal{O}_{\mathfrak{q}}}(-t)\chi_{\mathfrak{q}}(t)|t|^{-2s}\, \mathrm{d}^{\times}t \\ =&\mu_{\mathfrak{q}}(a)|a|^{s+\frac{1}{2}} \sum_{j=0}^{m+\mathrm{ord}_{\mathfrak{q}}(\mathfrak{d})} \chi_{\mathfrak{q}}(\varpi_{\mathfrak{q}}^{j})q_{\mathfrak{q}}^{2sj}. \end{aligned}$$

If $\mathfrak{q} \mid \mathfrak{N}\mathfrak{c}$, then $\mu_{\mathfrak{q}}$ is unramified by assumption. It is easy to verify that

$$\widehat{\phi}_{\nu_{\mathfrak{q}}}(x) = \begin{cases} \mathbb{I}_{\mathcal{O}_{\mathfrak{q}}}(x) - q_{\mathfrak{q}}^{-1}\mathbb{I}_{\mathfrak{q}^{-1}\mathcal{O}_{\mathfrak{q}}}(x) & \text{if } \mathfrak{q} \mid \mathfrak{N}, \\ \varepsilon(1,\nu_{\mathfrak{q}}^{-1})\nu_{\mathfrak{q}}(x^{-1})\mathbb{I}_{\varpi_{\mathfrak{q}}^{-c(\nu_{\mathfrak{q}})}\mathcal{O}_{\mathfrak{q}}^{\times}}(x) & \text{if } \mathfrak{q} \mid \mathfrak{c}. \end{cases}$$

One can readily prove the case $\mathfrak{q} \mid \mathfrak{N}$. If $\mathfrak{c}$ is divisible by $\mathfrak{q}$, then

$$\begin{aligned} \mu_{\mathfrak{q}}(a)^{-1}|a|^{-s-\frac{1}{2}}W(\mathbf{t}(a), f_{\mathcal{D},s,\mathfrak{q}}) &= \int_{F_{\mathfrak{q}}^{\times}} \mathbb{I}_{C\mathcal{O}_{\mathfrak{q}}}(at)\widehat{\phi}_{\nu_{\mathfrak{q}}}(-t^{-1})(\mu_{\mathfrak{q}}\nu_{\mathfrak{q}}^{-1})(t)|t|^{2s}\, \mathrm{d}^{\times}t \\ &= \varepsilon(1,\nu_{\mathfrak{q}}^{-1})\nu_{\mathfrak{q}}(-1)\mu_{\mathfrak{q}}\big(\varpi_{\mathfrak{q}}^{c(\nu_{\mathfrak{q}})}\big)q_{\mathfrak{q}}^{-2sc(\nu_{\mathfrak{q}})}\mathbb{I}_{C\varpi_{\mathfrak{q}}^{-c(\nu_{\mathfrak{q}})}\mathcal{O}_{\mathfrak{q}}}(a). \end{aligned}$$

Note that $C\mathcal{O}_{\mathfrak{q}} = \varpi_{\mathfrak{q}}^{-c(\nu_{\mathfrak{q}})}\mathcal{O}_{\mathfrak{q}}$ for $\mathfrak{q} \mid \mathfrak{c}$ by our assumption on the conductor of ν and that

$$\varepsilon(1,\nu_{\mathfrak{q}}^{-1})\nu_{\mathfrak{q}}(-1)\mu_{\mathfrak{q}}\big(\varpi_{\mathfrak{q}}^{c(\nu_{\mathfrak{q}})}\big)q_{\mathfrak{q}}^{-2sc(\nu_{\mathfrak{q}})} = \nu_{\mathfrak{q}}(-1)\varepsilon(1+2s,\chi_{\mathfrak{q}}^{-1}) = \varepsilon(-2s,\chi_{\mathfrak{q}})^{-1}$$

by (2.9). If $\mathfrak{q} \mid \bar{\mathfrak{c}}$, then $W(\mathbf{t}(a), f_{\mathcal{D},s,\mathfrak{q}})$ equals

$$\mu_{\mathfrak{q}}(a)|a|^{s+\frac{1}{2}} \int_{F_{\mathfrak{q}}^{\times}} \mathbb{I}_{\mathcal{O}_{\mathfrak{q}}}(at)\phi_{\nu_{\mathfrak{q}}^{-1}}(t^{-1})(\mu_{\mathfrak{q}}\nu_{\mathfrak{q}}^{-1})(t)|t|^{2s}\, \mathrm{d}^{\times}t = \mu_{\mathfrak{q}}(a)|a|^{s+\frac{1}{2}}\mathbb{I}_{\mathcal{O}_{\mathfrak{q}}}(a).$$

Finally, if $v = \mathfrak{p}|p$, then we find that $W(\mathbf{t}(a), f_{\mathcal{D},s,\mathfrak{p}})$ equals

$$\mu_{\mathfrak{p}}(a)|a|^{s+\frac{1}{2}} \int_{F_{\mathfrak{p}}^{\times}} \phi_{\mu_{\mathfrak{p}}^{-1}}(at)\phi_{\nu_{\mathfrak{p}}^{-1}}(t^{-1})(\mu_{\mathfrak{p}}\nu_{\mathfrak{p}}^{-1})(t)|t|^{2s}\, \mathrm{d}^{\times}t = \mathbb{I}_{\mathcal{O}_{\mathfrak{p}}^{\times}}(a)$$

by a similar calculation. □

For each non-zero element $\beta \in F^{\times}$ we define the polynomials $\mathcal{P}_{\beta,\mathfrak{q}}$ and $\mathcal{Q}_{\chi,\mathfrak{q}}$ in $\mathbf{Z}_{(q_{\mathfrak{q}})}[X, X^{-1}]$ by

(3.5)

$$\begin{aligned} \mathcal{P}_{\beta,\mathfrak{q}}(X) &= \begin{cases} \sum_{j=0}^{\mathrm{ord}_{\mathfrak{q}}(\beta\mathfrak{d})} q_{\mathfrak{q}}^{-j}X^{j} & \text{if } \mathfrak{q} \nmid p\mathfrak{N}\mathfrak{c}, \\ \sum_{j=0}^{\mathrm{ord}_{\mathfrak{q}}(\beta\mathfrak{N}^{-1})} q_{\mathfrak{q}}^{-j}X^{j} - \sum_{j=-1}^{\mathrm{ord}_{\mathfrak{q}}(\beta\mathfrak{N}^{-1})} q_{\mathfrak{q}}^{-(j+1)}X^{j} & \text{if } \mathfrak{q} \mid \mathfrak{N}, \end{cases} \\ \mathcal{Q}_{\chi,\mathfrak{q}}(X) &= \varepsilon(0,\chi_{\mathfrak{q}})^{-1} \cdot (q_{\mathfrak{q}}X^{-1})^{c(\chi_{\mathfrak{q}})}. \end{aligned}$$

Let $\beta \in F$. We write $\beta > 0$ if $\sigma_i(\beta) > 0$ for $i = 1, 2$.

Corollary 3.8. *We have the following Fourier expansion around the infinity cusp:*

$$E_k^{\pm}(\mu,\nu)(\tau_1,\tau_2) = \sum_{0<\beta\in\mathfrak{d}^{-1},\,(p,\beta)=1} \sigma_\beta^{\pm}(\mu,\nu,k)\cdot e^{2\pi\sqrt{-1}(\tau_1\sigma_1(\beta)+\tau_2\sigma_2(\beta))},$$

where

$$\sigma_\beta^{+}(\mu,\nu,k) = \mu_p^{-1}(\beta)\prod_{\mathfrak{q}\nmid\mathfrak{c}p}\mathcal{P}_{\beta,\mathfrak{q}}(\gamma_\mathfrak{q}\cdot q_\mathfrak{q}^k)\prod_{\mathfrak{q}\mid(\mathfrak{c},\beta)}\mathcal{Q}_{\mu^{-1}\nu,\mathfrak{q}}(q_\mathfrak{q}^k),$$

$$\sigma_\beta^{-}(\mu,\nu,k) = \mathrm{N}_{F/\mathbf{Q}}(\beta)^{k-1}\cdot\mu_p^{-1}(\beta)\prod_{\mathfrak{q}\nmid\mathfrak{c}p}\mathcal{P}_{\beta,\mathfrak{q}}(\gamma_\mathfrak{q}\cdot q_\mathfrak{q}^{2-k})\prod_{\mathfrak{q}\mid(\mathfrak{c},\beta)}\mathcal{Q}_{\mu^{-1}\nu,\mathfrak{q}}(q_\mathfrak{q}^{2-k}).$$

Proof. Note that if $\Phi = \phi_1\otimes\phi_2 \in \mathcal{S}(F_v^2)$, then $\widehat{\Phi}(x,y) = \widehat{\phi}_2(-x)\widehat{\phi}_1(y)$. Since $\Phi_{\mathcal{D},\mathfrak{p}}(0,y) = 0$ and $\widehat{\Phi}_{\mathcal{D},\mathfrak{p}}(0,y) = \phi_{\nu_\mathfrak{p}^{-1}}(0)\widehat{\phi}_{\mu_\mathfrak{p}^{-1}}(y) = 0$ for a prime $\mathfrak{p}$ lying above the distinguished prime p, we see that

$$f_{\mathcal{D},s,\mathfrak{p}}(g) = f_{\nu_\mathfrak{p},\mu_\mathfrak{p},\widehat{\Phi}_{\mathcal{D},\mathfrak{p}},-s}(g) = 0 \text{ for } g\in B(F_\mathfrak{p}). \tag{3.6}$$

This in particular implies that

$$f_{\mathcal{D},s}\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right) = f_{\nu_\mathfrak{p},\mu_\mathfrak{p},\widehat{\Phi}_{\mathcal{D},\mathfrak{p}},-s}\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right) = 0.$$

In view of (3.2) and Lemma 3.6, we find that

$$\sigma_\beta^{\pm}(\mu,\nu,k) = \mathrm{N}(\beta)^{\frac{k}{2}}\prod_{\mathfrak{q}<\infty} W(\mathbf{t}(\beta), f_{\mathcal{D},s,\mathfrak{q}})|_{s=\pm\frac{k-1}{2}}.$$

The assertion follows from Lemma 3.7 by noting that $\mu_\mathfrak{q}^{-1}\nu_\mathfrak{q}(\varpi_\mathfrak{q}) = \mu\nu^{-1}(\mathfrak{q})$ if $\mathfrak{q}$ is the prime induced by v. ☐

4. Restriction of Eisenstein series

4.1. **Optimal embeddings.** Let F be a real quadratic field whose discriminant is denoted by Δ_F. Define $\theta\in F$ by $\theta = \frac{D'-\sqrt{\Delta_F}}{2}$, where $D' = \Delta_F$ or $\frac{\Delta_F}{2}$ according to whether Δ_F is odd or even. Then $\mathcal{O}_F = \mathbf{Z}+\mathbf{Z}\theta$, and if q is ramified in F, then θ is a local uniformizer of $\mathcal{O}_q$. Denote by $x\mapsto\overline{x}$ the unique automorphism of $\mathrm{Gal}(F/\mathbf{Q})$. Put

$$\delta := \overline{\theta}-\theta = \sqrt{\Delta_F}.$$

We choose an embedding $\sigma_1 : F \hookrightarrow \mathbf{R}$ such that $\sigma_1(\delta) > 0$. Define an algebraic group T over $\mathbf{Q}$ by $T(R) = (F\otimes R)^\times$ for any commutative field R of characteristic zero. We view T as a maximal torus of GL_2 via the embedding $\Psi\colon F\hookrightarrow \mathrm{M}_2(\mathbf{Q})$ defined by

$$\Psi(\theta) = \begin{pmatrix}\mathrm{T}(\theta) & -\mathrm{N}(\theta)\\ 1 & 0\end{pmatrix}.$$

Put

$$\eta := \begin{pmatrix} 1 & -\theta \\ -1 & \overline{\theta} \end{pmatrix} \delta^{-1} = \begin{pmatrix} \overline{\theta} & \theta \\ 1 & 1 \end{pmatrix}^{-1} \in \mathrm{GL}_2(F).$$

It is important to note that for $t \in F$

$$\eta\Psi(t)\eta^{-1} = \begin{pmatrix} \bar{t} & 0 \\ 0 & t \end{pmatrix}. \tag{4.1}$$

Let N and C be positive integers such that

- C and $N\Delta_F$ are coprime;
- Every prime factor of NC is split in F.

Fix decompositions $N\mathcal{O}_F = \mathfrak{N}\overline{\mathfrak{N}}$ and $C\mathcal{O}_F = \mathfrak{c}\overline{\mathfrak{c}}$ once and for all. Fix a prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$ lying above p.

We define special elements ς, $\varsigma^{(C)}$ and $\varsigma^{(Cp^n)}$ in $\mathrm{GL}_2(\mathbf{A})$ as follows:

- At the archimedean place, put
$$\varsigma_\infty = \begin{pmatrix} \sigma_2(\theta) & \sigma_1(\theta) \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{R}).$$
- For each rational prime q we fix a prime ideal $\mathfrak{q}$ of $\mathcal{O}_F$ above q and define $\varsigma_q \in \mathrm{GL}_2(\mathbf{Q}_q)$ by
$$\varsigma_q = \begin{pmatrix} \overline{\theta} & \theta \\ 1 & 1 \end{pmatrix} \delta^{-1} \in \mathrm{GL}_2(F_{\mathfrak{q}}) = \mathrm{GL}_2(\mathbf{Q}_q) \text{ if } q = \mathfrak{q}\overline{\mathfrak{q}} \text{ is split,}$$
$$\varsigma_q = 1 \text{ otherwise.}$$
- Put
$$\varsigma_q^{(C)} = \begin{pmatrix} C & -1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_q);$$
$$\varsigma_p^{(n)} = \begin{cases} \begin{pmatrix} p^n & -1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(F_{\mathfrak{p}}) & \text{if } p = \mathfrak{p}\overline{\mathfrak{p}} \text{ is split in } F \\ \begin{pmatrix} 0 & 1 \\ -p^n & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_p) & \text{if } p \text{ is inert in } F. \end{cases}$$
Finally, we define
$$\varsigma = \prod_v \varsigma_v, \quad \varsigma^{(C)} := \varsigma \prod_{q|C} \varsigma_q^{(C)}; \quad \varsigma^{(Cp^n)} := \varsigma^{(C)}\varsigma_p^{(n)}.$$

Let $\mathcal{O}_C = \mathbf{Z} + C\mathcal{O}_F$ be the order of F of conductor C. It is not difficult to verify immediately that the inclusion map $\Psi : K \hookrightarrow \mathrm{M}_2(\mathbf{Q})$ is an optimal embedding of $\mathcal{O}_C$ into the Eichler order $R_N := \mathrm{M}_2(\mathbf{Q}) \cap \varsigma^{(C)}\mathrm{M}_2(\widehat{\mathbf{Z}})(\varsigma^{(C)})^{-1}$ of level N. In other words,

$$\Psi^{-1}(R_N) \cap F = \mathcal{O}_C. \tag{4.2}$$

4.2. **A result of Keaton and Pitale.** Let $\pi \simeq \otimes'_v \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ generated by $\Phi(f) \in \mathcal{A}^0_{2k}(N,\omega)$. Let μ and ν be unitary Hecke characters of $\mathbf{A}_F^\times$ such that μ has p-power conductor and such that the restriction of $\mu\nu$ to $\mathbf{A}^\times$ is ω. Define the Hecke character $\chi : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ by

$$\chi(x) := \mu(x)\nu(\bar{x}).$$

Given $\varphi \in \pi$, we define the global zeta integral by

$$Z_{\mathcal{D}}(s,\varphi) = \int_{\mathbf{A}^\times \mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A})} E_{\mathbf{A}}(g, f_{\mathcal{D},s})\varphi(g)\omega(\det g)^{-1}\, \mathrm{d}^\tau g,$$

where $f_{\mathcal{D},s}$ is the section defined in Definition 3.2 associated with the datum $\mathcal{D} = (\mu,\nu,k,\mathfrak{N},\mathfrak{c})$. This integral converges absolutely for all s away from the poles of $E_{\mathbf{A}}(g, f_{\mathcal{D},s})$ and defines a meromorphic function in s.

We define the Tamagawa measures $\mathrm{d}^\times x$ of $\mathbf{A}_F^\times$ and $\mathrm{d}^\times a$ of $\mathbf{A}^\times$ in §2.6.1. We define the Tamagawa measure $\mathrm{d}t$ of $T(\mathbf{A})$ as the quotient measure of $\mathrm{d}^\times x$ and $\mathrm{d}^\times a$. Let $\mathrm{d}g$ denote the quotient measure of $\mathrm{d}^\tau g$ and $\mathrm{d}t$. Given $\varphi \in \pi$, we define the toric period integral by

$$B^\chi_\varphi(g) = \int_{\mathbf{A}^\times F^\times \backslash \mathbf{A}_F^\times} \varphi(\Psi(t)g)\chi(t)^{-1}\, \mathrm{d}t. \tag{4.3}$$

Theorem 4.1 (Keaton and Pitale)**.** *Let $\varphi \in \pi$. Then*

$$Z_{\mathcal{D}}(s,\varphi) = \int_{T(\mathbf{A})\backslash \mathrm{GL}_2(\mathbf{A})} B^\chi_\varphi(g)\, \mathrm{d}g.$$

Proof. This is nothing but Proposition 2.3 of [KP19]. □

4.3. **Global setting.** Now we let $f = \sum_{n=1}^\infty \mathbf{a}(n,f)q^n \in \mathcal{S}_{2k}(Np^r,\omega^{-1})$ be a p-stablized newform and $\varphi = \Phi(f) \in \mathcal{A}^0_{2k}(N,\omega)$ be the automorphic form associated with f in (2.3). For each prime factor q of C we choose a root $\alpha_q(f)$ of the Hecke polynomial $X^2 - \mathbf{a}(q,f)X + \omega^{-1}(q)q^{2k-1}$. Let $\breve{f}$ be the unique form in $\mathcal{S}_{2k}(NCp^r,\omega^{-1})[f]$ such that $\mathbf{a}(1,\breve{f}) = 1$ and $\mathbf{U}_q\breve{f} = \alpha_q(f)\breve{f}$. Let $\breve{\varphi} = \Phi(\breve{f})$ be the adelic lift of $\breve{f}$. We impose the following assumptions:

- ω has a square root $\omega^{\frac{1}{2}}$;
- μ and ω are unramified outside p;
- $C\mathcal{O}_F$ is the conductor of $\chi\omega_F^{-\frac{1}{2}}$ ($\omega_F^{\frac{1}{2}} := \omega^{\frac{1}{2}} \circ \mathrm{N}$).

Note that these assumptions imply that the $C\mathcal{O}_F$ is the prime-to-p part of the conductor of ν. Define the matrices $\mathcal{J}_\infty$ and t_n for each integer n in $\mathrm{GL}_2(\mathbf{A})$ by

$$\mathcal{J}_\infty = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{R}), \quad t_n = \begin{pmatrix} 0 & p^{-n} \\ -p^n & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_p). \tag{4.4}$$

4.4. **Local zeta integrals.** For each place v of $\mathbf{Q}$ we set $f_{\mathcal{D},s,v}=\bigotimes_{\mathbf{v}|v} f_{\mathcal{D},s,\mathbf{v}}$. Assume that φ has the factorizable Whittaker function $W_\varphi(g)=\prod_v W_v(g_v)$ for $g=(g_v)\in \mathrm{GL}_2(\mathbf{A})$. We associate to each Whittaker function $W_v\in\mathcal{W}(\pi_v,\boldsymbol{\psi}_v)$ a Bessel function $B_{W_v}:\mathrm{GL}_2(\mathbf{Q}_v)\to\mathbf{C}$ by

$$B_{W_v}(g_v)=\int_{\mathbf{Q}_v^\times\backslash F_v^\times} W_v(\varsigma_v^{-1}\Psi(t_v)g_v)\chi_v(t_v)^{-1}\,\mathrm{d}t_v$$

unless $v=p$ is inert in F. Here $\mathrm{d}t_v$ is the quotient measure of $\mathrm{d}^\times x_v$ and $\mathrm{d}^\times a_v$ (see §2.6.1). This integral is absolutely convergent (see the proof of Proposition 4.3). If $v=p$ is inert in F, then we will explicitly choose a Whittaker function $\widetilde{W}\in\mathcal{W}(\pi_p^\vee,\boldsymbol{\psi}_p^{-1})$ in the proof of Proposition 4.5 so that $\rho(t)\widetilde{W}=\chi_p(t)^{-1}\widetilde{W}$. Recall the standard $\mathrm{GL}_2(\mathbf{Q}_p)$-invariant pairing $\langle\ ,\ \rangle:\mathcal{W}(\pi_p,\boldsymbol{\psi}_p)\times\mathcal{W}(\pi_p^\vee,\boldsymbol{\psi}_p^{-1})\to\mathbf{C}$ defined by

$$\langle W_1,W_2\rangle=\int_{\mathbf{Q}_p^\times} W_1(\mathbf{t}(a_p))W_2(\mathbf{t}(a_p))\,\mathrm{d}^\times a_p.$$

Define the Bessel function $B_{W_p}:\mathrm{GL}_2(\mathbf{Q}_p)\to\mathbf{C}$ by $B_{W_p}(g):=\langle\rho(g)W_p,\widetilde{W}\rangle$. The integral

$$Z_{\mathcal{D}}(s,B_{W_v})=\int_{T(\mathbf{Q}_v)\backslash\mathrm{GL}_2(\mathbf{Q}_v)} f_{\mathcal{D},s,v}(\eta g_v)\omega_v(\det g_v)^{-1}B_{W_v}(g_v)\,\mathrm{d}g_v \tag{4.5}$$

makes sense by (4.1), where $\mathrm{d}g_v$ is the quotient measure of $\mathrm{d}^\tau g_v$ and $\mathrm{d}t_v$.

4.5. **Convergence.** In this and next subsections we fix a place v of $\mathbf{Q}$ and suppress the subscript v from the notation. Thus

$$F=F\otimes\mathbf{Q}_v,\quad \boldsymbol{\psi}=\boldsymbol{\psi}_v,\qquad |\cdot|=|\cdot|_v,\quad \mu=\mu_v,\quad \nu=\nu_v,$$
$$\pi=\pi_v,\qquad \varPhi_{\mathcal{D}}=\otimes_{\mathbf{v}|v}\varPhi_{\mathcal{D},\mathbf{v}}\in\mathcal{S}(F^2),\dots.$$

Lemma 4.2. *The integral defining $Z_{\mathcal{D}}(s,B_W)$ converges absolutely for $\mathrm{Re}\,s\gg 0$.*

Proof. Put $T_q=T(\mathbf{Q}_q)$. For $W\in\mathcal{W}(\pi)$ we have

$$\begin{aligned}Z_{\mathcal{D}}(s,B_W)&=\int_{T_q\backslash\mathrm{GL}_2(\mathbf{Q}_q)} f_{\mathcal{D},s,q}(\eta g)\omega(\det g)^{-1}B_W(g)\,\mathrm{d}g\\&=\int_{T_q\backslash\mathrm{GL}_2(\mathbf{Q}_q)} f_{\mathcal{D},s,q}(\eta g)\omega(\det h)^{-1}\int_{\mathbf{Q}_q^\times\backslash T_q} W(\varsigma_q^{-1}tg)\chi(t)^{-1}\,\mathrm{d}t\mathrm{d}h\\&=\int_{T_q\backslash\mathrm{GL}_2(\mathbf{Q}_q)}\int_{\mathbf{Q}_q^\times\backslash T_q} f_{\mathcal{D},s,q}(\eta tg)\omega(\det(tg))^{-1}W(\varsigma_q^{-1}tg)\,\mathrm{d}t\mathrm{d}g\end{aligned}$$

by definition. We combine the iterated integral to obtain

$$Z_{\mathcal{D}}(s,B_W)=\int_{\mathrm{PGL}_2(\mathbf{Q}_q)} f_{\mathcal{D},s,q}(\eta g)\omega(\det g)^{-1}W(\varsigma_q^{-1}g)\,\mathrm{d}^\tau g.$$

First assume that $v = q = \mathfrak{q}\bar{\mathfrak{q}}$ is split in F. Since $\eta\varsigma_q = \delta^{-1}$ and $\bar{\eta}\varsigma_q = \delta^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we get

$$
\begin{aligned}
Z_{\mathcal{D}}(s, B_W) &= \int_{\mathrm{PGL}_2(\mathbf{Q}_q)} f_{\mathcal{D},s,\mathfrak{q}}(g)\omega(\det g)^{-1} f_{\mathcal{D},s,\bar{\mathfrak{q}}}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g\right) W(g)\,\mathrm{d}^\tau g \\
&= \int_{N(\mathbf{Q}_q)\backslash \mathrm{PGL}_2(\mathbf{Q}_q)} f_{\mathcal{D},s,\mathfrak{q}}(g)\omega(\det g)^{-1} W_{\bar{\mathfrak{q}}}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g\right) W(g)\,\mathrm{d}^\tau g, \qquad (4.6)
\end{aligned}
$$

where $W_{\bar{\mathfrak{q}}}(g) := W(g, f_{\mathcal{D},s,\bar{\mathfrak{q}}})$. This is nothing but the local Rankin-Selberg integral for $\mathrm{GL}_2 \times \mathrm{GL}_2$, which is absolutely convergent for $\mathrm{Re}\, s \geq 0$.

Next assume that $v = q$ remains prime in F. It suffices to show that the integral

$$
\int_{\mathbf{Q}_q^\times} \int_{\mathbf{Q}_q} \frac{1}{\omega(a)\,|a|} f_{\mathcal{D},s,q}(\eta \mathbf{n}(x)\mathbf{t}(a))\boldsymbol{\psi}(x)\,\mathrm{d}x \cdot W(\mathbf{t}(a))\mathrm{d}^\times a \tag{4.7}
$$

converges absolutely in view of the Iwasawa decomposition. Since $\eta = \delta^{-1}\begin{pmatrix} 1 & -\theta \\ -1 & \bar{\theta} \end{pmatrix}$, the inner integral is

$$
\mu(\delta^{-1}a)\frac{|\delta^{-1}a|_F^{s+\frac{1}{2}}}{\omega(a)\,|a|} \int_{\mathbf{Q}_q} \int_{F^\times} (\mu\nu^{-1})(t)|t|_F^{2s+1} \Phi\left((0,t)\begin{pmatrix} 1 & -\theta \\ -1 & \bar{\theta} \end{pmatrix}\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right)\boldsymbol{\psi}(x)\,\mathrm{d}^\times t \mathrm{d}x.
$$

Put $\xi := \mu\nu^{-1}\boldsymbol{\alpha}_F^{2s+1}$. Let $\Phi = \Phi_1 \otimes \Phi_2$. We may assume that $|\Phi_1(x)| \leq 1$ and $\Phi_2(xc) = \Phi_2(x)$ for $x \in F$ and $c \in \mathcal{O}_F^\times$. Since the integral

$$
\int_{\mathbf{Q}_q} \int_{\mathbf{Q}_q^\times} |\xi(t)\Phi_1(-at)\Phi_2(t(\bar{\theta} - x))|\,\mathrm{d}^\times t \mathrm{d}x \leq \int_{\mathbf{Q}_q} \int_{\mathbf{Q}_q^\times} |\xi(t)\Phi_2(t\bar{\theta} - x)|\,\mathrm{d}^\times t \frac{\mathrm{d}x}{|t|}
$$

converges for $\mathrm{Re}\, s \gg 0$, the double integral (4.7) is absolutely convergent for $\mathrm{Re}\, s \gg 0$. □

4.6. **Local calculations.** We shall compute the local zeta integrals $Z_{\mathcal{D}}(s, B_{W_v})$ occurring in the factorization of the global integral $Z_{\mathcal{D}}(s, \rho(\mathcal{J}_\infty t_n)\breve{\varphi}_f)$. Put $\nu_+ := \nu|_{\mathbf{Q}_v^\times}$. Recall the normalized Whittaker newform $W_\pi \in \mathcal{W}(\pi, \boldsymbol{\psi})$ (see §2.6.3). For each prime factor $v = q$ of C, if we write $\pi = \varrho_q \boxplus \upsilon_q$ with $\varrho_q(q) = \alpha_q(f)q^{\frac{1-2k}{2}}$, then

$$
\breve{W}_\pi := W_\pi - \upsilon_q(q)\,|q|^{\frac{1}{2}}\,\pi(\mathbf{t}(q^{-1}))W_\pi.
$$

Then $\breve{W}_\pi$ is characterized uniquely by $\breve{W}_\pi(\mathbf{1}_2) = 1$ and $\mathbf{U}_q\breve{W}_\pi = \varrho_q(q)\,|q|^{-\frac{1}{2}}\,\breve{W}_\pi$. In the case $v = p$, we denote by W_π^{ord} an ordinary vector of eigenvalue $\mathbf{a}(p, f)p^{1-k}$. By our assumptions,

μ and ν_+ are unramified outside p; $\chi\omega_F^{-\frac{1}{2}}$ is only ramified at primes dividing C.

Proposition 4.3. *Let $v \neq p$ be a place of $\mathbf{Q}$ which is split in F. We have*

- *If $v=\infty$ is the archimedean place, then* $B_{\rho(\mathcal{J}_\infty)W_\pi}(\varsigma_\infty)\neq 0$, *and*
$$Z_{\mathcal{D}}(s,B_{\rho(\mathcal{J}_\infty)W_\pi})=4(-4\sqrt{-1})^{-k}\nu_{\sigma_1}(-1)\Gamma_{\mathbf{C}}(2s+k)\cdot B_{\rho(\mathcal{J}_\infty)W_\pi}(\varsigma_\infty).$$
- *If $v=q$ and NC are coprime, then*
$$Z_{\mathcal{D}}(s,B_{W_\pi})=L\biggl(2s+\frac{1}{2},\pi\otimes\nu_+^{-1}\biggr)B_{W_\pi}(\varsigma_q).$$
- *If $v=q$ is a prime factor of N, then* $B_{W_\pi}(\varsigma_q)\neq 0$ *and*
$$Z_{\mathcal{D}}(s,B_{W_\pi})=\frac{\zeta_q(2)\,|NC|_{\mathbf{Q}_q}}{\zeta_q(1)}\cdot L\biggl(2s+\frac{1}{2},\pi\otimes\nu_+^{-1}\biggr)B_{W_\pi}(\varsigma_q).$$
- *When $v=q$ is a prime factor of C, then* $B_{\breve{W}_\pi}(\varsigma_q\varsigma_q^{(C)})\neq 0$ *and*
$$Z_{\mathcal{D}}(s,B_{W_\pi})=\frac{\zeta_q(2)\,|NC|_{\mathbf{Q}_q}}{\zeta_q(1)}\cdot L\biggl(2s+\frac{1}{2},\pi\otimes\nu_+^{-1}\biggr)\frac{\varepsilon\bigl(0,\chi_{\bar{\mathfrak{q}}}^{-1}\bigr)}{\zeta_q(1)}\cdot B_{\breve{W}_\pi}(\varsigma_q\varsigma_q^{(C)}).$$

Proof. We first treat the archimedean case. Let $W=\rho(\mathcal{J}_\infty)W_{\pi_\infty}$. By definition, $B_W(\varsigma_\infty\mathbf{n}(x))$ equals
$$\begin{aligned}&\int_{\mathbf{R}^\times}W_{\pi_\infty}(\mathbf{t}(a)\mathbf{n}(x)\mathcal{J}_\infty)\mu_{\sigma_2}(a)^{-1}\nu_{\sigma_1}(a)^{-1}\,\mathrm{d}^\times a\\ =&(\mu_{\sigma_2}\nu_{\sigma_1})(-1)\int_0^\infty e^{-2\pi a(1+x\sqrt{-1})}a^k\,\mathrm{d}^\times a=(\mu_{\sigma_2}\nu_{\sigma_1})(-1)(2\pi)^{-k}(1+x\sqrt{-1})^{-k}\Gamma(k)\end{aligned}$$
by (2.7), where we have shifted the coutour of integration. By the Iwasawa decomposition $\mathrm{GL}_2(\mathbf{R})=B(\mathbf{R})\mathbf{K}_\infty$ and Remark 3.3, the local integral $Z_{\mathcal{D}}(s,B_W)$ equals
$$\begin{aligned}&\int_{\mathbf{R}}f_{\mathcal{D},s,\sigma_1}(\mathbf{n}(x))f_{\mathcal{D},s,\sigma_2}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\mathbf{n}(x)\right)B_W(\varsigma_\infty\mathbf{n}(x))\,\mathrm{d}x\\ =&\frac{\Gamma(k)}{(2\pi\sqrt{-1})^k}\cdot 4^{-k}(-1)^k\frac{\Gamma\bigl(s+\frac{k+1}{2}\bigr)^2}{\pi^{2s+k+1}}\\ &\times\int_{\mathbf{R}}\mu_{\sigma_2}(-1)(1+x^2)^{-(s+1/2)}\left(\frac{x-\sqrt{-1}}{\sqrt{1+x^2}}\right)^k\frac{\mathrm{d}x}{(x-\sqrt{-1})^k}\\ =&\frac{\Gamma(k)}{(2\pi\sqrt{-1})^k}\cdot 2(2\pi)^{-(2s+k)}\Gamma(2s+k)\cdot 4(-4)^{-k}\mu_{\sigma_2}(-1).\end{aligned}$$

Let $v=q=\mathfrak{q}\bar{\mathfrak{q}}$ be a finite split prime. Then
$$B_{W_\pi}(\varsigma_q)=L\biggl(\frac{1}{2},\pi\otimes\chi_{\bar{\mathfrak{q}}}^{-1}\biggr)=L\biggl(\frac{1}{2},\pi\otimes\mu_{\bar{\mathfrak{q}}}^{-1}\nu_{\mathfrak{q}}^{-1}\biggr).\tag{4.8}$$
If q and Nc are coprime, then
$$Z_{\mathcal{D}}(s,B_{W_\pi})=L\biggl(2s+\frac{1}{2},\pi_q\otimes\nu_{\mathfrak{q}}^{-1}\nu_{\bar{\mathfrak{q}}}^{-1}\biggr)L\biggl(\frac{1}{2},\pi_q\otimes\mu_{\bar{\mathfrak{q}}}^{-1}\nu_{\mathfrak{q}}^{-1}\biggr)$$
by (4.6). The unramified case follows from (4.8).

Suppose that $v = q$ divides NC. Then $\mu_{\mathfrak{q}}, \mu_{\bar{\mathfrak{q}}}$ are unramified, the conductors of $\nu_{\mathfrak{q}}$ and $\nu_{\bar{\mathfrak{q}}}$ are $C\mathbf{Z}_q$, and

$$W_{\bar{\mathfrak{q}}}(\mathbf{t}(a)) = \begin{cases} \mu_{\bar{\mathfrak{q}}}(a)\,|a|^{s+\frac{1}{2}}\,\mathbb{I}_{\mathbf{Z}_q}(a) & \text{if } q \mid C, \\ W_{\Pi_{\bar{\mathfrak{q}}}} & \text{if } q \nmid C. \end{cases}$$

Here $W_{\Pi_{\bar{\mathfrak{q}}}}$ is the spherical vector for $\Pi_{\bar{\mathfrak{q}}} = \mu_{\bar{\mathfrak{q}}}\boldsymbol{\alpha}^s \boxplus \nu_{\bar{\mathfrak{q}}}\boldsymbol{\alpha}^{-s}$. Put

$$U_0(NC\mathbf{Z}_q) = \left\{ \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_q) \;\middle|\; c \in NC\mathbf{Z}_q \right\}.$$

We claim that $f_{\mathcal{D},s,\mathfrak{q}}$ is supported in $B(\mathbf{Q}_q)U_0(NC\mathbf{Z}_q)$. Indeed, if $f_{\mathcal{D},s,\mathfrak{q}}(g) \neq 0$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_q)$, then $\Phi_{\mathcal{D},\mathfrak{q}}((0,t)g) \neq 0$ for some $t \in \mathbf{Q}_q^\times$. According to the recipe in Definition 3.2, we find that $(tc, td) \in NC\mathbf{Z}_q \oplus \mathbf{Z}_q^\times$, and hence $cd^{-1} \in NC\mathbf{Z}_q$. Since $f_{\mathcal{D},s,\mathfrak{q}}(\mathbf{1}_2) = 1$, we see that

$$\begin{aligned} Z_{\mathcal{D}}(s, B_{W_\pi}) =& [\mathrm{GL}_2(\mathbf{Z}_q) : U_0(NC\mathbf{Z}_q)] \int_{\mathbf{Q}_q^\times} (\mu_{\mathfrak{q}}\omega^{-1})(a)|a|^{s+\frac{1}{2}} W_{\bar{\mathfrak{q}}}(\mathbf{t}(a)) W_\pi(\mathbf{t}(a))\mathrm{d}^\times a \\ =& \frac{\zeta_q(2)\,|NC|}{\zeta_q(1)} L\left(2s + \frac{1}{2}, \pi \otimes \nu_+^{-1}\right) L\left(\frac{1}{2}, \pi \otimes \mu_{\bar{\mathfrak{q}}}^{-1}\nu_{\mathfrak{q}}^{-1}\right). \end{aligned}$$

The case of a prime factor q of N follows from (4.8).

Finally, we assume that C is divisible by q. Since $\breve{W}_\pi(\mathbf{t}(a)) = \varrho_q(a)\,|a|^{\frac{1}{2}}\,\mathbb{I}_{\mathbf{Z}_q}(a)$, we have

$$\begin{aligned} B_{\breve{W}_\pi}(\varsigma_q\varsigma_q^{(C)}) &= \int_{\mathbf{Q}_q^\times} W_\pi^\dagger\left(\mathbf{t}(a)\begin{pmatrix} C & -1 \\ 0 & 1 \end{pmatrix}\right)\chi_{\bar{\mathfrak{q}}}^{-1}(a)\mathrm{d}^\times a \\ &= |C|^{\frac{1}{2}}\,\varrho_q(C)\int_{\mathbf{Q}_q^\times} |a|^{\frac{1}{2}}\,\chi_{\bar{\mathfrak{q}}}^{-1}\varrho_q(a)\varPhi(a)\mathrm{d}^\times a, \end{aligned}$$

where $\varPhi(a) = \boldsymbol{\psi}_q(-a)\mathbb{I}_{C^{-1}\mathbf{Z}_q}(a)$. The integral above equals

$$\begin{aligned} &\gamma\left(\frac{1}{2}, \chi_{\bar{\mathfrak{q}}}^{-1}\varrho_q\right)^{-1}\int_{\mathbf{Q}_q^\times} |a|^{\frac{1}{2}}\widehat{\varPhi}(a)(\chi_{\bar{\mathfrak{q}}}\varrho_q^{-1})(a)\mathrm{d}^\times a \\ =& \frac{\mathrm{vol}(C^{-1}\mathbf{Z}_q, \mathrm{d}a)}{\varepsilon\left(\frac{1}{2}, \chi_{\bar{\mathfrak{q}}}^{-1}\varrho_q\right)}\,\mathrm{vol}(1 + C\mathbf{Z}_q, \mathrm{d}^\times a) \\ =& \frac{\zeta_q(1)}{\varepsilon\left(\frac{1}{2}, \chi_{\bar{\mathfrak{q}}}^{-1}\varrho_q\right)} = \frac{\zeta_q(1)}{\varepsilon\left(0, \chi_{\bar{\mathfrak{q}}}^{-1}\right)\varrho_q(C)\,|C|^{\frac{1}{2}}} \end{aligned}$$

by the local functional equation (2.8) for GL_1. In the final stage we utilized (2.9). □

Proposition 4.4. *If $v = q$ remains a prime in F and does not divide pNC, then*

$$Z_{\mathcal{D}}(s, B_{W_\pi}) = \mu(\delta)^{-1}\,|\delta|_F^{-s-\frac{1}{2}}\,L\left(2s + \frac{1}{2}, \pi_q \otimes \nu_+^{-1}\right)B_{W_\pi}(\mathbf{1}_2).$$

Proof. By assumption π and χ are both unramified. Thus $W_\pi = W^0$ is the normalized spherical Whittaker function, and so by the Iwasawa decomposition $\mathrm{GL}_2(\mathbf{Q}_q) = B(\mathbf{Q}_q)\,\mathrm{GL}_2(\mathbf{Z}_q)$, we have

$$Z_{\mathcal{D}}(s, B_{W^0}) = \int_{\mathbf{Q}_q^\times} G(a) \cdot W^0(\mathbf{t}(a))\,\mathrm{d}^\times a,$$

where

$$G(a) := \frac{1}{\omega(a)\,|a|}\int_{\mathbf{Q}_q} f_{\mathcal{D},s,q}(\eta\mathbf{n}(x)\mathbf{t}(a))\boldsymbol{\psi}(x)\,\mathrm{d}x.$$

Recall that $\Phi_{\mathcal{D},q} = |\delta|_F^{\frac{1}{2}}\,\mathbb{I}_{\delta^{-1}\mathcal{O}_q\oplus\delta^{-1}\mathcal{O}_q}$ and $\eta = \delta^{-1}\begin{pmatrix}1 & -\theta\\ -1 & \overline{\theta}\end{pmatrix}$. Put $\Phi^0 = \mathbb{I}_{\mathcal{O}_q\oplus\mathcal{O}_q}$ and $\xi := \mu\nu^{-1}\boldsymbol{\alpha}_F^{2s+1}$. The computation in the proof of Lemma 4.2 shows that

$$\mu(\delta)\,|\delta|_F^s\,G(a) = \frac{|a|_F^s}{\nu(a)}\int_{\mathbf{Q}_q}\sum_{m\in\mathbf{Z}}\xi(q^m)\Phi^0(-aq^m, q^m(\overline{\theta}-x))\boldsymbol{\psi}(x)\,\mathrm{d}x.$$

If $F/\mathbf{Q}_q$ is unramified, then $B_{W^0}(\mathbf{1}_2) = 1$ and

$$G(a) = \frac{|a|_F^s}{\nu(a)}\sum_{m=0}^{\infty}\mathbb{I}_{\mathcal{O}_F}(aq^m)\xi(q^m)\int_{q^{-m}\mathbf{Z}_q}\boldsymbol{\psi}(x)\,\mathrm{d}x = \frac{|a|^{2s}}{\nu(a)}\mathbb{I}_{\mathbf{Z}_q}(a).$$

It follows that

$$Z_{\mathcal{D}}(s, B_{W^0}) = \int_{\mathbf{Q}_q^\times}\frac{|a|^{2s}}{\nu(a)}\mathbb{I}_{\mathbf{Z}_q}(a)W^0(\mathbf{t}(a))\,\mathrm{d}^\times a = L\left(2s+\frac{1}{2}, \pi\otimes\nu_+^{-1}\right).$$

Next we consider the case where q is ramified in F. Then θ is a uniformizer. We see that

$$B_{W^0}(\mathbf{1}_2) = W^0(\mathbf{1}_2)\cdot|\Delta_F|^{\frac{1}{2}} + W^0(\Psi(\theta))\chi^{-1}(\theta)\,|\Delta_F|^{\frac{1}{2}}$$

from the decomposition $F^\times = \mathbf{Q}_q^\times\mathcal{O}_F^\times \sqcup \mathbf{Q}_q^\times\mathcal{O}_F^\times\theta$ and $\mathrm{vol}(\mathcal{O}_F^\times, \mathrm{d}t_q) = |\Delta_F|^{\frac{1}{2}}$. Writing $\pi = \varrho\boxplus\upsilon$, $\alpha = \varrho(q)$ and $\beta = \upsilon(q)$, we get

$$|\delta|_F^{-\frac{1}{2}}\,B_{W^0}(\mathbf{1}_2) = 1 + \chi(\theta)^{-1}(\alpha+\beta)\,|q|^{\frac{1}{2}} = 1 + (\mu\nu)(\theta^{-1})\,|q|^{\frac{1}{2}}\,(\alpha+\beta)$$

by the Iwasawa decomposition of $\Psi(\theta)$. On the other hand, $\mu(\delta)\,|\delta|_F^s\,G(a)$ equals

$$\frac{|a|_F^s}{\nu(a)}\int_{\mathbf{Q}_q}\sum_{m\in\mathbf{Z}}\left(\xi(\theta^{2m})\Phi^0(a\theta^{2m}, \theta^{2m}(\overline{\theta}-x)) + \xi(\theta^{2m-1})\Phi^0(a\theta^{2m-1}, \theta^{2m-1}(\overline{\theta}-x))\right)\boldsymbol{\psi}(x)\,\mathrm{d}x$$
$$=\nu(a)^{-1}\,|a|^{2s}\left(\mathbb{I}_{\mathbf{Z}_q}(a) + \xi(\theta)\mathbb{I}_{\mathbf{Z}_q}(a) + \xi(\theta^{-1})\,|q|\cdot\mathbb{I}_{q\mathbf{Z}_q}(a)\right).$$

Since

$$\sum_{m=1}^{\infty}W^0\left(\begin{pmatrix}q^m & 0\\ 0 & 1\end{pmatrix}\right)q^{-2ms}\nu(q)^{-m} = |q|^{2s+\frac{1}{2}}\,\nu(q)^{-1}\left(\alpha+\beta-\alpha\beta\nu(q)^{-1}\,|q|^{2s+\frac{1}{2}}\right),$$

we conclude that

$$Z_{\mathcal{D}}(s, B_{W^0}) = \mu(\delta^{-1})\,|\delta|_F^{-s}\, L\left(2s+\frac{1}{2}, \pi\otimes\nu_+^{-1}\right) \\ \times \left\{1+\xi(\theta)+\xi(\theta^{-1})\,|q|\cdot|q|^{2s+\frac{1}{2}}\,\nu(q)^{-1}\left(\alpha+\beta-\alpha\beta\nu(q)^{-1}\,|q|^{2s+\frac{1}{2}}\right)\right\}.$$

Since $(\mu\nu)|_{\mathbf{Q}_q^\times} = \omega$, the second factor equals

$$1+(\mu\nu^{-1})(\theta)\,|q|^{2s+1}+(\mu\nu)(\theta^{-1})\,|q|^{\frac{1}{2}}\left(\alpha+\beta-(\nu^{-1}\omega)(q)\,|q|^{2s+\frac{1}{2}}\right) \\ = 1+(\mu\nu)(\theta^{-1})\,|q|^{\frac{1}{2}}\,(\alpha+\beta) = B_{W^0}(\mathbf{1}_2)\,|\delta|_F^{-\frac{1}{2}},$$

which finishes the proof of the ramified case. □

Proposition 4.5. *In the p-adic case, if $\pi = \varrho\boxplus\upsilon$ with ϱ unramified and $\upsilon(-1)=1$, then for $n\gg 0$, we have*

$$B_{W_\pi^{\mathrm{ord}}}(\varsigma_p^{(n)})\neq 0,\quad Z_{\mathcal{D}}(s, B_{\rho(t_n)W_\pi^{\mathrm{ord}}}) = \frac{B_{W_\pi^{\mathrm{ord}}}(\varsigma_p^{(n)})}{\gamma\left(2s+\frac{1}{2},\varrho\nu_+^{-1}\right)}(\omega^{-1}\varrho)(p^n)\,|p^n|^{\frac{1}{2}}\,\frac{\zeta_p(2)}{\zeta_{F_p}(1)}.$$

Proof. We first assume that $p=\mathfrak{p}\overline{\mathfrak{p}}$ is split. Then $F=\mathbf{Q}_p\oplus\mathbf{Q}_p$ and $\Phi_p = \Phi_{\mathfrak{p}}\otimes\Phi_{\overline{\mathfrak{p}}}$, where $\Phi_v = \phi_{\mu_v^{-1}}\otimes\widehat{\phi}_{\nu_v^{-1}}$ with $v=\mathfrak{p}$ or $\overline{\mathfrak{p}}$. From (4.6)

$$Z_{\mathcal{D}}(s, B_{\rho(t_n)W_\pi^{\mathrm{ord}}}) = \int_{N(\mathbf{Q}_p)\backslash\mathrm{PGL}_2(\mathbf{Q}_p)} f_{\mathcal{D},s,\mathfrak{p}}(g)\omega(\det g)^{-1}W_{\overline{\mathfrak{p}}}\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix}g\right)W_\pi^{\mathrm{ord}}(gt_n)\,\mathrm{d}^\tau g.$$

Put $\mathbf{u}(x)=\begin{pmatrix}0&-1\\-1&-x\end{pmatrix}$. Using the integration formula

$$\int_{\mathrm{PGL}_2(\mathbf{Q}_p)} h(g)\,\mathrm{d}^\tau g = \frac{\zeta_p(2)}{\zeta_p(1)}\int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p^\times}\int_{\mathbf{Q}_p} h(\mathbf{n}(y)\mathbf{t}(a)J_1\mathbf{n}(x))|a|^{-1}\,\mathrm{d}y\mathrm{d}^\times a\mathrm{d}x$$

for an integrable function h on $\mathrm{PGL}_2(\mathbf{Q}_p)$, we see that $Z_{\mathcal{D}}(s, B_{\rho(t_n)W_\pi^{\mathrm{ord}}})$ equals

$$\frac{\zeta_p(2)}{\zeta_p(1)}\int_{\mathbf{Q}_p^\times}\int_{\mathbf{Q}_p}(\omega^{-1}\mu_{\mathfrak{p}})(a)\,|a|^{s-\frac{1}{2}}\,f_{\mathcal{D},s,\mathfrak{p}}(J_1\mathbf{n}(x))W_{\overline{\mathfrak{p}}}(\mathbf{t}(a)\mathbf{u}(x)) \\ \times W_\pi^{\mathrm{ord}}\left(\begin{pmatrix}ap^n&0\\0&p^{-n}\end{pmatrix}\begin{pmatrix}1&0\\-p^{2n}x&1\end{pmatrix}\right)\mathrm{d}x\mathrm{d}^\times a.$$

Since $f_{\mathcal{D},s,\mathfrak{p}}(J_1\mathbf{n}(x)) = \widehat{\phi}_{\nu_{\mathfrak{p}}^{-1}}(x)$ by (3.4), if $n\gg 0$, then $Z_{\mathcal{D}}(s, B_{\rho(t_n)W_\pi^{\mathrm{ord}}})$ equals

$$\frac{\zeta_p(2)}{\zeta_p(1)\omega(p^n)}\int_{\mathbf{Q}_p^\times}\int_{\mathbf{Q}_p}(\omega^{-1}\mu_{\mathfrak{p}})(a)\,|a|^{s-\frac{1}{2}}\,\widehat{\phi}_{\nu_{\mathfrak{p}}^{-1}}(x)W_{\overline{\mathfrak{p}}}(\mathbf{t}(a)\mathbf{u}(x))W_\pi^{\mathrm{ord}}(\mathbf{t}(ap^{2n}))\,\mathrm{d}x\mathrm{d}^\times a.$$

Since the function $a \mapsto \widehat{\phi}_{\nu_{\mathfrak{p}}^{-1}}(x) W_{\overline{\mathfrak{p}}}(\mathbf{t}(a)\mathbf{u}(x))$ has a bounded support uniformly with respect to x, if $n \gg 0$, then the integral is equal to

$$\begin{aligned}&\varrho(p^{2n})\,|p^n| \int_{\mathbf{Q}_p^\times}\int_{\mathbf{Q}_p} (\omega^{-1}\mu_{\mathfrak{p}})(a)\,|a|^s\,\widehat{\phi}_{\nu_{\mathfrak{p}}^{-1}}(x) W_{\overline{\mathfrak{p}}}(\mathbf{t}(a)\mathbf{u}(x))\varrho(a)\mathbb{I}_{p^{-2n}\mathbf{Z}_p}(a)\,\mathrm{d}x\mathrm{d}^\times a\\ =&\varrho(p^{2n})\,|p^n| \int_{\mathbf{Q}_p^\times}\int_{\mathbf{Q}_p} (\upsilon^{-1}\mu_{\mathfrak{p}})(a)\,|a|^s\,\widehat{\phi}_{\nu_{\mathfrak{p}}^{-1}}(x) W_{\overline{\mathfrak{p}}}(\mathbf{t}(a)\mathbf{u}(x))\,\mathrm{d}x\mathrm{d}^\times a.\end{aligned}$$

Put $\Pi_{\overline{\mathfrak{p}}} = \mu_{\overline{\mathfrak{p}}}\boldsymbol{\alpha}^s \boxplus \nu_{\overline{\mathfrak{p}}}\boldsymbol{\alpha}^{-s}$. We use the local functional equation (2.10) for GL_2 to see that the last integral equals the ratio of

$$\int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p^\times} (\mu_{\mathfrak{p}}^{-1}\upsilon)(a)\,|a|^{-s}\,\widehat{\phi}_{\nu_{\mathfrak{p}}^{-1}}(x) W_{\overline{\mathfrak{p}}}(\mathbf{t}(a)J_1^{-1}\mathbf{u}(x))\mu_{\overline{\mathfrak{p}}}\nu_{\overline{\mathfrak{p}}}(a^{-1})\,\mathrm{d}^\times a\mathrm{d}x$$

divided by

$$\gamma\left(s+\frac{1}{2}, \mu_{\mathfrak{p}}\upsilon^{-1}\otimes\Pi_{\overline{\mathfrak{p}}}\right) = \gamma\left(2s+\frac{1}{2}, \varrho\nu_+^{-1}\right)\gamma\left(\frac{1}{2}, \varrho\chi_{\overline{\mathfrak{p}}}^{-1}\right).$$

Since

$$\omega = \varrho\upsilon = \mu_{\mathfrak{p}}\nu_{\mathfrak{p}}\mu_{\overline{\mathfrak{p}}}\nu_{\overline{\mathfrak{p}}}, \quad \mathbf{t}(a)J_1^{-1}\mathbf{u}(x) = \mathbf{n}(-ax)\mathbf{t}(-a), \quad W_{\overline{\mathfrak{p}}}(\mathbf{t}(a)) = \mathbb{I}_{\mathbf{Z}_p^\times}(a)$$

by Lemma 3.7, this integral equals

$$\begin{aligned}&\int_{\mathbf{Q}_p^\times} (\varrho^{-1}\nu_{\mathfrak{p}})(a)\,|a|^{-s}\,W_{\overline{\mathfrak{p}}}(\mathbf{t}(-a))\int_{\mathbf{Q}_p}\widehat{\phi}_{\nu_{\mathfrak{p}}^{-1}}(x)\boldsymbol{\psi}(-ax)\,\mathrm{d}x\mathrm{d}^\times a\\ =&\int_{\mathbf{Q}_p^\times}\varrho(a)^{-1}\,|a|^{-s}\,\mathbb{I}_{\mathbf{Z}_p^\times}(a)W_{\overline{\mathfrak{p}}}(\mathbf{t}(-a))\mathrm{d}^\times a = 1.\end{aligned}$$

On the other hand, we see by (2.8) that

$$\begin{aligned}B_{W_\pi^{\mathrm{ord}}}(\varsigma_p^{(n)}) &= \int_{\mathbf{Q}_p^\times} W_\pi^{\mathrm{ord}}\left(\mathbf{t}(a)\begin{pmatrix}p^n & -1\\ 0 & 1\end{pmatrix}\right)\chi_{\overline{\mathfrak{p}}}(a)^{-1}\,\mathrm{d}^\times a\\ &= \int_{\mathbf{Q}_p^\times}\varrho(ap^n)|ap^n|^{1/2}\boldsymbol{\psi}(-a)\mathbb{I}_{\mathbf{Z}_p}(ap^n)\chi_{\overline{\mathfrak{p}}}(a)^{-1}\,\mathrm{d}^\times a\\ &=\varrho(p^n)\,|p^n|^{1/2}\,\frac{\mathrm{vol}(p^{-n}\mathbf{Z}_p, \mathrm{d}a)}{\gamma\left(\frac{1}{2}, \varrho\chi_{\overline{\mathfrak{p}}}^{-1}\right)}\,\mathrm{vol}(1+p^n\mathbf{Z}_p, \mathrm{d}^\times a) = \varrho(p^n)\,|p^n|^{1/2}\,\frac{\zeta_p(1)}{\gamma\left(\frac{1}{2}, \varrho\chi_{\overline{\mathfrak{p}}}^{-1}\right)}.\end{aligned}$$

Now we consider the case where p is inert in F and π is a principal series. Using the decomposition $\mathrm{GL}_2(\mathbf{Q}_p) = \Psi(F^\times)\cdot B(\mathbf{Q}_p)$, we have

$$Z_{\mathcal{D}}(s, B_{\rho(t_n)W_\pi^{\mathrm{ord}}}) = \int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p^\times} f_{\mathcal{D},s,p}(\eta\mathbf{t}(a)\mathbf{n}(x))\omega(a)^{-1}B_{W_\pi^{\mathrm{ord}}}(\mathbf{t}(a)\mathbf{n}(x)t_n)\,|a|\,\mathrm{d}^\times a\mathrm{d}x.$$

We proceed to compute

$$f_{\mathcal{D},s,p}(\eta\mathbf{t}(a)\mathbf{n}(x)) = (\mu\nu)(\delta^{-1})f_{\mathcal{D},s,p}\left(\begin{pmatrix}1 & -\theta\\ -1 & \overline{\theta}\end{pmatrix}\begin{pmatrix}a & ax\\ 0 & 1\end{pmatrix}\right)$$
$$= \nu(\delta^{-1})\mu(a)\,|a|_F^{s+\frac{1}{2}}\int_{F^\times}\Phi_{\mathcal{D},p}(-ta, t(\overline{\theta}-xa))(\mu\nu^{-1})(t)\,|t|_F^{2s+1}\,\mathrm{d}^\times t$$
$$= \nu(\delta^{-1})\nu(a)\,|a|_F^{-s-\frac{1}{2}}\int_{F^\times}\Phi_{\mathcal{D},p}(-t, a^{-1}t\overline{\theta}-xt)(\mu\nu^{-1})(t)\,|t|_F^{2s+1}\,\mathrm{d}^\times t.$$

Since $\Phi_{\mathcal{D},p} = \phi_{\mu^{-1}}\otimes\widehat{\phi}_{\nu^{-1}}$, we find that

$$f_{\mathcal{D},s,p}(\eta\mathbf{t}(a)\mathbf{n}(x)) = \mu(-1)\nu(a\delta^{-1})\,|a|_F^{-s-\frac{1}{2}}\,\widehat{\phi}_{\nu^{-1}}(a^{-1}\overline{\theta}-x).$$

In particular, the function $x\mapsto\widehat{\phi}_{\nu^{-1}}(a^{-1}\overline{\theta}-x)$ has a bounded support with respect to a. Hence for $n\gg_\nu 0$

$$\mu(-1)\nu(\delta)Z_{\mathcal{D}}(s, B_{\rho(t_n)W_\pi^{\mathrm{ord}}}) = \int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p^\times}\nu(a)\,|a|^{-2s}\,\widehat{\phi}_{\nu^{-1}}(a^{-1}\overline{\theta}-x)\omega(a)^{-1}B_{W_\pi^{\mathrm{ord}}}(\mathbf{t}(a)t_n)\,\mathrm{d}^\times a\mathrm{d}x$$
$$= \int_{\mathbf{Q}_p^\times}\nu(a)^{-1}\,|a|^{2s}\,\varPhi_2(a)\omega(a)B_{W_\pi^{\mathrm{ord}}}(\mathbf{t}(a^{-1})t_n)\,\mathrm{d}^\times a,$$

where $\varPhi_2(a)\in\mathcal{S}(\mathbf{Q}_p)$ is defined by

$$\varPhi_2(a) := \int_{\mathbf{Q}_p}\widehat{\phi}_{\nu^{-1}}(x+a\overline{\theta})\,\mathrm{d}x.$$

Observe that if $\varPhi_2(a)\neq 0$, then

$$\omega(a)B_{W_\pi^{\mathrm{ord}}}(\mathbf{t}(a^{-1})t_n) = \omega(p^n)^{-1}B_{W_\pi^{\mathrm{ord}}}(J_1^{-1}\mathbf{t}(ap^{2n})) = \omega(p^n)^{-1}\varrho(ap^{2n})\left|ap^{2n}\right|^{\frac{1}{2}}\mathcal{Z}(\widetilde{W}),$$

where

$$\mathcal{Z}(\widetilde{W}) = \int_{\mathbf{Q}_p^\times}\varrho(t)\,|t|^{\frac{1}{2}}\,\mathbb{I}_{\mathbf{Z}_p}(atp^{2n})\widetilde{W}(\mathbf{t}(t)J_1)\,\mathrm{d}^\times t = \int_{\mathbf{Q}_p^\times}\varrho(t)\,|t|^{\frac{1}{2}}\,\widetilde{W}(\mathbf{t}(t)J_1)\,\mathrm{d}^\times t$$

for $n\gg_{\nu,\widetilde{W}} 0$. Thus we find that

$$\mu(-1)\nu(\delta)Z_{\mathcal{D}}(s, B_{\rho(t_n)W_\pi^{\mathrm{ord}}}) = (\omega^{-1}\varrho^2)(p^n)\,|p^n|\,\mathcal{Z}(\widetilde{W})\int_{\mathbf{Q}_p^\times}(\nu^{-1}\varrho)(a)\,|a|^{2s+\frac{1}{2}}\,\varPhi_2(a)\,\mathrm{d}^\times a.$$

The last integral equals

$$\gamma\left(2s+\frac{1}{2}, \varrho\nu_+^{-1}\right)^{-1}\int_{\mathbf{Q}_p^\times}(\nu\varrho^{-1})(a)\,|a|^{\frac{1}{2}-2s}\,\widehat{\varPhi}_2(a)\,\mathrm{d}^\times a$$

by (2.8), where

$$\widehat{\varPhi}_2(a) = \int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p}\widehat{\phi}_{\nu^{-1}}(x+y\overline{\theta})\boldsymbol{\psi}(ay)\,\mathrm{d}x\mathrm{d}y = \int_F\widehat{\phi}_{\nu^{-1}}(z)\boldsymbol{\psi}_F(a\delta^{-1}z)\,\mathrm{d}z = \phi_{\nu^{-1}}(-a\delta^{-1}).$$

We conclude that

$$Z_{\mathcal{D}}(s, B_{\rho(t_n)W_\pi^{\mathrm{ord}}}) = \gamma\left(2s+\frac{1}{2}, \varrho\nu_+^{-1}\right)^{-1}(\omega^{-1}\varrho^2)(p^n)\,|p^n|\,\omega(-1)\cdot\mathcal{Z}(\widetilde{W}).$$

On the other hand, for $n \gg 0$,

$$B_{W_\pi^{\rm ord}}(\varsigma_p^{(n)}) = \varrho(p^n)\,|p^n|^{\frac{1}{2}}\,\mathcal{Z}(\widetilde{W}).$$

The following lemma will complete our proof. □

Lemma 4.6. $\mathcal{Z}(\widetilde{W}) \neq 0$.

Proof. Let $\xi := \chi^{-1}\upsilon_F$. If ξ is unramified, then so is $\xi|_{\mathbf{Q}_p^\times} = \omega^{-1}\upsilon^2 = \varrho^{-1}\upsilon$, which implies that both π and χ are unramified, so that $\widetilde{W}$ is the spherical Whittaker function, and

$$\mathcal{Z}(\widetilde{W}) = L(1, \pi^\vee \otimes \varrho) \neq 0.$$

Suppose that ξ is a ramified character. Since $\chi\omega_F^{-\frac{1}{2}}$ is assumed to be unramified, we find that $c(\xi|_{\mathbf{Q}_p^\times}) = c(\varrho^{-1}\upsilon) = c(\omega) > 0$. Let $\widetilde{f} \in \upsilon^{-1} \boxplus \varrho^{-1}$ be the unique section such that

$$\widetilde{f}\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \Psi(t)\right) = \upsilon(a)^{-1}\varrho(d)^{-1}\left|\frac{a}{d}\right|^{\frac{1}{2}}\chi^{-1}(t)$$

for $a, d \in \mathbf{Q}_p^\times$, $b \in \mathbf{Q}_p$ and $t \in F^\times$. Then we can choose $\widetilde{W}(g) := W(g, \widetilde{f})$, and $\widetilde{W}(\mathbf{t}(a)J_1)$ equals

$$\int^{\rm st}_{\mathbf{Q}_p} \widetilde{f}\left(\begin{pmatrix} -1 & 0 \\ x & -a \end{pmatrix}\right)\boldsymbol{\psi}(-x)\,\mathrm{d}x = \varrho(a)^{-1}\,|a|^{\frac{1}{2}} \int^{\rm st}_{\mathbf{Q}_p} \widetilde{f}\left(\begin{pmatrix} -1 & 0 \\ x & -1 \end{pmatrix}\right)\boldsymbol{\psi}(-ax)\,\mathrm{d}x.$$

Since

$$\begin{pmatrix} -1 & 0 \\ x & -1 \end{pmatrix} = \begin{pmatrix} \mathrm{N}(x\theta-1)^{-1} & * \\ 0 & 1 \end{pmatrix}\Psi(x\theta-1),$$

we find that

$$\widetilde{W}(\mathbf{t}(a)J_1) = \varrho(a)^{-1}\,|a|^{\frac{1}{2}} \int^{\rm st}_{\mathbf{Q}_p} \frac{\xi(x\theta-1)}{|x\theta-1|_F^{1/2}}\boldsymbol{\psi}(-ax)\,\mathrm{d}x.$$

Put $\Phi_N(x) := \frac{\xi(x\theta-1)}{|x\theta-1|_F^{1/2}}\mathbb{I}_{p^{-N}\mathbf{Z}_p}(x)$. We have seen that

$$\varrho(a)\,|a|^{-\frac{1}{2}}\,\widetilde{W}(\mathbf{t}(a)J_1) = \lim_{N\to\infty}\widehat{\Phi}_N(a).$$

Take an integer $B > c(\omega)$. Then we have

$$\widehat{\Phi}_N(a) = \widehat{\Phi}_B(a) + \sum_{j=B+1}^{N} I_j$$

for $N > B$, where

$$I_j = \int_{p^{-j}\mathbf{Z}_p^\times} \frac{\xi(x\theta-1)}{|x\theta-1|_F^{1/2}}\boldsymbol{\psi}(-ax)\,\mathrm{d}x.$$

Note that $I_j = 0$ unless $j = \mathrm{ord}_p(a) + c(\omega)$. Recall an additive character $\boldsymbol{\psi}^a$ defined by $\boldsymbol{\psi}^a(x) = \boldsymbol{\psi}(ax)$. Then

$$I_{ord_p(a)+c(\omega)} = \xi(\theta)\varepsilon\left(-\frac{1}{2}, \varrho\upsilon^{-1}, \boldsymbol{\psi}^{-a}\right) = \xi(\theta)(\varrho\upsilon^{-1})(-a)|a|^{-1}\varepsilon\left(-\frac{1}{2}, \varrho\upsilon^{-1}\right)$$

by (2.9). Since $\varrho\upsilon^{-1}$ is ramified, we get

$$\begin{aligned}\mathcal{Z}(\widetilde{W}) =& \int_{\mathbf{Q}_p}\left(\widehat{\Phi}_B(a) + \xi(\theta)(\varrho\upsilon^{-1})(-a)|a|^{-1}\varepsilon\left(-\frac{1}{2}, \varrho\upsilon^{-1}\right)\mathbb{I}_{p^{B+1-c(\omega)}\mathbf{Z}_p}(a)\right)|a|\, \mathrm{d}^\times a \\ =& \Phi_B(0)\zeta_p(1) \neq 0.\end{aligned}$$

□

4.7. **The explicit pull-back formula.** Now we are ready to give the explicit formula of $Z_{\mathcal{D}}(s, \rho(\mathcal{J}_\infty t_n)\varphi_f)$. The notation is as in §4.3. Let τ_F be the quadratic Dirichlet character associated to the extension $F/\mathbf{Q}$.

Theorem 4.7. *Let λ be a Hecke character of $\mathbf{A}^\times$ of p-power conductor and ϕ be a finite order Hecke character of $\mathbf{A}_F^\times$ with $\phi|_{\mathbf{A}^\times} = 1$ and the conductor $C\mathcal{O}_F$. Put $\chi = \omega_F^{\frac{1}{2}}\phi$ and*

$$\mathcal{D} = (\omega_F^{\frac{1}{2}}\lambda_F, \phi^{-1}\lambda_F^{-1}, k, \mathfrak{N}, \mathfrak{c}).$$

For $n \gg 0$, we have

$$\frac{Z_{\mathcal{D}}(s, \rho(\mathcal{J}_\infty t_n)\varphi_f)}{B^\chi_{\rho(\mathcal{J}_\infty)\breve{\varphi}_f}(\varsigma^{(Cp^n)})} = L^{\{p\}}\left(2s + \frac{1}{2}, \pi \otimes \nu_+^{-1}\right)\gamma\left(2s + \frac{1}{2}, \varrho_p\nu_{+,p}^{-1}\right)^{-1}$$
$$\times (\omega_p^{-1}\varrho_p)(p^n)\,|p^n|^{\frac{1}{2}}_{\mathbf{Q}_p}\frac{\zeta_p(2)}{\zeta_{F_p}(1)} \cdot \frac{L(1, \tau_F)\mathfrak{f}_\infty\mathfrak{f}_{\mathrm{ram}}\mathfrak{f}_C}{\zeta_{\mathbf{Q}}(2)[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(NC)]},$$

where $\mathfrak{f}_\infty$, $\mathfrak{f}_{\mathrm{ram}}$ and $\mathfrak{f}_C$ are local fudge factors given by

$$\mathfrak{f}_\infty := 4(-4\sqrt{-1})^{-k}(\lambda_\infty\phi_{\sigma_1})(-1), \quad \mathfrak{f}_{\mathrm{ram}} := \prod_{q|\Delta_F}\omega_q^{\frac{1}{2}}\lambda_q(\Delta_F^{-1})\,|\Delta_F|^{-s-\frac{1}{2}}_{\mathbf{Q}_q},$$

$$\mathfrak{f}_C := \prod_{q|C}\omega_q^{\frac{1}{2}}(C^{-1})\frac{\varepsilon(0, \phi_{\overline{\mathfrak{q}}}^{-1})}{\zeta_q(1)}.$$

Proof. There exists a nonzero constant c such that $B^\chi_\varphi(g) = c\prod_v B_{W_v}(g_v)$ for $\varphi \in \pi$ with $W_\varphi(g) = \prod_v W_v(g_v)$ by the uniqueness and the existence of the Waldspurger models. Put

$$\varphi^\star = \rho(\mathcal{J}_\infty t_n)\breve{\varphi}_f, \qquad W_\infty^\star = \rho(\mathcal{J}_\infty)W_{\pi_\infty}, \qquad W_p^\star = \rho(t_n)W^{\mathrm{ord}}_{\pi_p}.$$

The Whittaker function of $\varphi^\star$ is given by

$$W_{\varphi^\star}(g) = W_\infty^\star(g_\infty) \cdot W_p^\star(g_p) \cdot \prod_{q|C}\breve{W}_{\pi_q}(g_q)\prod_{\ell\nmid pC}W_{\pi_\ell}(g_\ell).$$

It follows that

$$B^{\chi}_{\rho(\mathcal{J}_\infty)\breve{\varphi}_f}(\varsigma^{(Cp^n)}) = cB_{W^\star_\infty}(\varsigma_\infty)\cdot B_{W^{\rm ord}_{\pi_p}}(\varsigma_p^{(n)})\cdot\prod_{q|C}B_{\breve{W}_{\pi_q}}(\varsigma_q\varsigma_q^{(C)})\prod_{\ell\nmid pC}W_{\pi_\ell}(\varsigma_\ell).$$

On the other hand, Theorem 4.1 gives

$$Z_{\mathcal{D}}(s,\rho(\mathcal{J}_\infty t_n)\varphi_f) = c\frac{L(1,\tau_F)}{\zeta_{\mathbf{Q}}(2)}\cdot Z_{\mathcal{D}}(s,B_{W^\star_\infty})\cdot Z_{\mathcal{D}}(s,B_{W^\star_p})\prod_{q\neq p}Z_{\mathcal{D}}(s,B_{W_{\pi_q}}).$$

Theorem 4.7 now follows from Propositions 4.3, 4.4 and 4.5 with $\mu=\omega_F^{\frac{1}{2}}\lambda_F$, $\nu=\phi^{-1}\lambda_F^{-1}$ and $\chi=\omega_F^{\frac{1}{2}}\phi$. □

5. The construction of p-adic twisted triple product L-functions

5.1. **Notation.** Define the p-adic cyclotomic character by

$$\boldsymbol{\varepsilon}_{\rm cyc}:\mathbf{Q}^\times\backslash\mathbf{A}^\times\to\mathbf{Z}_p^\times,\quad \boldsymbol{\varepsilon}_{\rm cyc}(a)=|a|_{\mathbf{A}}\,a_\infty^{-1}a_p.$$

Let $\boldsymbol{\omega}:\mathbf{Q}^\times\backslash\mathbf{A}^\times\to\mu_{p-1}(\mathbf{C}_p)$ be the Teichmüller character. Fix isomorphisms $\iota_\infty:\overline{\mathbf{Q}}\hookrightarrow\mathbf{C}$ and $\iota_p:\overline{\mathbf{Q}}\hookrightarrow\mathbf{C}_p$ once and for all. For every arithmetit point $Q\in\mathfrak{X}^+_{\mathbf{I}}$, we shall view the finite part ϵ_Q as a Hecke character of $\mathbf{A}^\times$ via $\epsilon_Q(a):=\iota_\infty\iota_p^{-1}(\epsilon_Q(\boldsymbol{\varepsilon}_{\rm cyc}(a)\boldsymbol{\omega}^{-1}(a)))$. The set of embeddings $\Sigma_{\mathbf{R}}=\{\sigma_1,\sigma_2\}$ from F to $\mathbf{R}$ is identified with $\mathrm{Gal}(F/\mathbf{Q})$ via ι_∞.

Let $\mathcal{O}=\mathcal{O}_L$ for some finite extension L of $\mathbf{Q}_p$ containing $\iota_p(F)$. Let $\Lambda=\mathcal{O}[\![1+p\mathbf{Z}_p]\!]$ and write $[\cdot]:1+p\mathbf{Z}_p\to\mathcal{O}[\![1+p\mathbf{Z}_p]\!]^\times$ for the inclusion of group-like elements. Let $\mathbf{u}=1+p$. For a variable X, let $\langle\cdot\rangle_X:\mathbf{Z}_p^\times\to\mathbf{Z}_p[\![X]\!]^\times$ be the character defined by

$$\langle a\rangle_X:=(1+X)^{\frac{\log_p a}{\log_p\mathbf{u}}}. \tag{5.1}$$

Write $\mathrm{N}=\mathrm{N}_{F/\mathbf{Q}}:F\to\mathbf{Q}$ for the norm map. If $\mathfrak{a}$ is a fractional ideal of F coprime to p, put $\langle\mathfrak{a}\rangle_X=\langle\mathrm{N}(\mathfrak{a})\rangle_X$. If $\mathbf{I}$ is a finite extension of Λ, recall that a point $Q\in\operatorname{Spec}\mathbf{I}(\mathbf{C}_p)$ is called a locally algebraic point of weight k and finite part ϵ if the map $Q|_\Lambda:1+p\mathbf{Z}_p\xrightarrow{[\cdot]}\Lambda^\times\xrightarrow{Q}\overline{\mathbf{Q}}_p^\times$ is given by $Q(x)=x^k\epsilon(x)$ for some integer $k\geq 1$ and a finite order character $\epsilon:1+p\mathbf{Z}_p\to\mu_{p^\infty}(\overline{\mathbf{Q}}_p)$. For a locally algebraic Q, denote by k_Q the weight of Q and ϵ_Q the finite part of Q. Let $\mathfrak{X}^+_{\mathbf{I}}$ be the set of locally algebraic points Q in $\operatorname{Spec}\mathbf{I}(\mathbf{C}_p)$ with $k_Q\geq 1$. A locally algebraic point $Q\in\mathfrak{X}^+_{\mathbf{I}}$ is called arithmetic if $k_Q\geq 2$.

If A and B are two complete $\mathcal{O}$-modules, we write $A\widehat{\otimes}B$ for $A\widehat{\otimes}_{\mathcal{O}}B$ for simplicity.

5.2. **Preliminaries on Hida theory for modular forms.** Let $\mathbf{I}$ be a normal domain finite flat over Λ. Let N be a positive integer prime to p and let $\chi:(\mathbf{Z}/Np\mathbf{Z})^\times\to\mathcal{O}^\times$ be a Dirichlet character modulo Np. Denote by $\mathbf{M}(N,\chi,\mathbf{I})$ the space of $\mathbf{I}$-adic modular forms of tame level N and (even) branch character χ, consisting of formal power series $\boldsymbol{f}(q)=$

$\sum_{n\geq 1}\mathbf{a}(n,\boldsymbol{f})q^n \in \mathbf{I}[\![q]\!]$ with the following property: there exists an integer $a_{\boldsymbol{f}}$ such that for every point $Q\in\mathfrak{X}_{\mathbf{I}}^{+}$ with $k_Q\equiv 0\,(\mathrm{mod}\,2)$ and $k_Q\geq a_{\boldsymbol{f}}$, the specialization $\iota_p(\boldsymbol{f}_Q(q))$ is the q-expansion of a cusp form $\boldsymbol{f}_Q\in\mathcal{M}_{k_Q}(Np^e,\chi\boldsymbol{\omega}^{2-k_Q}\epsilon_Q)$. For a positive integer d prime to p, define $V_d:\mathbf{M}(N,\chi,\mathbf{I})\to\mathbf{M}(Nd,\chi,\mathbf{I})$ by $V_d(\sum_n\mathbf{a}(n,\boldsymbol{f})q^n)=d\sum_n\mathbf{a}(n,\boldsymbol{f})q^{dn}$. Let $\mathbf{S}(N,\chi,\mathbf{I})\subset\mathbf{M}(N,\chi,\mathbf{I})$ be the space of $\mathbf{I}$-adic cusp forms, consisting of elements $\boldsymbol{f}\in\mathbf{M}(N,\chi,\mathbf{I})$ such that $\boldsymbol{f}_Q$ is a cusp form for a Zariski dense subset $Q\in\mathfrak{X}_{\mathbf{I}}^{+}$.

The space $\mathbf{M}(N,\chi,\mathbf{I})$ is equipped with the action of the usual Hecke operators T_ℓ for $\ell\nmid Np$ as in [Wil88, page 537] and the operators $\mathbf{U}_\ell$ for $\ell\mid pN$ given by $\mathbf{U}_\ell(\sum_n\mathbf{a}(n,\boldsymbol{f})q^n)=\sum_n\mathbf{a}(n\ell,\boldsymbol{f})q^n$. Recall that Hida's ordinary projector e is defined by

$$e:=\lim_{n\to\infty}\mathbf{U}_p^{n!}.$$

This ordinary projector e is a convergent operator on the space of classical modular forms preserving the cuspidal part as well as on the spaces $\mathbf{M}(N,\chi,\mathbf{I})$ and $\mathbf{S}(N,\chi,\mathbf{I})$(*cf.* [Wil88, page 537 and Proposition 1.2.1]).

For a divisor $M\mid N$, let $\mathbf{T}(N,M)\subset\operatorname{End}\mathbf{S}(N,\chi,\mathbf{I})$ be the $\mathbf{I}$-algebra generated by Hecke operators $\{T_q\}_{q\nmid Np}$ and $\{\mathbf{U}_q\}_{q\mid Mp}$. The space $e\mathbf{S}(N,\chi,\mathbf{I})$ is called the space of ordinary $\mathbf{I}$-adic forms defined over $\mathbf{I}$. A key result in Hida's theory of ordinary $\mathbf{I}$-adic cusp forms is that if $\boldsymbol{f}\in e\mathbf{S}(N,\chi,\mathbf{I})$, then for *every* arithmetic points $Q\in\mathfrak{X}_{\mathbf{I}}^{+}$, we have $\boldsymbol{f}_Q\in e\mathcal{S}_{k_Q}(Np^e,\chi\boldsymbol{\omega}^{2-k_Q}\epsilon_Q)$. We say $\boldsymbol{f}\in e\mathbf{S}(N,\chi,\mathbf{I})$ is a *primitive Hida family* if for every arithmetic points $Q\in\mathfrak{X}_{\mathbf{I}}$, $\boldsymbol{f}_Q$ is a p-stabilized cuspidal newform of tame conductor N.

Let $\mathbf{T}^{\mathrm{ord}}(N)$ be the image of $\mathbf{T}(N,N)$ in $\operatorname{End}(e\mathbf{S}(N,\chi,\mathbf{I}))$. A classical result in Hida theory for modular forms asserts that $\mathbf{T}^{\mathrm{ord}}(N)$ is a free of finite frank over $\mathbf{I}$. Let $\boldsymbol{f}\in e\mathbf{S}(N,\chi,\mathbf{I})$ be a primitive Hida family. Then $\boldsymbol{f}$ induces the $\mathbf{I}$-algebra homomorphism $\lambda_{\boldsymbol{f}}:\mathbf{T}^{\mathrm{ord}}(N)\to\mathbf{I}$ with $\lambda_{\boldsymbol{f}}(T_q)=\mathbf{a}(q,\boldsymbol{f})$ for $q\nmid Np$ and $\lambda_{\boldsymbol{f}}(\mathbf{U}_q)=\mathbf{a}(q,\boldsymbol{f})$ for $q\mid Np$. By the primitiveness of $\boldsymbol{f}$, there exists an unique idempotent $1_{\boldsymbol{f}}$ in $\mathbf{T}^{\mathrm{ord}}(N)\otimes_{\mathbf{I}}\operatorname{Frac}\mathbf{I}$ such that $\lambda_{\boldsymbol{f}}(1_{\boldsymbol{f}})=1$.

Remark 5.1. Recall that the congruence ideal $C(\boldsymbol{f})$ of $\boldsymbol{f}$ is defined by

$$C(\boldsymbol{f}):=\lambda_{\boldsymbol{f}}(\{t\in\mathbf{T}^{\mathrm{ord}}(N)\mid 1_{\boldsymbol{f}}t=t\})\subset\mathbf{I}.$$

By definition, $C(\boldsymbol{f})\cdot 1_{\boldsymbol{f}}\subset\mathbf{T}^{\mathrm{ord}}(N)$ and $C(\boldsymbol{f})$ is the annihilator of the congruence module of $\lambda_{\boldsymbol{f}}$. For each arithmetic point $Q\in\mathfrak{X}_{\mathbf{I}}^{+}$, let $\wp_Q=\ker Q$. By control theorem for the Hecke algebras and the congruence modules (*cf.* [Hid88b, (0.4b), (5.8a)]), we find that $Q(C(\boldsymbol{f}))$ is the congruence ideal for $\lambda_{\boldsymbol{f}_Q}:\mathbf{T}^{ord}(N)/\wp_Q\to\mathbf{I}/\wp_Q$. In particular, this implies $Q(C(\boldsymbol{f}))\neq 0$ and hence $1_{\boldsymbol{f}}$ belongs to the localization $\mathbf{T}^{\mathrm{ord}}(N)_{\wp_Q}$ at $\wp_Q$.

5.3. **A two-variable p-adic family of Eisenstein series.** We shall make the identification

$$\Lambda\widehat{\otimes}\Lambda=\mathcal{O}[\![X,T]\!],\quad X=([\mathbf{u}]-1)\otimes 1,\,T=1\otimes([\mathbf{u}]-1). \tag{5.2}$$

Let (χ_1, χ_2) be a pair of finite order Hecke characters of $\mathbf{A}_F^\times$ of level $p\mathcal{O}_F$ and $pC\mathcal{O}_F$ respectively. We assume that (χ_1, χ_2) satisfies Hypothesis 3.1 and $\chi_1\chi_2$ is totally even. A Hecke character χ of $\mathbf{A}_F^\times$ will be viewed as a ideal class character by

$$\chi(\mathfrak{q}) := \chi(\varpi_{\mathfrak{q}})^{-1}$$

for any prime ideal $\mathfrak{q}$ away from the conductor of χ. Define the $\Lambda\widehat{\otimes}\Lambda$-adic q-expansion by

$$\boldsymbol{E}(\chi_1, \chi_2)(X, T) := \sum_{\beta \in \mathfrak{d}_+^{-1}, (p, (\beta))=1} \mathcal{A}_\beta(\chi_1, \chi_2) q^\beta \in \Lambda\widehat{\otimes}\Lambda[\![q^{\mathfrak{d}_+^{-1}}]\!],$$

where $\mathcal{A}_\beta(\chi_1, \chi_2) \in \Lambda\widehat{\otimes}\Lambda$ is defined by

$$\mathcal{A}_\beta(\chi_1, \chi_2) = \langle(\beta)\rangle_X \langle(\beta)\rangle_T^{-1} \chi_1^{-1}((\beta)) \prod_{\mathfrak{q}\nmid \mathfrak{c}p} \mathcal{P}_{\beta,\mathfrak{q}}(\chi_1\chi_2^{-1}(\mathfrak{q}) \langle\mathfrak{q}\rangle_X^{-1} \langle\mathfrak{q}\rangle_T^2)$$
$$\times \prod_{\mathfrak{q}|(\mathfrak{c},\beta)} \mathcal{Q}_{\chi_1\chi_2^{-1},\mathfrak{q}}(\langle\mathfrak{q}\rangle_X^{-1} \langle\mathfrak{q}\rangle_T^2),$$

where $\mathcal{P}_{\beta,\mathfrak{q}}$ and $\mathcal{Q}_{\chi_1\chi_2^{-1},\mathfrak{q}}$ are polynomials defined in (3.5). If R is an $\mathcal{O}_F$-algebra, the theta operator $\theta_\sigma \in \mathrm{End}(R[\![q^{\mathfrak{d}_+^{-1}}]\!])$ for $\sigma \in \mathrm{Gal}(F/\mathbf{Q})$ is defined by

$$\theta_\sigma(\sum_\beta a_\beta q^\beta) = \sum_\beta \sigma(\beta) a_\beta q^\beta. \tag{5.3}$$

For $Q \in \mathfrak{X}_\Lambda$, let ξ_Q be the finite order Hecke character of $\mathbf{A}_F^\times$ given by

$$\xi_Q := \epsilon_Q \boldsymbol{\omega}^{-k_Q} \circ \mathrm{N}.$$

Proposition 5.2. *For every $(Q, P) \in \mathfrak{X}_\Lambda^+ \times \mathfrak{X}_\Lambda^+$ with $k_Q \le k_P$, we have the interpolation*

$$\boldsymbol{E}(\chi_1, \chi_2)(Q, P) = \begin{cases} \theta^{k_Q - k_P} E^+_{2k_P - k_Q}(\chi_1\xi_Q^{-1}\xi_P,\, \chi_2\xi_P^{-1}) & \text{if } 2k_P > k_Q \\ \theta^{k_Q-1} E^-_{k_Q - 2k_P + 2}(\chi_1\xi_Q^{-1}\xi_P,\, \chi_2\xi_P^{-1}) & \text{if } 2k_P \le k_Q, \end{cases}$$

where $\theta = \theta_{\sigma_1}\theta_{\sigma_2}$ is the theta operator $\theta(\sum_\beta a_\beta q^\beta) = \sum_\beta \mathrm{N}(\beta) a_\beta q^\beta$.

Proof. Let $\mu = \chi_1\xi_Q^{-1}\xi_P$ and $\nu = \chi_2\xi_P^{-1}$. Put $\mathbf{k} = 2k_P - k_Q$. For an integer n prime to p, we have

$$\mathcal{A}_\beta(\chi_1, \chi_2)(Q, P) = \mathrm{N}(\beta)^{k_Q - k_P} \mu^{-1}((\beta)) \prod_{\mathfrak{q}\nmid \mathfrak{c}p} \mathcal{P}_{\beta,\mathfrak{q}}(\mu\nu^{-1}(\mathfrak{q}) q_{\mathfrak{q}}^{\mathbf{k}})$$
$$\times \prod_{\mathfrak{q}|(\mathfrak{c},\beta)} \mathcal{Q}_{\chi_1^{-1}\chi_2,\mathfrak{q}}(\chi_1^{-1}\chi_2\mu\nu^{-1}(\mathfrak{q}) q_{\mathfrak{q}}^{\mathbf{k}}).$$

Since $\chi_1^{-1}\chi_2\mu\nu^{-1}$ is unramified outside p, one verifies that

$$\mathcal{Q}_{\chi_1^{-1}\chi_2,\mathfrak{q}}(\chi_1^{-1}\chi_2\mu\nu^{-1}(\mathfrak{q}) X) = \mathcal{Q}_{\mu^{-1}\nu,\mathfrak{q}}(X).$$

By Corollary 3.8, we find that

$$\mathcal{A}_\beta(\chi_1,\chi_2)(Q,P)=\begin{cases}\mathrm{N}(\beta)^{k_Q-k_P}\cdot\sigma_\beta^+(\mu,\nu,\mathbf{k}) & \text{if } \mathbf{k}>0,\\ \mathrm{N}(\beta)^{1-\mathbf{k}+k_P-1}\cdot\sigma_\beta^-(\mu,\nu,2-\mathbf{k}) & \text{if } \mathbf{k}\le 0.\end{cases}$$

The proposition follows immediately. □

5.4. **The construction of the twisted triple p-adic L-function.** For any $\mathcal{O}_F$-algebra R, define the *diagonal restriction map* by

$$\mathrm{res}_{F/\mathbf{Q}}:R[\![q^{\mathfrak{d}_+^{-1}}]\!]\to R[\![q]\!],\quad \mathrm{res}_{F/\mathbf{Q}}\Big(\sum_{\beta\in\mathfrak{d}_+^{-1}}a_\beta q^\beta\Big)=\sum_{n>0}\Big(\sum_{\substack{\beta\in\mathfrak{d}_+^{-1}\\ \mathrm{Tr}_{F/\mathbf{Q}}(\beta)=n}}a_\beta\Big)q^\beta.$$

For an even integer a and a finite order Hecke character

$$\phi:F^\times\backslash\mathbf{A}_F^\times/\widehat{\mathcal{O}}_C^\times\to\mathcal{O}^\times \text{ such that } \phi_\sigma(-1)=(-1)^{\frac{j}{2}} \text{ for } \sigma\in\Sigma_{\mathbf{R}},\tag{5.4}$$

we define the two-variable q-expansion $\boldsymbol{E}_\phi^{[a]}(X,T)\in\Lambda\widehat{\otimes}\Lambda[\![q^{\mathfrak{d}_+^{-1}}]\!]$ by

$$\boldsymbol{E}_\phi^{[a]}(X,T)=\boldsymbol{E}(\boldsymbol{\omega}_F^{\frac{a-j}{2}},\boldsymbol{\omega}_F^{-\frac{a}{2}}\phi)(1+X)^{1/2}-1,(1+T)^{1/2}-1)$$

and define its diagonal restriction $\boldsymbol{G}^{[a]}(X,T)\in\Lambda\widehat{\otimes}\Lambda[\![q]\!]$ by

$$\boldsymbol{G}_\phi^{[a]}(X,T):=\mathrm{res}_{F/\mathbf{Q}}\big(\boldsymbol{E}_\phi^{[a]}(X,T)\big).$$

We regard Λ as a subring of $\Lambda\widehat{\otimes}\Lambda$ via $x\mapsto x\otimes 1$. Let

$$\mathfrak{X}_{\mathbf{I}}^{++}:=\big\{Q\in\mathfrak{X}_{\mathbf{I}}^+\mid k_Q\equiv 0\ (\mathrm{mod}\ 2)\big\}\subset\mathfrak{X}_{\mathbf{I}}^+.$$

Lemma 5.3. *The q-expansion $\boldsymbol{G}_\phi^{[a]}$ belongs to $\mathbf{M}(NC,\boldsymbol{\omega}^{j-2},\Lambda)\widehat{\otimes}_\Lambda(\Lambda\otimes\Lambda)$.*

Proof. Let $Z=(1+T)(1+X)^{-1}-1$ and write $\boldsymbol{G}(X,Z)=\boldsymbol{G}^{[a]}(X,(1+X)(1+Z)-1)$. If $\zeta\in\mu_{p^\infty}(\mathbf{C})$ is a p-power root of unity, let $\boldsymbol{\alpha}_\zeta:\mathbf{A}_F^\times\to\mathbf{C}^\times$ be the Hecke character $\boldsymbol{\alpha}_\zeta(a)=\langle\mathrm{N}(a)\rangle_X|_{X=\zeta-1}$. By Proposition 5.2, for any point $Q\in\mathfrak{X}_{\mathbf{I}}^{++}$, we have

$$\boldsymbol{G}(Q,\zeta-1)=E_{k_Q/2}^+(\mu_{Q,\zeta},\nu_{Q,\zeta})|_{\mathfrak{H}}\in\mathcal{M}_{k_Q}(CN,\boldsymbol{\omega}^{j-2}\xi_Q),$$

where $\mu_{Q,\zeta}=\boldsymbol{\omega}_F^{\frac{a-j}{2}}\boldsymbol{\alpha}_\zeta$ and $\nu_{Q,\zeta}=\boldsymbol{\omega}_F^{-\frac{a}{2}}\boldsymbol{\alpha}_\zeta^{-1}\xi_Q^{-\frac{1}{2}}$. This shows that $\boldsymbol{G}(X,\zeta-1)\in\mathbf{M}(NC,\boldsymbol{\omega}^{j-2},\Lambda)\otimes_{\mathcal{O}}\mathcal{O}[\zeta]$ for every $\zeta\in\mu_{p^\infty}(\mathbf{C}_p)$. By [Hid93, Lemma 1 in page 328], we see that $\boldsymbol{G}\in\mathbf{M}(NC,\boldsymbol{\omega}^{j-2},\Lambda)\widehat{\otimes}\mathcal{O}[\![Z]\!]=\mathbf{M}(NC,\boldsymbol{\omega}^{j-2},\Lambda)\widehat{\otimes}_\Lambda(\Lambda\widehat{\otimes}\Lambda)$. □

In view of the above lemma, we can apply the ordinary projector $e\otimes 1$ to $\boldsymbol{G}^{[a]}$ and obtain an Λ-adic ordinary modular form $e\boldsymbol{G}_\phi^{[a]}:=(e\otimes 1)\boldsymbol{G}_\phi^{[a]}$ with coefficients in $\Lambda\widehat{\otimes}\Lambda$.

Lemma 5.4. *We have $e\boldsymbol{G}_\phi^{[a]}\in e\mathbf{S}(N,\boldsymbol{\omega}^{j-2},\Lambda)\widehat{\otimes}(\Lambda\widehat{\otimes}\Lambda)$.*

Proof. Notation is as the above Lemma 5.3. For $(Q,\zeta)\in\mathfrak{X}_{\mathbf{I}}^{++}\times\mu_{p^\infty}(\mathbf{C})$ as above, let $\mu=\mu_{Q,\zeta}$ and $\nu=\nu_{Q,\zeta}$. Then $\boldsymbol{G}(Q,\zeta-1)$ is the diagonal restriction of the holomorphic Eisenstein series $E^+_{k_Q/2}(\mu,\nu)$. The adelic lift of $E^+_{k_Q/2}(\mu,\nu)$ is given by $E_{\mathbf{A}}(g,f_{\mathcal{D},s})|_{s=\frac{k_Q/2-1}{2}}$ with $\mathcal{D}=(\mu,\nu,k_Q/2,\mathfrak{N},\mathfrak{c})$. By (3.2), the constant term function of $E_{\mathbf{A}}(g,f_{\mathcal{D},s})$ is given by $f_{\mu,\nu,\Phi_{\mathcal{D}},s}+f_{\mu,\nu,\widehat{\Phi}_{\mathcal{D}},s}$, and by (3.6) its values at $g\in\mathrm{GL}_2(\mathbf{A}_F)$ all vanish whenever g_p is upper triangular. The lemma follows from [HY20, Lemma 6.7]. □

Definition 5.5. Let

$$\boldsymbol{f}=\sum_{n>0}\mathbf{a}(n,\boldsymbol{f})q^n\in e\mathbf{S}(N,\boldsymbol{\omega}^{j-2},\mathbf{I})$$

be a primitive Hida family. The p-adic twisted triple product L-series $\mathcal{L}_{\boldsymbol{E}_\phi^{[a]},\boldsymbol{f}}$ is defined by

$$\mathcal{L}_{\boldsymbol{E}_\phi^{[a]},\boldsymbol{f}}:=\text{ the first Fourier coefficient of }1_{\boldsymbol{f}}(e\boldsymbol{G}_\phi^{[a]})\in(\mathbf{I}\widehat{\otimes}\Lambda)\otimes_{\mathbf{I}}\operatorname{Frac}\mathbf{I}.$$

By Remark 5.1, $\mathcal{L}_{\boldsymbol{E}_\phi^{[a]},\boldsymbol{f}}(Q,P)$ is finite at every arithmetic point $Q\in\mathfrak{X}_{\mathbf{I}}^+$ and $P\in\operatorname{Spec}\Lambda(\mathbf{C}_p)$.

5.5. **The interpolation formula of the p-adic twisted triple product L-series.** Define the weight space of critical points by

$$\mathfrak{X}^{\rm crit}:=\left\{(Q,P)\in\mathfrak{X}_{\mathbf{I}}^+\times\mathfrak{X}_\Lambda^+\mid k_Q\geq k_P,\,k_Q\equiv k_P\equiv 0\,(\mathrm{mod}\ 2)\right\}. \tag{5.5}$$

The purpose of this subsection is to give the precise formula of $\mathcal{L}_{\boldsymbol{E}_\phi^{[a]},\boldsymbol{f}}(Q,P)$. We begin with some notation. For an arithmetic point Q, denote by $\boldsymbol{f}_Q^\circ$ the normalized newform of weight k_Q and conductor $N_Q=Np^{n_Q}$ corresponding to $\boldsymbol{f}_Q$. Let $\|\boldsymbol{f}_Q^\circ\|^2_{\Gamma_0(N_Q)}$ be the usual Petersson norm of $\boldsymbol{f}_Q^\circ$ and let $\mathcal{E}_p(\boldsymbol{f}_Q,\mathrm{Ad})\in\mathbf{C}^\times$ be the modified p-Euler factor for the adjoint motive associated with $\boldsymbol{f}_Q$ defined in [Hsi20, (3.10)]. Define the modified period

$$\mathrm{Per}^\dagger(\boldsymbol{f}_Q):=(-2\sqrt{-1})^{k_Q+1}\|\boldsymbol{f}_Q^\circ\|^2_{\Gamma_0(N_Q)}\cdot\mathcal{E}_p(\boldsymbol{f}_Q,\mathrm{Ad})\in\mathbf{C}^\times. \tag{5.6}$$

Let $\varrho_{\boldsymbol{f}_Q,p}:\mathbf{Q}_p^\times\to\mathbf{C}^\times$ be the unique unramified character with

$$\varrho_{\boldsymbol{f}_Q,p}(p)=\mathbf{a}(p,\boldsymbol{f}_Q)p^{\frac{1-k_Q}{2}}. \tag{5.7}$$

Definition 5.6 (The test vector)**.** Let $e\mathbf{S}(NC,\boldsymbol{\omega}^{j-2},\mathbf{I})[\boldsymbol{f}]$ be the subspace of $e\mathbf{S}(NC,\boldsymbol{\omega}^{j-2},\mathbf{I})$ consisting of ordinary $\mathbf{I}$-adic forms $\boldsymbol{h}$ such that $t\boldsymbol{h}=\lambda_{\boldsymbol{f}}(t)\boldsymbol{h}$ for all $t\in\mathbf{T}(NC,N)$. For each prime $q\mid C$, let $\{\alpha_q(\boldsymbol{f}),\beta_q(\boldsymbol{f})\}$ be two roots of the q-th Hecke polynomial $H_q(x,\boldsymbol{f}):=x^2-\mathbf{a}(q,\boldsymbol{f})x+q^{-1}\boldsymbol{\omega}^j(q)\langle q\rangle_X$. We fix a choice of roots $\{\alpha_q(\boldsymbol{f})\}_{q|C}$. Enlarging the coefficient ring $\mathcal{O}$ if necessary, we can assume $\alpha_q(\boldsymbol{f})\in\mathbf{I}$. Let $\breve{\boldsymbol{f}}$ be the unique Hida family in $e\mathbf{S}(NC,\boldsymbol{\omega}^{j-2},\mathbf{I})[\boldsymbol{f}]$ such that $\mathbf{a}(1,\breve{\boldsymbol{f}})=1$ and $\mathbf{U}_q\breve{\boldsymbol{f}}=\alpha_q(\boldsymbol{f})\breve{\boldsymbol{f}}$ for $q\mid C$.

Proposition 5.7. *Let* $\mathfrak{f}(X,T) := \langle \Delta_F C\rangle_X^{-\frac{1}{2}} \langle \Delta_F\rangle_T^{\frac{1}{2}} \in (\Lambda\widehat{\otimes}\Lambda)^\times$. *For every* $(Q,P) \in \mathfrak{X}^{\rm crit}$, *we have*

$$\mathcal{L}_{\boldsymbol{E}_\phi^{[a]},\boldsymbol{f}}(Q,P) = \frac{(-2)(-C\delta\sqrt{-1})^{\frac{k_Q}{2}} L(1,\tau_F)}{\prod_{q|C}\zeta_q(1)} \cdot B^{\chi_Q}_{\breve{\boldsymbol{f}}_Q}(\varsigma^{(Cp^n)})$$
$$\times (-\sqrt{-1})^{k_P-1}\frac{L^{\{p\}}(k_P - \frac{k_Q+1}{2}, \pi_{\boldsymbol{f}_Q}\otimes \boldsymbol{\omega}^{a-k_P}\epsilon_P)}{\mathrm{Per}^\dagger(\boldsymbol{f}_Q)}\gamma(k_P - \frac{k_Q+1}{2}, \varrho_{\boldsymbol{f}_Q,p}\otimes\boldsymbol{\omega}_p^{a-k_P}\epsilon_{P,p})^{-1}$$
$$\times\, \mathfrak{f}(Q,P)c_1,$$

where $\chi_Q := \phi\cdot\epsilon_Q^{-\frac{1}{2}}\boldsymbol{\omega}^{\frac{k_Q-2}{2}}\circ \mathrm{N}_{F/\mathbf{Q}}$ *and* c_1 *is the constant*

$$c_1 = 4(-1)^{\frac{a-j}{2}}\boldsymbol{\omega}_p^{-\frac{j}{2}}(C)\boldsymbol{\omega}_p^{\frac{a-j}{2}}(\Delta_F)\prod_{\mathfrak{q}|\mathfrak{c}}\varepsilon(0,\phi_{\mathfrak{q}}) \in \overline{\mathbf{Z}}^\times_{(p)}.$$

Proof. Since $\boldsymbol{f}_Q$ is a p-stabilized newform of tame conductor N, by the multiplicity one for new and ordinary vectors, we have

$$1_{\boldsymbol{f}_Q}\mathrm{Tr}_{Nc/N}(e(\boldsymbol{G}^{[a]}(Q,P))) = \mathcal{L}_{\boldsymbol{E}_\phi^{[a]},\boldsymbol{f}}(Q,P)\cdot \boldsymbol{f}_Q. \tag{5.8}$$

We put

$$\omega^{\frac{1}{2}} = \epsilon_Q^{-\frac{1}{2}}\boldsymbol{\omega}^{\frac{k_Q-j}{2}} \text{ and } \lambda = \epsilon_P^{\frac{1}{2}}\boldsymbol{\omega}^{\frac{a-k_P}{2}}. \tag{5.9}$$

Put $k_1 = k_Q/2$ and $k_2 = k_P/2$. By Proposition 5.2, we have

$$\boldsymbol{E}_\phi^{[a]}(Q,P) = \begin{cases} \theta^{k_1-k_2}E^+_{2k_2-k_1}(\omega_F^{\frac{1}{2}}\lambda_F, \lambda_F^{-1}\phi) & \text{if } 2k_2 > k_1,\\ \theta^{k_2-k_1-1}E^-_{k_1-2k_2+2}(\omega_F^{\frac{1}{2}}\lambda_F,\lambda_F^{-1}\phi) & \text{if } 2k_2\le k_1.\end{cases}$$

Applying the argument in the proof of [Hid88a, Lemma 6.5(iv)], it is not difficult to see that for a Hilbert modular form h over F of weight (k_1,k_2) and non-negative integers a,b,

$$e\,\mathrm{Hol}\left((\delta^a_{k_1,\sigma_1}\delta^b_{k_2,\sigma_2}h)|_{\mathfrak{H}}\right) = e\left((\theta^a_{\sigma_1}\theta^b_{\sigma_2}h)|_{\mathfrak{H}}\right), \tag{5.10}$$

where $\delta^a_{k_1,\sigma_1}\delta^b_{k_2,\sigma_2}$ is the Maass-Shimura differential operator and Hol is the holomorphic projection as in [Hid93, (8a), page 314]. It follows that

$$e\boldsymbol{G}^{[a]}(Q,P) = e(\boldsymbol{E}(Q,P)|_{\mathfrak{H}}) = e\mathrm{Hol}(E^\dagger|_{\mathfrak{H}}), \tag{5.11}$$

where

$$E^\dagger := \begin{cases}\delta^{k_1-k_2}_{2k_2-k_1}E^+_{2k_2-k_1}(\omega_F^{\frac{1}{2}}\lambda_F,\lambda_F^{-1}\phi) & \text{if } 2k_2>k_1\\ \delta^{k_2-k_1-1}_{k_1-2k_2+2}E^-_{k_1-2k_2+2}(\omega_F^{\frac{1}{2}}\lambda_F,\lambda_F^{-1}\phi) & \text{if } 2k_2\le k_1.\end{cases} \tag{5.12}$$

where $\delta^m_k = \delta^m_{k,\sigma_1}\delta^m_{k,\sigma_2}$. Let $f := \boldsymbol{f}_Q \in \mathcal{S}_{k_Q}(Np^r, \epsilon_Q\boldsymbol{\omega}^{j-k_Q})$ and let

$$\varphi_f = \varPhi(\boldsymbol{f}_Q) \in \mathcal{A}^0_{k_Q}(Np^r,\omega),\quad \omega = \epsilon_Q^{-1}\boldsymbol{\omega}^{k_Q-j}.$$

Let n be a sufficiently large positive integer. Let $\mathcal{J}_\infty$ and $t_n \in \mathrm{GL}_2(\mathbf{A})$ be the matrices introduced in (4.4). Let $[-,-] : \mathcal{A}^0_{k_Q}(Np^n,\omega) \times \mathcal{A}_{k_Q}(Np^n,\omega) \to \mathbf{C}$ be the pairing defined by

$$\big[\varphi_1,\varphi_2\big] := \langle \rho(\mathcal{J}_\infty t_n)\varphi_1 \otimes \omega^{-1}, \varphi_2\rangle,$$

where $\langle\, ,\, \rangle$ is the pairing defined in (2.2). Pairing with the form $\varphi_f\otimes$ on the adelic lifts on both sides of (5.8), we obtain that

$$\mathcal{L}_{\boldsymbol{E}^{[a]}_\phi,\boldsymbol{f}}(Q,P)\cdot\big[\varphi_f,\varphi_f\big] = \big[\varphi_f, 1_{\boldsymbol{f}_Q}\mathrm{Tr}_{cN/N}e\,\varPhi(\mathrm{Hol}(E^\dagger|_{\mathfrak{H}}))\big], \tag{5.13}$$

where $1_{\boldsymbol{f}_Q} \in (\mathbf{T}^{\mathrm{ord}}(N)/\wp_Q)\otimes\mathbf{C} \subset \mathrm{End}\, e\mathcal{S}_{k_Q}(Np^n,\omega^{-1})$ is the specialization of $1_{\boldsymbol{f}}$ at Q. Since the Hecke operators $\{T_q\}_{q\nmid Np}$ and $\mathbf{U}_p$, the holomorphic projection Hol and the trace map $\mathrm{Tr}_{CN/N}$ are self-adjoint operators with respect to the pairing $[-,-]$ (*cf.* the proof of [Hsi20, Proposition 3.7]), we thus obtain

$$\mathcal{L}_{\boldsymbol{E}^{[a]}_\phi,\boldsymbol{f}}(Q,P)\cdot\big[\varphi_f,\varphi_f\big] = [U_0(CN):U_0(N)]\cdot\big[\varphi_f,\varPhi(E^\dagger|\mathfrak{H})\big]. \tag{5.14}$$

On the other hand, according to (5.12) and Proposition 3.5, we have

$$\varPhi(E^\dagger|_{\mathfrak{H}}) = E_{\mathbf{A}}(g,f_{\mathcal{D},s})\big|_{s=\frac{2k_2-k_1-1}{2}}, \quad g\in\mathrm{GL}_2(\mathbf{A}),$$

where $\mathcal{D}$ is the Eisenstein datum

$$\mathcal{D} = (\omega_F^{\frac{1}{2}}\lambda_F,\, \phi^{-1}\lambda_F^{-1},\, \frac{k_Q}{2},\, \mathfrak{c},\, \mathfrak{N}). \tag{5.15}$$

Therefore, from (5.14) we see that

$$\begin{aligned}\mathcal{L}^{[\boldsymbol{f}]}_{\boldsymbol{E}}(Q,P)\cdot\big[\varphi_f,\varphi_f\big] =& [\Gamma_0(CN):\Gamma_0(N)]\cdot\langle\rho(\mathcal{J}_\infty t_n)\varphi_f, E_{\mathbf{A}}(-,f_{\mathcal{D},s})\otimes\omega^{-1}\rangle\big|_{s=\frac{2k_2-k_1-1}{2}}\\ =& [\Gamma_0(CN):\Gamma_0(N)]\cdot Z_{\mathcal{D}}(s,\rho(\mathcal{J}_\infty t_n)\varphi_f)\big|_{s=\frac{2k_2-k_1-1}{2}}.\end{aligned}$$

By [Hsi20, Lemma 3.6], we have

$$\begin{aligned}\big[\varphi_f,\varphi_f\big] &= \langle\rho(\mathcal{J}_\infty t_n)\varphi_f\otimes\omega^{-1},\varphi_f\rangle\\ &= \frac{\zeta_{\mathbf{Q}}(2)^{-1}}{[\mathrm{SL}_2(\mathbf{Z}):\Gamma_0(N)]}\cdot(-2\sqrt{-1})^{-k_Q-1}\cdot\mathrm{Per}^\dagger(f)\cdot\frac{\omega_p^{-1}\alpha_f^2(p^n)\,|p^n|_{\mathbf{Q}_p}\,\zeta_p(2)}{\zeta_p(1)}.\end{aligned}$$

Then we have the interpolation formula

$$\begin{aligned}\mathcal{L}_{\boldsymbol{E}^{[a]}_\phi,\boldsymbol{f}}(Q,P) =& Z_{\mathcal{D}}(s,\rho(\mathcal{J}_\infty t_n)\varphi_f))\big|_{s=\frac{k_P-k_Q/2-1}{2}}\\ &\times\frac{\zeta_{\mathbf{Q}}(2)[\mathrm{SL}_2(\mathbf{Z}):\Gamma_0(CN)](-2\sqrt{-1})^{k_Q+1}}{\mathrm{Per}^\dagger(\boldsymbol{f}_Q)}\cdot\frac{\zeta_p(1)}{\omega_p^{-1}\varrho_f^2(p^n)\,|p^n|_{\mathbf{Q}_p}\,\zeta_p(2)}\end{aligned}$$

for any sufficiently large positive n. From the above equation and the formula of $Z_{\mathcal{D}}(s,\rho(\mathcal{J}_\infty t_n)\varphi_f)$ in Theorem 4.7 with the fudge factors given by

$$\begin{aligned}
\mathfrak{f}_\infty &= 4(-4\sqrt{-1})^{-k_Q/2}(-1)^{\frac{-j+a-k_P}{2}},\\
\mathfrak{f}_{\mathrm{ram}} &= \omega_p^{\frac{1}{2}}\lambda_p(\Delta_F)\Delta_F^{\frac{k_P-k_Q}{2}}\delta^{\frac{k_Q}{2}} = \langle\Delta_F\rangle_X^{-\frac{1}{2}}(Q)\,\langle\Delta_F\rangle^{\frac{1}{2}}(P)\cdot\boldsymbol{\omega}_p^{\frac{a-j}{2}}(\Delta_F)\cdot\delta^{\frac{k_Q}{2}},\\
\mathfrak{f}_C &= \omega_p^{\frac{1}{2}}(C)\prod_{\mathfrak{q}|\mathfrak{c}}\frac{\varepsilon(0,\phi_{\mathfrak{q}})}{\zeta_q(1)} = \langle C\rangle_X^{-\frac{1}{2}}(Q)\cdot\boldsymbol{\omega}_p^{-\frac{j}{2}}(C)\prod_{\mathfrak{q}|\mathfrak{c}}\frac{\varepsilon(0,\phi_{\mathfrak{q}})}{\zeta_q(1)}\cdot C^{\frac{k_Q}{2}},
\end{aligned}$$

we get the desired interpolation formula by noting that

$$\mathfrak{f}_\infty\mathfrak{f}_{\mathrm{ram}}\mathfrak{f}_C = \frac{\mathfrak{f}(Q,P)c_1}{\prod_{q|C}\zeta_q(1)}\cdot(-\sqrt{-1}C\delta)^{\frac{k_Q}{2}}\cdot(-2\sqrt{-1})^{-k_Q-1}(\sqrt{-1})^{-k_P}.$$

□

6. p-adic L-functions for modular forms over real quadratic fields

In [BD09], the authors construct a square root p-adic L-functions for Hida families over real quadratic fields, interpolating the toric periods integrals of Hida families over real quadratic fields. The purpose of this section is to give a mild improvement of this construction and give more general interpolation formulae.

6.1. Preliminaries on modular symbols.

We review the theory of classical modular symbols in the *semi-adelic* language. Let $\mathbf{P} := \mathbf{P}^1(\mathbf{Q})$ and let $\mathfrak{D}_0 := \mathbf{Z}[\mathrm{Div}^0\,\mathbf{P}]\times\mathrm{GL}_2(\widehat{\mathbf{Q}})$. For each $r\in\mathbf{P}$, denote by $\{r\}$ its image in the divisor group of $\mathbf{P}$. Let $\gamma\in\mathrm{GL}_2(\mathbf{Q})$ and $u\in\mathrm{GL}_2(\widehat{\mathbf{Q}})$ act on $D=(\{r\}-\{s\},g_{\mathrm{f}})\in\mathfrak{D}_0$ by

$$\gamma Du := (\{\gamma\cdot r\}-\{\gamma\cdot s\},\gamma g_{\mathrm{f}}u).$$

For a ring R, let $L_n(R)$ be the space of two-variable homogeneous polynomials of dergee n with coefficients in R. For $P=P(X,Y)\in L_n(R)$ and $g\in\mathrm{GL}_2(R)$, define

$$P \mid \begin{pmatrix} a & b\\ c & d\end{pmatrix}(X,Y) = P(aX+bY,cX+dY).$$

Let $L_n^*(R)=\mathrm{Hom}_R(L_n(R),R)$. If R is a p-adic ring, let $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ acts on $L_n^*(R)$ by $(\rho_n(u)\xi)(P)=\xi(P|u_p)$. For an integer N and a Hecke character χ modulo N valued in R, we denote by $\mathcal{MS}_k(N,\chi,R)$ the space of p-adic modular symbols of weight k, level N and character χ, consisting of maps $\xi:\mathfrak{D}_0\to L_{k-2}^*(R)$ such that

$$\xi(\gamma Du)=\chi^{-1}(u)\cdot\rho_{k-2}(u_p^{-1})\xi(D)\text{ for }\gamma\in\mathrm{GL}_2^+(\mathbf{Q}),u\in U_1(N).$$

This space $\mathcal{MS}_k(N,\chi,R)$ is known to be a finitely generated R-module equipped with the Hecke action. The Hecke operators T_q for $q \nmid Np$ act on $\mathcal{MS}_k(N,\chi,R)$ by

$$T_q\xi(D) = \xi\left(D\begin{pmatrix}1 & 0\\ 0 & q\end{pmatrix}\right) + \sum_{b\in\mathbf{Z}_q/q\mathbf{Z}_q} \xi\left(D\begin{pmatrix}q & b\\ 0 & 1\end{pmatrix}\right), \tag{6.1}$$

the operator $\mathbf{U}_q$ for $q \mid N$, $q \neq p$ is given by

$$\mathbf{U}_q\xi(D) = \sum_{b\in\mathbf{Z}_q/q\mathbf{Z}_q} \xi\left(D\begin{pmatrix}q & b\\ 0 & 1\end{pmatrix}\right) \text{ for } q \mid N. \tag{6.2}$$

and the operator $\mathbf{U}_p$ is given by by

$$\mathbf{U}_p\xi(D) = \sum_{a\in\mathbf{Z}_p/p\mathbf{Z}_p} \rho_{k-2}\left(\begin{pmatrix}p & a\\ 0 & 1\end{pmatrix}\right)\xi\left(D\begin{pmatrix}p & a\\ 0 & 1\end{pmatrix}\right).$$

The ordinary projector $e := \lim_{n\to\infty}\mathbf{U}_p^{n!}$ is a convergent operator on $\mathcal{MS}_k(N,\chi,R)$. Choosing any element $\gamma \in \mathrm{GL}_2(\mathbf{Q})$ with $\det\gamma < 0$, we define an involution $[\mathbf{c}]$ on $\xi \in \mathcal{MS}_k(N,\chi,A)$ by

$$[\mathbf{c}]\xi(D) := \xi(\gamma\cdot D).$$

This definition does not depend on the choice of such γ. We define

$$\xi^+ := \left(\frac{1+[\mathbf{c}]}{2}\right)\xi;\quad \xi^- := \left(\frac{1-[\mathbf{c}]}{2}\right)\xi.$$

6.2. **Modular symbols associated with modular forms.** To each classical cusp from $f = f(z,g_{\mathrm{f}}) \in \mathcal{S}_k(N,\chi)$, we associate a classical modular symbol $\eta_f : \mathfrak{D}_0 \to L_{k-2}^*(\mathbf{C})$ defined by

$$\eta_f(\{r\}-\{s\},g_{\mathrm{f}})(P) := \int_r^s f(z,g_{\mathrm{f}})P(z,1)\mathrm{d}z.$$

It is easy to see that for $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$ and $u \in U_0(N)$,

$$\eta_f(\alpha Du) = \rho_{k-2}(\alpha)\eta_f(D)\chi^{-1}(u).$$

The involution $[\mathbf{c}]$ acts on the classical modular symbol η_f by $[\mathbf{c}]\eta_f(D) = \rho_{k-2}(\gamma)\eta_f(\gamma D)$, where $\gamma \in \mathrm{GL}_2(\mathbf{Q})$ is any element with $\det\gamma < 0$. By definition,

$$[\mathbf{c}]\eta_f(D) = -\overline{\eta_{f_\rho}(D)},$$

where $f_\rho(z,g_{\mathrm{f}}) = \overline{f(-\overline{z},\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}g_{\mathrm{f}})}$. On the other hand, the associated p-adic modular symbol $\xi_f \in \mathcal{MS}_k(N,\chi,\mathbf{C}_p)$ is defined by

$$\xi_f(D)(P) = \iota_p(\eta_f(D)(P|g_p^{-1})) \text{ for } D = (d,g_{\mathrm{f}}) \in \mathfrak{D}_0. \tag{6.3}$$

If f is a $\mathbf{U}_p$-eigenform with the eigenvalue $\alpha \in \mathbf{Z}_p^\times$, then ξ_f is also an eigenvector of $\mathbf{U}_p$ with eigenvalue α. Following the discussion in [Kit94, p.95], for each $D \in \mathfrak{D}_0$ we define the p-adic measure $\mu_f(D)(x)$ on $\mathbf{Z}_p$ by the rule

$$\int_{a+p^n\mathbf{Z}_p} \mu_f(D)(x) = \alpha^{-n}\xi_f(D\begin{pmatrix} p^n & a \\ 0 & 1\end{pmatrix})(Y^{k-2}) \text{ for } n \in \mathbf{Z}_{\geq 0}. \tag{6.4}$$

Lemma 6.1. *For any* $P \in L_{k-2}(\overline{\mathbf{Z}}_p)$,

$$\int_{a+p^n\mathbf{Z}_p} P(x,1)\mu_f(D)(x) = \alpha^{-n}\xi_f(D\begin{pmatrix} p^n & a \\ 0 & 1\end{pmatrix})(P|\begin{pmatrix} p^n & a \\ 0 & 1\end{pmatrix}).$$

Proof. This is [Kit94, Lemma 4.6]. We paraphrase the computation there in our semi-adelic formulation. Note that ξ_f has bounded denominators in the sense that $p^A \cdot \xi_f \in \mathcal{MS}_k(N,\chi,\overline{\mathbf{Z}}_p)$ for some $A \gg 0$. Let $0 \leq j \leq k-2$ be an integer. For every $m > A+n$, we have

$$\begin{aligned}
&\alpha^{-n}\xi_f\left(D\begin{pmatrix} p^n & a \\ 0 & 1\end{pmatrix}\right)\left(X^jY^{k-2-j}|\begin{pmatrix} p^n & a \\ 0 & 1\end{pmatrix}\right)\\
=&\alpha^{-m}\sum_{c=0}^{p^{m-n-1}}\xi_f\left(D\begin{pmatrix} p^m & a+p^nc \\ 0 & 1\end{pmatrix}\right)\left(X^jY^{k-2-j}|\begin{pmatrix} p^m & a+p^nc \\ 0 & 1\end{pmatrix}\right)\\
\equiv&\alpha^{-m}\sum_{c}(a+p^nc)^j\xi_f\left(D\begin{pmatrix} p^m & a+p^nc \\ 0 & 1\end{pmatrix}\right)(Y^{k-2}) \pmod{p^{m-A}\overline{\mathbf{Z}}_p}.
\end{aligned}$$

Therefore, we find that

$$\begin{aligned}
\int_{a+p^n\mathbf{Z}_p} x^j\mu_f(D)(x) =& \sum_{m\to\infty}\alpha^{-m}\sum_{c=0}^{p^{m-n-1}}(a+p^nc)^j\xi_f\left(D\begin{pmatrix} p^m & a+p^nc \\ 0 & 1\end{pmatrix}\right)(Y^{k-2})\\
=&\alpha^{-n}\xi_f\left(D\begin{pmatrix} p^n & a \\ 0 & 1\end{pmatrix}\right)\left(X^jY^{k-2-j}|\begin{pmatrix} p^n & a \\ 0 & 1\end{pmatrix}\right).
\end{aligned}$$

This shows the lemma. □

6.3. **Hida theory for modular symbols.** We review the $\mathbf{I}$-adic symbols developed in [Kit94] in the semi-adelic formulation. Let $\mathbf{I}$ be a normal and finite domain over $\Lambda = \mathcal{O}[\![X]\!]$ with $X = [\mathbf{u}] - 1$ and let N be a positive integer coprime to p. Put

$$U_1(Np^\infty) = \left\{u \in U_1(N) \middle| u_p = \begin{pmatrix} a & b \\ 0 & 1\end{pmatrix},\, a \in \mathbf{Z}_p^\times, b \in \mathbf{Z}_p\right\}.$$

For each non-negative integer n, let $\wp^{(n)}$ be the principal ideal of $\mathbf{I}$ generated by $(\mathbf{u}^{-2}(1+X)-1)^{p^n}-1$. Define the Λ-adic Hecke character $\boldsymbol{\alpha}_X : \mathbf{Q}^\times\backslash\mathbf{A}^\times \to \Lambda^\times$ by

$$\boldsymbol{\alpha}_X(z) = \langle\boldsymbol{\varepsilon}_{\rm cyc}(z)\rangle_X \langle\boldsymbol{\varepsilon}_{\rm cyc}(z)\rangle^{-2}.$$

Definition 6.2. Define the space of $\mathbf{I}$-adic modular symbols of tame level N by

$$\mathcal{MS}(N,\mathbf{I}) := \varprojlim_{n} \varinjlim_{m} \mathcal{MS}_2(Np^m, \boldsymbol{\alpha}_X, \mathbf{I}/\wp^{(n)}).$$

In other words, $\mathcal{MS}(N,\mathbf{I})$ consists of *continuous* functions $\Xi : \mathfrak{D}_0 \to \mathbf{I}$ such that

- $\Xi(\gamma D u) = \Xi(D)$ for $\gamma \in \mathrm{GL}_2^+(\mathbf{Q})$ and $u \in U_1(Np^\infty)$;
- $\Xi(Dz) = \boldsymbol{\alpha}_X(z^{-1}) \cdot \Xi(D)$ for $z \in \widehat{\mathbf{Q}}^\times$;
- Ξ is continuous in the sense that for any n, there exists r_n such that the function $\Xi : \mathfrak{D}_0 \to \mathbf{I}/\wp^{(n)}$ factors through $\mathfrak{D}_0/U_1(Np^{r_n})$.

The space $\mathcal{MS}(N,\mathbf{I})$ is an $\mathbf{I}$-module equipped with the action of Hecke operators $\{T_q\}_{q\nmid Np}$ abd $\{U_q\}_{q|N}$ as in (6.1) and (6.2), while the $\mathbf{U}_p$-operator is defined by

$$\mathbf{U}_p\Xi(D) = \sum_{a\in\mathbf{Z}_p/p\mathbf{Z}_p} \Xi\left(D\begin{pmatrix} p & a \\ 0 & 1\end{pmatrix}\right).$$

For $(d,pN)=1$, define the level-raising operator $V_d : \mathcal{MS}(N,\mathbf{I}) \to \mathcal{MS}(Nd,\mathbf{I})$ by

$$V_d\Xi(D) = d^{-1}\cdot \Xi\left(D\begin{pmatrix} d^{-1} & 0 \\ 0 & 1\end{pmatrix}\right). \tag{6.5}$$

The involution $[\mathbf{c}]$ on $\mathcal{MS}(N,\mathbf{I})$ is defined by $[\mathbf{c}]\Xi(D) := \Xi(\gamma D)$ for any $\gamma \in \mathrm{GL}_2(\mathbf{Q})$ with $\det\gamma < 0$. Put

$$e\mathcal{MS}(N,\mathbf{I})^{\pm} := (1 \pm [\mathbf{c}])e\mathcal{MS}(N,\mathbf{I}).$$

The ordinary project $e = \lim_{n\to\infty} \mathbf{U}_p$ exists in $\mathrm{End}_{\mathbf{I}}\,\mathcal{MS}(N,\mathbf{I})$. The space $e\mathcal{MS}(N,\mathbf{I})$ is called the space of the ordinary $\mathbf{I}$-adic modular symbols. We remark that $e\mathcal{MS}(N,\mathbf{I})$ is nothing but $MS^{ord}(\mathbf{I}) = \mathrm{Hom}_\Lambda(UM^{ord}(\mathcal{O}),\mathbf{I})$ defined in [Kit94, §5.5]. The following is proved in [Kit94, Proposition 5.7].

Theorem 6.3. *The space $e\mathcal{MS}(N,\mathbf{I})$ is free of finite rank over $\mathbf{I}$.*

We recall the $\mathbf{I}$-adic measure associated with ordinary $\mathbf{I}$-adic modular symbols. Let $\mathcal{C}(\mathbf{Z}_p,\mathbf{I})$ be the space of continuous $\mathbf{I}$-valued functions on $\mathbf{Z}_p$ and $\mathcal{D}(\mathbf{Z}_p,\mathbf{I}) := \mathrm{Hom}_{\mathbf{I}}(\mathcal{C}(\mathbf{Z}_p,\mathbf{I}),\mathbf{I})$ be the space of $\mathbf{I}$-adic measures on $\mathbf{Z}_p$. To each ordinary $\mathbf{I}$-adic modular symbol $\Xi \in e\mathcal{MS}(N,\mathbf{I})$, we associate a unique linear map $D \mapsto \mu_\Xi(D)(x)$ in $\mathrm{Hom}(\mathfrak{D}_0,\mathcal{D}(\mathbf{Z}_p,\mathbf{I}))$ such that for $D \in \mathfrak{D}_0$ and $P \in \mathcal{C}(\mathbf{Z}_p,\mathbf{I})$

$$\int_{\mathbf{Z}_p} P(x)\mu_\Xi(D)(x) := \lim_{m\to\infty}\sum_{a=0}^{p^m-1} P(a)\mathbf{U}_p^{-m}\Xi\left(D\begin{pmatrix} p^m & a \\ 0 & 1\end{pmatrix}\right) \in \mathbf{I}. \tag{6.6}$$

It is straightforward to verify that the right hand side is a p-adically convergent Riemann sum valued in $\mathbf{I}$. For $P \in \mathcal{C}(\mathbf{Z}_p,\mathbf{I})$ and $u \in U_0(p)$ with

$u_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define

$$P \,|\, u(x) = P\left(\frac{ax+b}{cx+d}\right)\boldsymbol{\alpha}_X(cx+d). \tag{6.7}$$

Lemma 6.4. *For $P \in \mathcal{C}(\mathbf{Z}_p, \mathbf{I})$, we have*

(1) *For $m \in \mathbf{Z}^{\geq 0}$,*

$$\int_{p^m\mathbf{Z}_p} P(x)\mu_\Xi(D)(x) = \int_{\mathbf{Z}_p} P(p^m x)\mu_{\mathbf{U}_p^{-m}\Xi}\left(D\begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}\right).$$

(2) *For $u \in U_0(pN)$, we have*

$$\int_{\mathbf{Z}_p} P(x)\mu_\Xi(Du)(x) = \int_{\mathbf{Z}_p} P \,|\, u^{-1}(x)\mu_\Xi(D)(x).$$

Proof. The verification of part (1) is straightforward by (6.6). To see part (2), it suffices to show the equation for u_p of the form $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Let $u_p = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ with $c \in p\mathbf{Z}_p$. By definition, the left hand side equals

$$\begin{aligned} &\lim_{m\to\infty}\sum_{a=0}^{p^m-1} P(a)\mathbf{U}_p^{-m}\Xi\left(D\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} p^m & a \\ 0 & 1 \end{pmatrix}\right) \\ =&\lim_{m\to\infty}\sum_{a=0}^{p^m-1} P(a)\mathbf{U}_p^{-m}\Xi\left(D\begin{pmatrix} p^m & a(1+ac)^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} (1+ac)^{-1} & 0 \\ cp^m & 1+ac \end{pmatrix}\right). \end{aligned}$$

Making change of variable $a = z(1-cz)^{-1}$, we find that the last Riemann sum equals

$$\begin{aligned} &\lim_{m\to\infty}\sum_{z=0}^{p^m-1} P(z(1-cz)^{-1})\mathbf{U}_p^{-m}\Xi\left(D\begin{pmatrix} p^m & z \\ 0 & 1 \end{pmatrix}\right)\boldsymbol{\alpha}_X(1-cz)^{-1} \\ =&\int_{\mathbf{Z}_p} P \,|\, \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}(x)\mu_\Xi(D)(x). \end{aligned}$$

The case for $u_p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is similar. We omitted the details. □

For an arithmetic point Q in $\mathfrak{X}_{\mathbf{I}}^+$, we denote by $\wp_Q$ the kernel of the specialization $Q : \mathbf{I} \to \mathbf{C}_p$. Let $\mathcal{O}(Q) = \mathbf{I}/\wp_Q$ and let $r_Q = \max\{1, c_p(\epsilon_Q)\}$. Here $c_p(\epsilon_Q)$ is the exponent of the p-conductor of ϵ_Q. For any $\mathcal{O}(Q)$-algebra A, we put

$$\mathcal{MS}_Q^{\mathrm{ord}}(A) := e\mathcal{MS}_{k_Q}(Np^{r_Q}, \boldsymbol{\omega}^{2-k_Q}\epsilon_Q, A).$$

Theorem 6.5 (Control Theorem)**.** *For each arithmetic point Q, there is a Hecke-equivariant specialization isomorphism*

$$\begin{aligned}\mathrm{sp}_Q\colon e\mathcal{MS}(N,\mathbf{I})/\wp_Q &\simeq \mathcal{MS}_Q^{\mathrm{ord}}(\mathcal{O}(Q)),\\ \Xi\,(\mathit{mod}\ \wp_Q) &\mapsto \mathrm{sp}_Q(\Xi):=\Xi_Q,\end{aligned}$$

where Ξ_Q is the p-adic modular symbol of weight k_Q defined by

$$\Xi_Q(D)(P)=Q\biggl(\int_{\mathbf{Z}_p}P(x,1)\mu_\Xi(D)(x)\biggr),\quad P(X,Y)\in L_{k_Q-2}(\mathcal{O}(Q)).$$

We call Ξ_Q the specialization of Ξ at Q.

Proof. First we note that Ξ_Q is a p-adic modular symbol of weight k_Q and character $\boldsymbol{\omega}^{2-k_Q}\epsilon_Q$ by Lemma 6.4. It is straightforward to verify that the map sp_Q is Hecke-equivariant, so Ξ_Q belongs to $\mathcal{MS}_Q^{\mathrm{ord}}(\mathcal{O}(Q))$. We proceed to show sp_Q is an isomorphism. Let $n=k_Q-2$, $\mathcal{O}=\mathcal{O}(Q)$ and $\chi_Q=\boldsymbol{\alpha}_X\ (\mathrm{mod}\ \wp_Q)=\boldsymbol{\varepsilon}_{\mathrm{cyc}}^n\boldsymbol{\omega}^{-n}\epsilon_Q$. We have

$$e\mathcal{MS}(N,\mathbf{I})/\wp_Q=\varprojlim_t\varinjlim_r e\mathcal{MS}_2(Np^r,\chi_Q,\mathcal{O}/p^t).$$

For any $\mathbf{Z}_p$-module R, define $\iota_n: L_n^*(R)\to R,\quad \iota_n(\ell)=\ell(Y^n)$. By [Kit94, Corollary 5.2], ι_n induces a Hecke-equivariant isomorphism

$$\iota_n\colon e\mathcal{MS}_{k_Q}(Np^r,\epsilon_Q\boldsymbol{\omega}^{-n},\mathcal{O}/p^t)\simeq e\mathcal{MS}_2(Np,\chi_Q,\mathcal{O}/p^t)\text{ for }r\geq t.$$

Note that $\iota_n(\Xi_Q(D))=\Xi_Q(D)(Y^n)=Q(\Xi(D))$. We deduce that sp_Q is indeed given by the isomorphism

$$\begin{aligned}e\mathcal{MS}(N,\mathbf{I})/\wp_Q&=\varprojlim_t\varinjlim_r e\mathcal{MS}_2(Np^r,\chi_Q,\mathcal{O}/p^t)\\ &\overset{\iota_n^{-1}}{\simeq}\varprojlim_t\varinjlim_r e\mathcal{MS}_{k_Q}(Np^r,\epsilon_Q\boldsymbol{\omega}^{-n},\mathcal{O}/p^t)=e\mathcal{MS}_{k_Q}(Np^{r_Q},\epsilon_Q\boldsymbol{\omega}^{-n},\mathcal{O}),\end{aligned}$$

where the last equality is the base change property [Hid88b, Lemma 1.8 and Corollary 2.2] for ordinary p-adic modular symbols. This completes the proof. □

Remark 6.6. Let L_0' be the set of primitive elements in $\mathbf{Z}_p\times\mathbf{Z}_p$, consisting of elements in $\mathbf{Z}_p\times\mathbf{Z}_p$ which are not divisible by p. We recall the connection of Λ-adic symbols and the modular symbols with valued in the space $\mathcal{D}(L_0')$ of p-adic measures on L_0' described in [GS93, §5]. For each $k\in\mathbf{C}_p$ with $|k|_p\leq 1$, let $Q_k\in\operatorname{Spec}\Lambda(\mathbf{C}_p)$ be the unique point with $Q_k([\mathbf{u}])=\mathbf{u}^k$ and let $\mathscr{F}_k$ be the set of homogenous functions of degree k on L_0', i.e. continuous functions $h: L_0'\to\mathbf{Z}_p$ such that $h(ax,ay)=\langle a\rangle^k h(x,y)$ for all $a\in\mathbf{Z}_p^\times$. Then to each $\Xi\in e\mathcal{MS}(N,\Lambda)$, we can associate a modular symbol $\mu_\Xi^{\mathrm{GS}}\in\mathrm{Hom}_{U_0(N)}(\mathfrak{D}_0,\mathcal{D}(L_0'))$ characterized by the property that any $k\in\mathbf{Z}_p$ and

$h \in \mathscr{F}_{k-2}$, we have

$$\int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} h(x,y)\mu_\Xi^{\mathrm{GS}}(D)(x,y) = Q_k\left(\int_{\mathbf{Z}_p} h(x,1)\mu_\Xi(D)(x)\right);$$

$$\int_{\mathbf{Z}_p^\times \times p\mathbf{Z}_p} h(x,y)\mu_\Xi^{\mathrm{GS}}(D)(x,y) = Q_k\left(\int_{\mathbf{Z}_p} h(1,-py)\mu_{\mathbf{U}_p^{-1}\Xi}\left(D\begin{pmatrix}0 & 1\\ -p & 0\end{pmatrix}\right)(y)\right).$$

By a similar computation in Lemma 6.4, one verifies that the map μ_Ξ^{GS} is $U_0(N)$-invariant, namely for any $u \in U_0(N)$

$$\int_{L_0'} h \mid u^{-1}(x,y)\mu_\Xi^{\mathrm{GS}}(D)(x,y) = \int_{L_0'} h(x,y)\mu_\Xi^{\mathrm{GS}}(Du)(x,y). \tag{6.8}$$

6.4. **The Mazur-Kitagawa two variable p-adic L-functions.** Let $\boldsymbol{f} \in e\mathbf{S}(N,1,\mathbf{I})$ be a primitive Hida family of tame conductor N and let $\lambda_{\boldsymbol{f}} : \mathbf{T}(N,\mathbf{I}) \to \mathbf{I}$ be the corresponding homomorphism. For any integer C prime to N, let $e\mathcal{MS}(NC,\mathbf{I})^\pm[\boldsymbol{f}]$ be the space of $\mathbf{I}$-adic ordinary modular symbols $\Xi \in e\mathcal{MS}(NC,\mathbf{I})^\pm$ such that $t \cdot \Xi = \lambda_{\boldsymbol{f}}(t)\Xi$ for all $t \in \mathbf{T}(NC,N)$. The space $e\mathcal{MS}(N,\mathbf{I})^\pm[\boldsymbol{f}] \otimes_{\mathbf{I}} \operatorname{Frac}\mathbf{I}$ has a rank one over $\operatorname{Frac}\mathbf{I}$ as $\boldsymbol{f}$ is primitive of tame conductor N. For an arithmetic point Q, the space $\mathcal{MS}_Q^{\mathrm{ord}}(\mathcal{O}(Q))^\pm[\boldsymbol{f}_Q]$ is free of rank one over $\mathcal{O}(Q)$. On the other hand, Shimura in [Shi77] proved that $0 \neq \xi_{\boldsymbol{f}_Q}^\pm \in \mathcal{MS}_Q^{\mathrm{ord}}(\mathbf{C}_p)^\pm[\boldsymbol{f}_Q]$. Therefore, having fixed a basis $\beta_{\boldsymbol{f}_Q}^\pm$ of $\mathcal{MS}_Q^{\mathrm{ord}}(\mathcal{O}(Q))$, we can define the period $\Omega_{\boldsymbol{f}_Q}^\pm \in \mathbf{C}_p^\times$ associated with the p-stablized newform $\boldsymbol{f}_Q$ by

$$\xi_{\boldsymbol{f}_Q}^\pm = \Omega_{\boldsymbol{f}_Q}^\pm \beta_{\boldsymbol{f}_Q}^\pm.$$

Definition 6.7 (p-adic error terms)**.** Let $\Xi \in e\mathcal{MS}(N,\mathbf{I})[\boldsymbol{f}]$. We define the plus/minus error terms $\mathrm{Er}^\pm(\Xi_Q) \in \mathbf{C}_p$ by the equation

$$\Xi_Q^\pm = \frac{\mathrm{Er}^\pm(\Xi_Q)}{\Omega_{\boldsymbol{f}_Q}^\pm} \cdot \xi_{\boldsymbol{f}_Q}^\pm.$$

To each $\Xi \in e\mathcal{MS}(N,\mathbf{I})[\boldsymbol{f}]$ and a finite order Hecke character χ with $\chi(-1) = (-1)^i$, Kitagawa in [Kit94, Theorem 1.1] associates the two-variable p-adic L-function $L_p(\Xi,\chi) \in \mathbf{I}\widehat{\otimes}\Lambda$ satisfying the interpolation property: for every pair of arithmetic points $(Q,P) \in \mathfrak{X}_{\mathbf{I}}^+ \times \mathfrak{X}_\Lambda^+$ with $k_Q \geq k_P$,

$$\begin{aligned} L_p(\Xi,\chi)(Q,P) = {} & (-\sqrt{-1})^{k_P-1} \cdot \frac{L^{\{p\}}(k_P - \frac{k_Q+1}{2}, \pi_{\boldsymbol{f}_Q} \otimes \chi\boldsymbol{\omega}^{-k_P}\epsilon_P)}{\Omega_{\boldsymbol{f}_Q}^{(-1)^i}} \\ & \times \gamma\left(k_P - \frac{k_Q+1}{2}, \varrho_{\boldsymbol{f}_Q,p} \otimes \boldsymbol{\omega}_p^{-k_P}\epsilon_{P,p}\right)^{-1} \mathrm{Er}^{(-1)^i}(\Xi_Q), \end{aligned} \tag{6.9}$$

Note that the relation between L-functions associated with modular forms and the automorphic L-functions is given by

$$L\left(k_P - \frac{k_Q+1}{2}, \pi_{\boldsymbol{f}_Q} \otimes \chi\right) = 2(2\pi)^{1-k_P}\Gamma(k_P-1) \cdot L(k_P - 1, \boldsymbol{f}_Q \otimes \chi).$$

6.5. **The square root p-adic L-functions associated with Hida families over real quadratic fields.** We review the construction of the square root of p-adic L-functions for Hida families over real quadratic fields in [BD09]. Let F_+ be the group of totally positive elements in F and let $\mathrm{Cl}^+(\mathcal{O}_C) := F_+\backslash\widehat{F}^\times/\widehat{\mathcal{O}}_C^\times$ denote the narrow ring class group of conductor C. For $t \in \widehat{F}^\times$, write $[t] = F_+t\widehat{\mathcal{O}}_C^\times$ for the class represented by t. Let ϵ_C be a generator of the units group $F_+ \cap \widehat{\mathcal{O}}_C^\times$. Let $P_\Psi(X,Y) = (X-\theta Y)(X-\overline{\theta}Y)$ and $\delta = \overline{\theta} - \theta = \sqrt{\Delta_F}$. Define $\vartheta_X \colon \mathbf{Z}_p^\times \to \Lambda^\times$ by

$$\vartheta_X(x) = \langle x\rangle_X^{\frac{1}{2}} \langle x\rangle^{-1}.$$

So $\vartheta_X^2 = \boldsymbol{\alpha}_X|_{\mathbf{Z}_p^\times}$. Let ϕ be a finite order Hecke character of $\mathbf{A}_F^\times$ as in (5.4). Equivalently, $\phi|_{\widehat{F}^\times}$ is an even/odd character of $\mathrm{Cl}^+(\mathcal{O}_C)$, depending on the sign of $\phi_\infty(\delta) = (-1)^{\frac{j}{2}}$ or the parity of $\frac{j}{2}$.

Definition 6.8. Let $\Xi \in e\mathcal{MS}(NC,\mathbf{I})^\pm[\boldsymbol{f}]$. For $D \in \mathfrak{D}_0$, we define $\mathcal{L}_\Xi(D) \in \mathbf{I}$ as follows: if p is split in F, put

$$\mathcal{L}_\Xi(D) = \int_{\mathbf{Z}_p^\times} \vartheta_X(x)\mu_\Xi(D)(x) \in \mathbf{I};$$

if p is inert in F, put

$$\begin{aligned}\mathcal{L}_\Xi(D) = &\int_{\mathbf{Z}_p} \vartheta_X(P_\Psi(x,1))\mu_\Xi(D)(x)\\ &+ \alpha_{\boldsymbol{f}}^{-1}\int_{\mathbf{Z}_p} \vartheta_X(P_\Psi(1,-px))\mu_\Xi\left(D\begin{pmatrix}0 & 1\\ -p & 0\end{pmatrix}\right)(x).\end{aligned}$$

Fixing any base point $r \in \mathbf{P}$, we define the (square root) p-adic L-function $\mathcal{L}_{\Xi^\pm/F\otimes\phi} \in \mathbf{I}$ for $\boldsymbol{f}/F$ by

$$\mathcal{L}_{\Xi^\pm/F\otimes\phi} := \sum_{[t]\in\mathrm{Cl}^+(\mathcal{O}_C)} \phi(t)\vartheta_X(\boldsymbol{\varepsilon}_{\mathrm{cyc}}(\mathrm{N}(t)))\cdot\mathcal{L}_{\Xi^\pm}(\{r\}-\{\Psi(\epsilon_C)r\},\Psi(t)\varsigma_{\mathrm{f}}^{(C)}).$$

Note that the above definition does not depend on the choice of r and does not depend the representatives $[t]$ in $\mathrm{Cl}^+(\mathcal{O}_C)$.

6.6. **The interpolation formulae.** For an elliptic modular form $f \in \mathcal{S}_k(Np^r,\omega^{-1})$ and a finite order Hecke character χ of $\mathbf{A}_F^\times$ with $\chi|_{\mathbf{A}^\times} = \omega$, writing $\varphi_f := \Phi(f)$ for the adelic lift of f, define the global toric period by

$$B_f^\chi(g) := B_{\varphi_f}^\chi(g) = \int_{\mathbf{A}^\times F^\times\backslash\mathbf{A}_F^\times} \varphi_f(\Psi(t)g)\chi(t)\mathrm{d}t.$$

Let $\breve{\boldsymbol{f}} \in e\mathbf{S}(NC,1,\mathbf{I})[\boldsymbol{f}]$ be the *test vector* in Definition 5.6. Then $\breve{\boldsymbol{f}}$ can be expressed as

$$\breve{\boldsymbol{f}}(q) = \prod_{q|C}(1-\beta_q(\boldsymbol{f})V_q)\cdot\boldsymbol{f},$$

where $\beta_q(\boldsymbol{f})$ is the fixed choice of roots of the Hecke polynomial $H_q(x,\boldsymbol{f})$ of $\boldsymbol{f}$ at q. Let $\Xi \in e\mathcal{MS}(N,\mathbf{I})[\boldsymbol{f}]$ and define

$$\breve{\Xi} := \prod_{q|C}(1-\beta_q(\boldsymbol{f})V_q)\cdot\Xi \in e\mathcal{MS}(NC,\mathbf{I})[\boldsymbol{f}], \tag{6.10}$$

where V_d is the level-raising operator defined in (6.5). The next result shows that $\mathcal{L}_{\breve{\Xi}/F\otimes\phi}$ interpolates p-adically the toric period associated with $\breve{\boldsymbol{f}}_Q$ for $Q \in \mathfrak{X}_{\mathbf{I}}^{++}$.

Proposition 6.9. *For arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}}^{++}$ with even k_Q, set $\chi_Q := \phi\cdot\epsilon_Q^{-\frac{1}{2}}\boldsymbol{\omega}^{\frac{k_Q-2}{2}}\circ \mathrm{N}_{F/\mathbf{Q}}$. Let $\pm = \phi_\infty(\delta) = (-1)^{\frac{j}{2}}$. We have*

$$\mathcal{L}_{\breve{\Xi}^{\pm}/F\otimes\phi}(Q) = \frac{(-2)(-C\delta\sqrt{-1})^{\frac{k_Q}{2}}L(1,\tau_F)}{\prod_{q|C}\zeta_q(1)}\cdot\frac{B^{\chi_Q}_{\breve{\boldsymbol{f}}_Q}(\varsigma^{(Cp^n)})}{\Omega^{\pm}_{\boldsymbol{f}_Q}}\cdot \mathrm{Er}^{\pm}(\Xi_Q)$$
$$\times\frac{\zeta_p(1)}{\zeta_{F_p}(1)\alpha^n_{\boldsymbol{f}_Q}\,|p^n|_{\mathbf{Q}_p}^{\frac{k_Q}{2}}},$$

where $n \geq \max\{c_p(\chi_Q),1\}$ is any sufficiently large integer.

Proof. For simplicity, we write $f = \breve{\boldsymbol{f}}_Q$ and $\varphi = \Phi(f)$ and put

$$k = k_Q,\quad \omega^{\frac{1}{2}} = \epsilon_Q^{-\frac{1}{2}}\boldsymbol{\omega}^{\frac{k_Q-2}{2}}.$$

Then $\chi_Q = \phi\omega_F^{-\frac{1}{2}}$. Let $\mathbf{m}(y) = \begin{pmatrix} y & 0\\ 0 & y^{-1}\end{pmatrix}$ for $y \in \mathbf{R}^\times$. For $t \in \widehat{F}^\times$, define the partial period by

$$L_{[t]}(\varphi) := \int_{\mathbf{R}_+/\epsilon_C^{\mathbf{Z}}}\sum_{[u]\in\widehat{\mathcal{O}}_C^\times/\widehat{\mathcal{O}}_{Cp^n}^\times}\varphi(\varsigma_\infty\mathbf{m}(y)\Psi(tu)\varsigma_{\mathrm{f}}^{(Cp^n)})\chi_Q(u)\mathrm{d}^\times y.$$

Then we see that the toric period $B_{\breve{\boldsymbol{f}}_Q,\chi_Q}(\varsigma^{(Cp^n)})$ equals

$$\int_{\mathbf{A}^\times F^\times\backslash\mathbf{A}_F^\times}\varphi(\Psi(t)\varsigma^{(Cp^n)})\chi_Q(t)\mathrm{d}t = \mathrm{vol}(\mathcal{O}_C^\times)\frac{\zeta_{F_p}(1)}{p^n\zeta_p(1)}\sum_{[t]\in\mathrm{Cl}^+(\mathcal{O}_C)}\chi_Q(t)L_{[t]}(\varphi),$$

where $\mathrm{vol}(\widehat{\mathcal{O}}_C^\times)$ is the volume of the image of $\widehat{\mathcal{O}}_C^\times$ in $\widehat{\mathbf{Q}}^\times\backslash\widehat{F}^\times$ with respect to the quotient measure $\mathrm{d}t/\mathrm{d}^\times t_\infty$ explicitly given by

$$\mathrm{vol}(\widehat{\mathcal{O}}_C^\times)^{-1} = \sqrt{\Delta_F}L(1,\tau_F)\#(\mathbf{Z}/C\mathbf{Z})^\times = L(1,\tau_F)\delta C\prod_{q|C}(1-q^{-1}).$$

By a direct computation, if $z = \varsigma_\infty\mathbf{m}(y)\cdot\sqrt{-1} = \varsigma_\infty\cdot y^2\sqrt{-1}$, then

$$J(\varsigma_\infty\mathbf{m}(y),\sqrt{-1})^{-2} = P_\Psi(z,1)\cdot(-\sqrt{-1}\Delta_F)^{-1},$$

and $\mathrm{d}z = (2\delta\sqrt{-1})\cdot J(\varsigma_\infty\mathbf{m}(y),\sqrt{-1})^{-2}\mathrm{d}^\times y$. It follows that

$$\begin{aligned}L_{[t]}(\varphi) &= (2\sqrt{-1})^{-1}(-\sqrt{-1}\Delta_F)^{\frac{2-k}{2}}\delta^{\frac{k-2}{2}}\left|Cp^n\mathrm{N}(t)\right|_{\widehat{\mathbf{Q}}}^{\frac{k-2}{2}} \\ &\times\int_r^{\Psi(\epsilon_C)r}\sum_{[u]\in\widehat{\mathcal{O}}_C^\times/\widehat{\mathcal{O}}_{Cp^n}}\chi_Q(u)\cdot f(z,\Psi(tu)\varsigma_{\mathrm{f}}^{(Cp^n)})P_\Psi(z,1)^{\frac{k-2}{2}}\mathrm{d}z \\ &= \ell_1\cdot|\mathrm{N}(t)|_{\mathbf{A}}^{\frac{k-2}{2}}\sum_{[u]\in\widehat{\mathcal{O}}_C^\times/\widehat{\mathcal{O}}_{Cp^n}^\times}\chi_Q(u)\eta_f(\{r\}-\{\Psi(\epsilon_C)r\},\Psi(tu)\varsigma_{\mathrm{f}}^{(Cp^n)})(P_\Psi^{\frac{k-2}{2}}),\end{aligned}$$

where r can be chosen to be any point in $\mathbf{P}$ and

$$\ell_1 := (2\sqrt{-1})^{-1}(-C\delta\sqrt{-1})^{\frac{2-k}{2}}\left|p^n\right|_{\mathbf{Q}_p}^{\frac{k-2}{2}}.$$

For $t\in\widehat{F}^\times$, we set

$$D_t := (\{r\}-\{\Psi(\epsilon_C)r\},\Psi(t)\varsigma_{\mathrm{f}}^{(C)})\in\mathfrak{D}_0.$$

Putting

$$\ell_2 := \delta^{-1}L(1,\tau_F)^{-1}C^{-1}\prod_{q|C}\zeta_q(1)\cdot\frac{\zeta_{F_p}(1)}{p^n\zeta_p(1)},$$

we have

$$\begin{aligned}B_{\breve{\boldsymbol{f}}_Q,\chi_Q}(\varsigma^{(Cp^n)}) &= \ell_2\sum_{[t]\in\mathrm{Cl}^+(\mathcal{O}_C)}\chi_Q(t)L_{[t]}(\varphi) \\ &= \ell_1\ell_2\sum_{[t]\in\mathrm{Cl}^+(\mathcal{O}_C)}\chi_Q(t)\left|\mathrm{N}(t)\right|_{\mathbf{A}}^{\frac{k-2}{2}}\sum_{[u]\in\widehat{\mathcal{O}}_C^\times/\widehat{\mathcal{O}}_{Cp^n}^\times}\chi_Q(u)\eta_f(D_{tu}\varsigma_p^{(n)})(P_\Psi^{\frac{k-2}{2}}).\end{aligned}$$

On the other hand, if we replace the base point r by $\Psi(\delta)r$, noting that $\mathrm{N}(\delta)<0$, we obtain that

$$\begin{aligned}B_{\breve{\boldsymbol{f}}_Q,\chi_Q}(\varsigma^{(Cp^n)}) &= \ell_1\ell_2\sum_{[t]\in\mathrm{Cl}^+(\mathcal{O}_C)}\chi_Q(t\delta_{\mathrm{f}})\left|\mathrm{N}(t)\right|_{\mathbf{A}}^{\frac{k-2}{2}} \\ &\times\sum_{[u]\in\widehat{\mathcal{O}}_C^\times/\widehat{\mathcal{O}}_{Cp^n}^\times}\chi_Q(u)(-1)^{\frac{k-2}{2}}[\mathbf{c}]\eta_f(D_{tu}\cdot\varsigma_p^{(n)})(P_\Psi^{\frac{k-2}{2}}),\end{aligned}$$

where $[\mathbf{c}]$ is the involution on classical modular symbols. Since $\chi_Q(\delta_{\mathrm{f}}) = (-1)^{\frac{k-2}{2}}\phi_\infty(\delta) = (-1)^{\frac{k+j}{2}-1}$, we conclude that

$$\begin{aligned}B_\varphi^{\chi_Q}(\varsigma^{(Cp^n)}) =&\ell_1\ell_2\sum_{[t]\in\mathrm{Cl}^+(\mathcal{O}_C)}\chi_Q(t)\left|\mathrm{N}(t)\right|_{\mathbf{A}}^{\frac{k-2}{2}} \\ &\times\sum_{[u]\in\widehat{\mathcal{O}}_C^\times/\widehat{\mathcal{O}}_{Cp^n}^\times}\chi_Q(u)\eta_f^{(-1)^{\frac{j}{2}}}(D_{tu}\varsigma_p^{(n)})(P_\Psi^{\frac{k-2}{2}}).\end{aligned}\tag{6.11}$$

Now we proceed to work on the left hand side of the assertion

$$\mathcal{L}_{\breve{\Xi}^\pm/F}(Q)=\sum_{[t]\in F^\times\backslash\widehat{F}^\times/\widehat{\mathcal{O}}_C^\times}\chi_Q(t)\,|\mathrm{N}(t)|_{\mathbf{A}}^{\frac{k-2}{2}}\,\mathrm{N}(t_p)^{\frac{k-2}{2}}\cdot\mathscr{L}_{\breve{\Xi}^\pm}(D_t)(Q).$$

Put $\mho_Q=\frac{\mathrm{Er}^\pm(\Xi_Q)}{\Omega^\pm_{\boldsymbol{f}_Q}}$. In view of (6.11), we need to verify the following interpolation formula

$$(6.12)\qquad \begin{aligned}\mathcal{L}_{\breve{\Xi}^\pm}(D_t)(Q)=&\mho_Q\alpha_{\boldsymbol{f}_Q}^{-n}\mathrm{N}(t_p)^{\frac{2-k}{2}}\\ &\times\sum_{[u]\in\widehat{\mathcal{O}}_C^\times/\widehat{\mathcal{O}}_{Cp^n}^\times}\omega^{-\frac{1}{2}}(\mathrm{N}(u))\eta_f^+(D_{tu}\varsigma_p^{(n)})(P_\Psi^{\frac{k-2}{2}}),\end{aligned}$$

where $\varsigma_p^{(n)}=\begin{pmatrix}p^n&-1\\0&1\end{pmatrix}$ if p is split, and $\varsigma_p^{(n)}=\begin{pmatrix}0&1\\-p^n&0\end{pmatrix}$ if p is inert. For $d\mid C$, it is straightforward to verify that $V_d\Xi_Q^\pm=\mho_Q\cdot\xi^\pm_{V_d\boldsymbol{f}_Q}$, and hence $\breve{\Xi}_Q^\pm=\mho_Q\cdot\xi_f^\pm$. It follows that for $D=(\{r\}-\{s\},g_{\mathrm{f}})\in\mathfrak{D}_0$ and $P(X,Y)\in L_{k_Q-2}(\mathbf{Z}_p)$, we have

$$(6.13)\qquad \begin{aligned}&Q\left(\int_{a+p^n\mathbf{Z}_p}P(x,1)\mu_{\breve{\Xi}^\pm}(D)(x)\right)\\ =&\alpha_{\boldsymbol{f}_Q}^{-n}\cdot\breve{\Xi}_Q^\pm\left(D\begin{pmatrix}p^n&a\\0&1\end{pmatrix}\right)\left(P|\begin{pmatrix}p^n&a\\0&1\end{pmatrix}\right)\ \text{ by (6.6)}\\ =&\mho_Q\cdot\alpha_{\boldsymbol{f}_Q}^{-n}\eta_f^\pm\left(D\begin{pmatrix}p^n&a\\0&1\end{pmatrix}\right)(P|g_p^{-1})\qquad\text{by (6.3).}\end{aligned}$$

Now we verify (6.12) in the case where p is split in F. By Definition 6.8, $\mathcal{L}_{\breve{\Xi}^\pm}(D_t)(Q)$ equals

$$\begin{aligned}&\sum_{a\in(\mathbf{Z}_p/p^n\mathbf{Z}_p)^\times}\omega_p^{-\frac{1}{2}}(a)Q\left(\int_{a+p^n\mathbf{Z}_p}x^{\frac{k-2}{2}}\mu_{\breve{\Xi}^\pm}(D_t)(x)\right)\quad(Q(\vartheta_X(x))=\omega_p^{-\frac{1}{2}}(x)x^{\frac{k-2}{2}})\\ =&\mho_Q\cdot\alpha_{\boldsymbol{f}_Q}^{-n}\sum_{a\in(\mathbf{Z}_p/p^n\mathbf{Z}_p)^\times}\omega_p^{-\frac{1}{2}}(-a)\eta_f^\pm\left(D_t\cdot\begin{pmatrix}p^n&-a\\0&1\end{pmatrix}\right)((XY)^{\frac{k-2}{2}}|\varsigma_p^{-1}\Psi(t_p^{-1}))\end{aligned}$$

by (6.13). Then (6.12) follows from the equations $\omega_p^{\frac{1}{2}}(-1)=(-1)^{\frac{k-2}{2}}$, and

$$(XY)|\varsigma_p^{-1}\Psi(t_p^{-1})=(XY)|\begin{pmatrix}t_{\overline{\mathfrak{p}}}&0\\0&t_{\mathfrak{p}}\end{pmatrix}\begin{pmatrix}1&-\theta\\-1&\overline{\theta}\end{pmatrix}=-\mathrm{N}(t_p^{-1})\cdot P_\Psi(X,Y).$$

In the inert case, $\mathcal{L}_{\breve{\Xi}^{\pm}}(D_t)(Q)$ equlas

$$\begin{aligned}&\sum_{a=0}^{p^n-1}\omega^{-\frac{1}{2}}(\mathrm{N}(a-\theta))\int_{a+p^n\mathbf{Z}_p}\mathrm{N}(x-\theta)^{\frac{k-2}{2}}\mu_{\breve{\Xi}^{\pm}_Q}(D_t)(x)\\ &+\alpha_{\boldsymbol{f}_Q}^{-1}\sum_{a=0}^{p^{n-1}-1}\omega^{-\frac{1}{2}}(\mathrm{N}(1+pa\theta))\int_{a+p^{n-1}\mathbf{Z}_p}\mathrm{N}(1-px\theta)^{\frac{k-2}{2}}\mu_{\breve{\Xi}^{\pm}_Q}\left(D_t\begin{pmatrix}0&1\\-p&0\end{pmatrix}\right)(x)\\ =&\mathrm{N}(t_p)^{\frac{2-k}{2}}\mho_Q\alpha_{\boldsymbol{f}_Q}^{-n}\sum_{a=0}^{p^n-1}\omega^{-\frac{1}{2}}(\mathrm{N}(a-\overline{\theta}))\eta_f^{\pm}\left(D_t\begin{pmatrix}p^n&a\\0&1\end{pmatrix}\right)(P_\Psi^{\frac{k-2}{2}})\\ &+\mathrm{N}(t_p)^{\frac{2-k}{2}}\alpha_{\boldsymbol{f}_Q}^{-n}\mho_Q\sum_{a=0}^{p^{n-1}-1}\omega^{-\frac{1}{2}}(\mathrm{N}(1+pa\overline{\theta}))\eta_f^{\pm}\left(D_t\begin{pmatrix}0&1\\-p&0\end{pmatrix}\begin{pmatrix}p^{n-1}&a\\0&1\end{pmatrix}\right)(P_\Psi^{\frac{k-2}{2}}).\end{aligned}$$

We thus obtain (6.12) from the observations below

$$\begin{aligned}\begin{pmatrix}p^n&a\\0&1\end{pmatrix}U_1(p^n)&=\Psi(a-\overline{\theta})\begin{pmatrix}0&1\\-p^n&0\end{pmatrix}U_1(p^n),\\ \begin{pmatrix}0&1\\-p&0\end{pmatrix}\begin{pmatrix}p^{n-1}&a\\0&1\end{pmatrix}U_1(p^n)&=\Psi(1+pa\overline{\theta})\begin{pmatrix}0&1\\-p^n&0\end{pmatrix}U_1(p^n).\end{aligned}$$

This verifies (6.12) in both cases and finishes the proof. □

7. The factorization of p-adic L-functions and Stark-Heegner points

7.1. In this section, we show the twisted triple product p-adic L-function $\mathcal{L}_{\boldsymbol{E}_\phi^{[a]},\boldsymbol{f}}$ in Definition 5.5 can be essentially factorized into a product of the square-root p-adic L-function $\mathcal{L}_{\breve{\Xi}^-/F\otimes\phi}$ for $\boldsymbol{f}$ over F and the Mazur-Kitagawa p-adic L-function $L_p(\Xi^+,\boldsymbol{\omega}^a)$.

Theorem 7.1. *Let a be an even integer. Let $\boldsymbol{f}\in e\mathbf{S}(N,1,\mathbf{I})$ be a primitive Hida family of tame conductor N and $\phi:\mathrm{Cl}^+(\mathcal{O}_C)\to\mathcal{O}^\times$ be an odd character of the exact conductor C. For every $\Xi\in e\mathcal{MS}(N,\mathbf{I})[\boldsymbol{f}]$ and an even integer a, there is an element $C_\Xi\in\mathrm{Frac}\,\mathbf{I}$ such that*

$$C_\Xi\cdot\mathcal{L}_{\boldsymbol{E}_\phi^{[a]},\boldsymbol{f}}=\mathcal{L}_{\breve{\Xi}^-/F\otimes\phi}\cdot L_p(\Xi^+,\boldsymbol{\omega}^a)\cdot\mathfrak{f}c_1,$$

where $\mathfrak{f}\in(\Lambda\widehat{\otimes}\Lambda)^\times$ and the constant $c_1\in\overline{\mathbf{Z}}_{(p)}^\times$ are defined in Proposition 5.7 with $j=2$. Moreover, $C_\Xi\in\mathrm{Frac}\,\mathbf{I}$ is holomorphic at every arithmetic point $Q\in\mathfrak{X}_{\mathbf{I}}^+$ with the value $C_\Xi(Q)=\frac{\mathrm{Per}^\dagger(\boldsymbol{f}_Q)}{\Omega^+_{\boldsymbol{f}_Q}\Omega^-_{\boldsymbol{f}_Q}}\cdot\mathrm{Er}^+(\Xi_Q)\mathrm{Er}^-(\Xi_Q)$.

Proof. For a primitive Hida family $\boldsymbol{g}\in\mathbf{J}[\![q]\!]$ for some normal domain $\mathbf{J}$ finite over Λ, let

$$L_p^{\boldsymbol{f}}(\boldsymbol{f}\otimes\boldsymbol{g})\in(\mathbf{I}\widehat{\otimes}\mathbf{J}\widehat{\otimes}\Lambda)\otimes_{\mathbf{I}}\mathrm{Frac}\,\mathbf{I}$$

be the primitive Hida's three-variable Rankin-Selberg p-adic L-function associated with $\boldsymbol{f}$ and $\boldsymbol{g}$. For each point $(Q_1, Q_2, P) \in \mathfrak{X}_{\mathbf{I}}^+ \times \mathfrak{X}_{\mathbf{J}}^+ \times \mathfrak{X}_{\Lambda}^+$ with $k_{Q_2} < k_P \le k_{Q_1}$, this p-adic L-function enjoys the interpolation formula:

$$\begin{aligned} &L_p^{\boldsymbol{f}}(\boldsymbol{f} \otimes \boldsymbol{g})(Q_1, Q_2, P) \\ &= (\sqrt{-1})^{1+k_{Q_2}-2k_P} \cdot \frac{L^{\{p\}}(k_P - \frac{k_{Q1}+k_{Q_2}}{2}, \pi_{\boldsymbol{f}_{Q_1}} \times \pi_{\boldsymbol{g}_{Q_2}} \otimes \boldsymbol{\omega}^{-k_P})}{\mathrm{Per}^\dagger(\boldsymbol{f}_{Q_1})} \\ &\times \gamma\left(k_P - \frac{k_{Q1}+k_{Q_2}}{2}, \varrho_{\boldsymbol{f}_{Q_1},p} \otimes \pi_{\boldsymbol{g}_{Q_2}}\right)^{-1}, \end{aligned} \tag{7.1}$$

where $\varrho_{\boldsymbol{f}_{Q_1},p} : \mathbf{Q}_p^\times \to \mathbf{C}^\times$ is the unramified character defined by (5.7). (See [CH20, Theorem 7.1] for the above form of the interpolation formula). Choose a Dirichlet character χ with $\chi(-1) = -1$ and an imaginary quadratic field K where p is split. Let $\chi_K := \chi \circ \mathrm{N}_{K/\mathbf{Q}}$ be a finite order Hecke character of $\mathbf{A}_K^\times$. Let $\boldsymbol{g}$ denote a primitive Hida family such that the weight one specialization $\boldsymbol{g}_{Q_0}$ is a p-stablized theta series $\theta_{\chi_K}^\dagger$ associated with χ_K. Define the two-variable p-adic L-function $L_p(\boldsymbol{f}_{/K} \otimes \chi_K)$ by

$$L_p(\boldsymbol{f}_{/K} \otimes \chi_K) := (1 \otimes Q_0 \otimes 1)\big(L_p^{\boldsymbol{f}}(\boldsymbol{f} \otimes \boldsymbol{g})\big) \in \mathbf{I}\widehat{\otimes}\Lambda.$$

Let $\mathfrak{X}_\Lambda^{(2)}$ be the set of arithmetic points P of weight $k_P = 2$. For $P \in \mathfrak{X}_\Lambda^{(2)}$, define

$$C_{\Xi,P} := (1 \otimes P)\left(\frac{L_p(\Xi^-, \chi)L_p(\Xi^+, \chi\tau_{K/\mathbf{Q}})}{L_p(\boldsymbol{f}_{/K} \otimes \chi_K)}\right) \in \mathrm{Frac}(\mathbf{I} \otimes_{\mathcal{O}} \mathcal{O}(P)).$$

Let P_0 be the point with $k_{P_0} = 2$ and $\epsilon_{P_0} = 1$ and set $C_\Xi = C_{\Xi,P_0} \in \mathrm{Frac}\,\mathbf{I}$. Let $P \in \mathfrak{X}_\Lambda^{(2)}$. From the interpolation formulae (6.9) and (7.1), we see that

$$C_\Xi(Q) = C_{\Xi,P}(Q) = \frac{\mathrm{Per}^\dagger(\boldsymbol{f}_Q)}{\Omega_{\boldsymbol{f}_Q}^+ \Omega_{\boldsymbol{f}_Q}^-} \cdot \mathrm{Er}^+(\Xi_Q)\mathrm{Er}^-(\Xi_Q)$$

for all $Q \in \mathfrak{X}_{\mathbf{I}}^+$ with $k_Q > 2$, so we can conclude that $C_\Xi = C_{\Xi,P}$ for all $P \in \mathfrak{X}_\Lambda^{(2)}$. Thanks to a result of Rohrlich [Roh84], for any arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}}^+$, there exists a point $P_0 \in \mathfrak{X}_\Lambda^{(2)}$ such that $L_p(\boldsymbol{f}_{/K} \otimes \chi_K)(Q, P_0) \neq 0$. This implies that $C_\Xi = C_{\Xi,P_0}$ is holomorphic at Q. Now the theorem follows immediately from the interpolation formulae in Propositions 5.7, 6.9 and (6.9). □

Remark 7.2. If the residual Galois representation associated with $\boldsymbol{f}$ is absolutely irreducible and p-distinguished, then the Gorensteiness of the local component of the Hecke algebra $\mathbf{T}^{\mathrm{ord}}(N)$ corresponding to $\boldsymbol{f}$ is known thanks to the work of Wiles, et.al. It follows that the $\mathbf{I}$-module $e\mathcal{MS}(N, \mathbf{I})^\pm[\boldsymbol{f}]$ is free of rank one by [Kit94, Lemma 5.11]. there exist a $\mathbf{I}$-adic modular symbol $\Xi \in e\mathcal{MS}(N, \mathbf{I})^\pm[\boldsymbol{f}]$. Choose a basis $\Xi^\pm$ in each space and put $\Xi = \Xi^+ + \Xi^-$. Then p-adic error terms $\mathrm{Er}^\pm(\Xi_Q)$ are p-adic units for all $Q \in \mathfrak{X}_{\mathbf{I}}^+$ by [Kit94, Proposition 5.12], and C_Ξ is a generator of the congruence ideal $C(\boldsymbol{f})$ by a result of Hida [Hid88b, Theorem 0.1].

7.2. **The derivative of the twisted triple product p-adic L-functions.** We shall keep the notation in Remark 6.6. Let E be an elliptic curve over $\mathbf{Q}$ of conductor pN. There exists a primitive Hida family $\boldsymbol{f} \in \mathbf{I}[\![q]\!]$ such that $\boldsymbol{f}_Q$ is the elliptic newform f associated with E for some weight two point $Q \in \mathfrak{X}_{\mathbf{I}}^+$. Here $\mathbf{I}$ is the local component of $\mathbf{T}^{\mathrm{ord}}(N)$ corresponding to λ_f. Let $\mathscr{X} = \{k \in \mathbf{C}_p \mid |k|_p \leq 1\}$ and write $j : \mathfrak{X} \hookrightarrow \operatorname{Spec} \Lambda(\mathbf{C}_p)$ for the map $k \mapsto (Q_k : [x] \mapsto x^k)$. Let $\wp_2$ be the kernel of Q_2. Then we have $\mathbf{I} \subset \mathbf{I}_{\wp_Q} = \Lambda_{\wp_2}$ since f has rational coefficients. This implies that there exists a neighborhood $\mathscr{U}$ around $2 \in \mathscr{X}$ such that $j : \mathscr{U} \hookrightarrow \operatorname{Spec} \mathbf{I}(\mathbf{C}_p)$. We put

$$\mathcal{L}_{\boldsymbol{E}_\phi^{[a]},\boldsymbol{f}}(k,s) = \mathcal{L}_{\boldsymbol{E}_\phi^{[a]},\boldsymbol{f}}(Q_k, Q_s), \quad (k,s) \in \mathscr{U} \times \mathscr{X}.$$

Corollary 7.3. *Suppose that p is inert in F and $\phi : \mathrm{Cl}^+(\mathcal{O}_F) \to \mathcal{O}^\times$ is an odd narrow ideal class character, i.e. $C = 1$. Let $w_N \in \{\pm 1\}$ be the sign of the Fricke involution at N. Then we have $\mathcal{L}_{\boldsymbol{E}_\phi^{[a]}}(2,s) = 0$ and*

$$\frac{d}{dk}\big(\mathcal{L}_{\boldsymbol{E}_\phi^{[2]},\boldsymbol{f}}(k,s+1)\big)|_{k=2} = \frac{1}{2}(1+\phi(\mathfrak{N})^{-1}w_N)\log_E P_\phi \cdot \frac{c_f \cdot L_p(E,s)\,\langle \Delta_F \rangle^{\frac{s-1}{2}}}{m_E^2 2^{\alpha(E)}},$$

where

- $P_\phi \in E(F_p) \otimes \mathbf{Q}(\phi)$ *is the Stark-Heegner point in* [Dar01, (182)],
- $\sigma_{\mathfrak{N}} \in \widehat{F}^\times$ *is some finite idele such that* $(\sigma_{\mathfrak{N}}\widehat{\mathcal{O}}_F \cap F) = \mathfrak{N}$,
- $L_p(E,s)$ *is the Mazur-Tate-Titelbaum p-adic L-function for E.*
- $c_f \in \mathbf{Z}_{>0}$ *is the congruence number for f, $m_E \in \mathbf{Q}^\times$ is the Mainn constant for E and* $2^{\alpha(E)} = [\mathrm{H}_1(E(\mathbf{C}),\mathbf{Z}) : \mathrm{H}_1(E(\mathbf{C}),\mathbf{Z})^+ \oplus \mathrm{H}_1(E(\mathbf{C}),\mathbf{Z})^-]$.

Proof. For each $\Xi \in e\mathcal{MS}(N,\Lambda) \otimes_\Lambda A(\mathscr{U})[\boldsymbol{f}]$, put

$$\mathcal{L}_p(\Xi/F,\phi,k) = \mathcal{L}_{\Xi^-/F\otimes\phi}(Q_k), \quad k \in \mathfrak{X}.$$

Shrinking $\mathscr{U}$ if necessary, we may assume that the function $\mathcal{L}_p(\Xi/F,\phi,k)$ is analytic at $k \in \mathscr{U}$. Since π_f is special at p and p is inert in F, it is well-known that the local root number of the base change $\mathrm{BC}_F(\pi_f) \otimes \phi$ is -1, and hence the toric period $B_f^\phi = 0$ must vanish by a classic theorem of Saito and Tunnell. We obtain $\mathcal{L}_p(\Xi/F,\phi,2) = 0$ in view of Proposition 6.9, and hence $\mathcal{L}_{\boldsymbol{E}_\phi^{[a]},\boldsymbol{f}}(2,s) = 0$ for all even a. By Theorem 7.1,

$$\begin{aligned} &C_\Xi(2)\frac{d}{dk}\big(\mathcal{L}_{\boldsymbol{E}_\phi^{[2]},\boldsymbol{f}}(k,s+1)\big)|_{k=2} \\ =&\frac{d}{dk}\big(\mathcal{L}_p(\Xi/F,\phi,k)\big)|_{k=2} \cdot L_p(\Xi^+,\boldsymbol{\omega}^2)(2,s+1). \end{aligned} \tag{7.2}$$

To get the derivative formula, we first compute the derivative of $\mathcal{L}_p(\Xi/F,\phi,k)$ at $k=2$ for suitable normalized Ξ. Let $\mu_{\Xi^-}^{\mathrm{GS}}(x,y)$ be the p-adic measure on

L'_0 attached to Ξ^- introduced in Remark 6.6. By definition, we have the expression

$$\mathcal{L}_p(\Xi/F,\phi,k)=\sum_{[t]\in\mathrm{Cl}^+(\mathcal{O}_C)}\phi(t)\left\langle\boldsymbol{\varepsilon}_{\mathrm{cyc}}(\mathrm{N}(t))\right\rangle^{\frac{k-2}{2}}$$
$$\times\int_{L'_0}\left\langle(x-\theta y)(x-\overline{\theta}y)\right\rangle^{\frac{k-2}{2}}\mu^{\mathrm{GS}}_{\Xi^-}(\{r\}-\{\Psi(\epsilon_1)r\},\Psi(t)\varsigma_{\mathrm{f}})(x,y).$$

Here ϵ_1 is the totally positive fundamental unit in $\mathcal{O}_F^\times$ and ς_{f} is the finite part of $\varsigma\in\mathrm{GL}_2(\widehat{\mathbf{Q}})$ defined in §4.1. Choosing a branch of p-adic logarithm $\log:F_p^\times\to F_p$, we obtain

$$\frac{d}{dk}\big(\mathcal{L}_p(\Xi/F,\phi,k)\big)|_{k=2}=\frac{1}{2}\sum_{[t]\in\mathrm{Cl}^+(\mathcal{O}_F)}\phi(t)(J_\theta[t]+J_{\overline{\theta}}[t]),\tag{7.3}$$

where for $\tau\in\mathbf{C}_p$ with $\tau\notin\mathbf{Q}_p$,

$$J_\tau[t]:=\int_{L'_0}\log(x-\tau y)\mu^{\mathrm{GS}}_{\Xi^-}(\{r\}-\{\Psi(\epsilon_1)r\},\Psi(t)\varsigma_{\mathrm{f}})(x,y).$$

Let $\mathcal{J}=\begin{pmatrix}-1&\mathrm{T}(\theta)\\0&1\end{pmatrix}\in\mathrm{GL}_2(\mathbf{Q})\hookrightarrow\mathrm{GL}_2(\widehat{\mathbf{Q}})$. Write $\mathcal{J}_p$ and $\mathcal{J}^{(p)}$ for its image in $\mathrm{GL}_2(\mathbf{Q}_p)$ and $\mathrm{GL}_2(\widehat{\mathbf{Q}}^{(p)})$ respectively and let $\tau_N=\begin{pmatrix}0&1\\-N&0\end{pmatrix}\in\mathrm{GL}_2(\widehat{\mathbf{Q}}^{(p)})$ be the Fricke involution at N. Since $\mathcal{J}^2=1$ and $\varsigma_p=1$, one verifies that

$$\varsigma\mathcal{J}_p=\mathcal{J}\mathcal{J}^{(p)}\varsigma_{\mathrm{f}}=\mathcal{J}\Psi(\sigma_{\mathfrak{N}})\varsigma_{\mathrm{f}}\cdot\tau_N$$

for an appropriate choice of $\sigma_{\mathfrak{N}}$. It follows from $\mathcal{J}(\theta)=\overline{\theta}$ and the $U_0(N)$-invariance (6.8) that

$$\begin{aligned}J_{\overline{\theta}}[t]&=\int_{L'_0}\log_p(x-\theta y)\mu^{\mathrm{GS}}_{\Xi^-}(\{r\}-\{\Psi(\epsilon_1)r\},\Psi(t)\varsigma\mathcal{J}_p)(x,y)\\&=\int_{L'_0}\log(x-\theta y)\mu^{\mathrm{GS}}_{\Xi^-}([\mathbf{c}](\{r\}-\left\{\Psi(\epsilon_1^{-1})r\right\},\Psi(t\sigma_{\mathfrak{N}})\varsigma\tau_N))(x,y)\\&=(-1)\cdot(-1)\cdot w_N\cdot J_\theta[t\sigma_{\mathfrak{N}}\epsilon_1].\end{aligned}\tag{7.4}$$

Now we fix the normalization of Ξ. The $\Lambda_{\wp_2}$-module

$$e\mathcal{MS}(N,\mathbf{I})^\pm[\boldsymbol{f}]\otimes_{\mathbf{I}}\mathbf{I}_{\wp_Q}=(e\mathcal{MS}(N,\Lambda)^\pm\otimes\Lambda_{\wp_2})[\boldsymbol{f}]$$

is free of rank one. Let $\Xi^\pm$ be the basis normalized so that the weight two specialization $\Xi^\pm_Q=\frac{\xi_f^\pm}{\Omega^\pm}$ with the periods $\Omega^\pm=(2\pi\sqrt{-1})^{-1}\Omega_E^\pm$, where $\Omega_E^\pm$ are the plus/minus periods for E such that Ω_E^+ and $(\sqrt{-1})^{-1}\Omega_E^-$ are real and positive. With this choice of periods, it straightforward to deduce from

[BD09, Corollary 2.6] that the p-adic logarithm of Stark-Heegner point P_ϕ is given by

$$\log_E P_\phi = \sum_{[t]\in \mathrm{Cl}^+(\mathcal{O}_F)} \phi(t) J_\theta[t].$$

We thus obtain from (7.3) and (7.4) that

$$\frac{d}{dk}\mathcal{L}_p(\Xi/F,\phi,k)|_{k=2} = 2^{-1}(1+\phi(\sigma_{\mathfrak{N}})w_N)\log_E P_\phi.$$

By the inspection on the interpolation (6.9), we see easily that the associated Mazur-Kitagawa p-adic L-function $L_p(\Xi^+,\boldsymbol{\omega}^2)(2,s+1)$ is the cyclotomic p-adic L-function $2L_p(E,s)$ for the elliptic curve E. This extra 2 comes from the factor 2 in the definition of the archimedean Γ-factor $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. On the other hand, it is clear that $\mathfrak{f}(2,s+1)c_1 = 4\langle \Delta_F\rangle^{\frac{s-1}{2}}$ with $a=j=2$, and by the formulae in [Hid81, p.255],

$$\|f\|^2_{\Gamma_0(N)} = c_f m_E^{-2} 2^{-2-\alpha(E)}\pi^{-2}(\sqrt{-1})^{-1}\Omega_E^+\Omega_E^-.$$

We thus obtain

$$C_\Xi(2) = \frac{\mathrm{Per}^\dagger(f)}{\Omega^+\Omega^-} = \frac{-(-2\sqrt{-1})^3\|f\|^2_{\Gamma_0(N)}}{(-4\pi^2)^{-1}\Omega_E^+\Omega_E^-} = \frac{8c_f}{m_E^2 2^{\alpha(E)}}.$$

Putting these together, we get the corollary from (7.2). □

Remark 7.4. The same argument applies to more general ring class characters with split conductor (i.e. $C \neq 1$ is a product of primes split in F), but the formulae are more complicated due to the non-canonical choice of the test vector $\breve{\Xi}$ in the construction of $\mathcal{L}_{\breve{\Xi}^-/F\otimes\phi}$.

References

[BD09] Massimo Bertolini and Henri Darmon, *The rationality of Stark-Heegner points over genus fields of real quadratic fields*, Ann. of Math. (2) **170** (2009), no. 1, 343–370. MR 2521118

[Bum98] D. Bump, *Automorphic forms and representations*, first ed., Cambridge Studies in Advanced Mathematics 55, Cambridge University Press, 1998.

[Cas73] W. Casselman, *On some results of Atkin and Lehner*, Mathematische Annalen **201** (1973), 301–314.

[CH20] Shih-Yu Chen and M.-L. Hsieh, *On primitive p-adic rankin-selberg l-functions*, To appear in Development of Iwasawa theory - The Centennial of K. Iwasawa's Birth, Advanced Studies in Pure Mathematics,.

[Dar01] Henri Darmon, *Integration on $\mathscr{H}_p \times \mathscr{H}$ and arithmetic applications*, Ann. of Math. (2) **154** (2001), no. 3, 589–639. MR 1884617

[DD06] Henri Darmon and Samit Dasgupta, *Elliptic units for real quadratic fields*, Ann. of Math. (2) **163** (2006), no. 1, 301–346. MR 2195136

[DPV19] Henri Darmon, Alice Pozzi, and Jan Vonk, *Gross-Stark units, Stark-heegner points, and derivatives of p-adic Eisenstein series*, Preprint. Available at http://www.math.mcgill.ca/darmon/pub/pub.html.

[GS93] R. Greenberg and G. Stevens, *p-adic L-functions and p-adic periods of modular forms*, Inventiones mathematiace **111** (1993), 407–447.

[Hid81] Haruzo Hida, *Congruence of cusp forms and special values of their zeta functions*, Invent. Math. **63** (1981), no. 2, 225–261. MR 610538

[Hid88a] H. Hida, *A p-adic measure attached to the zeta functions associated with two elliptic modular forms. II*, Annales de l'institut Fourier **38** (1988), no. 3, 1–83.

[Hid88b] ———, *Modules of congruence of Hecke algebras and L-functions associated with cusp forms*, American Journal of Mathematics **110** (1988), no. 2, 323–382.

[Hid93] ———, *Elementary Theory of L-functions and Eisenstein series*, London Mathematical Society Student Texts, vol. 26, Cambridge University Press, 1993.

[Hsi20] M.-L. Hsieh, *Hida families and p-adic triple product L-functions*, To appear in American Journal of Mathematics, arXiv:1705.02717.

[HY20] M.-L. Hsieh and S. Yamana, *Four variable p-adic triple product L-functions and the trivial zero conjecture*, submitted. arXiv:1906.10474.

[Jac72] H. Jacquet, *Automorphic Forms on* $\mathrm{GL}(2)$ *II*, Lecture Notes in Mathematics, vol. 278, Springer-Verlag, Berlin and New York, 1972.

[JL70] H. Jacquet and R. Langlands, *Automorphic forms on* $\mathrm{gl}(2)$, Lecture Notes in Mathematics, vol. 114, Springer-Verlag, Berlin and New York, 1970.

[Kit94] Koji Kitagawa, *On standard p-adic L-functions of families of elliptic cusp forms*, p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), Contemp. Math., vol. 165, Amer. Math. Soc., Providence, RI, 1994, pp. 81–110. MR 1279604

[KP19] Rodney Keaton and Ameya Pitale, *Restrictions of Eisenstein series and Rankin-Selberg convolution*, Doc. Math. **24** (2019), 1–45. MR 3935491

[Roh84] David E. Rohrlich, *On L-functions of elliptic curves and cyclotomic towers*, Invent. Math. **75** (1984), no. 3, 409–423. MR 735333

[Sah16] Abhishek Saha, *Large values of newforms on* $\mathrm{GL}(2)$ *with highly ramified central character*, Int. Math. Res. Not. IMRN (2016), no. 13, 4103–4131. MR 3544630

[Sch02] R. Schmidt, *Some remarks on local newforms for* $\mathrm{GL}(2)$, Journal of Ramanujan Mathematical Society **17** (2002), no. 2, 115–147.

[Shi77] G. Shimura, *On the periods of modular forms*, Mathematische Annalen **229** (1977), 211–221.

[Wil88] A. Wiles, *On ordinary λ-adic representations associated to modular forms*, Inventiones mathematiace **94** (1988), no. 3, 529–573.

Institute of Mathematics, Academia Sinica, Taipei 10617, Taiwan
Email address: mlhsieh@math.sinica.edu.tw

Osaka City University Advanced Mathematical Institute, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585 JAPAN
Email address: yamana@sci.osaka-cu.ac.jp