SHARP HARDY-LERAY INEQUALITY FOR SOLENOIDAL FIELDS

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ABSTRACT. This paper refines the former work by Costin-Maz'ya [4], who computed the best constant of Hardy-Leray inequality for solenoidal vector fields on $\mathbb{R}^N$ under the additional assumption of axisymmetry for $N \geq 3$. We derive the same best constant without any symmetry assumption; this is also a higher-dimensional extension of the previous work [5] in the three-dimensional case. Moreover, we provide some information about the non-attainability of the equality sign.

1. INTRODUCTION

Throughout this paper, N denotes an integer and $N \geq 3$. From the viewpoint of standard vector calculus on $\mathbb{R}^N$, we study the functional inequality for vector fields together with its improvement, called the Hardy-Leray inequality.

We use bold letters to denote vectors, say $\boldsymbol{x} = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$. The notation $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{k=1}^N x_k y_k$ denotes the standard inner product of two vectors, and we set $|\boldsymbol{x}| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$ as the length of $\boldsymbol{x}$. By writing $\boldsymbol{u} \in C_c^\infty(\Omega)^N$ for any open subset Ω of $\mathbb{R}^N$, we mean that

$$\boldsymbol{u} : \Omega \to \mathbb{R}^N, \qquad \boldsymbol{x} \mapsto \boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), \cdots, u_N(\boldsymbol{x}))$$

is a smooth vector field with compact support on Ω.

§1.1. **Preceding results and motivation.** The classical Hardy-Leray inequality (or shortly H-L inequality) on $\mathbb{R}^N$ is given by

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2} dx \leq \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 dx$$

for a vector field $\boldsymbol{u}$ together with its gradient field $\nabla \boldsymbol{u}$, where the constant number $\left(\frac{N-2}{2}\right)^2$ is known to be sharp as the test field $\boldsymbol{u}$ runs over $C_c^\infty(\mathbb{R}^N)^N$. This inequality was shown by J. Leray [9] for $N = 3$ along his study on the Navier-Stokes equations, as an extension of the 1-dimensional inequality by G. H. Hardy [8].

Now, we are interested in the problem whether the best constant of the H-L inequality can be changed to exceed $\left(\frac{N-2}{2}\right)^2$, by imposing $\boldsymbol{u}$ to be *solenoidal* (namely divergence-free). This is a natural question in the context of hydrodynamics, as asked by O. Costin and V. G. Maz'ya [4]; they derived the improved H-L inequality

$$\left(\frac{N-2}{2}\right)^2 \left(1 + \frac{8}{N^2 + 4N - 4}\right) \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2} dx \leq \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 dx$$

2010 *Mathematics Subject Classification.* Primary 35A23; Secondary 26D10.

Key words and phrases. Hardy-Leray inequality, Solenoidal, Poloidal, Toroidal, Spherical harmonics, Best constant.

for solenoidal fields $\boldsymbol{u}$ with the new best constant on the left-hand side, under the additional assumption that $\boldsymbol{u}$ is axisymmetric. Here, by saying that a vector field is axisymmetric, we mean that all its components along the cylindrical coordinates depend only on the axial distance and the height. The addition of such a symmetry assumption to the solenoidal condition on $\boldsymbol{u}$ simplifies and helps the calculation of the new best constant, without affecting the "core" part: one can easily check, in the original H-L inequality, that the condition of axisymmetry alone has no effect on changing the best constant from $\left(\frac{N-2}{2}\right)^2$. In that sense, the axisymmetry assumption seems to play a technical rather than essential role. Hence we may also think that it can be weakened or removed, in order to get a "pure" solenoidal improvement of H-L inequality.

In view of this observation, there was an advance in the three-dimensional case: The author of the present paper, in his recent joint work [7] with F. Takahashi, proved (as a corollary of their main theorem) that the case $N = 3$ of Costin-Maz'ya's inequality

$$\frac{25}{68}\int_{\mathbb{R}^3}\frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2}dx \le \int_{\mathbb{R}^3}|\nabla\boldsymbol{u}|^2dx$$

still holds for solenoidal fields $\boldsymbol{u}$ on $\mathbb{R}^3$, by only assuming the azimuthal component (not the full components) of $\boldsymbol{u}$ to be axisymmetric; so to speak, they succeeded in relaxing the axisymmetry assumption. Moreover, this result was further refined in [5], where it was shown that the above inequality does hold for solenoidal fields without any symmetry assumption at all. Hence it follows that the axisymmetry assumption for $N = 3$ is completely removable from the solenoidal improvement of H-L inequality. As a matter of course, then it is expected that the same also applies to the higher dimensional case $N > 3$; this is the main theme of our study.

As a side note, there is another type of improvement: it is also natural to consider the curl-free condition (in place of the solenoidal one) in the treatment of H-L inequality. Some topics related to this issue can be found in [6].

§1.2. **Main result.** In the same fashion as the preceding works, we concern a solenoidal improvement of the H-L inequality with weight,

$$\left(\gamma + \tfrac{N}{2} - 1\right)^2\int_{\mathbb{R}^N}\frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2}|\boldsymbol{x}|^{2\gamma}dx \le \int_{\mathbb{R}^N}|\nabla\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx \qquad (\gamma\in\mathbb{R}),$$

which includes the classical H-L inequality as the special case $\gamma = 0$; historically, the case $\gamma \neq 0$ was found by Caffarelli-Kohn-Nirenberg [3] in a more generalized form.

Now, we state our main result as follows:

Theorem 1.1. *Let $\boldsymbol{u} \in C_c^\infty(\mathbb{R}^N)^N$ be a solenoidal field. We assume the additional condition that $\boldsymbol{u}(\boldsymbol{0}) = \boldsymbol{0}$ if $\gamma \le 1 - \frac{N}{2}$. Then the inequality*

$$C_{N,\gamma}\int_{\mathbb{R}^N}\frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2}|\boldsymbol{x}|^{2\gamma}dx \le \int_{\mathbb{R}^N}|\nabla\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx \tag{1.1}$$

holds with the best constant $C_{N,\gamma}$ expressed as

$$C_{N,\gamma} = \left(\gamma + \frac{N}{2} - 1\right)^2 + \min\left\{N-1,\ 2 + \min_{\tau\ge 0}\left(\tau + \frac{4(N-1)(\gamma-1)}{\tau + N - 1 + (\gamma - \frac{N}{2})^2}\right)\right\}. \tag{1.2}$$

Remark 1.2. *Let us restrict ourselves to the case $\gamma \leq 1$ in Theorem 1.1. Then the inequality* (1.1)*, under the same assumption on $\boldsymbol{u}$, can be strengthened into*

$$C_{N,\gamma}\int_{\mathbb{R}^N}\frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2}|\boldsymbol{x}|^{2\gamma}dx+\mathcal{R}_{N,\gamma}[\boldsymbol{u}]\leq\int_{\mathbb{R}^N}|\nabla\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx \tag{1.3}$$

for the same constant $C_{N,\gamma}=\left(\gamma+\frac{N}{2}-1\right)^2\frac{\left(\gamma-\frac{N}{2}\right)^2+N+1}{\left(\gamma-\frac{N}{2}\right)^2+N-1}$ as (1.2)*, together with the additional nonnegative term $\mathcal{R}_{N,\gamma}[\boldsymbol{u}]$ given by the expression:*

$$\begin{aligned}\mathcal{R}_{N,\gamma}[\boldsymbol{u}]=&\int_{\mathbb{R}^N}\left|\boldsymbol{x}\cdot\nabla\left(|\boldsymbol{x}|^{\gamma+\frac{N}{2}-1}\boldsymbol{u}\right)\right|^2|\boldsymbol{x}|^{-N}dx\\&+\frac{4(1-\gamma)(N-1)}{\left(\gamma-\frac{N}{2}\right)^2+N-1}\int_{\mathbb{R}^N}\left|(\boldsymbol{x}\cdot\nabla)\nabla_\sigma\triangle_\sigma^{-1}\left(|\boldsymbol{x}|^{\gamma+\frac{N}{2}-2}\boldsymbol{x}\cdot\boldsymbol{u}\right)\right|^2|\boldsymbol{x}|^{-N}dx.\end{aligned}$$

Here ∇_σ and $\triangle_\sigma$ are respectively the spherical gradient and spherical Laplacian (§2.1). Moreover, the equality sign of (1.3) *is attained if and only if the equations*

$$-\triangle_\sigma(\boldsymbol{x}\cdot\boldsymbol{u})=(N-1)\boldsymbol{x}\cdot\boldsymbol{u}\quad\text{and}\quad\begin{cases}-\triangle_\sigma\boldsymbol{u}_T=2\,\boldsymbol{u}_T & \text{for } (N,\gamma)=(3,1)\\ \boldsymbol{u}_T=\boldsymbol{0} & \text{otherwise}\end{cases}$$

hold on $\mathbb{R}^N\backslash\{\boldsymbol{0}\}$, where $\boldsymbol{u}_T$ denotes the toroidal part (§3.2) of $\boldsymbol{u}$.

As an easy consequence of this remark, it follows that the equality sign in the inequality (1.1) for $\gamma \leq 1$ is never attained by any solenoidal field $\boldsymbol{u} \not\equiv \boldsymbol{0}$. For $\gamma > 1$, however, we do not have much knowledge about the attainability.

The proof of Theorem 1.1 is parallel to the previous work [5] where the special case $N = 3$ was proved by applying a so-called poloidal-toroidal (or shortly PT) decomposition theorem of solenoidal fields. The PT theorem in our study, which originates from G. Backus [1] on $\mathbb{R}^3$, is still applicable to the case of $\mathbb{R}^N$ ($N \geq 3$), and it enables us to separate the calculation of the best constant $C_{N,\gamma}$ into two computable parts. However, some techniques on $\mathbb{R}^3$, employed in the previous work, is not allowed in the higher-dimensional case: we cannot use the "cross product" of vectors in general $\mathbb{R}^N$, and furthermore, there is no way to represent every toroidal field in terms of a single-scalar potential. To avoid such a difficulty, we derive with a simple proof the spherical zero-mean property of toroidal fields, from which one can easily deduce such as a Poincaré-type estimate.

Incidentally, we also point out that there is an advanced formalization by N. Weck [10], who gave a very general PT theorem in the framework of differential forms. Then the PT theorem in our discussion can be viewed as a simple case of his one, by identifying solenoidal fields with coclosed 1-forms. However, our approach is based on the standard vector calculus and does not need such as differential forms.

The remaining content of this paper is organized as follows: Section 2 reviews vector calculus on $\mathbb{R}^N\backslash\{\boldsymbol{0}\}$ in terms of radial-spherical variables. Section 3 gives a systematic introduction to the concept of PT fields and establishes the PT decomposition theorem on $\mathbb{R}^N$, together with some formulae or estimates. Section 4 gives the proof of Theorem 1.1 (and Remark 1.2), where we compute the best constant $C_{N,\gamma}$ by making full use of the content of Section 3.

2. Standard Vector Calculus on $\dot{\mathbb{R}}^N \cong \mathbb{R}_+ \times \mathbb{S}^{N-1}$

In what follows, we basically use the notations

$$\dot{\mathbb{R}}^N = \{\boldsymbol{x} \in \mathbb{R}^N;\ \boldsymbol{x} \neq \boldsymbol{0}\} \quad \text{and} \quad \mathbb{S}^{N-1} = \big\{\boldsymbol{x} \in \mathbb{R}^N;\ |\boldsymbol{x}| = 1\big\}.$$

for the subsets of $\mathbb{R}^N$. We review gradient or Laplace operators acting on vector fields on $\dot{\mathbb{R}}^N$ and derive some basic formulae, in terms of radial-spherical variables.

§2.1. **Radial-spherical decomposition of operators.** From the viewpoint of differential geometry, $\dot{\mathbb{R}}^N$ is a smooth manifold diffeomorphic to the product of the half line $\mathbb{R}_+ = \{r \in \mathbb{R}\ ;\ r > 0\}$ and the $(N-1)$-dimensional unit sphere $\mathbb{S}^{N-1}$, which we denote by $\dot{\mathbb{R}}^N \cong \mathbb{R}_+ \times \mathbb{S}^{N-1}$. Indeed, every $\boldsymbol{x} \in \dot{\mathbb{R}}^N$ can be uniquely written as

$$\boldsymbol{x} = r\boldsymbol{\sigma}$$

in terms of the radius $r > 0$ and the unit vector $\boldsymbol{\sigma} \in \mathbb{S}^{N-1}$ given by

$$r = |\boldsymbol{x}| \quad \text{and} \quad \boldsymbol{\sigma} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}. \tag{2.1}$$

Now let $\boldsymbol{u} = (u_1, u_2, \cdots, u_N) : \dot{\mathbb{R}}^N \to \mathbb{R}^N$ be a vector field, and let $\boldsymbol{\sigma} : \dot{\mathbb{R}}^N \to \mathbb{S}^{N-1}$ be the unit vector field given by the second equation of (2.1). Then there exists an unique pair of scalar field $u_R \in C^\infty(\dot{\mathbb{R}}^N)$ and vector field $\boldsymbol{u}_S \in C^\infty(\dot{\mathbb{R}}^N)^N$ satisfying

$$\boldsymbol{u} = \boldsymbol{\sigma} u_R + \boldsymbol{u}_S \qquad \text{and} \qquad \boldsymbol{\sigma} \cdot \boldsymbol{u}_S = 0 \qquad \text{on } \dot{\mathbb{R}}^N,$$

which we call the radial-spherical decomposition of $\boldsymbol{u}$.

Here let us consider the following two derivative operators. The gradient operator $\nabla = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_N})$ resp. Laplacian $\triangle = \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2}$ maps every scalar field f to the vector field ∇f resp. scalar field $\triangle f$. In order to extract only the spherical part of them, we introduce two derivatives: the spherical gradient ∇_σ and spherical Laplacian $\triangle_\sigma$ (known as the Laplace-Beltrami operator) are defined for all $f \in C^\infty(\mathbb{S}^{N-1})$ by the formulae

$$\nabla_\sigma f = \nabla \dot{f} \quad \text{and} \quad \triangle_\sigma f = \triangle \dot{f} \qquad \text{on } \mathbb{S}^{N-1},$$

where $\dot{f}(\boldsymbol{x}) = f(\boldsymbol{x}/|\boldsymbol{x}|)$ is the degree-zero homogeneous extension of f. When ∇_σ or $\triangle_\sigma$ acts on any $f \in C^\infty(\dot{\mathbb{R}}^N)$, such an operation is understood by regarding $f(\boldsymbol{x}) = f(r\boldsymbol{\sigma})$ as a function of $\boldsymbol{\sigma} \in \mathbb{S}^{N-1}$ for every fixed radius r. Then it turns out that those operators are related by the well-known identities

$$\nabla = \boldsymbol{\sigma}\partial_r + r^{-1}\nabla_\sigma \quad \text{and} \quad \triangle = \partial_r'\partial_r + r^{-2}\triangle_\sigma. \tag{2.2}$$

Here

$$\partial_r := \boldsymbol{\sigma} \cdot \nabla = \sum_{k=1}^N \frac{x_k}{|\boldsymbol{x}|}\frac{\partial}{\partial x_k} \quad \text{resp.} \quad \partial_r' := \partial_r + \frac{N-1}{r}$$

denotes the radial derivative resp. its skew L^2 adjoint, in the sense that

$$\int_{\mathbb{R}^N} f\partial_r g\, dx = -\int_{\mathbb{R}^N} g\partial_r' f\, dx$$

holds for all $f, g \in C_c^\infty(\dot{\mathbb{R}}^N)$. As a simple application of (2.2), we get the formulae $\nabla r = \boldsymbol{\sigma}$ and

$$\triangle r^\lambda = \alpha_\lambda r^{\lambda-2}, \quad \text{where} \quad \alpha_\lambda := \lambda(\lambda + N - 2) \qquad \forall \lambda \in \mathbb{R}. \tag{2.3}$$

When the gradient or Laplace operator acts on vector fields, such an operation is componentwise: for $\boldsymbol{u} \in C^\infty(\dot{\mathbb{R}}^N)^N$,

$$\begin{aligned} &\nabla \boldsymbol{u} = (\nabla u_1, \cdots, \nabla u_N) \in C^\infty(\dot{\mathbb{R}}^N)^{N\times N} \\ \text{resp.}\quad &\triangle \boldsymbol{u} = (\triangle u_1, \cdots, \triangle u_N) \in C^\infty(\dot{\mathbb{R}}^N)^N, \\ \big(\text{as well as}\quad &\partial_r \boldsymbol{u} = (\partial_r u_1, \cdots, \partial_r u_N) \in C^\infty(\dot{\mathbb{R}}^N)^N, \big) \end{aligned}$$

and the same also applies to ∇_σ resp. $\triangle_\sigma$. The divergence of $\boldsymbol{u}$ is given by $\operatorname{div} \boldsymbol{u} = \nabla \cdot \boldsymbol{u} = \sum_{k=1}^N \partial u_k/\partial x_k$ as the trace part of the matrix field $\nabla \boldsymbol{u}$. The spherical divergence of $\boldsymbol{u}$, which we denote by $\nabla_\sigma \cdot \boldsymbol{u}_S$, is defined as the trace part of $\nabla_\sigma \boldsymbol{u}_S$. Then a direct calculation by using (2.2) yields

$$\nabla_\sigma \cdot \boldsymbol{\sigma} = r^{-1}\nabla_\sigma \cdot (r\boldsymbol{\sigma}) = \nabla \cdot \boldsymbol{x} - \boldsymbol{\sigma}\partial_r \cdot (r\boldsymbol{\sigma}) = N-1,$$

from which we further get

$$\begin{aligned} \operatorname{div} \boldsymbol{u} &= \big(\boldsymbol{\sigma}\partial_r + r^{-1}\nabla_\sigma\big) \cdot (\boldsymbol{\sigma} u_R + \boldsymbol{u}_S) \\ &= \partial_r u_R + r^{-1}(\nabla_\sigma \cdot \boldsymbol{\sigma}) u_R + r^{-1}\nabla_\sigma \cdot \boldsymbol{u}_S \\ &= \partial_r' u_R + r^{-1}\nabla_\sigma \cdot \boldsymbol{u}_S \qquad \text{on } \dot{\mathbb{R}}^N \end{aligned} \tag{2.4}$$

as a radial-spherical representation of the divergence. We can deduce from this result the following elementary fact:

Lemma 2.1. *For all $f \in C^\infty(\dot{\mathbb{R}}^N)$, the identity*

$$\nabla_\sigma \cdot \nabla_\sigma f = \triangle_\sigma f \qquad \text{on } \dot{\mathbb{R}}^N$$

and the spherical integration by parts formula

$$\int_{\mathbb{S}^{N-1}} \boldsymbol{u} \cdot \nabla_\sigma f \, \mathrm{d}\sigma = -\int_{\mathbb{S}^{N-1}} (\nabla_\sigma \cdot \boldsymbol{u}_S) f \, \mathrm{d}\sigma$$

hold for all $\boldsymbol{u} \in C^\infty(\dot{\mathbb{R}}^N)^N$. Here the integrals are taken for any fixed radius.

Proof. Since the operations are relevant to only the spherical variable $\boldsymbol{\sigma}$, it suffices to check the case where f is independent of the radius r. Apply (2.4) to the case of the spherical gradient field $\nabla f = r^{-1}\nabla_\sigma f$, and we get

$$\operatorname{div} \nabla f = r^{-2} \nabla_\sigma \cdot \nabla_\sigma f.$$

This together with $\operatorname{div}\nabla f = \triangle f = r^{-2}\triangle_\sigma f$ gives the first identity of the lemma. To check the integral formula, let $\zeta \in C_c^\infty(\dot{\mathbb{R}}^N)$ be any radially symmetric scalar field. Then integration by parts of $(\boldsymbol{u} \cdot \nabla_\sigma f)\zeta = \boldsymbol{u}_S \cdot \nabla(rf\zeta)$ yields

$$\int_{\mathbb{R}^N} (\boldsymbol{u} \cdot \nabla_\sigma f)\zeta \, dx = -\int_{\mathbb{R}^N} (\operatorname{div} \boldsymbol{u}_S) r f \zeta \, dx = -\int_{\mathbb{R}^N} (\nabla_\sigma \cdot \boldsymbol{u}_S) f \zeta \, dx,$$

where the last equality follows from (2.4). Since the choice of ζ is arbitrary in the radial direction, we get the desired formula, with the aid of the measure transformation formula $dx = r^{N-1} dr \, \mathrm{d}\sigma$. □

For later use, we also show the following:

Lemma 2.2. *The identities*

$$\begin{aligned} &\triangle_\sigma(\boldsymbol{\sigma} f) = \boldsymbol{\sigma}(\triangle_\sigma - N + 1) f + 2\nabla_\sigma f, \\ &\triangle_\sigma \nabla_\sigma f = \nabla_\sigma \triangle_\sigma f + (N-3)\nabla_\sigma f - 2\boldsymbol{\sigma}\triangle_\sigma f \end{aligned}$$

hold for all $f \in C^\infty(\dot{\mathbb{R}}^N)$.

Proof. It suffices to check the case where f is independent of r. Then a direct calculation by using (2.2) together with the Leibniz rule yields

$$\begin{aligned}\triangle_\sigma(\boldsymbol{\sigma} f) &= (1/r)\triangle_\sigma(r\boldsymbol{\sigma} f) = r\triangle(\boldsymbol{x} f) - r\partial_r'\partial_r(r\boldsymbol{\sigma} f)\\ &= 2r\nabla\boldsymbol{x}\cdot\nabla f + r\boldsymbol{x}\triangle f - r\partial_r'(\boldsymbol{\sigma} f)\\ &= 2r\nabla f + \boldsymbol{\sigma}\triangle_\sigma f - (N-1)\boldsymbol{\sigma} f\\ &= 2\nabla_\sigma f + \boldsymbol{\sigma}(\triangle_\sigma - N + 1)f\end{aligned}$$

to get the first identity of the lemma. A similar calculation also yields

$$\begin{aligned}\triangle_\sigma\nabla_\sigma f &= (1/r)\triangle_\sigma(r\nabla_\sigma f) = (r\triangle - r\partial_r'\partial_r)(r\nabla_\sigma f)\\ &= r\triangle\big(\nabla(r^2 f) - 2\boldsymbol{x} f\big) - r\partial_r'\nabla_\sigma f\\ &= r\nabla\triangle(r^2 f) - 2r\triangle(\boldsymbol{x} f) - (N-1)\nabla_\sigma f\\ &= \nabla_\sigma\big((\triangle r^2)f + r^2\triangle f\big) - 4r\nabla f - 2r\boldsymbol{x}\triangle f - (N-1)\nabla_\sigma f\\ &= (N-3)\nabla_\sigma f + \nabla_\sigma\triangle_\sigma f - 2\boldsymbol{\sigma}\triangle_\sigma f\end{aligned}$$

to obtain the second identity of the lemma. □

3. Poloidal-toroidal fields

After introducing the definition of pre-poloidal and toroidal fields on $\dot{\mathbb{R}}^N$, we construct the so-called PT decomposition theorem of solenoidal fields on $\mathbb{R}^N$, by using a generator of poloidal fields.

§3.1. **Pre-poloidal fields and toroidal fields on $\dot{\mathbb{R}}^N$.** We say that a vector field $\boldsymbol{u} \in C^\infty(\dot{\mathbb{R}}^N)^N$ is *pre-poloidal* if there exist two scalar fields $f, g \in C^\infty(\dot{\mathbb{R}}^N)$ satisfying

$$\boldsymbol{u} = \boldsymbol{x} g + \nabla f \quad \text{ on } \dot{\mathbb{R}}^N,$$

and we denote by $\mathcal{P}(\dot{\mathbb{R}}^N)$ the set of all pre-poloidal fields. Then it is clear from (2.2) that this condition is equivalent to the existence of f, g satisfying

$$\boldsymbol{u} = \boldsymbol{\sigma} g + \nabla_\sigma f \quad \text{ on } \dot{\mathbb{R}}^N.$$

By using the two equivalent conditions, one can easily check that

$$\{\zeta\boldsymbol{u},\ \partial_r\boldsymbol{u},\ \triangle\boldsymbol{u},\ \triangle_\sigma\boldsymbol{u}\} \subset \mathcal{P}(\dot{\mathbb{R}}^N) \qquad \forall\boldsymbol{u} \in \mathcal{P}(\dot{\mathbb{R}}^N), \tag{3.1}$$

where $\zeta \in C^\infty(\dot{\mathbb{R}}^N)$ is any radially symmetric scalar field. Hence the pre-poloidal property is invariant under the operations of radial multiplication, radial derivative and (spherical) Laplacian.

A vector field $\boldsymbol{u} \in C^\infty(\dot{\mathbb{R}}^N)^N$ is said to be *toroidal* if it is spherical and divergence-free:

$$\left.\begin{aligned} \boldsymbol{x}\cdot\boldsymbol{u} = \operatorname{div}\boldsymbol{u} = 0\\ \text{or equivalently} \quad u_R = \nabla_\sigma\cdot\boldsymbol{u} = 0\end{aligned}\right\} \text{ on } \dot{\mathbb{R}}^N.$$

We denote by $\mathcal{T}(\dot{\mathbb{R}}^N)$ the set of all toroidal fields; then the same invariant property (3.1) also applies to the case of toroidal fields $\mathcal{T}(\dot{\mathbb{R}}^N)$ $\big($in place of $\mathcal{P}(\dot{\mathbb{R}}^N)\big)$. Here let us show that every toroidal field has zero spherical mean:

$$\int_{\mathbb{S}^{N-1}} \boldsymbol{u}(r\boldsymbol{\sigma})\,\mathrm{d}\sigma = 0 \qquad \forall r > 0, \quad \forall\boldsymbol{u} \in \mathcal{T}(\dot{\mathbb{R}}^N). \tag{3.2}$$

To this end, let $\zeta \in C_c^\infty(\dot{\mathbb{R}}^N)$ be any radially symmetric scalar field with compact support on $\dot{\mathbb{R}}^N$. We set $\boldsymbol{w} := \zeta \boldsymbol{u}$ and notice that $\boldsymbol{w} \in \mathcal{T}(\dot{\mathbb{R}}^N)$; then integration by parts of the k-th component of $\boldsymbol{w}$ yields

$$\int_{\mathbb{R}^N} u_k \zeta dx = \int_{\mathbb{R}^N} w_k dx = -\int_{\mathbb{R}^N} x_k \frac{\partial w_k}{\partial x_k} dx = \sum_{j\neq k} \int_{\mathbb{R}^N} x_k \frac{\partial w_j}{\partial x_j} dx = 0$$

for all $k = 1, 2, \cdots, N$; where the third equality follows from $\operatorname{div} \boldsymbol{w} = 0$. Since the choice of ζ is arbitrary in the radial direction, we arrive at $\int_{\mathbb{S}^{N-1}} u_k \mathrm{d}\sigma = 0$ and hence (3.2), with the aid of the measure transformation formula $dx = r^{N-1} dr\, \mathrm{d}\sigma$.

The following lemma summarizes some basic properties of the sets (or spaces) of pre-poloidal fields and toroidal fields:

Lemma 3.1. *All pre-poloidal fields are $L^2(\mathbb{S}^{N-1})$-orthogonal to all toroidal fields, in the sense that*

$$\int_{\mathbb{S}^{N-1}} \boldsymbol{v} \cdot \boldsymbol{w}\, \mathrm{d}\sigma = \int_{\mathbb{S}^{N-1}} \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{w}\, \mathrm{d}\sigma = 0$$

for all $\boldsymbol{v} \in \mathcal{P}(\dot{\mathbb{R}}^N)$ and $\boldsymbol{w} \in \mathcal{T}(\dot{\mathbb{R}}^N)$, where the integrals are taken for any radius. Moreover, these fields satisfy

$$\{\zeta\boldsymbol{v},\, \partial_r \boldsymbol{v},\, \triangle_\sigma \boldsymbol{v}\} \subset \mathcal{P}(\dot{\mathbb{R}}^N) \quad \text{and} \quad \{\zeta\boldsymbol{w},\, \partial_r \boldsymbol{w},\, \triangle_\sigma \boldsymbol{w}\} \subset \mathcal{T}(\dot{\mathbb{R}}^N),$$

where $\zeta \in C^\infty(\dot{\mathbb{R}}^N)$ is any radially symmetric scalar field; namely, the two spaces $\mathcal{P}(\dot{\mathbb{R}}^N)$ and $\mathcal{T}(\dot{\mathbb{R}}^N)$ are invariant under the operations of ζ, ∂_r and $\triangle_\sigma$.

Proof. It suffices to check the orthogonality formulae. The pre-poloidal property of $\boldsymbol{v}$ says that $\boldsymbol{v} = \boldsymbol{\sigma} g + \nabla_\sigma f$ for some $f, g \in C^\infty(\dot{\mathbb{R}}^N)$, and hence

$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \nabla_\sigma f$$

follows from the spherical property $w_R = 0$ of the toroidal field $\boldsymbol{w}$. Then integration by parts yields

$$\int_{\mathbb{S}^{N-1}} \boldsymbol{v} \cdot \boldsymbol{w}\, \mathrm{d}\sigma = -\int_{\mathbb{S}^{N-1}} (\nabla_\sigma \cdot \boldsymbol{w})\, f\, \mathrm{d}\sigma = 0$$

due to $\nabla_\sigma \cdot \boldsymbol{w} = 0$. This proves the first orthogonality formula. To prove the second, by using (2.2) and Lemma 2.1, integration by parts yields

$$\begin{aligned}
\int_{\mathbb{S}^{N-1}} \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{w}\, \mathrm{d}\sigma &= \int_{\mathbb{S}^{N-1}} \Big(\partial_r \boldsymbol{v} \cdot \partial_r \boldsymbol{w} + r^{-2} \nabla_\sigma \boldsymbol{v} \cdot \nabla_\sigma \boldsymbol{w}\Big) \mathrm{d}\sigma \\
&= \int_{\mathbb{S}^{N-1}} \partial_r \boldsymbol{v} \cdot \partial_r \boldsymbol{w}\, \mathrm{d}\sigma - r^{-2} \int_{\mathbb{S}^{N-1}} \boldsymbol{v} \cdot \triangle_\sigma \boldsymbol{w}\, \mathrm{d}\sigma = 0,
\end{aligned}$$

where the last equality follows by applying the first orthogonality formula to the fields $\{\partial_r \boldsymbol{v}, \boldsymbol{v}\} \subset \mathcal{P}(\dot{\mathbb{R}}^N)$ and $\{\partial_r \boldsymbol{w}, \triangle_\sigma \boldsymbol{w}\} \subset \mathcal{T}(\dot{\mathbb{R}}^N)$. □

§3.2. **PT decomposition of solenoidal fields on $\mathbb{R}^N$.** A vector field is said to be solenoidal if it is divergence-free. In view of §3.1, all toroidal fields are solenoidal, while pre-poloidal fields are not necessarily so; we say that a pre-poloidal field is *poloidal* whenever it is solenoidal.

Now let $\boldsymbol{u} \in C^\infty(\mathbb{R}^N)^N$ be a solenoidal field smoothly defined on the *entire* space $\mathbb{R}^N$. Notice that the surface integral of $\boldsymbol{u}$ over $\mathbb{S}^{N-1}$ gives

$$\int_{\mathbb{S}^{N-1}} \boldsymbol{\sigma} \cdot \boldsymbol{u}\, \mathrm{d}\sigma = 0 \qquad \text{(for any radius)}$$

by use of Gauss' divergence theorem; hence the scalar field $u_R = \boldsymbol{\sigma} \cdot \boldsymbol{u}$ has zero spherical mean. Then it is well known that the Poisson-Beltrami equation

$$\triangle_\sigma f = u_R \quad \text{on } \dot{\mathbb{R}}^N$$

has an unique solution $f \in C^\infty(\dot{\mathbb{R}}^N)$ with zero spherical mean; we denote such a solution by $f = \triangle_\sigma^{-1} u_R$, and we call it the *poloidal potential* of $\boldsymbol{u}$. To understand this naming, let us introduce the second-order derivative operator

$$\boldsymbol{D} := \boldsymbol{\sigma} \triangle_\sigma - r\partial_r' \nabla_\sigma \tag{3.3}$$

which we call the poloidal generator. It maps every scalar field to a poloidal field on $\dot{\mathbb{R}}^N$; indeed, it is clear that $\boldsymbol{D}f \in \mathcal{P}(\dot{\mathbb{R}}^N)$ for every $f \in C^\infty(\dot{\mathbb{R}}^N)$, and that $\mathrm{div}\boldsymbol{D} = \partial_r' \triangle_\sigma - \partial_r' \nabla_\sigma \cdot \nabla_\sigma = 0$ follows from (2.4). Moreover,

$$\boldsymbol{u} - \boldsymbol{D}\triangle_\sigma^{-1} u_R = \boldsymbol{u}_S + \nabla_\sigma \triangle_\sigma^{-1}(r\partial_r' u_R)$$

is a toroidal field whenever $\boldsymbol{u}$ is solenoidal. Hence we have obtained the following:

Proposition 3.2 (PT theorem)**.** *Let $\boldsymbol{u} \in C^\infty(\mathbb{R}^N)^N$ be a solenoidal field. Then there exists an unique pair of poloidal-toroidal fields $(\boldsymbol{u}_P, \boldsymbol{u}_T) \in \mathcal{P}(\dot{\mathbb{R}}^N) \times \mathcal{T}(\dot{\mathbb{R}}^N)$ satisfying*

$$\boldsymbol{u} = \boldsymbol{u}_P + \boldsymbol{u}_T \qquad \text{on } \dot{\mathbb{R}}^N.$$

Here the poloidal part of $\boldsymbol{u}$ has the explicit expression $\boldsymbol{u}_P = \boldsymbol{D}f$ in terms of the poloidal potential $f = \triangle_\sigma^{-1} u_R$ and the poloidal generator (3.3)*.*

For later use, we show some $L^2(\mathbb{S}^{N-1})$-deviation estimates for a perturbation of poloidal potential by radial multiplication:

Lemma 3.3. *Let $f \in C^\infty(\dot{\mathbb{R}}^N)$. Then there exists some $C > 0$ depending only on N such that the inequalities*

$$C\int_{\mathbb{S}^{N-1}} |\boldsymbol{D}(\zeta f) - \zeta \boldsymbol{D}f|^2 \mathrm{d}\sigma \le (r\zeta')^2 \int_{\mathbb{S}^{N-1}} |\boldsymbol{D}f|^2 \mathrm{d}\sigma,$$
$$C\int_{\mathbb{S}^{N-1}} |\nabla \boldsymbol{D}(\zeta f) - \zeta \nabla \boldsymbol{D}f|^2 \mathrm{d}\sigma \le \Big((r\zeta')^2 + (r^2\zeta'')^2 \Big) \int_{\mathbb{S}^{N-1}} \frac{|\boldsymbol{D}f|^2}{r^2} \mathrm{d}\sigma$$

hold for any radially symmetric scalar field $\zeta \in C^\infty(\dot{\mathbb{R}}^N)$ together with the notation for its radial derivatives $\zeta' = \partial_r \zeta$ and $\zeta'' = \partial_r^2 \zeta$. Here the integrals are taken for every fixed radius.

Proof. A direct calculation by using the Leibniz rule gives

$$\boldsymbol{D}(\zeta f) - \zeta \boldsymbol{D}f = -r\zeta' \nabla_\sigma f, \tag{3.4}$$
$$\nabla \boldsymbol{D}(\zeta f) - \zeta \nabla \boldsymbol{D}f = \boldsymbol{\sigma}\zeta' \boldsymbol{D}f - \boldsymbol{\sigma}(r\zeta')' \nabla_\sigma f - r\zeta' \nabla\nabla_\sigma f, \tag{3.5}$$

where the second identity follows by taking the gradient of the first. We aim to estimate these two fields. First of all, The $L^2(\mathbb{S}^{N-1})$ integration of (3.4) yields

$$\int_{\mathbb{S}^{N-1}} |\boldsymbol{D}(\zeta f) - \zeta \boldsymbol{D}f|^2 \mathrm{d}\sigma = (r\zeta')^2 \int_{\mathbb{S}^{N-1}} |\nabla_\sigma f|^2 \mathrm{d}\sigma \le \frac{(r\zeta')^2}{N-1} \int_{\mathbb{S}^{N-1}} |\boldsymbol{D}f|^2 \, \mathrm{d}\sigma.$$

Here the last inequality follows by combining

$$|\triangle_\sigma f| = |\boldsymbol{\sigma} \cdot \boldsymbol{D}f| \le |\boldsymbol{D}f|$$

with the spectral estimate

$$\int_{\mathbb{S}^{N-1}} |\nabla_\sigma f|^2 \mathrm{d}\sigma \le \frac{1}{N-1} \int_{\mathbb{S}^{N-1}} (\triangle_\sigma f)^2 \,\mathrm{d}\sigma$$

which can be easily verified by using the spherical harmonics expansion of f. Therefore, we have proved the first inequality of the lemma. To prove the second, we begin to estimate the last term of (3.5): the identity

$$|\nabla\nabla_\sigma f|^2 = |\partial_r \nabla_\sigma f|^2 + r^{-2} |\nabla_\sigma \nabla_\sigma f|^2$$

follows from (2.2), and integration by parts on both sides gives

$$\begin{aligned}
\int_{\mathbb{S}^{N-1}} |\nabla\nabla_\sigma f|^2 \mathrm{d}\sigma &= \int_{\mathbb{S}^{N-1}} \big(|\partial_r \nabla_\sigma f|^2 - r^{-2} \nabla_\sigma f \cdot \triangle_\sigma \nabla_\sigma f\big) \mathrm{d}\sigma \\
&= \int_{\mathbb{S}^{N-1}} \big(|\nabla_\sigma \partial_r f|^2 - r^{-2} \nabla_\sigma f \cdot \nabla_\sigma \big(\triangle_\sigma + N - 3\big) f\big)\, \mathrm{d}\sigma \quad \text{(due to Lemma 2.2)} \\
&= \int_{\mathbb{S}^{N-1}} \left(\left| \nabla_\sigma \partial_r' f - \frac{N-1}{r} \nabla_\sigma f \right|^2 + r^{-2} \Big((\triangle_\sigma f)^2 + (N-3) |\nabla_\sigma f|^2 \Big) \right) \mathrm{d}\sigma \\
&= \int_{\mathbb{S}^{N-1}} \left(\left| r^{-1} (\boldsymbol{D} f)_S - \frac{N-1}{r} \nabla_\sigma f \right|^2 + r^{-2} \Big((\boldsymbol{\sigma} \cdot \boldsymbol{D} f)^2 + (N-3) |\nabla_\sigma f|^2 \Big) \right) \mathrm{d}\sigma \\
&\lesssim \frac{1}{r^2} \int_{\mathbb{S}^{N-1}} \big(|\boldsymbol{D} f|^2 + |\triangle_\sigma f|^2 \big) \mathrm{d}\sigma \lesssim \frac{1}{r^2} \int_{\mathbb{S}^{N-1}} |\boldsymbol{D} f|^2 \mathrm{d}\sigma,
\end{aligned}$$

where the notation "$\lesssim$" means that

$$x \lesssim y \quad :\Longleftrightarrow \quad x \le Cy \qquad \text{for some constant } C > 0 \text{ depending only on } N$$

as a transitive relation between two nonnegative real numbers. By using this result, the $L^2(\mathbb{S}^{N-1})$ integration of (3.5) yields

$$\begin{aligned}
\int_{\mathbb{S}^{N-1}} |\nabla \boldsymbol{D}(\zeta f) - \zeta \nabla \boldsymbol{D} f|^2 \mathrm{d}\sigma &= \int_{\mathbb{S}^{N-1}} |\boldsymbol{\sigma} \zeta' \boldsymbol{D} f - \boldsymbol{\sigma} (r\zeta')' \nabla_\sigma f - r\zeta' \nabla\nabla_\sigma f|^2 \mathrm{d}\sigma \\
&\lesssim (\zeta')^2 \int_{\mathbb{S}^{N-1}} |\boldsymbol{D} f|^2 \mathrm{d}\sigma + ((r\zeta')')^2 \int_{\mathbb{S}^{N-1}} |\triangle_\sigma f|^2 \mathrm{d}\sigma + (r\zeta')^2 \int_{\mathbb{S}^{N-1}} |\nabla\nabla_\sigma f|^2 \mathrm{d}\sigma \\
&\lesssim \big((r\zeta')^2 + (r^2 \zeta'')^2 \big) \int_{\mathbb{S}^{N-1}} \frac{|\boldsymbol{D} f|^2}{r^2} \mathrm{d}\sigma
\end{aligned}$$

to arrive at the desired result. □

4. Proof of main theorem

In the following, we always assume that the test solenoidal fields $\boldsymbol{u}$ satisfy

$$\boldsymbol{u} \not\equiv \boldsymbol{0} \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx < \infty,$$

since otherwise there is nothing to prove. This integrability together with the smoothness of $|\nabla \boldsymbol{u}|^2$ tells us that there must be an integer $k > -\gamma - \frac{N}{2}$ such that $\nabla \boldsymbol{u} = O(|\boldsymbol{x}|^k)$ as $|\boldsymbol{x}| \to 0$. Then, by using the "additional condition" stated in Theorem 1.1, we get $\boldsymbol{u} = O(|\boldsymbol{x}|^{k+1})$ for $\gamma \le 1 - \frac{N}{2}$, and hence

$$\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma - 2} dx < \infty$$

due to the support compactness of $\boldsymbol{u}$ on $\mathbb{R}^N$.

§4.1. **Reduction to the case of PT fields with compact support on $\dot{\mathbb{R}}^N$.** Recall that the formula $\boldsymbol{u} = \boldsymbol{u}_P + \boldsymbol{u}_T$ in Proposition 3.2 is an $L^2(\mathbb{S}^{N-1})$-direct sum in the sense of Lemma 3.1. Then the ratio of the two integrals in inequality (1.1), which we simply call the *H-L quotient*, can be expressed as

$$\frac{\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx} = \frac{\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}_P|^2 |\boldsymbol{x}|^{2\gamma} dx + \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}_T|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}_P|^2 |\boldsymbol{x}|^{2\gamma-2} dx + \int_{\mathbb{R}^N} |\boldsymbol{u}_T|^2 |\boldsymbol{x}|^{2\gamma-2} dx}.$$

Taking the infimum on both sides over the test solenoidal fields $\boldsymbol{u}$, we get

$$C_{N,\gamma} = \inf_{\operatorname{div}\boldsymbol{u}\equiv 0} \frac{\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx} = \min\{C_{P,N,\gamma}, C_{T,N,\gamma}\} \tag{4.1}$$

as the best constant of H-L inequality for solenoidal fields, in terms of the notation

$$C_{P,N,\gamma} := \inf_{\substack{\boldsymbol{u}_P \not\equiv \boldsymbol{0},\\ \operatorname{div}\boldsymbol{u}\equiv 0}} \frac{\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}_P|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}_P|^2 |\boldsymbol{x}|^{2\gamma-2} dx} = \inf_{\boldsymbol{u}\in\mathcal{P}} \frac{\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx}$$

$$\text{resp.}\quad C_{T,N,\gamma} := \inf_{\substack{\boldsymbol{u}_T \not\equiv \boldsymbol{0},\\ \operatorname{div}\boldsymbol{u}= 0}} \frac{\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}_T|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}_T|^2 |\boldsymbol{x}|^{2\gamma-2} dx} = \inf_{\boldsymbol{u}\in\mathcal{T}} \frac{\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx}$$

denoting the best constant of H-L inequality for poloidal resp. toroidal fields. Here the abbreviation "$\boldsymbol{u} \in \mathcal{P}$" resp. "$\boldsymbol{u} \in \mathcal{T}$" on the right-hand side means that $\boldsymbol{u}$ is poloidal resp. toroidal (as well as $\boldsymbol{u} \not\equiv \boldsymbol{0}$). Therefore, the computation of $C_{N,\gamma}$ is reduced to that of the individual $C_{P,N,\gamma}$ and $C_{T,N,\gamma}$.

To compute the best constants, we can further assume that all the test solenoidal fields are compactly supported on $\dot{\mathbb{R}}^N$, for the following reason: Let $f \in C^\infty(\dot{\mathbb{R}}^N)$ be the poloidal potential of any solenoidal field $\boldsymbol{u}$, and hence we have

$$\boldsymbol{u} = \boldsymbol{u}_P + \boldsymbol{u}_T, \qquad \boldsymbol{u}_P = \boldsymbol{D} f.$$

Define $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\dot{\mathbb{R}}^N)^N$ as a sequence of solenoidal fields by the formula

$$\boldsymbol{u}_n = \boldsymbol{D}(\zeta_n f) + \zeta_n \boldsymbol{u}_T \qquad \forall n \in \mathbb{N},$$

where $\{\zeta_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\dot{\mathbb{R}}^N)$ are radially symmetric scalar fields given by

$$\zeta_n(\boldsymbol{x}) = \zeta_0(|\boldsymbol{x}|^{\frac{1}{n}}) \qquad \forall \boldsymbol{x} \in \mathbb{R}^N, \quad \forall n \in \mathbb{N}$$

for some 1-variable smooth function $\zeta_0 \in C^\infty(\mathbb{R}_+)$ with compact support on $\mathbb{R}_+$ such that $\zeta_0(1) = 1$. Then a direct calculation by applying Lemma 3.3 to $\zeta = \zeta_n$ yields

$$C \int_{\mathbb{R}^N} |\boldsymbol{u}_n - \zeta_n \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx \le \int_{\mathbb{R}^N} (r\zeta_n')^2 |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx,$$
$$C \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}_n - \zeta_n \nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \le \int_{\mathbb{R}^N} \left((r\zeta_n')^2 + (r^2\zeta_n'')^2 \right) |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx$$

for some constant $C > 0$ depending only on N. Notice on the right-hand sides that the radial factors have the estimates

$$|r\zeta_n'| = \left| \frac{1}{n} r^{\frac{1}{n}} \zeta_0'(r^{\frac{1}{n}}) \right| \le \frac{C}{n},$$
$$|r^2\zeta_n''| = \left| \frac{1}{n}\left(\frac{1}{n} - 1\right) r^{\frac{1}{n}} \zeta_0'(r^{\frac{1}{n}}) + \frac{1}{n^2} r^{\frac{2}{n}} \zeta_0''(r^{\frac{1}{n}}) \right| \le \frac{C}{n}$$

for some constant $C > 0$ depending only on ζ_0, and hence we have

$$\left.\begin{aligned} &\int_{\mathbb{R}^N} |\boldsymbol{u}_n - \zeta_n \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx \to 0, \\ &\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}_n - \zeta_n \nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \to 0 \end{aligned}\right\} \quad \text{as} \quad n \to \infty$$

by using the integrability $\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx < \infty$. Since the dominated convergence theorem says that

$$\zeta_n \boldsymbol{u} \to \boldsymbol{u} \quad \text{resp.} \quad \zeta_n \nabla \boldsymbol{u}_n \to \nabla \boldsymbol{u} \qquad (n \to \infty)$$

holds in $L^2(|\boldsymbol{x}|^{2\gamma-2}dx)$ resp. $L^2(|\boldsymbol{x}|^{2\gamma}dx)^N$, we obtain

$$\int_{\mathbb{R}^N} |\boldsymbol{u}_n - \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}_n - \nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \to 0$$

through the L^2-triangle inequality. Therefore, the two integrals in H-L inequality for solenoidal fields on $\mathbb{R}^N$ can be approximated by those with compact support on $\dot{\mathbb{R}}^N$.

§4.2. **Estimation for poloidal fields: evaluation of $C_{P,N,\gamma}$.** Throughout this subsection, $\boldsymbol{u}$ is a assumed to be poloidal with compact support on $\dot{\mathbb{R}}^N$. Notice from Proposition 3.2 that

$$\boldsymbol{u} = \boldsymbol{u}_P = \boldsymbol{D} f = \boldsymbol{\sigma} \triangle_\sigma f - r\partial_r' \nabla_\sigma f \quad \text{on } \dot{\mathbb{R}}^N$$

for the poloidal potential $f = \triangle_\sigma^{-1} u_R$. Now, let us transform $\boldsymbol{u}$ resp. f into a vector field $\boldsymbol{v}$ resp. scalar field g by the formula

$$\left.\begin{aligned} &\boldsymbol{v}(\boldsymbol{x}) := |\boldsymbol{x}|^{\gamma+\frac{N}{2}-1} \boldsymbol{u}(\boldsymbol{x}) \\ \text{resp.} \quad &g(\boldsymbol{x}) := |\boldsymbol{x}|^{\gamma+\frac{N}{2}-1} f(\boldsymbol{x}) = \triangle_\sigma^{-1} v_R(\boldsymbol{x}) \end{aligned}\right\} \quad \forall \boldsymbol{x} \in \dot{\mathbb{R}}^N, \tag{4.2}$$

which stems from an idea of Brezis-Vázquez [2]. Then $\boldsymbol{v}$ can be expressed in terms of g by the following calculation:

$$\begin{aligned} \boldsymbol{v} &= r^{\gamma+\frac{N}{2}-1} \boldsymbol{D}\big(r^{1-\gamma-\frac{N}{2}} g\big) = \boldsymbol{\sigma} \triangle_\sigma g - r^{\gamma+\frac{N}{2}} \partial_r' \nabla_\sigma \big(r^{1-\gamma-\frac{N}{2}} g\big) \\ &= \boldsymbol{\sigma} \triangle_\sigma g - \nabla_\sigma \left((r\partial_r + \tfrac{N}{2} - \gamma) g \right) \\ &= \boldsymbol{\sigma} \triangle_\sigma g - \nabla_\sigma \left((\partial_t - \gamma + \tfrac{N}{2}) g \right). \end{aligned} \tag{4.3}$$

Here and hereafter we employ the notation $t := \log |\boldsymbol{x}|$, which serves as an alternative radial coordinate obeying the differential chain rule:

$$\partial_t = r\partial_r = \boldsymbol{x} \cdot \nabla, \qquad dt = r^{-1} dr. \tag{4.4}$$

Taking the derivatives of (4.3) also yields the calculation:

$$
\begin{aligned}
\partial_t \boldsymbol{v} &= \boldsymbol{\sigma} \triangle_\sigma \partial_t g - \nabla_\sigma \left((\partial_t - \gamma + \tfrac{N}{2}) \partial_t g \right), \qquad (4.5)\\
\triangle_\sigma \boldsymbol{v} &= \triangle_\sigma (\boldsymbol{\sigma} \triangle_\sigma g) - \triangle_\sigma \nabla_\sigma \left((\partial_t - \gamma + \tfrac{N}{2}) g \right)\\
&= \boldsymbol{\sigma} \left(\triangle_\sigma^2 g - (N-1) \triangle_\sigma g \right) + 2 \nabla_\sigma \triangle_\sigma g\\
&\quad + 2\boldsymbol{\sigma} \triangle_\sigma \left(\left(\partial_t - \gamma + \tfrac{N}{2}\right) g \right) - \nabla_\sigma (\triangle_\sigma + N - 3) \left((\partial_t - \gamma + \tfrac{N}{2}) g \right)\\
&= \boldsymbol{\sigma} \triangle_\sigma^2 g + \boldsymbol{\sigma} \left(2\partial_t - 2\gamma - N + 4 \right) \triangle_\sigma g + (N-3) \boldsymbol{\sigma} \triangle_\sigma g\\
&\quad + \nabla_\sigma \left(\left(-\partial_t + \gamma - \tfrac{N}{2} + 2 \right) \triangle_\sigma g \right) - (N-3) \nabla_\sigma \left(\left(\partial_t - \gamma + \tfrac{N}{2} \right) g \right)\\
&= \boldsymbol{\sigma} \left(\triangle_\sigma^2 g + 2\partial_t \triangle_\sigma g - 2 \left(\gamma + \tfrac{N}{2} - 2 \right) \triangle_\sigma g \right)\\
&\quad + \nabla_\sigma \left(\left(-\partial_t + \gamma - \tfrac{N}{2} + 2 \right) \triangle_\sigma g \right) + (N-3) \boldsymbol{v}, \qquad (4.6)
\end{aligned}
$$

where the equality in the third line follows by using Lemma 2.2. On the other hand, to express in terms of $\boldsymbol{v}$ the integrals in (1.1), we have the following calculation:

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx &= \int_{\mathbb{R}^N} |\boldsymbol{v}|^2 |\boldsymbol{x}|^{-N} dx, \qquad (4.7)\\
\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx &= \int_{\mathbb{R}^N} \left| \nabla \left(r^{1-\gamma-\frac{N}{2}} \boldsymbol{v} \right) \right|^2 r^{2\gamma} dx\\
&= \int_{\mathbb{R}^N} \left| \left(1 - \gamma - \tfrac{N}{2}\right) r^{-\gamma-\frac{N}{2}} \boldsymbol{\sigma} \boldsymbol{v} + r^{1-\gamma-\frac{N}{2}} \nabla \boldsymbol{v} \right|^2 r^{2\gamma} dx\\
&= \int_{\mathbb{R}^N} \left(\left(\gamma + \tfrac{N}{2} - 1 \right)^2 |\boldsymbol{v}|^2 + \left(1 - \gamma - \tfrac{N}{2} \right) r \partial_r |\boldsymbol{v}|^2 + |r \nabla \boldsymbol{v}|^2 \right) r^{-N} dx\\
&= \left(\gamma + \tfrac{N}{2} - 1 \right)^2 \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx + \int_{\mathbb{R}^N} |r \nabla \boldsymbol{v}|^2 r^{-N} dx, \qquad (4.8)
\end{aligned}
$$

where the last equality follows from the first and the support compactness of $\boldsymbol{v}$ on $\dot{\mathbb{R}}^N$. In particular, taking the ratio of (4.8) to (4.7) gives

$$
\frac{\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx} = \left(\gamma + \tfrac{N}{2} - 1 \right)^2 + \frac{\int_{\mathbb{R}^N} |r \nabla \boldsymbol{v}|^2 r^{-N} dx}{\int_{\mathbb{R}^N} |\boldsymbol{v}|^2 r^{-N} dx}. \qquad (4.9)
$$

Hence the evaluation of the H-L quotient is further reduced to that of the quotient on the right-hand side. To this end, let us compute in terms of g the L^2 integrals of $\boldsymbol{v}$ and $r\nabla \boldsymbol{v}$. First of all, with respect to the measure

$$
r^{-N} dx = dt \, \mathrm{d}\sigma \quad \text{over} \quad \dot{\mathbb{R}}^N \cong \mathbb{R} \times \mathbb{S}^{N-1}, \qquad (4.10)
$$

the L^2 integration by parts of (4.3) and (4.5) yields

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\boldsymbol{v}|^2 r^{-N} dx &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left((\triangle_\sigma g)^2 + \left| \left(\partial_t - \gamma + \tfrac{N}{2} \right) \nabla_\sigma g \right|^2 \right) dt \, \mathrm{d}\sigma\\
&= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left((\triangle_\sigma g)^2 + |\partial_t \nabla_\sigma g|^2 + \left(\gamma - \tfrac{N}{2} \right)^2 |\nabla_\sigma g|^2 \right) dt \, \mathrm{d}\sigma, \qquad (4.11)\\
\int_{\mathbb{R}^N} |\partial_t \boldsymbol{v}|^2 r^{-N} dx &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left((\partial_t \triangle_\sigma g)^2 + |\partial_t^2 \nabla_\sigma g|^2 + \left(\gamma - \tfrac{N}{2} \right)^2 |\partial_t \nabla_\sigma g|^2 \right) dt \, \mathrm{d}\sigma \qquad (4.12)
\end{aligned}
$$

by using the support compactness of g. Next, in order to compute the L^2 integral of $\nabla_\sigma \boldsymbol{v}$, taking the scalar product of (4.3) and (4.6) yields

$$-\boldsymbol{v}\cdot(\triangle_\sigma \boldsymbol{v}) = -(\triangle_\sigma g)\left(\triangle_\sigma^2 g + 2\partial_t\triangle_\sigma g - 2(\gamma+\tfrac{N}{2}-2)\triangle_\sigma g\right) \\ + \left(\left(\partial_t-\gamma+\tfrac{N}{2}\right)\nabla_\sigma g\right)\cdot\nabla_\sigma\left(\left(-\partial_t+\gamma-\tfrac{N}{2}+2\right)\triangle_\sigma g\right) - (N-3)|\boldsymbol{v}|^2.$$

Then integration by parts on both sides with respect to the measure (4.10) gives

$$\begin{aligned}
\int_{\mathbb{R}^N}|\nabla_\sigma \boldsymbol{v}|^2 r^{-N}dx &= -(N-3)\int_{\mathbb{R}^N}|\boldsymbol{v}|^2 r^{-N}dx \\
&\quad + \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\begin{pmatrix} -(\triangle_\sigma g)\left(\triangle_\sigma^2 g + 2\partial_t\triangle_\sigma g - 2\left(\gamma+\frac{N}{2}-2\right)\triangle_\sigma g\right) \\ +\left(\left(-\partial_t+\gamma-\frac{N}{2}\right)\triangle_\sigma g\right)\left(-\partial_t+\gamma-\frac{N}{2}+2\right)\triangle_\sigma g \end{pmatrix} dt\,\mathrm{d}\sigma \\
&= -(N-3)\int_{\mathbb{R}^N}|\boldsymbol{v}|^2 r^{-N}dx \\
&\quad + \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\begin{pmatrix} -(\triangle_\sigma g)\triangle_\sigma^2 g + 2\left(\gamma+\frac{N}{2}-2\right)(\triangle_\sigma g)^2 \\ +(\partial_t\triangle_\sigma g)^2 + \left(\gamma-\frac{N}{2}\right)\left(\gamma-\frac{N}{2}+2\right)(\triangle_\sigma g)^2 \end{pmatrix} dt\,\mathrm{d}\sigma \\
&= -(N-3)\int_{\mathbb{R}^N}|\boldsymbol{v}|^2 r^{-N}dx + 4(\gamma-1)\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}(\triangle_\sigma g)^2 dt\,\mathrm{d}\sigma \\
&\quad + \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\left(|\nabla_\sigma\triangle_\sigma g|^2 + (\partial_t\triangle_\sigma g)^2 + \left(\gamma-\tfrac{N}{2}\right)^2(\triangle_\sigma g)^2\right)dt\,\mathrm{d}\sigma,
\end{aligned} \tag{4.13}$$

where the second equality follows again from the support compactness. To further proceed, let us consider separately the two cases $\gamma \le 1$ and $\gamma > 1$.

§4.2.1. *The case* $\gamma \le 1$. In order to estimate the last two integrals in (4.13), we express the spherical harmonics expansion of g as

$$g = \sum_{\nu\in\mathbb{N}} g_\nu, \quad -\triangle_\sigma g_\nu = \alpha_\nu g_\nu \quad (\forall \nu\in\mathbb{N}),$$

by using the same notation $\alpha_\nu = \nu(\nu+N-2)$ as in (2.3). Then a direct calculation gives the following estimate:

$$\begin{aligned}
4(\gamma-1)&\int_{\mathbb{S}^{N-1}}(\triangle_\sigma g)^2\mathrm{d}\sigma + \int_{\mathbb{S}^{N-1}}\left(|\nabla_\sigma\triangle_\sigma g|^2 + \left(\gamma-\tfrac{N}{2}\right)^2(\triangle_\sigma g)^2\right)\mathrm{d}\sigma \\
&= \sum_{\nu\in\mathbb{N}}\alpha_\nu\left(\alpha_\nu^2 + \left(\gamma-\tfrac{N}{2}\right)^2\alpha_\nu + 4(\gamma-1)\alpha_\nu\right)g_\nu^2 \\
&\ge \sum_{\nu\in\mathbb{N}}\alpha_1\left(\alpha_\nu^2 + \left(\gamma-\tfrac{N}{2}\right)^2\alpha_\nu + 4(\gamma-1)\alpha_\nu\right)g_\nu^2 \\
&\ge \sum_{\nu\in\mathbb{N}}\alpha_1\left(\alpha_\nu^2 + \left(\gamma-\tfrac{N}{2}\right)^2\alpha_\nu + 4(\gamma-1)\frac{\left(\gamma-\frac{N}{2}\right)^2\alpha_\nu+\alpha_\nu^2}{\left(\gamma-\frac{N}{2}\right)^2+\alpha_1}\right)g_\nu^2 \\
&= \alpha_1\left(1-\frac{4(1-\gamma)}{\left(\gamma-\frac{N}{2}\right)^2+\alpha_1}\right)\sum_{\nu\in\mathbb{N}}\left(\alpha_\nu^2 + \left(\gamma-\tfrac{N}{2}\right)^2\alpha_\nu\right)g_\nu^2 \\
&= \alpha_1\left(1-\frac{4(1-\gamma)}{\left(\gamma-\frac{N}{2}\right)^2+\alpha_1}\right)\int_{\mathbb{S}^{N-1}}\left((\triangle_\sigma g)^2 + \left(\gamma-\tfrac{N}{2}\right)^2|\nabla_\sigma g|^2\right)\mathrm{d}\sigma.
\end{aligned}$$

Here the second inequality follows from that the coefficients of g_ν^2 are all nonnegative, and the third inequality follows from $\gamma \le 1$; notice that both the equalities in these inequalities are simultaneously attained if and only if $g_\nu = 0\ \ \forall\nu \ge 2$, namely

$-\triangle_\sigma g = \alpha_1 g$. Considering also the spherical harmonics expansion of $\partial_t g$, we have the estimate:

$$\int_{\mathbb{S}^{N-1}} (\partial_t \triangle_\sigma g)^2 \mathrm{d}\sigma \geq \alpha_1 \int_{\mathbb{S}^{N-1}} |\partial_t \nabla_\sigma g|^2 \mathrm{d}\sigma.$$

Combine the above two estimates with the right-hand side of (4.13), and we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla_\sigma \boldsymbol{v}|^2 r^{-N} dx \\
&\quad \geq \alpha_1 \Big(1 - \frac{4(1-\gamma)}{(\gamma - \frac{N}{2})^2 + \alpha_1}\Big) \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big((\triangle_\sigma g)^2 + \big(\gamma - \tfrac{N}{2}\big)^2 |\nabla_\sigma g|^2\Big)\, dt\, \mathrm{d}\sigma \\
&\qquad + \alpha_1 \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} |\partial_t \nabla_\sigma g|^2 dt\, \mathrm{d}\sigma - (N-3) \int_{\mathbb{R}^N} |\boldsymbol{v}|^2 r^{-N} dx \\
&\quad = \Big(2 - \frac{4(1-\gamma)\alpha_1}{(\gamma - \frac{N}{2})^2 + \alpha_1}\Big) \int_{\mathbb{R}^N} |\boldsymbol{v}|^2 r^{-N} dx + \frac{4(1-\gamma)\alpha_1}{(\gamma - \frac{N}{2})^2 + \alpha_1} \int_{\mathbb{R}^N} |\partial_t \nabla_\sigma g|^2 r^{-N} dx
\end{aligned}$$

by use of (4.11) $\big($and (4.10)$\big)$. Add $\int_{\mathbb{R}^N} |\partial_t \boldsymbol{v}|^2 r^{-N} dx$ to both sides, and we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |r\nabla \boldsymbol{v}|^2 r^{-N} dx &= \int_{\mathbb{R}^N} \left(|\partial_t \boldsymbol{v}|^2 + |\nabla_\sigma \boldsymbol{v}|^2\right) r^{-N} dx \\
&\geq \Big(2 - \frac{4(1-\gamma)\alpha_1}{(\gamma - \frac{N}{2})^2 + \alpha_1}\Big) \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx + \mathcal{R}_{N,\gamma}[\boldsymbol{u}]
\end{aligned}$$

to estimate the last integral in (4.8). Here we have defined

$$\mathcal{R}_{N,\gamma}[\boldsymbol{u}] := \int_{\mathbb{R}^N} |\partial_t \boldsymbol{v}|^2 r^{-N} dx + \frac{4(1-\gamma)\alpha_1}{(\gamma - \frac{N}{2})^2 + \alpha_1} \int_{\mathbb{R}^N} |\partial_t \nabla_\sigma g|^2 r^{-N} dx$$

as a nonnegative functional; it coincides with that given in Remark 1.2, as one can easily check by recalling (4.2) and (4.4). Therefore, we have obtained the inequality

$$\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \geq C_{P,N,\gamma} \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx + \mathcal{R}_{N,\gamma}[\boldsymbol{u}] \tag{4.14}$$

together with the constant number

$$C_{P,N,\gamma} = \big(\gamma + \tfrac{N}{2} - 1\big)^2 + 2 - \frac{4(1-\gamma)\alpha_1}{(\gamma - \frac{N}{2})^2 + \alpha_1} = \big(\gamma + \tfrac{N}{2} - 1\big)^2 \frac{(\gamma - \frac{N}{2})^2 + N + 1}{(\gamma - \frac{N}{2})^2 + N - 1},$$

where the equality of (4.14) holds if and only if $-\triangle_\sigma g = \alpha_1 g$ or equivalently $-\triangle_\sigma u_R = \alpha_1 u_R$ on $\dot{\mathbb{R}}^N$.

§4.2.2. *The case* $\gamma > 1$. In a similar way as the previous case, a calculation by using the spherical harmonics expansion of g yields

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big(|\nabla_\sigma \triangle_\sigma g|^2 + (\partial_t \triangle_\sigma g)^2 + \big(\gamma - \tfrac{N}{2}\big)^2 (\triangle_\sigma g)^2\Big)\, dt\, \mathrm{d}\sigma \\
&\qquad \geq \alpha_1 \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big((\triangle_\sigma g)^2 + |\partial_t \nabla_\sigma g|^2 + \big(\gamma - \tfrac{N}{2}\big)^2 |\nabla_\sigma g|^2\Big)\, dt\, \mathrm{d}\sigma \\
&\qquad = (N-1) \int_{\mathbb{R}^N} |\boldsymbol{v}|^2 r^{-N} dx
\end{aligned}$$

to estimate the last integral in (4.13); hence we get

$$\int_{\mathbb{R}^N} |\nabla_\sigma \boldsymbol{v}|^2 r^{-N} dx \geq 2 \int_{\mathbb{R}^N} |\boldsymbol{v}|^2 r^{-N} dx + 4\,(\gamma - 1) \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} (\triangle_\sigma g)^2 dt\, \mathrm{d}\sigma,$$

where the equality holds if and only if $-\triangle_\sigma g = \alpha_1 g$ on $\dot{\mathbb{R}}^N$. This estimate together with the equations (4.11) and (4.12) further yields

$$\begin{aligned}
\frac{\int_{\mathbb{R}^N} |r\nabla \boldsymbol{v}|^2 r^{-N} dx}{\int_{\mathbb{R}^N} |\boldsymbol{v}|^2 r^{-N} dx} &= \frac{(4.12) + \int_{\mathbb{R}^N} |\nabla_\sigma \boldsymbol{v}|^2 r^{-N} dx}{(4.11)} \\
&\geq 2 + \frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \begin{pmatrix} (\partial_t \triangle_\sigma g)^2 + |\partial_t^2 \nabla_\sigma g|^2 + \left(\gamma - \frac{N}{2}\right)^2 |\partial_t \nabla_\sigma g|^2 \\ + 4\,(\gamma - 1)\,(\triangle_\sigma g)^2 \end{pmatrix} dt\, \mathrm{d}\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \left((\triangle_\sigma g)^2 + |\partial_t \nabla_\sigma g|^2 + \left(\gamma - \frac{N}{2}\right)^2 |\nabla_\sigma g|^2 \right) dt\, \mathrm{d}\sigma} \\
&= 2 + \frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} g \left(-\partial_t^2 \triangle_\sigma^2 - \partial_t^4 \triangle_\sigma + \left(\gamma - \frac{N}{2}\right)^2 \partial_t^2 \triangle_\sigma + 4(\gamma - 1)\triangle_\sigma^2 \right) g\, dt\, \mathrm{d}\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} g \left(\triangle_\sigma^2 + \partial_t^2 \triangle_\sigma - \left(\gamma - \frac{N}{2}\right)^2 \triangle_\sigma \right) g\, dt\, \mathrm{d}\sigma} \\
&= 2 + \frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} g\, Q_1(-\partial_t^2, -\triangle_\sigma) g\, dt\, \mathrm{d}\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} g\, Q_0(-\partial_t^2, -\triangle_\sigma) g\, dt\, \mathrm{d}\sigma}, \qquad (4.15)
\end{aligned}$$

where we have defined the two polynomials Q_0 and Q_1 by the formulae

$$\begin{aligned}
Q_1(\tau, \alpha) &:= \tau\alpha^2 + \tau^2\alpha + \left(\gamma - \tfrac{N}{2}\right)^2 \tau\alpha + 4(\gamma - 1)\alpha^2, \\
Q_0(\tau, \alpha) &:= \tau\alpha + \alpha^2 + \left(\gamma - \tfrac{N}{2}\right)^2 \alpha\,.
\end{aligned}$$

In order to evaluate (4.15), we now introduce the 1-D Fourier transformation

$$g(\boldsymbol{x}) = g(e^t \boldsymbol{\sigma}) \longmapsto \widehat{g}(\tau, \boldsymbol{\sigma}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\tau t} g(e^t \boldsymbol{\sigma}) dt$$

in the radial direction, which commutes with the spherical derivatives and changes t-derivative into an imaginary scalar multiplication: $\widehat{\partial_t g} = i\tau \widehat{g}$, $i = \sqrt{-1}$. Then, by expressing the spherical harmonics expansion of $\widehat{g}$ as

$$\widehat{g} = \sum_{\nu \in \mathbb{N}} \widehat{g}_\nu, \quad -\triangle_\sigma \widehat{g}_\nu = \alpha_\nu \widehat{g}_\nu \quad (\forall \nu \in \mathbb{N}),$$

the $L^2(\mathbb{R})$ isometry of the Fourier integration yields the following estimate:

$$\begin{aligned}
\frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} g\, Q_1(-\partial_t^2, -\triangle_\sigma) g\, dt\, \mathrm{d}\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} g\, Q_0(-\partial_t^2, -\triangle_\sigma) g\, dt\, \mathrm{d}\sigma} &= \frac{\sum_{\nu=1}^\infty \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} Q_1(\tau^2, \alpha_\nu) |\widehat{g}_\nu|^2 d\tau \mathrm{d}\sigma}{\sum_{\nu=1}^\infty \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} Q_0(\tau^2, \alpha_\nu) |\widehat{g}_\nu|^2 d\tau \mathrm{d}\sigma} \\
&\geq \inf_{\tau \in \mathbb{R}} \inf_{\nu \in \mathbb{N}} \frac{Q_1(\tau^2, \alpha_\nu)}{Q_0(\tau^2, \alpha_\nu)} = \inf_{\tau \geq 0} \inf_{\nu \in \mathbb{N}} \frac{Q_1(\tau, \alpha_\nu)}{Q_0(\tau, \alpha_\nu)} \\
&= \inf_{\tau \geq 0} \inf_{\nu \in \mathbb{N}} \left(\tau + \frac{4(\gamma - 1)\alpha_\nu}{\tau + \alpha_\nu + \left(\gamma - \frac{N}{2}\right)^2} \right) \\
&= \min_{\tau \geq 0} \frac{Q_1(\tau, \alpha_1)}{Q_0(\tau, \alpha_1)} = \min_{\tau \geq 0} \left(\tau + \frac{4\alpha_1(\gamma - 1)}{\tau + \alpha_1 + \left(\gamma - \frac{N}{2}\right)^2} \right),
\end{aligned}$$

where the second last equality follows by using $\gamma > 1$. Combine this result with (4.15), and we obtain

$$\frac{\int_{\mathbb{R}} |r\nabla \boldsymbol{v}|^2 r^{-N} dx}{\int_{\mathbb{R}^N} |\boldsymbol{v}|^2 r^{-N} dx} \geq 2 + \min_{\tau \geq 0} \left(\tau + \frac{4(N-1)(\gamma - 1)}{\tau + N - 1 + \left(\gamma - \frac{N}{2}\right)^2} \right)$$

to evaluate the quotient on the right-hand side of (4.9); therefore, the inequality

$$\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \geq C_{P,N,\gamma} \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma - 2} dx$$

holds with the constant number

$$C_{P,N,\gamma} = \Big(\gamma + \frac{N}{2} - 1\Big)^2 + 2 + \min_{\tau \geq 0} \left(\tau + \frac{4(N-1)(\gamma-1)}{\tau + N - 1 + \big(\gamma - \frac{N}{2}\big)^2} \right). \tag{4.16}$$

§4.3. **Optimality of $C_{P,N,\gamma}$.** It follows from §4.2.1 and §4.2.2 that the inequality

$$\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \geq C_{P,N,\gamma} \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx$$

together with the constant number

$$C_{P,N,\gamma} = \Big(\gamma + \frac{N}{2} - 1\Big)^2 + 2 + \min_{\tau \geq 0} \frac{Q_1(\tau, \alpha_1)}{Q_0(\tau, \alpha_1)},$$
$$\text{where} \quad \frac{Q_1(\tau, \alpha_1)}{Q_0(\tau, \alpha_1)} = \tau + \frac{4(N-1)(\gamma-1)}{\tau + N - 1 + \big(\gamma - \frac{N}{2}\big)^2},$$

holds for all poloidal fields $\boldsymbol{u}$ in $C_c^\infty(\dot{\mathbb{R}}^N)^N$, regardless of the case $\gamma \leq 1$ or $\gamma > 1$. Let us show that this number is the best possible. To do so, choose $\tau_\gamma \geq 0$ to satisfy

$$\min_{\tau \geq 0} \frac{Q_1(\tau, \alpha_1)}{Q_0(\tau, \alpha_1)} = \frac{Q_1(\tau_\gamma^2, \alpha_1)}{Q_0(\tau_\gamma^2, \alpha_1)}.$$

Define $\{g_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\dot{\mathbb{R}}^N)$ as a sequence of scalar fields by

$$g_n(\boldsymbol{x}) = g_n(e^t \boldsymbol{\sigma}) = \zeta\left(\frac{t}{n}\right) \sigma_1 \cos(\tau_\gamma t) \qquad \forall n \in \mathbb{N},\ \forall \boldsymbol{x} \in \dot{\mathbb{R}}^N,$$

where $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_N) = \boldsymbol{x}/|\boldsymbol{x}| \in \mathbb{S}^{N-1}$ and $t = \log|\boldsymbol{x}| \in \mathbb{R}$, and where $\zeta : \mathbb{R} \to \mathbb{R}$ is a smooth function $\not\equiv 0$ with compact support on $\mathbb{R}$; notice that the g_n satisfies the eigenequation

$$-\triangle_\sigma g_n = \alpha_1 g_n \quad \text{on } \dot{\mathbb{R}}^N \quad (\forall n \in \mathbb{N})$$

since $-\triangle_\sigma \sigma_1 = \alpha_1 \sigma_1$. In this setting, define $\{\boldsymbol{v}_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\dot{\mathbb{R}}^N)$ by

$$\boldsymbol{v}_n = r^{\gamma + \frac{N}{2} - 1} \boldsymbol{D}\big(r^{1-\gamma-\frac{N}{2}} g_n\big) \qquad \forall\, n \in \mathbb{N}$$

in terms of the poloidal generator, and apply the same calculation in (4.15) to the case $\boldsymbol{v} = \boldsymbol{v}_n$:

$$\frac{\int_{\mathbb{R}^N} |r\nabla \boldsymbol{v}_n|^2 r^{-N} dx}{\int_{\mathbb{R}^N} |\boldsymbol{v}_n|^2 r^{-N} dx} = 2 + \frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} g_n Q_1(-\partial_t^2, \alpha_1) g_n \, dt \, \mathrm{d}\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} g_n Q_0(-\partial_t^2, \alpha_1) g_n \, dt \, \mathrm{d}\sigma}$$

thanks to the above eigenequation. In order to compute the quotient on the right-hand side, notice that the 1-D Fourier integration of g_n yields

$$\begin{aligned}
\widehat{g_n}(\tau, \boldsymbol{\sigma}) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\tau t} g_n(e^t \boldsymbol{\sigma}) dt = \frac{\sigma_1}{\sqrt{2\pi}} \int_{\mathbb{R}} \zeta\left(\frac{t}{n}\right) \frac{e^{-i(\tau-\tau_\gamma)t} + e^{-i(\tau+\tau_\gamma)t}}{2} dt \\
&= \frac{n\sigma_1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \zeta(t) \left(e^{-in(\tau-\tau_\gamma)t} + e^{-in(\tau+\tau_\gamma)t} \right) dt \\
&= \frac{n\sigma_1}{2} \left(\widehat{\zeta}\big(n(\tau - \tau_\gamma)\big) + \widehat{\zeta}\big(n(\tau + \tau_\gamma)\big) \right).
\end{aligned}$$

By using this formula, the $L^2(\mathbb{R})$ isometry of the Fourier integration yields the following calculation:

$$\frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} g_n\, Q_1(-\partial_t^2,\alpha_1) g_n\, dt\, \mathrm{d}\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} g_n\, Q_0(-\partial_t^2,\alpha_1) g_n\, dt\, \mathrm{d}\sigma} = \frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} Q_1(\tau^2,\alpha_1)|\widehat{g_n}|^2 d\tau\, \mathrm{d}\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} Q_0(\tau^2,\alpha_1)|\widehat{g_n}|^2 d\tau\, \mathrm{d}\sigma}$$

$$= \frac{\displaystyle\int_{\mathbb{R}} Q_1(\tau^2,\alpha_1)\begin{pmatrix} \left|\widehat{\zeta}\big(n(\tau-\tau_\gamma)\big)\right|^2 + \left|\widehat{\zeta}\big(n(\tau+\tau_\gamma)\big)\right|^2 \\ +2\,\mathrm{Re}\Big(\overline{\widehat{\zeta}\big(n(\tau-\tau_\gamma)\big)}\,\widehat{\zeta}\big(n(\tau+\tau_\gamma)\big)\Big)\end{pmatrix} d\tau}{\displaystyle\int_{\mathbb{R}} Q_0(\tau^2,\alpha_1)\begin{pmatrix} \left|\widehat{\zeta}\big(n(\tau-\tau_\gamma)\big)\right|^2 + \left|\widehat{\zeta}\big(n(\tau+\tau_\gamma)\big)\right|^2 \\ +2\,\mathrm{Re}\Big(\overline{\widehat{\zeta}\big(n(\tau-\tau_\gamma)\big)}\,\widehat{\zeta}\big(n(\tau+\tau_\gamma)\big)\Big)\end{pmatrix} d\tau}$$

$$= \frac{\displaystyle\int_{\mathbb{R}} \begin{pmatrix} \Big(Q_1\left((\tau_\gamma+\tfrac{\tau}{n})^2,\alpha_1\right) + Q_1\left((\tau_\gamma-\tfrac{\tau}{n})^2,\alpha_1\right)\Big)\left|\widehat{\zeta}(\tau)\right|^2 d\tau \\ +2\,\mathrm{Re}\Big(Q_1\left((\tau_\gamma+\tfrac{\tau}{n})^2,\alpha_1\right)\overline{\widehat{\zeta}(\tau)}\,\widehat{\zeta}(\tau+2n\tau_\gamma)\Big)\end{pmatrix} d\tau}{\displaystyle\int_{\mathbb{R}} \begin{pmatrix} \Big(Q_0\left((\tau_\gamma+\tfrac{\tau}{n})^2,\alpha_1\right) + Q_0\left((\tau_\gamma-\tfrac{\tau}{n})^2,\alpha_1\right)\Big)\left|\widehat{\zeta}(\tau)\right|^2 d\tau \\ +2\,\mathrm{Re}\Big(Q_0\left((\tau_\gamma+\tfrac{\tau}{n})^2,\alpha_1\right)\overline{\widehat{\zeta}(\tau)}\,\widehat{\zeta}(\tau+2n\tau_\gamma)\Big)\end{pmatrix} d\tau}$$

$$\longrightarrow \frac{Q_1(\tau_\gamma^2,\alpha_1)\int_{\mathbb{R}}|\zeta(\tau)|^2 d\tau}{Q_0(\tau_\gamma^2,\alpha_1)\int_{\mathbb{R}}|\zeta(\tau)|^2 d\tau} = \frac{Q_1(\tau_\gamma^2,\alpha_1)}{Q_0(\tau_\gamma^2,\alpha_1)} \qquad \text{as} \quad n\to\infty.$$

Here the convergence in the last line follows by using

$$\int_{\mathbb{R}} Q_j\left((\tau_\gamma+\tfrac{\tau}{n})^2,\alpha_1\right)\overline{\widehat{\zeta}(\tau)}\,\widehat{\zeta}(\tau+2n\tau_\gamma)d\tau \underset{(n\to\infty)}{\longrightarrow} \begin{cases} 0 & \text{if } \tau_\gamma\neq 0 \\ Q_j(\tau_\gamma^2,\alpha_1)\int_{\mathbb{R}}|\zeta(\tau)|^2 d\tau & \text{if } \tau_\gamma = 0 \end{cases}$$

for both $j=0$ and $j=1$; this fact is ensured by that $\widehat{\zeta}$ is rapidly decreasing. Combine the results above, and consequently

$$\frac{\int_{\mathbb{R}^N}|r\nabla \boldsymbol{v}_n|^2 r^{-N}dx}{\int_{\mathbb{R}^N}|\boldsymbol{v}_n|^2 r^{-N}dx} \underset{(n\to\infty)}{\longrightarrow} 2+\frac{Q_1(\tau_\gamma^2,\alpha_1)}{Q_0(\tau_\gamma^2,\alpha_1)} = 2+\min_{\tau\geq 0}\frac{Q_1(\tau,\alpha_1)}{Q_0(\tau,\alpha_1)}.$$

Therefore, it turns out from (4.9) that the sequence of poloidal fields

$$\boldsymbol{u}_n = r^{1-\gamma-\frac{N}{2}}\boldsymbol{v}_n = \boldsymbol{D}\big(r^{1-\gamma-\frac{N}{2}}g_n\big) \qquad (n=1,2,\cdots)$$

satisfies

$$\frac{\int_{\mathbb{R}^N}|\nabla\boldsymbol{u}_n|^2|\boldsymbol{x}|^{2\gamma}dx}{\int_{\mathbb{R}^N}|\boldsymbol{u}_n|^2|\boldsymbol{x}|^{2\gamma-2}dx} \to C_{P,N,\gamma} \qquad (n\to\infty),$$

as the desired optimality of $C_{P,N,\gamma}$.

§4.4. **Estimation for toroidal fields: evaluation and optimality of $C_{T,N,\gamma}$.** In this subsection, $\boldsymbol{u}$ is assumed to be toroidal with compact support on $\dot{\mathbb{R}}^N$. Let $\boldsymbol{v}$ be the toroidal field given by the same transformation

$$\boldsymbol{v}(\boldsymbol{x}) = |\boldsymbol{x}|^{\gamma+\frac{N}{2}-1}\boldsymbol{u}(\boldsymbol{x}) \qquad (\forall \boldsymbol{x}\in\dot{\mathbb{R}}^N)$$

as the first formula of (4.2). Applying the same calculation as in (4.7) and (4.8), we have

$$\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx = \left(\gamma + \tfrac{N}{2} - 1\right)^2 \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx + \int_{\mathbb{R}^N} |\partial_t \boldsymbol{v}|^2 r^{-N} dx + \int_{\mathbb{R}^N} |\nabla_\sigma \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx.$$

On the other hand, recall from (3.2) that every toroidal field has zero spherical mean; then, considering the spherical harmonics expansion of (every component of) the toroidal field $\boldsymbol{u}$, we easily get the Poincaré type inequality

$$\int_{\mathbb{S}^{N-1}} |\nabla_\sigma \boldsymbol{u}|^2 \mathrm{d}\sigma \geq \alpha_1 \int_{\mathbb{S}^{N-1}} |\boldsymbol{u}|^2 \mathrm{d}\sigma \qquad \text{(for any radius)},$$

$$\text{and hence} \quad \int_{\mathbb{R}^N} |\nabla_\sigma \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx \geq \alpha_1 \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx,$$

where the equality is attained if and only if $-\triangle_\sigma \boldsymbol{u} = \alpha_1 \boldsymbol{u}$ on $\dot{\mathbb{R}}^N$. Combine this integral inequality with the above integral equation, and we get

$$\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \geq C_{T,N,\gamma} \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx + \int_{\mathbb{R}^N} |\partial_t \boldsymbol{v}|^2 r^{-N} dx \tag{4.17}$$

together with the constant number

$$C_{T,N,\gamma} = \left(\gamma + \frac{N}{2} - 1\right)^2 + N - 1, \tag{4.18}$$

where the equality in (4.17) is attained if and only if $-\triangle_\sigma \boldsymbol{u} = \alpha_1 \boldsymbol{u}$ on $\dot{\mathbb{R}}^N$. In particular, we get the Hardy-Leray inequality

$$\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \geq C_{T,N,\gamma} \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx$$

for toroidal fields. To show that the constant number given by (4.18) is the best possible in this inequality, we set

$$\boldsymbol{v}_0(\boldsymbol{x}) := (-x_2, x_1, 0, \cdots, 0) \qquad \forall \boldsymbol{x} \in \dot{\mathbb{R}}^N$$

as a toroidal field satisfying the eigenequation

$$-\triangle_\sigma \boldsymbol{v}_0 = \alpha_1 \boldsymbol{v}_0 \quad \text{ on } \dot{\mathbb{R}}^N.$$

In this setting, define $\{\boldsymbol{v}_n\}_{n\in\mathbb{N}}$ and $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ as two sequences of toroidal fields by

$$\left.\begin{aligned} \boldsymbol{v}_n(\boldsymbol{x}) &= \zeta\left(\frac{\log|\boldsymbol{x}|}{n}\right) \boldsymbol{v}_0(\boldsymbol{x}/|\boldsymbol{x}|) \\ \boldsymbol{u}_n(\boldsymbol{x}) &= |\boldsymbol{x}|^{1-\gamma-\frac{N}{2}} \boldsymbol{v}_n(\boldsymbol{x}) \end{aligned}\right\} \quad \forall \boldsymbol{x} \in \dot{\mathbb{R}}^N,\ \forall n \in \mathbb{N},$$

where $\zeta : \mathbb{R} \to \mathbb{R}$ is a smooth function $\not\equiv 0$ with compact support on $\mathbb{R}$. Now apply (4.17) to the case $\boldsymbol{u} = \boldsymbol{u}_n$; then, thanks to the above eigenequation, we get

$$\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}_n|^2 |\boldsymbol{x}|^{2\gamma} dx = C_{T,N,\gamma} \int_{\mathbb{R}^N} |\boldsymbol{u}_n|^2 |\boldsymbol{x}|^{2\gamma-2} dx + \int_{\mathbb{R}^N} |\partial_t \boldsymbol{v}_n|^2 r^{-N} dx \quad (\forall n \in \mathbb{N}).$$

Dividing both sides by $\int_{\mathbb{R}^N}|\boldsymbol{u}_n|^2|\boldsymbol{x}|^{2\gamma-2}dx=\int_{\mathbb{R}^N}|\boldsymbol{v}_n|^2r^{-N}dx$, we obtain

$$\begin{aligned}\frac{\int_{\mathbb{R}^N}|\nabla\boldsymbol{u}_n|^2|\boldsymbol{x}|^{2\gamma}dx}{\int_{\mathbb{R}^N}|\boldsymbol{u}_n|^2|\boldsymbol{x}|^{2\gamma-2}dx}&=C_{T,N,\gamma}+\frac{\int_{\mathbb{R}^N}|\partial_t\boldsymbol{v}_n|^2r^{-N}dx}{\int_{\mathbb{R}^N}|\boldsymbol{v}_n|^2r^{-N}dx}\\&=C_{T,N,\gamma}+\frac{\int_{\mathbb{R}}(n^{-1}\zeta'(t/n))^2dt}{\int_{\mathbb{R}}(\zeta(t/n))^2dt}\longrightarrow C_{T,N,\gamma}\quad(n\to\infty)\end{aligned}$$

as the desired the optimality of $C_{T,N,\gamma}$.

§4.5. **Conclusion of the proof of main theorem and its remark.** Substituting (4.16) and (4.18) into (4.1), we get

$$\begin{aligned}C_{N,\gamma}&=\min\{C_{P,N,\gamma},C_{T,N,\gamma}\}\\&=\Big(\gamma+\frac{N}{2}-1\Big)^2+\min\left\{2+\min_{\tau\ge0}\left(\tau+\frac{4(N-1)(\gamma-1)}{\tau+N-1+\big(\gamma-\frac{N}{2}\big)^2}\right),\ N-1\right\}\end{aligned}$$

as the desired best constant (1.2) of the inequality (1.1) for all solenoidal fields $\boldsymbol{u}$.

Moreover, for the case $\gamma\le1$, we get the additional term of H-L inequality in the following way: Notice from a direct computation that

$$C_{N,\gamma}=C_{P,N,\gamma}\quad\text{and}\quad\begin{cases}C_{P,N,\gamma}=C_{T,N,\gamma}&\text{for }\ (N,\gamma)=(3,1),\\C_{P,N,\gamma}<C_{T,N,\gamma}&\text{otherwise.}\end{cases}$$

By using this fact, the calculation of $(4.14)_{\boldsymbol{u}=\boldsymbol{u}_P}$ plus $(4.17)_{\boldsymbol{u}=\boldsymbol{u}_T}$ gives

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx&=\int_{\mathbb{R}^N}|\nabla\boldsymbol{u}_P|^2|\boldsymbol{x}|^{2\gamma}dx+\int_{\mathbb{R}^N}|\nabla\boldsymbol{u}_T|^2|\boldsymbol{x}|^{2\gamma}dx\\&\ge C_{P,N,\gamma}\int_{\mathbb{R}^N}|\boldsymbol{u}_P|^2|\boldsymbol{x}|^{2\gamma-2}dx+C_{T,N,\gamma}\int_{\mathbb{R}^N}|\boldsymbol{u}_T|^2|\boldsymbol{x}|^{2\gamma-2}dx\\&\quad+\mathcal{R}_{N,\gamma}[\boldsymbol{u}_P]+\int_{\mathbb{R}^N}\Big|\partial_t\big(|\boldsymbol{x}|^{\gamma+\frac{N}{2}-1}\boldsymbol{u}_T\big)\Big|^2r^{-N}dx\\&=C_{N,\gamma}\int_{\mathbb{R}^N}|\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-2}dx+\big(C_{T,N,\gamma}-C_{P,N,\gamma}\big)\int_{\mathbb{R}^N}|\boldsymbol{u}_T|^2|\boldsymbol{x}|^{2\gamma-2}dx+\mathcal{R}_{N,\gamma}[\boldsymbol{u}]\\&\ge C_{N,\gamma}\int_{\mathbb{R}^N}|\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-2}dx+\mathcal{R}_{N,\gamma}[\boldsymbol{u}]\end{aligned}$$

to arrive at the desired inequality (1.3). Here the second resp. last equality sign is attained if and only if

$$-\triangle_\sigma u_R=\alpha_1u_R\quad\text{and}\quad-\triangle_\sigma\boldsymbol{u}_T=\alpha_1\boldsymbol{u}_T\qquad\text{on }\dot{\mathbb{R}}^N$$

resp. if and only if

$$\boldsymbol{u}_T\equiv\boldsymbol{0}\quad\text{or}\quad(N,\gamma)=(3,1);$$

hence both the equality signs are simultaneously attained if and only if

$$-\triangle_\sigma u_R=\alpha_1u_R\quad\text{and}\quad\begin{cases}-\triangle_\sigma\boldsymbol{u}_T=2\boldsymbol{u}_T&\text{for }(N,\gamma)=(3,1)\\\quad\boldsymbol{u}_T=\boldsymbol{0}&\text{otherwise}\end{cases}\quad\text{on }\dot{\mathbb{R}}^N,$$

which gives the same attainability condition as in Remark 1.2. The proof of Theorem 1.1 and its remark is now complete.

Acknowledgments

This research was partly supported by JSPS KAKENHI Grant-in-Aid for Scientific Research (B) 19H01800 (F. Takahashi). Additionally, this work was partly supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849).

References

[1] G. E. Backus, *A class of self-sustaining dissipative spherical dynamos*, Annals of Physics **4** (1958), no. 4, 372–447.

[2] H. Brezis and J. L. Vázquez, *Blow-up solutions of some nonlinear elliptic problems*, Revista Matemática de la Universidad Complutense de Madrid **10** (1997), no. 2, 443–469.

[3] L. A. Caffarelli, R. V. Kohn, and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Mathematica **53** (1984), no. 3, 259–275.

[4] O. Costin and V. G. Maz'ya, *Sharp Hardy–Leray inequality for axisymmetric divergence-free fields*, Calculus of Variations and Partial Differential Equations **32** (2008), no. 4, 523–532.

[5] N. Hamamoto, *Three-dimensional sharp Hardy-Leray inequality for solenoidal fields*, Nonlinear Analysis **191** (2020), 111634, 14 pp.

[6] N. Hamamoto and F. Takahashi, *Sharp Hardy-Leray and Rellich-Leray inequalities for curl-free vector fields*, Mathematische Annalen (2019).

[7] ———, *Sharp Hardy-Leray inequality for three-dimensional solenoidal fields with axisymmetric swirl*, Communications on Pure & Applied Analysis **19** (2020), no. 6, 3209–3222.

[8] G. H. Hardy, *Note on a theorem of Hilbert*, Mathematische Zeitschrift **6** (1920), no. 3–4, 314–317.

[9] J. Leray, *Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique*, Journal de Mathématiques Pures et Appliquées **12** (1933), 1–82 (French).

[10] N. Weck, *The poloidal toroidal decomposition of differential forms*, Analysis **17** (1997), no. 2-3, 265–285.

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