EXISTENCE OF BLOWING-UP SOLUTIONS TO SOME SCHRÖDINGER EQUATIONS INCLUDING NONLINEAR AMPLIFICATION WITH SMALL INITIAL DATA

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Abstract

We consider the existence of blowing-up solutions to some Schrödinger equations including nonlinear amplification. The blow-up is considered in $L^2(\mathbb{R})$. Even though initial data are taken so small, there exists some solutions blowing-up in finite time. The theorem in this paper is an extension of Cazenave-Martel-Zhao's result [2] from the points of making the lower bound of power of nonlinearity extended and ensuring that blowing-up solutions exist even for small initial data.

1 Introduction and Main Result

We consider the Cauchy problem of a nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u = -\frac{1}{2}\partial_x^2 u + (\lambda + i\kappa)|u|^{p-1}u, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where the complex-valued unknown function $u = u(t,x)$ is defined on $(t,x) \in [0,T) \times \mathbb{R}^1$. In the nonlinearity, the power satisfies $2 < p \leq 3$ and the coefficients $\lambda, \kappa \in \mathbb{R}$ satisfy

$$\kappa > 0, \quad (p-1)|\lambda| \leq 2\sqrt{p}\ \kappa. \tag{1.2}$$

In particular, the positivity of κ in (1.2) implies that the nonlinearity affects as an amplification. To see it, we refer to the idea of Zhang [3]. If the region of x is a bounded interval I and Dirichlet boundary condition is imposed, then it is easy to show that, for $u_0 \in L^2(\mathbb{R})$ and $u_0 \neq 0$, the solution to (1.1) blows up in finite time. In fact, we have

$$\begin{aligned} \frac{d\|u(t)\|_{L^2(I)}^2}{dt} &= 2\mathrm{Re}(u(t), \partial_t u(t))_{L^2(I)} \\ &= 2\kappa \|u(t)\|_{L^{p+1}(I)}^{p+1}, \end{aligned}$$

where $(f,g)_{L^2(I)} = \int_I f(x)\overline{g(x)}dx$ is the usual L^2-inner product. Applying Hölder's inequality : $|I|^{(p-1)/2}\|u(t)\|_{L^{p+1}(I)}^{p+1} \geq \|u(t)\|_{L^2(I)}^{p+1}$ where $|I|$ denotes the size of the interval, we see that

$$\frac{d\|u(t)\|_{L^2(I)}^2}{dt} \geq 2\kappa |I|^{-(p-1)/2}\|u(t)\|_{L^2(I)}^{p+1}.$$

2010 *Mathematics Subject Classification.* Primary 35Q55; Secondary 35B40.

Key Words and Phrases. nonlinear Schrödinger equation, nonlinear amplification, blowing-up solution, small initial data

Solving this differential inequality, we have

$$\|u(t)\|_{L^2(I)} \geq \frac{\|u_0\|_{L^2(I)}}{\left\{1 - \kappa(p-1)|I|^{-(p-1/2)}\|u_0\|_{L^2(I)}^{p-1} t\right\}^{1/(p-1)}},$$

and we know that $\|u(t)\|_{L^2(I)}$ blows up in finite time. However, this kind of estimate holds only in the case that x belongs to the bounded interval. Once the region becomes unbounded, the dispersion associated with $-\frac{1}{2}\partial_x^2$ will work so that the nonlinear amplification is suppressed, and it is difficult to presume that the nonlinear amplification surely generates a blowing-up solution. Actually when $3 < p$ and u_0 is sufficiently small in $H^1(\mathbb{R})$ with $xu_0 \in L^2(\mathbb{R})$ also small, the solution to (1.1) exists globally in time. This is because $|u(t,x)|^{p-1} \sim Ct^{-(p-1)/2}$ is integrable for large t, and the nonlinearity does not affect to the behavior of the solution. This observation suggests that, if we expect the blow-up for a small initial data, it is necessary to assume $p \leq 3$.

Our goal is to obtain blowing-up solutions to (1.1) even though the smallness is assumed on the initial data.

Theorem 1.1. *Let $2 < p \leq 3$. Also let λ and κ satisfy* (1.2). *Then, for any $\rho > 0$, there exists some initial data $u_0 \in L^2(\mathbb{R})$ such that*

(i) $\|u_0\|_{L^2(\mathbb{R})} < \rho$,

(ii) the solution u to (1.1) *with u_0 as the initial data satisfies*

$$\lim_{t \uparrow T*} \|u(t)\|_{L^2(\mathbb{R})} = \infty \tag{1.3}$$

for some $T^* > 0$.

Theorem 1.1 asserts the existence of a blowing-up solution only for some special small initial data. It remains open whether any small initial data except for $u_0 = 0$ give rise to the blow-up. In Theorem 1.1, the lower bound of p is required by the technical reason that the blowing-up profile must be integrable around the blowing-up time with respect to t. The upper bound of p is required to ensure the existence of blowing-up solution with small initial data. Precisely speaking, we will first construct a blowing-up profile, construct a solution to (1.1) which approaches to the profile while $t \uparrow T*$, and extend it backward in time. In order to guarantee the decay of the solution in the negative time-direction, the assumption of $p \leq 3$ is required.

The construction of a blowing-up solution to some Schrödinger equation with nonlinear source term was considered by Cazenave-Martel-Zhao [2]. They treated the N-dimensional nonlinear Scrödinger equation :

$$i\partial_t u = -\Delta u + i|u|^{p-1}u,$$

where $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ and $\Delta = \sum_{j=1}^N \partial_{x_j}^2$. In their idea, a profile $\varphi(t,x)$ of the blowing-up solution was firstly determined, which is subject to the ODE :

$$i\partial_t \varphi(t,x) = i|\varphi|^{p-1}\varphi(t,x).$$

They employed, for instance, $\varphi(t,x) = ((p-1)|t| + A|x|^k)^{-1/(p-1)}$ for some $A, k > 0$, which blows up at $t = 0$, and solve the nonlinear Schrödinger equation in $H^1(\mathbb{R}^N)$ by setting $u(t,x) = \varphi(t,x) + v(t,x)$ with $v(0,x) = 0$. In [2], the blow up of small initial data was not considered. In their argument, the condition $3 \leq p$ was assumed. We extend this restriction to $2 < p$ by somewhat sophisticated nonlinear estimate as well as the coefficient of nonlinearity is generalized as in (1.2). We will not consider N-dimensional problem since the p must be restricted into $p \leq 1 + 2/N$ and the $\varphi(t,x) = O(|t|^{-1/(p-1)})$ shows a non-integrable singularity if $N \geq 2$.

2 Blowing-Up Profiles

We expect that the blow-up of the solutions is caused by the nonlinearity, and so the dispersion associated with $-\frac{1}{2}\partial_x^2$ does not work so strongly just before the blowing-up time. This observation suggests that the blowing-up profile is subject to the ordinary differential equation :

$$i\partial_t \varphi(t,x) = (\lambda + i\kappa)|\varphi(t,x)|^{p-1}\varphi(t,x). \tag{2.1}$$

For (2.1), we impose an initial data $\varphi(-1,x) = \varphi_{-1}(x)$ at each $x \in \mathbb{R}$, where φ_{-1} satisfies

(A) *The assumption on* φ_{-1}:

(A.1) The $\varphi_{-1} \in C_0^\infty(\mathbb{R})$ is real valued.

(A.2) $0 \leq \varphi_{-1}(x) \leq (\kappa(p-1))^{-1/(p-1)}$.

(A.3) $\varphi_{-1}(x) = (\kappa(p-1))^{-1/(p-1)}$ if and only if $x = 0$.

(A.4) $\varphi_{-1}(x) = (\kappa(p-1))^{-1/(p-1)}(1 - x^{2N})^{1/(p-1)}$ for $|x| < 1/2$, where $N > 0$ is sufficiently large integer.

(A.5) $\varphi_{-1}(x) \leq \varphi_{-1}(1/2)$ for $|x| \geq 1/2$.

The ODE in (2.1) is easy to slove. In fact, by (2.1), we see that

$$\partial_t |\varphi(t,x)|^2 = 2\kappa|\varphi(t,x)|^{p+1},$$

which yields

$$\partial_t |\varphi(t,x)|^{-(p-1)} = -\kappa(p-1). \tag{2.2}$$

Integrating (2.2) from -1 to $t < 0$, we have

$$|\varphi(t,x)| = \frac{|\varphi_{-1}(x)|}{\{1 - \kappa(p-1)|\varphi_{-1}(x)|^{p-1}(t+1)\}^{1/(p-1)}}. \tag{2.3}$$

Substitute (2.3) into the $|\varphi(t,x)|^{p-1}$ on the right hand side of (2.1). Then we notice that it is a standard first order ODE of $\varphi(t,x)$, and we obtain

$$\varphi(t,x) = \varphi_{-1}(x)\left\{1 - \kappa(p-1)\varphi_{-1}^{p-1}(x)(t+1)\right\}^{(-1+i\frac{\lambda}{\kappa})\frac{1}{p-1}}. \tag{2.4}$$

We call $\varphi(t,x)$ in (2.4) *the blowing-up profile*. By the assumption (A) on φ_{-1}, the $\varphi(t,x)$ blows up at $t=0$, and, precisely speaking, $\lim_{t\uparrow 0}|\varphi(t,0)|=\infty$ occurs but $|\varphi(0,x)|<\infty$ for $x\neq 0$. The condition (A.4) suggests that the graph of $\varphi_{-1}(x)$ is so flat around $x=0$, which guarantees that the blowing-up rates of $\partial_x\varphi(t,x)$ and higher derivatives do not violate the integrability with respect to t around $t=0$ when $2<p$. We will see the detail on $\varphi(t,x)$ in next lemma.

Lemma 2.1. *Let φ_{-1} be such as defined in the assumption* (A)*, and let j be an integer satisfying* $0\le j\le N$*. Then there exist some $C_j>0$ such that the blowing-up profile* (2.4) *satisfies*

$$|\partial_x^j\varphi(t,x)|\le C_j|t|^{-1/(p-1)-j/(2N)} \tag{2.5}$$

for any $t\in(-1,0)$.

Proof of Lemma 2.1. It suffices to prove (2.5) for $x\in(-1/2,1/2)$, since the blow-up takes place at $x=0$ first. By the assumption (A), $\varphi_{-1}(x)=(\kappa(p-1))^{-1/(p-1)}(1-x^{2N})^{1/(p-1)}$ if $|x|<1/2$. Substitute it into (2.4). Then we have

$$\varphi(t,x)=(\kappa(p-1))^{-1/(p-1)}(1-x^{2N})^{1/(p-1)}\{(t+1)x^{2N}-t\}^{(-1+i\frac{\lambda}{\kappa})\frac{1}{p-1}}.$$

Applying Leibniz' rule and regarding $(1-x^{2N})^{1/(p-1)}\sim 1-\frac{1}{p-1}x^{2N}$, we have

$$\begin{aligned}|\partial_x^j\varphi(t,x)| \le\ & C(1-x^{2N})^{1/(p-1)}|\partial_x^j\{(t+1)x^{2N}-t\}^{(-1+i\frac{\lambda}{\kappa})\frac{1}{p-1}}| \\ & +C\sum_{k=0}^{j-1}|x|^{2N-(j-k)}|\partial_x^k\{(t+1)x^{2N}-t\}^{(-1+i\frac{\lambda}{\kappa})\frac{1}{p-1}}|.\end{aligned} \tag{2.6}$$

Note that, for the first term of (2.6), the chain rule yields

$$\begin{aligned}&|\partial_x^j\{(t+1)x^{2N}-t\}^{(-1+i\frac{\lambda}{\kappa})\frac{1}{p-1}}| \\ &\le C\sum_{\ell=1}^{j}\sum_{(\mu_1,\cdots,\mu_\ell)\in S_{j,\ell}}|\partial_x^{\mu_1}x^{2N}|\cdot|\partial_x^{\mu_2}x^{2N}|\cdots|\partial_x^{\mu_\ell}x^{2N}|\cdot|(t+1)x^{2N}-t|^{-1/(p-1)-\ell} \\ &\le C\sum_{\ell=1}^{j}|x|^{2N\ell-j}\cdot|(t+1)x^{2N}-t|^{-1/(p-1)-\ell},\end{aligned}$$

where $S_{j,\ell}=\{(\mu_1,\cdots,\mu_\ell)\in\mathbb{N}^\ell;\mu_1+\cdots+\mu_\ell=j\}$. We apply the similar estimate to the second term of (2.6). Then we have, for $x \in (-1/2, 1/2)$,

$$
\begin{aligned}
|\partial_x^j \varphi(t,x)| &\le C\sum_{\ell=1}^{j} |x|^{2N\ell-j}\cdot |(t+1)x^{2N}-t|^{-1/(p-1)-\ell} \\
&\quad +C\sum_{k=1}^{j-1}|x|^{2N-(j-k)}\sum_{\ell=1}^{k}|x|^{2N\ell-k}\cdot|(t+1)x^{2N}-t|^{-1/(p-1)-\ell} \\
&\quad +C|x|^{2N-j}\cdot|(t+1)x^{2N}-t|^{-1/(p-1)} \\
&= C\sum_{\ell=1}^{j} |x|^{2N\ell-j}\cdot |(t+1)x^{2N}-t|^{-1/(p-1)-\ell} \\
&\quad +C|x|^{2N}\sum_{k=1}^{j-1}\sum_{\ell=1}^{k}|x|^{2N\ell-j}\cdot|(t+1)x^{2N}-t|^{-1/(p-1)-\ell} \\
&\quad +C|x|^{2N-j}\cdot|(t+1)x^{2N}-t|^{-1/(p-1)} \qquad (2.7)
\end{aligned}
$$

Let $\xi = (t+1)x^{2N}/|t|$. Then, from (2.7), it follows that

$$
\begin{aligned}
|\partial_x^j \varphi(t,x)| &\le C\sum_{\ell=1}^{j}\frac{|t|^{-1/(p-1)-j/(2N)}}{(t+1)^{\ell-j/(2N)}}\xi^{\ell-j/(2N)}(\xi+1)^{-1/(p-1)-\ell} \\
&\quad +C|x|^{2N}\sum_{k=1}^{j-1}\sum_{\ell=1}^{k}\frac{|t|^{-1/(p-1)-j/(2N)}}{(t+1)^{\ell-j/(2N)}}\xi^{\ell-j/(2N)}(\xi+1)^{-1/(p-1)-\ell} \\
&\quad +C|x|^{2N-j}\cdot\frac{|t|^{-1/(p-1)}}{(t+1)^{-1/(p+1)}}(\xi+1)^{-1/(p-1)}. \qquad (2.8)
\end{aligned}
$$

Since $\sup_{\xi\ge 0}\xi^{\ell-j/(2N)}(\xi+1)^{-1/(p-1)-\ell} < \infty$, (2.8) yields

$$
\begin{aligned}
|\partial_x^j \varphi(t,x)| &\le C|t|^{-1/(p-1)-j/(2N)} + C|x|^{2N}\cdot|t|^{-1/(p-1)-j/(2N)} + C|x|^{2N-j}\cdot|t|^{-1/(p-1)} \\
&\le C(1+(1/2)^{2N}+(1/2)^{2N-j})|t|^{-1/(p-1)-j/(2N)}. \qquad \square
\end{aligned}
$$

In the subsequent section, we will use a modified profiles $\varphi_\nu(t,x) = \varphi(t-\nu, x)$ for $\nu \in (0,1]$ to consider approximate solutions around the blowing-up time. Applying the analogy in the proof of Lemma 2.1, we have some properties of $\varphi_\nu(t,x)$.

Corollary 2.2. *Let φ_{-1} be such as defined in the assumption* (A). *Let j be an integer satisfying $0 \le j \le N$, and $\varepsilon \in (0,1]$. Then there exist some $C_j > 0$, $C_{j,\varepsilon} > 0$ and $\delta > 0$ independent of $\nu, \nu' \in (0,1]$ such that*

$$
|\partial_x^j \varphi_\nu(t,x)| \le C_j |t|^{-1/(p-1)-j/(2N)}, \qquad (2.9)
$$

$$
|\partial_x^j(\varphi_\nu(t,x)-\varphi_{\nu'}(t,x))| \le C_{j,\varepsilon}|t|^{-1/(p-1)-j/(2N)-\varepsilon}(\nu^\varepsilon+\nu'^\varepsilon) \qquad (2.10)
$$

for any $t \in (-\delta, 0)$.

Proof of Corollary 2.2. By Lemma 2.1, we have

$$
\begin{aligned}
|\partial_x^j \varphi_\nu(t,x)| &= |\partial_x^j\varphi(t-\nu,x)| \\
&\le C|t-\nu|^{-1/(p-1)-j/(2N)} \\
&\le C|t|^{-1/(p-1)-j/(2N)},
\end{aligned}
$$

and we obtain (2.9). We next consider

$$\varphi_\nu(t,x)-\varphi(t,x)=-\int_{t-\nu}^{t}\partial_\tau\varphi(\tau,x)d\tau.$$

Since

$$\partial_\tau\varphi(\tau,x)=(\kappa-i\lambda)\varphi_{-1}^{p}(x)\left\{1-\kappa(p-1)\varphi_{-1}^{p-1}(x)(t+1)\right\}^{(-1+i\frac{\lambda}{\kappa})\frac{1}{p-1}-1},$$

we may retrace the estimate as we did in the proof of Lemma 2.1, replacing the power $(-1+i\frac{\lambda}{\kappa})\frac{1}{p-1}$ by $(-1+i\frac{\lambda}{\kappa})\frac{1}{p-1}-1$. Hence we have

$$\begin{aligned}|\partial_x^j(\varphi_\nu(t,x)-\varphi(t,x))| &\leq \int_{t-\nu}^{t}|\partial_x^j\partial_\tau\varphi(\tau,x)|d\tau\\ &\leq C\int_{t-\nu}^{t}|\tau|^{-1-1/(p-1)-j/(2N)}d\tau.\end{aligned}\tag{2.11}$$

The integrand is bounded by $|\tau|^{-1+\varepsilon}|\tau|^{-1/(p-1)-j/(2N)-\varepsilon}\leq|\tau|^{-1+\varepsilon}|t|^{-1/(p-1)-j/(2N)-\varepsilon}$. Then we see that

$$\begin{aligned}|\partial_x^j(\varphi_\nu(t,x)-\varphi(t,x))| &\leq C_j|t|^{-1/(p-1)-j/(2N)-\varepsilon}\int_{t-\nu}^{t}|\tau|^{-1+\varepsilon}d\tau\\ &\leq \frac{C_j}{\varepsilon}|t|^{-1/(p-1)-j/(2N)-\varepsilon}(|t-\nu|^{\varepsilon}-|t|^{\varepsilon})\\ &\leq C_{j,\varepsilon}|t|^{-1/(p-1)-j/(2N)-\varepsilon}\nu^{\varepsilon}.\end{aligned}$$

Since $|\partial_x^j(\varphi_\nu-\varphi_{\nu'})|\leq|\partial_x^j(\varphi_\nu-\varphi)|+|\partial_x^j(\varphi_{\nu'}-\varphi)|$, we obtain (2.10). □

3 A Solution Around the Blowing-Up Profile

We will construct a solution to (1.1) locally in negative time, which asymptotically tends to $\varphi(t,x)$ as $t\uparrow 0$. To this end, we write $u(t,x)=\varphi(t,x)+v(t,x)$. Then the equation that $v=v(t,x)$ satisfies is

$$\begin{cases} i\partial_t v=-\frac{1}{2}\partial_x^2 v-\frac{1}{2}\partial_x^2\varphi+(\lambda+i\kappa)(\mathcal{N}(\varphi+v)-\mathcal{N}(\varphi)),\\ v(0,x)=0,\end{cases}\tag{3.1}$$

where $\mathcal{N}(u)=|u|^{p-1}u$. One may first suppose to apply the contraction mapping priciple to (3.1) via Duhamel's priciple. But this apprach will not work so well, since the nonlinear estimate such as

$$|\mathcal{N}(\varphi+v)-\mathcal{N}(\varphi)|\leq C(|\varphi|^{p-1}+|v|^{p-1})|v|$$

contains the non-integrable singularity on $|\varphi|^{p-1}=O(|t|^{-1})$ around $t=0$. Thus we need to apply another approach so called the energy method. To derive a decay estimate of $\|u(t,\cdot)\|_{L^2(\mathbb{R})}$ as $t\to-\infty$, we must solve (3.1) in the weighted L^2 space. In this section, we will prove the next proposition.

Proposition 3.1. *Let $2 < p$, and let λ, κ satisfy* (1.2). *Then, for some $T_0 < 0$, there exists a unique solution $v = v(t,x)$ to* (3.1) *such that*

$$v \in C([T_0, 0]; H^1(\mathbb{R})) \cap C^1([T_0, 0); H^{-1}(\mathbb{R})), \tag{3.2}$$
$$xv \in C([T_0, 0]; L^2(\mathbb{R})). \tag{3.3}$$

Furthermore the solution satisfies

$$\|v(t,\cdot)\|_{L^2(\mathbb{R})} \le C|t|^{\alpha_0}, \quad \|\partial_x v(t,\cdot)\|_{L^2(\mathbb{R})} \le C|t|^{\alpha_1}, \tag{3.4}$$

where $\alpha_0 = 1 - 1/(p-1) - 2/(2N) > 0$ and $\alpha_1 = 1 - 1/(p-1) - 3/(2N) > 0$ with N defined in (A.4).

To prove Proposition 3.1, we begin to consider an approximate solution for $\varphi_\nu(t,x) = \varphi(t-\nu, x)$ with $0 < \nu < 1$, i.e.,

$$\begin{cases} i\partial_t v_\nu = -\frac{1}{2}\partial_x^2 v_\nu - \frac{1}{2}\partial_x^2 \varphi_\nu + (\lambda + i\kappa)(\mathcal{N}(\varphi_\nu + v_\nu) - \mathcal{N}(\varphi_\nu)), \\ v_\nu(0,x) = 0. \end{cases} \tag{3.5}$$

Since there is no singularity at $t = 0$ in $\varphi_\nu(t,x)$, the equation (3.5) can be solved locally in negative time by transforming it into the associated integral equation and by applying the contraction mapping principle [1]. Indeed we have a solution to (3.5) such that

$$v_\nu \in C([T_\nu, 0]; H^1(\mathbb{R})) \cap C^1([T_\nu, 0); H^{-1}(\mathbb{R})),$$
$$xv_\nu \in C([T_\nu, 0]; L^2(\mathbb{R})),$$

where $T_\nu < 0$ is given by

$$T_\nu = \inf\{T \in (-1, 0); \sup_{T<t\le 0} (\|v_\nu(t,\cdot)\|_{H^1(\mathbb{R})} + \|xv_\nu(t,\cdot)\|_{L^2(\mathbb{R})}) < 1\}.$$

Lemma 3.2. *Let $2 < p$, and let λ, κ satisfy* (1.2). *Then there exists some $T_0 < 0$ such that the next three assertions hold.*

(*i*) *We have $T_\nu \le T_0$ for any $\nu \in (0, 1]$.*

(*ii*) *We have*

$$\sum_{j=0}^{1} \|x^j v_\nu(t,\cdot)\|_{L^2(\mathbb{R})} \le C|t|^{1-1/(p-1)-2/(2N)}, \tag{3.6}$$
$$\|\partial_x v_\nu(t,\cdot)\|_{L^2(\mathbb{R})} \le C|t|^{1-1/(p-1)-3/(2N)} \tag{3.7}$$

for any $t \in [T_0, 0]$ and $\nu \in (0, 1]$.

(*iii*) *Let $\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon_0 > 0$ sufficiently small. Then there exists some constant $C_\varepsilon > 0$ such that*

$$\sum_{j=1}^{1} \|x^j (v_\nu(t,\cdot) - v_{\nu'}(t,\cdot))\|_{L^2(\mathbb{R})} \le C_\varepsilon |t|^{1-1/(p-1)-2/(2N)-\varepsilon}(\nu^\varepsilon + \nu'^\varepsilon), \tag{3.8}$$
$$\|\partial_x (v_\nu(t,\cdot) - v_{\nu'}(t,\cdot))\|_{L^2(\mathbb{R})} \le C_\varepsilon |t|^{1-1/(p-1)-3/(2N)-\varepsilon}(\nu^{(p-2)\varepsilon/2} + \nu'^{(p-2)\varepsilon/2}) \tag{3.9}$$

for any $t \in [T_0, 0]$ and $\nu, \nu' \in (0, 1]$.

Proof of Lemma 3.2. For the solution v_ν to (3.5), we have

$$\begin{aligned}\frac{d}{dt}\|v_\nu\|^2_{L^2(\mathbb{R})} &= -\mathrm{Im}(\partial_x^2\varphi_\nu, v_\nu)_{L^2(\mathbb{R})} + 2\mathrm{Im}\left\{(\lambda+i\kappa)(\mathcal{N}(\varphi_\nu+v_\nu)-\mathcal{N}(\varphi_\nu), v_\nu)_{L^2(\mathbb{R})}\right\} \\ &\equiv I + II, \end{aligned} \tag{3.10}$$

where $(f,g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\overline{g(x)}dx$ denotes the inner product. By Cauchy-Schwarz' inequality together with Corollary 2.2 (2.9), we see that

$$I \geq -C|t|^{-1/(p-1)-2/(2N)}\|v_\nu\|_{L^2(\mathbb{R})}. \tag{3.11}$$

Since $\mathcal{N}(\varphi_\nu+v_\nu)-\mathcal{N}(\varphi_\nu) = \int_0^1 \partial_\theta \mathcal{N}(\varphi_\nu+\theta v_\nu)d\theta$, we have

$$II = 2\int_0^1 \mathrm{Im}\left\{(\lambda+i\kappa)(\mathcal{N}_u(\varphi_\nu+\theta v_\nu)v_\nu + \mathcal{N}_{\bar{u}}(\varphi_\nu+\theta v_\nu)\overline{v_\nu}, v_\nu)_{L^2(\mathbb{R})}\right\} d\theta,$$

where $\mathcal{N}_u(u) = \partial_u\mathcal{N}(u) = \frac{p+1}{2}|u|^{p-1}$ and $\mathcal{N}_{\bar{u}}(u) = \partial_{\bar{u}}\mathcal{N}(u) = \frac{p-1}{2}|u|^{p-3}u^2$. Then it follows that

$$II \geq ((p+1)\kappa - (p-1)|\lambda+i\kappa|)\int_0^1\int_{\mathbb{R}} |\varphi_\nu+\theta v_\nu|^{p-1}|v_\nu|^2 dx d\theta.$$

Since $(p+1)\kappa-(p-1)|\lambda+i\kappa| \geq 0$ due to (1.2), we see that

$$II \geq 0, \tag{3.12}$$

which implies that the nonlinearity is dropped out on the right hand side of (3.10). Plugging (3.11) and (3.12) into (3.10), we have

$$\frac{d}{dt}\|v_\nu\|_{L^2(\mathbb{R})} \geq -C|t|^{-1/(p-1)-2/(2N)}. \tag{3.13}$$

Recall that $2<p$, and note that N is large enough as in (A.4). Then $-1 < -\frac{1}{p-1}-\frac{2}{2N}$ and so $|t|^{-1/(p-1)-2/(2N)}$ is integrable near $t=0$. Integrating (3.13) from t to 0, we see that there exists some constant $C>0$ independent of $\nu\in(0,1]$ such that , for $t\in(T_\nu,0]$,

$$\|v_\nu(t)\|_{L^2(\mathbb{R})} \leq C|t|^{1-1/(p-1)-2/(2N)}. \tag{3.14}$$

Note that $\varphi_\nu(t,x)$ is compactly supported. Then the similar estimate to derive (3.14) is applied to $\|xv_\nu(t)\|_{L^2(\mathbb{R})}$, and we have, for $t\in(T_\nu,0]$,

$$\|xv_\nu(t)\|_{L^2(\mathbb{R})} \leq C|t|^{1-1/(p-1)-2/(2N)}. \tag{3.15}$$

We next consider the estimate of $\|\partial_x v_\nu(t)\|_{L^2(\mathbb{R})}$. We see, formally, that

$$\begin{aligned}\frac{d}{dt}\|\partial_x v_\nu\|^2_{L^2(\mathbb{R})} &= -\mathrm{Im}(\partial_x^3\varphi_\nu, \partial_x v_\nu)_{L^2(\mathbb{R})} \\ &\quad +2\mathrm{Im}\left\{(\lambda+i\kappa)(\partial_x\mathcal{N}(\varphi_\nu+v_\nu)-\partial_x\mathcal{N}(\varphi_\nu), \partial_x v_\nu)_{L^2(\mathbb{R})}\right\} \\ &\equiv III + IV. \end{aligned} \tag{3.16}$$

By Cauchy-Schwarz' inequality together with Corollary 2.2 (2.9), we see that

$$III \geq -C|t|^{-1/(p-1)-3/(2N)}\|\partial_x v_\nu\|_{L^2(\mathbb{R})}. \tag{3.17}$$

Since

$$\begin{aligned}\partial_x\mathcal{N}(\varphi_\nu+v_\nu)-\partial_x\mathcal{N}(\varphi_\nu) &= \mathcal{N}_u(\varphi_\nu+v_\nu)\partial_x v_\nu+\mathcal{N}_{\bar u}(\varphi_\nu+v_\nu)\partial_x\overline{v_\nu}\\ &\quad+(\mathcal{N}_u(\varphi_\nu+v_\nu)-\mathcal{N}_u(\varphi_\nu))\partial_x\varphi_\nu\\ &\quad+(\mathcal{N}_{\bar u}(\varphi_\nu+v_\nu)-\mathcal{N}_{\bar u}(\varphi_\nu))\partial_x\overline{\varphi_\nu}\end{aligned}$$

and

$$|\mathcal{N}_u(\varphi_\nu+v_\nu)-\mathcal{N}_u(\varphi_\nu)|+|\mathcal{N}_{\bar u}(\varphi_\nu+v_\nu)-\mathcal{N}_{\bar u}(\varphi_\nu)|\leq C(|\varphi_\nu|^{p-2}+|v_\nu|^{p-2})|v_\nu|,$$

we have

$$\begin{aligned}IV &\geq ((p+1)\kappa-(p-1)|\lambda+i\kappa|)\int_{\mathbb{R}}|\varphi_\nu+v_\nu|^{p-1}|\partial_x v_\nu|^2dx\\ &\quad -C\int_{\mathbb{R}}(|\varphi_\nu|^{p-2}+|v_\nu|^{p-2})|v_\nu||\partial_x\varphi_\nu||\partial_x v_\nu|dx.\end{aligned} \tag{3.18}$$

By (1.2), we have $(p+1)\kappa-(p-1)|\lambda+i\kappa|\geq 0$, and so the first tern on the right hand side of (3.18) is dropped out. Applying Corollary 2.2 (2.9) to (3.18), we have

$$\begin{aligned}IV &\geq -C|t|^{-1-1/(2N)}\|v_\nu\|_{L^2(\mathbb{R})}\|\partial_x v_\nu\|_{L^2(\mathbb{R})}\\ &\quad -C|t|^{-1/(p-1)-1/(2N)}\|v_\nu\|^{p-1}_{L^{2(p-1)}(\mathbb{R})}\|\partial_x v_\nu\|_{L^2(\mathbb{R})}\\ &\geq C|t|^{-1-1/(2N)}\|v_\nu\|_{L^2(\mathbb{R})}\|\partial_x v_\nu\|_{L^2(\mathbb{R})}\\ &\quad -C|t|^{-1/(p-1)-1/(2N)}\|v_\nu\|^{p/2}_{L^2(\mathbb{R})}\|\partial_x v_\nu\|^{p/2}_{L^2(\mathbb{R})}.\end{aligned} \tag{3.19}$$

Note here that, to deduce the last inequality in (3.19), the Gagliardo-Nirenberg inequality : $\|v_\nu\|^{2(p-1)}_{L^{2(p-1)}(\mathbb{R})}\leq C\|v_\nu\|^p_{L^2(\mathbb{R})}\|\partial_x v_\nu\|^{p-2}_{L^2(\mathbb{R})}$ was applied. Plugging (3.17) and (3.19) into (3.16), and making use of (3.14), we have, for $t\in[T_\nu,0)$,

$$\frac{d}{dt}\|\partial_x v_\nu\|_{L^2(\mathbb{R})} \geq -C|t|^{-1/(p-1)-3/(2N)}-C|t|^{-1/(p-1)-1/(2N)}\|v_\nu\|^{p/2}_{L^2(\mathbb{R})}\|\partial_x v_\nu\|^{(p/2)-1}_{L^2(\mathbb{R})}.$$

Since $\|v_\nu\|_{L^2(\mathbb{R})}\leq C$ and $\|\partial_x v_\nu\|^{(p/2)-1}_{L^2(\mathbb{R})}\leq C(1+\|\partial_x v_\nu\|_{L^2(\mathbb{R})})$ due to Young's inequality, the above inequality turns out to be

$$\frac{d}{dt}\|\partial_x v_\nu\|_{L^2(\mathbb{R})} \geq -C|t|^{-1/(p-1)-3/(2N)}-C|t|^{-1/(p-1)-1/(2N)}\|\partial_x v_\nu\|_{L^2(\mathbb{R})}.$$

Then Gronwall's inequality yields, for $t\in[T_\nu,0)$,

$$\|\partial_x v_\nu(t)\|_{L^2(\mathbb{R})}\leq C|t|^{1-1/(p-1)-3/(2N)}, \tag{3.20}$$

where the constant C does not depend on $\nu\in(0,1]$. Combining (3.14), (3.15) and (3.20), we see that

$$\|v_\nu(t)\|_{H^1(\mathbb{R})}+\|xv_\nu(t)\|_{L^2(\mathbb{R})}\leq C(|t|^{1-1/(p-1)-2/(2N)}+|t|^{1-1/(p-1)-3/(2N)}).$$

Assume that $T_\nu \to 0$ as $\nu \downarrow 0$. Then, taking $t = T_\nu$ in the above and recalling the definition of T_ν, we have

$$1 \le C|T_\nu|^{1-1/(p-1)-2/(2N)} + C|T_\nu|^{1-1/(p-1)-3/(2N)}.$$

This is a contradiction, since $1 - \frac{1}{p-1} - \frac{3}{2N} > 0$ for large N. Hence there exists some $T_0 < 0$ such that $T_\nu \le T_0$ for any $\nu \in (0,1]$, and the proof for (i), (ii) is complete.

We are next going to prove (iii). We have

$$\begin{aligned}
\frac{d}{dt}\|v_\nu - v_{\nu'}\|^2_{L^2(\mathbb{R})} &= -\mathrm{Im}(\partial_x^2(\varphi_\nu - \varphi_\nu), v_\nu - v_{\nu'})_{L^2(\mathbb{R})} \\
&\quad +2\mathrm{Im}\left\{(\lambda + i\kappa)(\mathcal{N}(\varphi_\nu + v_\nu) - \mathcal{N}(\varphi_\nu), v_\nu - v_{\nu'})_{L^2(\mathbb{R})}\right\} \\
&\quad -2\mathrm{Im}\left\{(\lambda + i\kappa)(\mathcal{N}(\varphi_{\nu'} + v_{\nu'}) - \mathcal{N}(\varphi_{\nu'}), v_\nu - v_{\nu'})_{L^2(\mathbb{R})}\right\} \\
&\equiv V + VI - VI'. \qquad (3.21)
\end{aligned}$$

By Corollary 2.2 (2.10), we have

$$V \ge -C|t|^{-1/(p-1)-2/(2N)-\varepsilon}(\nu^\varepsilon + \nu'^\varepsilon)\|v_\nu - v_{\nu'}\|_{L^2(\mathbb{R})}. \qquad (3.22)$$

Since $\mathcal{N}(\varphi_\nu + v_\nu) - \mathcal{N}(\varphi_{\nu'}) = \int_0^1 \{\mathcal{N}_u(\varphi_\nu + \theta v_\nu)v_\nu + \mathcal{N}_{\bar{u}}(\varphi_\nu + \theta v_\nu)\overline{v_\nu}\}d\theta$ etc., we see that

$$\begin{aligned}
&VI - VI' \\
&= 2\mathrm{Im}\int_0^1 (\mathcal{N}_u(\varphi_\nu + \theta v_\nu)(v_\nu - v_{\nu'}) - \mathcal{N}_{\bar{u}}(\varphi_\nu + \theta v_\nu)\overline{(v_\nu - v_{\nu'})}, v_\nu - v_{\nu'})_{L^2(\mathbb{R})}d\theta \\
&\quad +2\mathrm{Im}\int_0^1 (\{\mathcal{N}_u(\varphi_\nu + \theta v_\nu) - \mathcal{N}_u(\varphi_{\nu'} + \theta v_{\nu'})\}v_{\nu'}, v_\nu - v_{\nu'})_{L^2(\mathbb{R})}d\theta \\
&\quad +2\mathrm{Im}\int_0^1 (\{\mathcal{N}_{\bar{u}}(\varphi_\nu + \theta v_\nu) - \mathcal{N}_{\bar{u}}(\varphi_{\nu'} + \theta v_{\nu'})\}\overline{v_{\nu'}}, v_\nu - v_{\nu'})_{L^2(\mathbb{R})}d\theta \\
&\ge ((p+1)\kappa - (p-1)|\lambda + i\kappa|)\int_0^1\int_{\mathbb{R}} |\varphi_\nu + \theta v_\nu|^{p-1}|v_\nu - v_{\nu'}|^2 dx d\theta \\
&\quad -C(\|\varphi_\nu\|^{p-2}_{L^\infty(\mathbb{R})} + \|\varphi_{\nu'}\|^{p-2}_{L^\infty(\mathbb{R})} + \|v_\nu\|^{p-2}_{L^\infty(\mathbb{R})} + \|v_{\nu'}\|^{p-2}_{L^\infty(\mathbb{R})}) \\
&\quad \times(\|\varphi_\nu - \varphi_{\nu'}\|_{L^\infty(\mathbb{R})}\|v_{\nu'}\|_{L^2(\mathbb{R})} + \|v_\nu - v_{\nu'}\|_{L^2(\mathbb{R})}\|v_{\nu'}\|_{L^\infty(\mathbb{R})})\|v_\nu - v_{\nu'}\|_{L^2(\mathbb{R})}.
\end{aligned}$$

By $(p+1)\kappa - (p-1)|\lambda + i\kappa| \ge 0$ due to (1.2) and the Gagliardo-Nirenberg inequality $\|v_\nu\|_{L^\infty(\mathbb{R})} \le C\|v_\nu\|^{1/2}_{L^2(\mathbb{R})}\|\partial_x v_\nu\|^{1/2}_{L^2(\mathbb{R})}$, we see that

$$\begin{aligned}
&VI - VI' \\
&\ge -C(|t|^{-1/(p-1)-2/(2N)-\varepsilon}(\nu^\varepsilon + \nu'^\varepsilon) + |t|^{-5/(4N)}\|v_\nu - v_{\nu'}\|_{L^2(\mathbb{R})}) \\
&\quad \times\|v_\nu - v_{\nu'}\|_{L^2(\mathbb{R})}. \qquad (3.23)
\end{aligned}$$

Plugging (3.22) and (3.23) into (3.21), we have

$$\frac{d}{dt}\|v_\nu - v_{\nu'}\|_{L^2(\mathbb{R})} \ge -C|t|^{-1/(p-1)-2/(2N)-\varepsilon}(\nu^\varepsilon + \nu'^\varepsilon) - C|t|^{-5/(4N)}\|v_\nu - v_{\nu'}\|_{L^2(\mathbb{R})}.$$

Then Gronwall's inequality yields

$$\|v_\nu - v_{\nu'}\|_{L^2(\mathbb{R})} \le -C|t|^{1-1/(p-1)-2/(2N)-\varepsilon}(\nu^\varepsilon + \nu'^\varepsilon). \qquad (3.24)$$

The estimate for $\|x(v_\nu - v_{\nu'})\|_{L^2(\mathbb{R})}$ similarly follows, and we have

$$\|x(v_\nu - v_{\nu'})\|_{L^2(\mathbb{R})} \le -C|t|^{1-1/(p-1)-2/(2N)-\varepsilon}(\nu^\varepsilon + \nu'^\varepsilon). \tag{3.25}$$

Finally we are going to consider the estimate of $\|\partial_x(v_\nu - v_{\nu'})\|_{L^2(\mathbb{R})}$. We have

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x(v_\nu - v_{\nu'})\|^2_{L^2(\mathbb{R})} \\
&= -\mathrm{Im}(\partial_x^3(\varphi_\nu - \varphi_\nu), \partial_x(v_\nu - v_{\nu'}))_{L^2(\mathbb{R})} \\
&\quad +2\mathrm{Im}\left\{(\lambda + i\kappa)(\partial_x\mathcal{N}(\varphi_\nu + v_\nu) - \partial_x\mathcal{N}(\varphi_\nu), \partial_x(v_\nu - v_{\nu'}))_{L^2(\mathbb{R})}\right\} \\
&\quad -2\mathrm{Im}\left\{(\lambda + i\kappa)(\partial_x\mathcal{N}(\varphi_{\nu'} + v_{\nu'}) - \partial_x\mathcal{N}(\varphi_{\nu'}), \partial_x(v_\nu - v_{\nu'}))_{L^2(\mathbb{R})}\right\} \\
&\equiv VII + VIII - VIII'.
\end{aligned} \tag{3.26}$$

By Corollary 2.2 (2.10), we have

$$VII \ \ge\ -C|t|^{-1/(p-1)-3/(2N)-\varepsilon}(\nu^\varepsilon + \nu'^\varepsilon)\|\partial_x(v_\nu - v_{\nu'})\|_{L^2(\mathbb{R})}. \tag{3.27}$$

Since $\partial_x\mathcal{N}(\varphi_\nu + v_\nu) = \mathcal{N}_u(\varphi_\nu + v_\nu)\partial_x(\varphi_\nu + v_\nu) + \mathcal{N}_{\bar{u}}(\varphi_\nu + v_\nu)\overline{\partial_x(\varphi_\nu + v_\nu)}$ and $\partial_x\mathcal{N}(\varphi_\nu) = \mathcal{N}_u(\varphi_\nu)\partial_x\varphi_\nu + \mathcal{N}_{\bar{u}}(\varphi_\nu)\overline{\partial_x\varphi_\nu}$ etc., it follows that

$$\begin{aligned}
&VIII - VIII' \\
&\ge 2\mathrm{Im}\left\{(\lambda + i\kappa)(\mathcal{N}_u(\varphi_\nu + v_\nu)\partial_x w - \mathcal{N}_{\bar{u}}(\varphi_\nu + v_\nu)\overline{\partial_x w}, \partial_x w)_{L^2(\mathbb{R})}\right\} \\
&\quad -C|(\{\mathcal{N}_u(\varphi_\nu + v_\nu) - \mathcal{N}_u(\varphi_{\nu'} + v_{\nu'})\}\partial_x v_{\nu'}, \partial_x w)_{L^2(\mathbb{R})}| \\
&\quad -C|(\{\mathcal{N}_{\bar{u}}(\varphi_\nu + v_\nu) - \mathcal{N}_{\bar{u}}(\varphi_{\nu'} + v_{\nu'})\}\overline{\partial_x v_{\nu'}}, \partial_x w)_{L^2(\mathbb{R})}| \\
&\quad -C|(\{\mathcal{N}_u(\varphi_\nu + v_\nu) - \mathcal{N}_u(\varphi_\nu)\}\partial_x(\varphi_\nu - \varphi_{\nu'}), \partial_x w)_{L^2(\mathbb{R})}| \\
&\quad -C|(\{\mathcal{N}_{\bar{u}}(\varphi_\nu + v_\nu) - \mathcal{N}_{\bar{u}}(\varphi_\nu)\}\overline{\partial_x(\varphi_\nu - \varphi_{\nu'})}, \partial_x w)_{L^2(\mathbb{R})}| \\
&\quad -C|(\mathcal{M}_1(\varphi_\nu, \varphi_{\nu'}, v_\nu, v_{\nu'})\partial_x\varphi_{\nu'}, \partial_x w)_{L^2(\mathbb{R})}| \\
&\quad -C|(\mathcal{M}_2(\varphi_\nu, \varphi_{\nu'}, v_\nu, v_{\nu'})\overline{\partial_x\varphi_{\nu'}}, \partial_x w)_{L^2(\mathbb{R})}|,
\end{aligned} \tag{3.28}$$

where $w = v_\nu - v_{\nu'}$ and

$$\begin{aligned}
\mathcal{M}_1(\varphi_\nu, \varphi_{\nu'}, v_\nu, v_{\nu'}) &= \mathcal{N}_u(\varphi_\nu + v_\nu) - \mathcal{N}_u(\varphi_\nu) - \mathcal{N}_u(\varphi_{\nu'} + v_{\nu'}) + \mathcal{N}_u(\varphi_{\nu'}), \\
\mathcal{M}_2(\varphi_\nu, \varphi_{\nu'}, v_\nu, v_{\nu'}) &= \mathcal{N}_{\bar{u}}(\varphi_\nu + v_\nu) - \mathcal{N}_{\bar{u}}(\varphi_\nu) - \mathcal{N}_{\bar{u}}(\varphi_{\nu'} + v_{\nu'}) + \mathcal{N}_{\bar{u}}(\varphi_{\nu'}).
\end{aligned}$$

Note that

$$\begin{aligned}
&2\mathrm{Im}\left\{(\lambda + i\kappa)(\mathcal{N}_u(\varphi_\nu + v_\nu)\partial_x w - \mathcal{N}_{\bar{u}}(\varphi_\nu + v_\nu)\overline{\partial_x w}, \partial_x w)_{L^2(\mathbb{R})}\right\} \\
&\quad = ((p+1)\kappa - (p-1)|\lambda + i\kappa|)\int_{\mathbb{R}} |\varphi_\nu + v_\nu|^{p-1}|\partial_x w|^2 dx \\
&\quad \ge 0,
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
&|\mathcal{N}_u(\varphi_\nu + v_\nu) - \mathcal{N}_u(\varphi_{\nu'} + v_{\nu'})| \\
&\quad \le C(\|\varphi_\nu\|^{p-2}_{L^\infty(\mathbb{R})} + \|\varphi_{\nu'}\|^{p-2}_{L^\infty(\mathbb{R})} + \|v_\nu\|^{p-2}_{L^\infty(\mathbb{R})} + \|v_{\nu'}\|^{p-2}_{L^\infty(\mathbb{R})}) \\
&\qquad \times(|\varphi_\nu - \varphi_{\nu'}| + |v_\nu - v_{\nu'}|).
\end{aligned} \tag{3.30}$$

We rewrite $\mathcal{M}_1$ in such a way that

$$
\begin{aligned}
&\mathcal{M}_1(\varphi_\nu, \varphi_{\nu'}, v_\nu, v_{\nu'}) \\
&= \int_0^1 \mathcal{N}_{uu}(\varphi_\nu + \theta v_\nu) v_\nu d\theta + \int_0^1 \mathcal{N}_{u\bar{u}}(\varphi_\nu + \theta v_\nu)\overline{v_{\nu'}} d\theta \\
&\quad - \int_0^1 \mathcal{N}_{uu}(\varphi_{\nu'} + \theta v_{\nu'}) v_{\nu'} d\theta - \int_0^1 \mathcal{N}_{u\bar{u}}(\varphi_{\nu'} + \theta v_{\nu'})\overline{\varphi_{\nu'}} d\theta \\
&= \int_0^1 \mathcal{N}_{uu}(\varphi_\nu + \theta v_\nu)(v_\nu - v_{\nu'}) d\theta + \int_0^1 \mathcal{N}_{u\bar{u}}(\varphi_\nu + \theta v_\nu)\overline{(v_\nu - v_{\nu'})} d\theta \\
&\quad + \int_0^1 (\mathcal{N}_{uu}(\varphi_\nu + \theta v_\nu) - \mathcal{N}_{uu}(\varphi_{\nu'} + \theta v_{\nu'})) v_{\nu'} d\theta \\
&\quad + \int_0^1 (\mathcal{N}_{u\bar{u}}(\varphi_\nu + \theta v_\nu) - \mathcal{N}_{u\bar{u}}(\varphi_{\nu'} + \theta v_{\nu'}))\overline{v_{\nu'}} d\theta \qquad (3.31)
\end{aligned}
$$

where $\mathcal{N}_{uu}(u) = \partial_u^2 \mathcal{N}(u)$ and $\mathcal{N}_{u\bar{u}}(u) = \partial_{\bar{u}}\partial_u \mathcal{N}(u)$. Apply, for instance, the simple inequalities :

$$
|\mathcal{N}_{uu}(\varphi_\nu + \theta v_\nu)| \le C(\|\varphi_\nu\|_{L^\infty(\mathbb{R})}^{p-2} + \|v_\nu\|_{L^\infty(\mathbb{R})}^{p-2})
$$

and

$$
|\mathcal{N}_{uu}(\varphi_\nu + \theta v_\nu) - \mathcal{N}_{uu}(\varphi_{\nu'} + \theta v_{\nu'})| \le C(\|\varphi_\nu - \varphi_{\nu'}\|_{L^\infty(\mathbb{R})}^{p-2} + \|v_\nu - v_{\nu'}\|_{L^\infty(\mathbb{R})}^{p-2})
$$

to (3.31). Then we have

$$
\begin{aligned}
|\mathcal{M}_1(\varphi_\nu, \varphi_{\nu'}, v_\nu, v_{\nu'})| \le\; & C(\|\varphi_\nu\|_{L^\infty(\mathbb{R})}^{p-2} + \|v_\nu\|_{L^\infty(\mathbb{R})}^{p-2})|v_\nu - v_{\nu'}| \\
& + C(\|\varphi_\nu - \varphi_{\nu'}\|_{L^\infty(\mathbb{R})}^{p-2} + \|v_\nu - v_{\nu'}\|_{L^\infty(\mathbb{R})}^{p-2})|v_{\nu'}|. \qquad (3.32)
\end{aligned}
$$

Plugging (3.29), (3.30) and (3.32) into (3.28), and making use of the similar estimates for $\mathcal{N}_{\bar{u}}$ and $\mathcal{M}_2$, we see that

$$
\begin{aligned}
& VIII - VIII' \\
&\ge -C(\|\varphi_\nu\|_{L^\infty(\mathbb{R})}^{p-2} + \|\varphi_{\nu'}\|_{L^\infty(\mathbb{R})}^{p-2} + \|v_\nu\|_{L^\infty(\mathbb{R})}^{p-2} + \|v_{\nu'}\|_{L^\infty(\mathbb{R})}^{p-2}) \\
&\qquad \times(\|\varphi_\nu - \varphi_{\nu'}\|_{L^\infty(\mathbb{R})} + \|v_\nu - v_{\nu'}\|_{L^\infty(\mathbb{R})})\|\partial_x v_{\nu'}\|_{L^2(\mathbb{R})}\|\partial_x w\|_{L^2(\mathbb{R})} \\
&\quad -C(\|\varphi_\nu\|_{L^\infty(\mathbb{R})}^{p-2} + \|v_\nu\|_{L^\infty(\mathbb{R})}^{p-2})\|v_\nu\|_{L^2(\mathbb{R})}\|\partial_x(\varphi_\nu - \varphi_{\nu'})\|_{L^\infty(\mathbb{R})}\|\partial_x w\|_{L^2(\mathbb{R})} \\
&\quad -C(\|\varphi_\nu\|_{L^\infty(\mathbb{R})}^{p-2} + \|v_\nu\|_{L^\infty(\mathbb{R})}^{p-2})\|v_\nu - v_{\nu'}\|_{L^2(\mathbb{R})}\|\partial_x \varphi_{\nu'}\|_{L^\infty(\mathbb{R})}\|\partial_x w\|_{L^2(\mathbb{R})} \\
&\quad -C(\|\varphi_\nu - \varphi_{\nu'}\|_{L^\infty(\mathbb{R})}^{p-2} + \|v_\nu - v_{\nu'}\|_{L^\infty(\mathbb{R})}^{p-2})\|v_{\nu'}\|_{L^2(\mathbb{R})}\|\partial_x \varphi_{\nu'}\|_{L^\infty(\mathbb{R})}\|\partial_x w\|_{L^2(\mathbb{R})}.
\end{aligned}
$$

Applying Corollary 2.2 to φ_ν, $\varphi_{\nu'}$ and $\varphi_\nu - \varphi_{\nu'}$, (3.6) - (3.8) to v_ν, $v_{\nu'}$ and $v_\nu - v_{\nu'}$, we have

$$
\begin{aligned}
& VIII - VIII' \\
&\ge -C(|t|^{-1/(p-1)-3/(2N)-\varepsilon}(\nu^\varepsilon + \nu'^\varepsilon) + |t|^{-3/(2N)}\|v_\nu - v_{\nu'}\|_{L^\infty(\mathbb{R})})\|\partial_x w\|_{L^2(\mathbb{R})} \\
&\quad -C|t|^{-1/(p-1)-3/(2N)-\varepsilon}(\nu^\varepsilon + \nu'^\varepsilon)\|\partial_x w\|_{L^2(\mathbb{R})} \\
&\quad -C|t|^{-1/(p-1)-3/(2N)-(p-2)\varepsilon}(\nu^{(p-2)\varepsilon} + \nu'^{(p-2)\varepsilon})\|\partial_x w\|_{L^2(\mathbb{R})} \\
&\quad -C|t|^{1-2/(p-1)-3/(2N)}\|v_\nu - v_{\nu'}\|_{L^\infty(\mathbb{R})}^{p-2}\|\partial_x w\|_{L^2(\mathbb{R})}. \qquad (3.33)
\end{aligned}
$$

Apply Gagliardo-Nirenberg's inequality : $\|f\|_{L^\infty(\mathbb{R})} \leq C\|f\|^{1/2}_{L^2(\mathbb{R})}\|\partial_x f\|_{L^2(\mathbb{R})}$ to $\|v_\nu - v_{\nu'}\|_{L^\infty(\mathbb{R})}$. Then we have

$$\begin{aligned}\|v_\nu - v_{\nu'}\|_{L^\infty(\mathbb{R})} &\leq C\|v_\nu - v_{\nu'}\|^{1/2}_{L^2(\mathbb{R})}\|\partial_x(v_\nu - v_{\nu'})\|^{1/2}_{L^2(\mathbb{R})} \\ &\leq C\|v_\nu - v_{\nu'}\|^{1/2}_{L^2(\mathbb{R})}(\|\partial_x v_\nu\|_{L^2(\mathbb{R})} + \|\partial_x v_{\nu'}\|_{L^2(\mathbb{R})})^{1/2} \\ &\leq C|t|^{1-1/(p-1)-5/(4N)-\varepsilon/2}(\nu^{\varepsilon/2} + \nu'^{\varepsilon/2}),\end{aligned}$$

where (3.7) and (3.8) were used. Plugging the above inequality to (3.33), we see that

$$\begin{aligned}&VIII - VIII' \\ &\geq -C|t|^{-1/(p-1)-3/(2N)-\varepsilon}(\nu^{\varepsilon} + \nu'^{\varepsilon})\|\partial_x w\|_{L^2(\mathbb{R})} \\ &\quad -C|t|^{1-1/(p-1)-11/(4N)-\varepsilon/2}(\nu^{\varepsilon/2} + \nu'^{\varepsilon/2})\|\partial_x w\|_{L^2(\mathbb{R})} \\ &\quad -C|t|^{-1/(p-1)-3/(2N)-(p-2)\varepsilon}(\nu^{(p-2)\varepsilon} + \nu'^{(p-2)\varepsilon})\|\partial_x w\|_{L^2(\mathbb{R})} \\ &\quad -C|t|^{p-2-1/(p-1)-(5p-4)/(4N)-(p-2)\varepsilon/2}(\nu^{(p-2)\varepsilon/2} + \nu'^{(p-2)\varepsilon/2})\|\partial_x w\|_{L^2(\mathbb{R})} \\ &\geq -C|t|^{-1/(p-1)-3/(2N)-\varepsilon}(\nu^{(p-2)\varepsilon/2} + \nu'^{(p-2)\varepsilon/2})\|\partial_x w\|_{L^2(\mathbb{R})} \qquad (3.34)\end{aligned}$$

for sufficiently large N and sufficiently small ε. Plugging (3.27) and (3.34) into (3.26), we have

$$\begin{aligned}&\frac{d}{dt}\|\partial_x(v_\nu - v_{\nu'})\|_{L^2(\mathbb{R})} \\ &\quad\geq -C|t|^{-1/(p-1)-3/(2N)-\varepsilon}(\nu^{(p-2)\varepsilon/2} + \nu'^{(p-2)\varepsilon/2}).\end{aligned}$$

Integrating from t to 0, we have

$$\begin{aligned}&\|\partial_x(v_\nu - v_{\nu'})\|_{L^2(\mathbb{R})} \\ &\quad\leq C|t|^{1-1/(p-1)-3/(2N)-\varepsilon}(\nu^{(p-2)\varepsilon/2} + \nu'^{(p-2)\varepsilon/2}). \qquad (3.35)\end{aligned}$$

This completes the proof of Lemma 3.2. □

Proof of Proposition 3.1 By Lemma 3.2 (3.8) and (3.9), there exists a limit $\lim_{\nu\downarrow 0} v_\nu = v$ in $C([T_0, 0]; H^1(\mathbb{R}))$ and in the weghted $L^2(\mathbb{R})$. Also we see that

$$\begin{aligned}&-\frac{1}{2}\partial_x^2 v_\nu - \frac{1}{2}\partial_x^2\varphi_\nu + (\lambda + i\kappa)(\mathcal{N}(\varphi_\nu + v_\nu) - \mathcal{N}(\varphi_\nu)) \\ &\overset{\nu\downarrow 0}{\to} -\frac{1}{2}\partial_x^2 v - \frac{1}{2}\partial_x^2\varphi + (\lambda + i\kappa)(\mathcal{N}(\varphi + v) - \mathcal{N}(\varphi))\end{aligned}$$

holds in $C([T_0, \tau]; H^{-1}(\mathbb{R}))$ for any $\tau \in (T_0, 0)$. It follows that $\lim_{\nu\downarrow 0}\partial_t v_\nu = \partial_t v$ in $C([T_0, 0); H^{-1}(\mathbb{R}))$, and hence $v \in C^1([T_0, 0); H^{-1}(\mathbb{R}))$. The uniqueness follows by deriving $\|v_1 - v_2\|_{L^2(\mathbb{R})} = 0$. □

4 Proof of Theorem 1.1

We need to prolong the solution $u = \varphi + v$ backward in negative time. It is easy to guess that the size of the solution tends to 0 as $t \to -\infty$, since the nonlinear amplification (i.e.,

$\kappa > 0$) works as the dissipation in negative time direction. However this observation fails when $3 < p$ since the dispersion caused by $-(1/2)\partial_x^2$ breaks down the nonlinearity. Hence the condition $p \le 3$ is required to ensure $\lim_{t\to-\infty} \|u(t)\|_{L^2(\mathbb{R})} = 0$.

Proposition 4.1. *Let $1 < p \le 3$ and λ, κ satisfy (1.2). Let $u(T_0, \cdot) \in H^1(\mathbb{R})$ and $xu(T_0, \cdot) \in L^2(\mathbb{R})$. Then the solution $u = u(t, x)$ to (1.1) exists globally in negative time. Furthermore we have*

$$\|u(t,\cdot)\|_{L^2(\mathbb{R})} \le C \left\{ \begin{array}{ll} (\log|t|)^{-1/3} & (p=3), \\ |t|^{-(2/3)(1/(p-1)-1/2)} & (2<p<3) \end{array} \right. \tag{4.1}$$

for $t \in (-\infty, T_0]$.

Proof of Proposition 4.1. By (1.1), we see that

$$\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R})} = \kappa \|u\|^{p+1}_{L^{p+1}(\mathbb{R})}.$$

Applying Hölder's inequality : $\|u\|^{2p}_{L^2(\mathbb{R})} \le \|u\|^{p+1}_{L^{p+1}(\mathbb{R})} \|u\|^{p-1}_{L^1(\mathbb{R})}$, we have

$$\frac{d}{dt}\|u\|_{L^2(\mathbb{R})} \ge \kappa \frac{\|u\|^{2p}_{L^2(\mathbb{R})}}{\|u\|^{p-1}_{L^1(\mathbb{R})}}.$$

Next apply (scale-invariant) Cauchy-Schwarz' inequality : $\|u\|_{L^1(\mathbb{R})} \le C\|u\|^{1/2}_{L^2(\mathbb{R})}\|xu\|^{1/2}_{L^2(\mathbb{R})}$. Then we have

$$\frac{d}{dt}\|u\|_{L^2(\mathbb{R})} \ge C \frac{\|u\|^{(3p+1)/2}_{L^2(\mathbb{R})}}{\|xu\|^{(p-1)/2}_{L^2(\mathbb{R})}}.$$

Since $xu = Ju + it\partial_x u$ where $J = x - it\partial_x$, it follows that

$$\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R})} \ge C \frac{\|u\|^{(3p+1)/2}_{L^2(\mathbb{R})}}{\|Ju\|^{(p-1)/2}_{L^2(\mathbb{R})} + t^{(p-1)/2}\|\partial_x u\|_{L^2(\mathbb{R})}}. \tag{4.2}$$

We here note that

$$\begin{aligned} \frac{d}{dt}\|\partial_x u\|^2_{L^2(\mathbb{R})} &= 2\mathrm{Im}\left\{(\lambda + i\kappa)(\mathcal{N}_u(u)\partial_x u + \mathcal{N}_{\bar u}(u)\overline{\partial_x u}, u)_{L^2(\mathbb{R})}\right\} \\ &\ge ((p+1)\kappa - (p-1)|\lambda + i\kappa|) \int_{\mathbb{R}} |u|^{p+1}|\partial_x u|^2 dx \\ &\ge 0. \end{aligned}$$

Then we have, for $t \in (-\infty, T_0]$,

$$\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})} \le \|u(T_0,\cdot)\|_{L^2(\mathbb{R})}. \tag{4.3}$$

Also, noting that the operator J and $i\partial_t+\frac{1}{2}\partial_x^2$ commute and applying $J\mathcal{N}(u)=\mathcal{N}_u(u)Ju-\mathcal{N}_{\bar{u}}(u)\overline{Ju}$, we see that

$$\begin{aligned}\frac{d}{dt}\|Ju\|_{L^2(\mathbb{R})}^2 &= 2\mathrm{Im}\left\{(\lambda+i\kappa)(\mathcal{N}_u(u)Ju-\mathcal{N}_{\bar{u}}(u)\overline{Ju},u)_{L^2(\mathbb{R})}\right\}\\ &\geq ((p+1)\kappa-(p-1)|\lambda+i\kappa|)\int_{\mathbb{R}}|u|^{p+1}|Ju|^2dx\\ &\geq 0,\end{aligned}$$

and so we have

$$\|Ju(t,\cdot)\|_{L^2(\mathbb{R})}\leq\|xu(T_0,\cdot)-iT_0\partial_x u(T_0,\cdot)\|_{L^2(\mathbb{R})}. \tag{4.4}$$

Plugging (4.3) and (4.4) into (4.2), we see that, for $t\in(-\infty,T_0]$,

$$\frac{d}{dt}\|u\|_{L^2(\mathbb{R})}^2\geq Ct^{-(p-1)/2}\|u\|_{L^2(\mathbb{R})}^{(3p+1)/2},$$

which is equivalent to

$$-\frac{2}{3(p-1)}\frac{d}{dt}\|u\|_{L^2(\mathbb{R})}^{-3(p-1)/2}\geq Ct^{-(p-1)/2}. \tag{4.5}$$

Integrating (4.5) from t to T_0, we have

$$\|u(t,\cdot)\|_{L^2(\mathbb{R})}\leq\begin{cases}\left(\|u(T_0,\cdot)\|_{L^2(\mathbb{R})}^{-3}+C\log\frac{|t|}{|T_0|}\right)^{-\frac{1}{3}} & (p=3),\\ \left(\|u(T_0,\cdot)\|_{L^2(\mathbb{R})}^{-\frac{3(p-1)}{2}}+C(|t|^{\frac{3-p}{2}}-|T_0|^{\frac{3-p}{2}})\right)^{-\frac{2}{3(p-1)}} & (2<p<3).\end{cases} \tag{4.6}$$

This completes the proof of Proposition 4.1. □

Proof of Theorem 1.1. By Proposition 3.1, there exists a solution to (1.1) in $[T_0,0]$ such as $u(t,x)=\varphi(t,x)+v(t,x)$ where $\varphi(t,x)$ denotes a blowing-up profile determined in § 2 and $v(t,x)$ satisfies $v(0,x)=0$. Since $u(T_0,\cdot)\in H^1(\mathbb{R})$ and $xu(T_0,\cdot)\in L^2(\mathbb{R})$, Proposition 4.1 is applied, and so we have a solution such that $\lim_{t\to-\infty}\|u(t)\|_{L^2(\mathbb{R})}=0$. This means that, for any $\rho>0$, there exists some $\tau<0$ such that $\|u(\tau,\cdot)\|_{L^2(\mathbb{R})}<\rho$. Take $u(\tau,x)=u_0(x)$ as a initial data of (1.1), and consider the positive time direction. Then, from the translation-invariance of (1.1) with respect to t and the uniqueness of the solution in $H^1(\mathbb{R})$, it follows that the solution u blows up at some $T^*(=|\tau|)$. □

Acknowledgement. N. Kita would like to express his appreciation to Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University for giving him opportunities to participate in the joint research activity titled "Reaction Diffusion Equations and Nonlinear Dispersive Equations" organized on 20-22 of February, 2019. He could obtain a chance for the collaboration with the other authors of this article. N. Kita also appreciates the spacial support of Osaka City University Advanced Mathematical Institute, MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849, for giving him a chance to organize a joint research on reaction diffusion equations and nonlinear dispersive equations.

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