# A SYSTEM OF NONLINEAR SCHRÖDINGER EQUATIONS WITH SINGLE DELTA-FUNCTIONS AS INITIAL DATA IN $n$ SPACE DIMENSION

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**Abstract**

This paper treats a system of nonlinear Schrödinger equations with single $\delta$-functions as initial data. By imposing $\delta$-functions on the initial data, the partial differential equations are reduced into a couple of ODEs, and the behaviors of the solutions are observed in detail. Doi-Shimizu [2] considered a similar problem in case that the powers of nonlinearities coincides in both equations. But this paper removes this coincidence, considers the global existence and finite time blow-up of the solutions.

# 1 Introduction and Main Results

We consider the Cauchy problem for the coupled nonlinear Schrödinger equations:

$$\begin{cases} i\partial_t u + \frac{1}{2m_1}\Delta u = \lambda_1 |v|^{p_1-1}u, \\ i\partial_t v + \frac{1}{2m_2}\Delta v = \lambda_2 |u|^{p_2-1}v, \\ u(0,x) = \mu\delta_a(x), v(0,x) = \nu\delta_b(x), \end{cases} \tag{1.1}$$

where the complex-valued unknown functions $u$ and $v$ are defined on $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$ with $n \geq 1$, $m_1$, $m_2$ are nonzero real numbers. The Laplacian is given by $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$. In the nonlinearities, the powers satisfy $p_1, p_2 \in (1, 1+2/n)$ and the coefficients $\lambda_1$, $\lambda_2$ takes values in $\mathbb{C}$. We will solve (1.1) with $\delta$-functions as initial data, where $\delta_c(x)$ denotes the Dirac $\delta$-function supported at $x = c \in \mathbb{R}^n$ and $\mu$, $\nu \in \mathbb{C}$ with $\mu\nu \neq 0$. In particular, when $\mathrm{Im}\lambda_1$ or $\mathrm{Im}\lambda_2$ is negative, the corresponding nonlinearity affects as dissipation. On the other hand, when it is positive, the corresponding nonlinearity affects as amplification.

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When the initial data are given by single $\delta$-functions, the problem in (1.1) is reduced into that of ODEs. In fact, assuming that $u(t,x) = A(t)U_{m_1}(t)\delta_a(x)$ and $v(t,x) = B(t)U_{m_2}(t)\delta_b(x)$ where $U_m(t) = \exp(it\Delta/2m)$ denotes the one-parameter group for the linear Schrödinger operator $-\frac{1}{2m}\Delta$ and $A(t)$, $B(t)$ are functions depending only on $t$-variable, we see that (1.1) is transformed into

$$\begin{cases} i\dfrac{dA}{dt}U_{m_1}(t)\delta_a = \lambda_1|BU_{m_2}(t)\delta_b|^{p_1-1}AU_{m_1}(t)\delta_a, \\ i\dfrac{dB}{dt}U_{m_2}(t)\delta_b = \lambda_2|AU_{m_1}(t)\delta_a|^{p_2-1}BU_{m_2}(t)\delta_b, \\ A(0) = \mu, B(0) = \nu. \end{cases}$$

Note here that $U_{m_1}(t)\delta_a = (m_1/2\pi i t)^{n/2}\exp(im_1|x-a|^2/2t)$ etc. Then, matching the coefficients on both hand sides, we have the following coupled ODEs :

$$\begin{cases} i\dfrac{dA}{dt} = \eta_1 t^{-d_1}|B|^{p_1-1}A, \\ i\dfrac{dB}{dt} = \eta_2 t^{-d_2}|A|^{p_2-1}B, \\ A(0) = \mu, B(0) = \nu, \end{cases} \tag{1.2}$$

where $\eta_1 = \lambda_1(m_2/2\pi)^{(p_1-1)n/2}$, $\eta_2 = \lambda_2(m_1/2\pi)^{(p_2-1)n/2}$ and $d_j = (p_j-1)n/2$ $(j=1,2)$. Since $p_1, p_2 < 1 + 2/n$ which implies that $d_1, d_2 < 1$, $t^{-d_1}$ and $t^{-d_2}$ are integrable around $t = 0$, it is easy to show the local well-posedness of the solution $(A(t), B(t))$ to (1.2) in $C([0,T); \mathbb{C}\times\mathbb{C}) \cap C^1((0,T); \mathbb{C}\times\mathbb{C})$ due to the simple application of the contraction mapping principle. Remark here that we have focused on the well-posedness on (1.2), and we do not consider the uniqueness of the solution to the original nonlinear Schrödinger equations (1.1) since it causes a very difficult problem in the nonlinear estimate under the function spaces of low regularity. Then what one can conclude for (1.1) is only the existence of a solution. The aim of this paper is to make sure whether the interval $[0,T)$ in which the solution to (1.2) exists can be extended to $[0,\infty)$ or not, and to classify the decay estimates of $u(t) = A(t)U_{m_1}(t)\delta_a$ and $v(t) = B(t)U_{m_2}(t)\delta_b$ if the solution exists globally in time. Doi-Shimizu [2] solved this kind of problem in the case that the nonlinear powers coincide, i.e., $p_1 = p_2 = p \in (1, 1+2/n)$ by deriving the conservative quantity :

$$\frac{|A(t)|^{p-1}}{\text{Im }\eta_1} - \frac{|B(t)|^{p-1}}{\text{Im }\eta_2}. \tag{1.3}$$

It is easy to make sure that (1.3) is conserved. In fact, multiplying $\text{Im}\eta_2|A(t)|^{p-3}\overline{A(t)}$ on the first equation of (1.2) and $\text{Im}\eta_1|B(t)|^{p-3}\overline{B(t)}$ on the second, taking subtraction and taking the imaginary part, we will find that the quantity of (1.3) is conserved. By the conservation of (1.3), the ODE system (1.2) is reduced into two single equations, and the standard approach based on the method of separation of variables works well. The conservation of (1.3) is , however, obtained in virtue of the coincidence of $p_1$ and $p_2$. Hence we need to employ another approach in the present case $p_1 \neq p_2$. Before stating our theorems, a rough sketch of the results on global existence or blow-up in finite time of the solution to (1.2) is exhibited on Table 1.1. The behaviors of the solutions $(A(t), B(t))$ are classified by the sign of $\text{Im}\lambda_1$ and $\text{Im}\lambda_2$.

|  | $\mathrm{Im}\lambda_2 < 0$ | $\mathrm{Im}\lambda_2 = 0$ | $\mathrm{Im}\lambda_2 > 0$ |
|---|---|---|---|
| $\mathrm{Im}\lambda_1 < 0$ | Global (Theorem 1.1) | Global (Theorem 1.3) | Global (Theorem 1.2) |
| $\mathrm{Im}\lambda_1 = 0$ | Global (Theorem 1.3) | Global (Theorem 1.3) | Global (Theorem 1.3) |
| $\mathrm{Im}\lambda_1 > 0$ | Global (Theorem 1.2) | Global (Theorem 1.3) | Blow-up (Theorem 1.4) |

Table 1.1: Classification of global existence or blow-up in finite time

Our goals are to obtain decay estimates of the global solutions, and to clarify the blowing-up rate of the non-global solutions. Theorem 1.1 treats the case that the both nonlinearities of (1.1) plays a role of dissipation. It assets that the relation of the coefficients $\mu$, $\nu$ in the initial data determines which unknown variable rapidly decays.

**Theorem 1.1.** *Let* $\mathrm{Im}\lambda_1 < 0$ *and* $\mathrm{Im}\lambda_2 < 0$ *which indicates* $\mathrm{Im}\eta_1 < 0$ *and* $\mathrm{Im}\eta_2 < 0$ *respectively in* (1.2). *Then there exist solutions to* (1.1) *described as* $u = A(t)U_{m_1}(t)\delta_a$ *and* $v = B(t)U_{m_2}(t)\delta_b$ *globally in time, where*

$$(A(t), B(t)) \in C([0,\infty); \mathbb{C}\times\mathbb{C}) \cap C^1((0,\infty); \mathbb{C}\times\mathbb{C}).$$

*Furthermore, let* $\alpha = 1/(p_2-1) - n/2$ *and* $\beta = 1/(p_1-1) - n/2$. *Then, for the solutions* $u$ *and* $v$, *we have*

*(i) if* $|\mu|$ *is small in comparison with* $|\nu|$ *in the sense that the inequality :*

$$|\mu|^{\beta}\left(\frac{\alpha}{|\mathrm{Im}\eta_1|e}\right)^{\alpha/(p_1-1)} < |\nu|^{\alpha}\left(\frac{\beta}{|\mathrm{Im}\eta_2|e}\right)^{\beta/(p_2-1)}$$

*holds, there exist some positive constant* $C_1$ *such that*

$$\|u(t,\cdot)\|_{L^\infty} = O(t^{-n/2}\exp(-C_1 t^{(p_1-1)\beta})), \quad (1.4)$$
$$\|v(t,\cdot)\|_{L^\infty} = O(t^{-n/2}) \quad (1.5)$$

*as* $t \to \infty$.

*(ii) if* $|\mu|$ *is large in comparison with* $|\nu|$ *in the sense that the inequality :*

$$|\mu|^{\beta}\left(\frac{\alpha}{|\mathrm{Im}\eta_1|e}\right)^{\alpha/(p_1-1)} > |\nu|^{\alpha}\left(\frac{\beta}{|\mathrm{Im}\eta_2|e}\right)^{\beta/(p_2-1)}$$

*holds, there exist some positive constant* $C_2$ *such that*

$$\|u(t,\cdot)\|_{L^\infty} = O(t^{-n/2}), \quad (1.6)$$
$$\|v(t,\cdot)\|_{L^\infty} = O(t^{-n/2}\exp(-C_2 t^{(p_2-1)\alpha})) \quad (1.7)$$

*as* $t \to \infty$.

(*iii*) *if* $|\mu|$ *and* $|\nu|$ *are balanced in the sense that the equality :*

$$|\mu|^{\beta}\left(\frac{\alpha}{|\mathrm{Im}\eta_1|e}\right)^{\alpha/(p_1-1)} = |\nu|^{\alpha}\left(\frac{\beta}{|\mathrm{Im}\eta_2|e}\right)^{\beta/(p_2-1)}$$

*holds, the* $\|u(t,\cdot)\|_{L^\infty}$ *and* $\|v(t,\cdot)\|_{L^\infty}$ *decay in polynomial order. Precisely speaking, we have*

$$\|u(t,\cdot)\|_{L^\infty} = O(t^{-1/(p_2-1)}), \tag{1.8}$$
$$\|v(t,\cdot)\|_{L^\infty} = O(t^{-1/(p_1-1)}) \tag{1.9}$$

*as* $t\to\infty$.

Theorem 1.2 below treats the case that one nonlinearity is dissipation and the other is amplification. It asserts that the solution does not blow up but exists globally in time.

**Theorem 1.2.** *Let* $\mathrm{Im}\lambda_1\mathrm{Im}\lambda_2 < 0$ *which indicates* $\mathrm{Im}\eta_1\mathrm{Im}\eta_2 < 0$ *in* (1.2). *Then there exist solutions to* (1.1) *described as* $u = A(t)U_{m_1}(t)\delta_a$ *and* $v = B(t)U_{m_2}(t)\delta_b$ *globally in time, where*

$$(A(t), B(t)) \in C([0,\infty);\mathbb{C}\times\mathbb{C}) \cap C^1((0,\infty);\mathbb{C}\times\mathbb{C}).$$

*Furthermore, we have*

(*i*) *if* $\mathrm{Im}\lambda_1 < 0$ *and* $\mathrm{Im}\lambda_2 > 0$, *there exist some positive constant* $C_1$ *such that*

$$\|u(t,\cdot)\|_{L^\infty} = O(t^{-n/2}\exp(-C_1t^{1-n(p_1-1)/2})), \tag{1.10}$$
$$\|v(t,\cdot)\|_{L^\infty} = O(t^{-n/2}) \tag{1.11}$$

*as* $t\to\infty$.

(*ii*) *if* $\mathrm{Im}\lambda_1 > 0$ *and* $\mathrm{Im}\lambda_2 < 0$, *there exist some positive constant* $C_2$ *such that*

$$\|u(t,\cdot)\|_{L^\infty} = O(t^{-n/2}), \tag{1.12}$$
$$\|v(t,\cdot)\|_{L^\infty} = O(t^{-n/2}\exp(-C_2t^{1-n(p_2-1)/2})) \tag{1.13}$$

*as* $t\to\infty$.

Theorem 1.3 treats the case that at least one nonlinearity is of mass-conservation.

**Theorem 1.3.** *Let either* $\mathrm{Im}\lambda_1 = 0$ *or* $\mathrm{Im}\lambda_2 = 0$ *hold. Then there exist solutions to* (1.1) *described as* $u = A(t)U_{m_1}(t)\delta_a$ *and* $v = B(t)U_{m_2}(t)\delta_b$ *globally in time, where*

$$(A(t), B(t)) \in C([0,\infty);\mathbb{C}\times\mathbb{C}) \cap C^1((0,\infty);\mathbb{C}\times\mathbb{C}).$$

*Furthermore the solutions* $u(t)$ *and* $v(t)$ *admit* (*i*) *if* $\mathrm{Im}\lambda_1 = 0$,

$$\begin{aligned} \|u(t,\cdot)\|_{L^\infty} &= O(t^{-n/2}), \qquad (1.14)\\ \|v(t,\cdot)\|_{L^\infty} &= O(t^{-n/2}\exp\left(\frac{2\mathrm{Im}\eta_2|\mu|^{p_2-1}}{2-n(p_2-1)}t^{1-n(p_2-1)/2}\right)) \qquad (1.15)\end{aligned}$$

*as* $t \to \infty$.

(*ii*) *if* $\mathrm{Im}\lambda_2 = 0$,

$$\begin{aligned} \|u(t,\cdot)\|_{L^\infty} &= O(t^{-n/2}\exp\left(\frac{2\mathrm{Im}\eta_1|\nu|^{p_1-1}}{2-n(p_1-1)}t^{1-n(p_1-1)/2}\right)), \qquad (1.16)\\ \|v(t,\cdot)\|_{L^\infty} &= O(t^{-n/2}) \qquad (1.17)\end{aligned}$$

*as* $t \to \infty$.

It remains to consider the case that both nonlinearities of (1.1) are amplification. Theorem 1.4 asserts that the solutions blows up in finite time. Of course, it is difficult to obtain the explicit descriptions of the solutions. However sharp blowing-up rate of the solution is determined.

**Theorem 1.4.** *Let* $\mathrm{Im}\lambda_1 > 0$ *and* $\mathrm{Im}\lambda_2 > 0$. *Then there exist solutions to* (1.1) *described as* $u = A(t)U_{m_1}(t)\delta_a$ *and* $v = B(t)U_{m_2}(t)\delta_b$, *where*

$$(A(t), B(t)) \in C([0,T^*); \mathbb{C}\times\mathbb{C}) \cap C^1((0,T^*); \mathbb{C}\times\mathbb{C})$$

*for some* $T^* > 0$. *Furthermore* $|A(t)|$ *and* $|B(t)|$ *blow up simultaneously at* $T^*$. *Precisely speaking, we have*

$$\lim_{t\uparrow T^*}(T^*-t)|A(t)|^{p_2-1} = \frac{(T^*)^{(p_2-1)n/2}}{(p_1-1)\mathrm{Im}\eta_2}, \qquad (1.18)$$

$$\lim_{t\uparrow T^*}(T^*-t)|B(t)|^{p_1-1} = \frac{(T^*)^{(p_1-1)n/2}}{(p_2-1)\mathrm{Im}\eta_1}, \qquad (1.19)$$

*where* $\alpha = 1/(p_2-1) - n/2$ *and* $\beta = 1/(p_1-1) - n/2$.

The single nonlinear Schrödinger equation with a $\delta$-function as initial data, i.e.,

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^{p-1}u, \\ u(0,x) = \mu\delta_a(x) \end{cases} \qquad (1.20)$$

was considered in [1, 3, 4, 5]. If $1 < p < 1+2/n$, Banica and Vega [1] constructed a solution of the form $u(t,x) = A(t)U(t)\delta_a(x)$, where $A(t)$ denotes an amplitude depending only on $t$-variable and $U(t) = \exp(it\Delta/2)$ denotes the Schrödinger group. In their work, solutions with the perturbed initial data described as $u(0,x) = \mu\delta_a(x) + v(x)$, where $v(x) \in L^2(\mathbb{R})$ were also investigated. Kita [5, 4] treated the case that the initial data is given by the superposition of multiple $\delta$-functions. The idea on the construction of solutions to (1.1) is based on these works. However, in coupled case, the solutions sometimes present the exponential decay or grow-up as in Theorem 1.1-1.3, which is distinguished from the single case. It is interesting to refer to Kenig-Ponce-Vega's work [3], which considered the case of $n = 1$ and $p = 3$ – their idea can be also applied to the case of $n \leq 2$ and $1 + 2/n < p$. They proved the ill-posedness of the solution to (1.20), and their theorem asserts that there exist no solution or more than two solutions to (1.20) in $C([0,T); \mathcal{S}'(\mathbb{R}))$, where $\mathcal{S}'(\mathbb{R})$ denotes the space of tempered distributions. Hence the condition $1 < p_1, p_2 < 1 + 2/n$ is required in this paper to avoid the ill-posedness on (1.1). As for another singular initial data, Wada [6] considered the Cauchy problem when the initial data consists of p.v.$x^{-1}$ + ($L^2(\mathbb{R})$-function), and the global existence of solutions was proved.

# 2 Deformation of the Coupled ODEs

Unlike Doi-Shimizu's approach [2], our method of the proofs is based on the change of variables. Let $A(t) = t^{-\alpha}\tilde{A}(t)$ and $B(t) = t^{-\beta}\tilde{B}(t)$ where $\alpha$ and $\beta$ will be found to be $\alpha = 1/(p_2 - 1) - n/2 > 0$ and $\beta = 1/(p_1 - 1) - n/2 > 0$ later. Substituting them into (1.2), we have

$$\begin{cases} \dfrac{d\tilde{A}(t)}{dt} = (\alpha t^{-1} - i\eta_1 t^{-d_1-(p_1-1)\beta}|\tilde{B}(t)|^{p_1-1})\tilde{A}(t), \\ \dfrac{d\tilde{B}(t)}{dt} = (\beta t^{-1} - i\eta_2 t^{-d_2-(p_2-1)\alpha}|\tilde{A}(t)|^{p_2-1})\tilde{B}(t). \end{cases}$$

Choosing $\alpha$ and $\beta$ so that $1 = d_1 + (p_1 - 1)\beta$ and $1 = d_2 + (p_2 - 1)\alpha$, we have

$$\begin{cases} \dfrac{d\tilde{A}(t)}{dt} = t^{-1}(\alpha - i\eta_1|\tilde{B}(t)|^{p_1-1})\tilde{A}(t), \\ \dfrac{d\tilde{B}(t)}{dt} = t^{-1}(\beta - i\eta_2|\tilde{A}(t)|^{p_2-1})\tilde{B}(t). \end{cases} \tag{2.1}$$

Let $\tilde{A}(t) = A^\sharp(s)$ and $\tilde{B}(t) = B^\sharp(s)$ with $s = \log t \in (-\infty, \infty)$. Then the $t^{-1}$ in (2.1) is dropped out, and we have

$$\begin{cases} \dfrac{dA^\sharp(s)}{ds} = (\alpha - i\eta_1|B^\sharp(s)|^{p_1-1})A^\sharp(s), \\ \dfrac{dB^\sharp(s)}{ds} = (\beta - i\eta_2|A^\sharp(s)|^{p_2-1})B^\sharp(s). \end{cases}$$

We are interested in the feature of $|A^\sharp(s)|$ and $|B^\sharp(s)|$, which satisfy

$$\begin{cases} \dfrac{d|A^\sharp(s)|}{ds} = (\alpha + \mathrm{Im}\eta_1|B^\sharp(s)|^{p_1-1})|A^\sharp(s)|, \\ \dfrac{d|B^\sharp(s)|}{ds} = (\beta + \mathrm{Im}\eta_2|A^\sharp(s)|^{p_2-1})|B^\sharp(s)|. \end{cases} \tag{2.2}$$

For $|A^\sharp|$ and $|B^\sharp|$ satisfying (2.2), an explicit constraint is derived, which is described in Lemma 2.1 below.

**Lemma 2.1.** *The solutions* $|A^\sharp(s)|$ *and* $|B^\sharp(s)|$ *to* (2.2) *vary under the constraint :*

$$\frac{|A^\sharp(s)|^\beta}{|\mu|^\beta}\exp\left(\frac{\mathrm{Im}\eta_2}{p_2-1}|A^\sharp(s)|^{p_2-1}\right)=\frac{|B^\sharp(s)|^\alpha}{|\nu|^\alpha}\exp\left(\frac{\mathrm{Im}\eta_1}{p_1-1}|B^\sharp(s)|^{p_1-1}\right). \tag{2.3}$$

**Proof of Lemma 2.1.** From (2.2), it follows that

$$\frac{d|B^\sharp|}{d|A^\sharp|}=\frac{(\beta+\mathrm{Im}\eta_1|A^\sharp|^{p_2-1})|B^\sharp|}{(\alpha+\mathrm{Im}\eta_2|B^\sharp|^{p_1-1})|A^\sharp|}.$$

Since this is the differential equation of separation of variables, we see that

$$\int\frac{\beta+\mathrm{Im}\eta_2|A^\sharp|^{p_2-1}}{|A^\sharp|}d|A^\sharp|=\int\frac{\alpha+\mathrm{Im}\eta_1|B^\sharp|^{p_1-1}}{|B^\sharp|}d|B^\sharp|,$$

which leads us to

$$|A^\sharp|^\beta\exp\left(\frac{\mathrm{Im}\eta_2}{p_2-1}|A^\sharp|^{p_2-1}\right)=C|B^\sharp|^\alpha\exp\left(\frac{\mathrm{Im}\eta_1}{p_1-1}|B^\sharp|^{p_1-1}\right) \tag{2.4}$$

with some constant $C$. To determine the constant $C$, we are going to use the profile of $|A^\sharp(s)|$ and $|B^\sharp(s)|$ as $s\to-\infty$. Since $|A^\sharp(s)|=t^\alpha|A(t)|$ and $|B^\sharp(s)|=t^\beta|B(t)|$, (2.4) yields

$$\begin{aligned}&t^{\alpha\beta}|A(t)|^\beta\exp\left(\frac{\mathrm{Im}\eta_2}{p_2-1}|t^\alpha A(t)|^{p_2-1}\right)\\&=\ Ct^{\alpha\beta}|B(t)|^\alpha\exp\left(\frac{\mathrm{Im}\eta_1}{p_1-1}|t^\beta B(t)|^{p_1-1}\right).\end{aligned} \tag{2.5}$$

Divide the both hand sides of (2.5) with $t^{\alpha\beta}$, and take the limit $t\to+0$. Then we see that $|\mu|^\beta=C|\nu|^\alpha$ and obtain (2.3). □

# 3 Proof of Theorem 1.1

From the view of the dynamical system, the presence of three kinds of classifications in Theorem 1.1 is easy to be understood. Before the rigorous proof is exhibited, we will overview how to observe the behavior of the solutions by applying the dynamical system approach to the ODE system (2.2). The stationary point of (2.2), i.e., the point where $d|A^\sharp|/ds=d|B^\sharp|/ds=0$ holds are $(|A^\sharp|,|B^\sharp|)=(0,0)$ or $((\beta/|\mathrm{Im}\eta_2|)^{1/(p_2-1)},(\alpha/|\mathrm{Im}\eta_1|)^{1/(p_1-1)})$. Let $(a_s,b_s)=((\beta/|\mathrm{Im}\eta_2|)^{1/(p_2-1)},(\alpha/|\mathrm{Im}\eta_1|)^{1/(p_1-1)})$. Then, observing the sign of the right hand side of (2.2), we know that

(i) if $0<|A^\sharp|<a_s$ and $0<|B^\sharp|<b_s$, both $|A^\sharp(s)|$ and $|B^\sharp(s)|$ are monotone increasing.

(ii) if $a_s<|A^\sharp|$ and $0<|B^\sharp|<b_s$, the $|A^\sharp(s)|$ is monotone increasing, and the $|B^\sharp(s)|$ is monotone decreasing.

(iii) if $0<|A^\sharp|<a_s$ and $b_s<|B^\sharp|$, the $|A^\sharp(s)|$ is monotone decreasing, and the $|B^\sharp(s)|$ are monotone increasing.

Combining these properties together with $\lim_{s\to-\infty}|A^\sharp(s)| = \lim_{s\to-\infty}|B^\sharp(s)| = 0$, the solution curves on the $|A^\sharp|$-$|B^\sharp|$ coordinate plane are expected to be the flows shown in Figure 3.1. The curves (i) suggest the rapid decay of $u(t)$ and slow decay of $v(t)$ as in the statement $(i)$ of Theorem 1.1, and the curves (ii) suggest the slow decay of $u(t)$ and rapid decay of $v(t)$ as in the statement $(ii)$. The curve (iii) which connects the origin O and stationary point $(a_s, b_s)$ suggests the polynomial decay of both $u(t)$ and $v(t)$ (but it presents more rapid decay than the free solutions) as in the statement $(iii)$. We also remark that the curve (iii) is the boundary between the regions of curves (i) and (ii). This observation let us presume that the situation as in the statement $(iii)$ emerges under the exquisite conditions on the initial data and so it scarcely takes place.

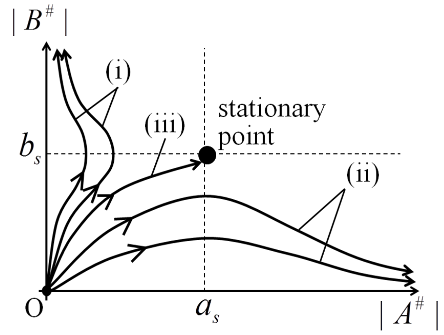


Figure 3.1: solution curves

We are now going to prove Theorem 1.1.

**Proof of Theorem 1.1.** We define two functions $f$ and $g$ by

$$f(\xi) = \frac{\xi^\beta}{|\mu|^\beta}\exp\left(\frac{\mathrm{Im}\eta_2}{p_2-1}\xi^{p_2-1}\right), \tag{3.1}$$

$$g(\xi) = \frac{\xi^\alpha}{|\nu|^\alpha}\exp\left(\frac{\mathrm{Im}\eta_1}{p_1-1}\xi^{p_1-1}\right). \tag{3.2}$$

Then, from Lemma 2.1, it follows that $|A^\sharp(s)|$ and $|B^\sharp(s)|$ vary while satisfying $f(|A^\sharp(s)|) = g(|B^\sharp(s)|)$ as in Figure 3.2. It is helpful in our proof to sketch graphs of $f$ and $g$. Since $\mathrm{Im}\eta_1 < 0$ and $\mathrm{Im}\eta_2 < 0$ are assumed, both $f$ and $g$ take critical values. Considering $f'(\xi) = 0$ and $g'(\xi) = 0$, we see that the function $f$ takes maximum value at $\xi = (\beta/|\mathrm{Im}\eta_2|)^{1/(p_2-1)}(= a_s)$ and so does $g$ at $\xi = (\alpha/|\mathrm{Im}\eta_1|)^{1/(p_1-1)}(= b_s)$. The function $f$ monotonically increases on the interval $(0, a_s)$ and monotonically decreases on $(a_s, \infty)$. The function $g$ monotonically increases on $(0, b_s)$ and monotonically decreases on $(b_s, \infty)$. Keeping these properties in our mind, we proceed in the proof.

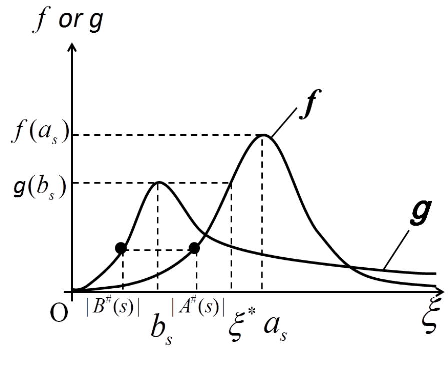


Figure 3.2: Transition of $|A^\sharp(s)|$ and $|B^\sharp(s)|$

$(i)$ (Step 1) We first show that $|B^\sharp(s)| \to \infty$ as $s \to \infty$. From (2.2), the global existence of $B^\sharp(s)$ follows by considering

$$\frac{d|B^\sharp(s)|}{ds} \le \beta|B^\sharp(s)|,$$

which yields $|B^\sharp(s)| \leq |B^\sharp(s_0)|e^{\beta(s-s_0)}$ for $s > s_0$. (The global existence for $A^\sharp(s)$ analogously follows.) Note that the assumption in $(i)$ suggests the relation of maximum values : $f(a_s) > g(b_s)$. Let $\xi^* = \min\{\xi \geq 0;\ f(\xi) = g(b_s)\}$. Then the solution $|A^\sharp(s)|$ never exceeds $\xi^*$ since two solutions $|A^\sharp(s)|$ and $|B^\sharp(s)|$ are continuous with respest to $s$ and must satisfy $f(|A^\sharp(s)|) = g(|B^\sharp(s)|)$. Then we have $|A^\sharp(s)| \leq \xi^* < a_s$. This implies that, for some $\rho > 0$, it holds that $\beta + \mathrm{Im}\eta_2|A^\sharp(s)|^{p_2-1} > \rho$. By the second equation in (2.2), we see that

$$\frac{d|B^\sharp(s)|}{ds} > \rho|B^\sharp(s)|,$$

which yields

$$|B^\sharp(s)| \;>\; |B^\sharp(s_0)|e^{\rho(s-s_0)} \tag{3.3}$$

for any $s > s_0$. Hence we see that $|B^\sharp(s)| \to \infty$ as $s \to \infty$.
(Step 2) We will show that $|A^\sharp(s)| \to 0$ as $s \to \infty$. In fact, by the first equation of (2.2), we have, for $s > s_0$,

$$|A^\sharp(s)| \;=\; |A^\sharp(s_0)|\exp\left(\int_{s_0}^{s}(\alpha + \mathrm{Im}\eta_1|B^\sharp(\sigma)|^{p_1-1})d\sigma\right).$$

Applying (3.3), we see that

$$\begin{aligned}
|A^\sharp(s)| \;&\leq\; |A^\sharp(s_0)|\exp\left(\int_{s_0}^{s}(\alpha - Ce^{\rho(p_1-1)(\sigma-s_0)})d\sigma\right)\\
&\leq\; C_1\exp\left(\alpha(s-s_0) - Ce^{\rho(p_1-1)(s-s_0)}\right)\\
&\leq\; C_2\exp(-C_3e^{C_4 s}) \qquad\qquad (3.4)\\
&\to\; 0 \quad (\text{as } s \to \infty).
\end{aligned}$$

(Step 3) We will show that $|B^\sharp(s)| = O(e^{\beta s})$. In fact, by the second equation of (2.2), we have, for $s > s_0$,

$$|B^\sharp(s)| \;=\; |B^\sharp(s_0)|\exp\left(\int_{s_0}^{s}(\beta + \mathrm{Im}\eta_2|A^\sharp(\sigma)|^{p_2-1})d\sigma\right).$$

Since (3.4) yields $\int_{s_0}^{\infty}|A^\sharp(\sigma)|^{p_2-1}d\sigma < \infty$, it follows that

$$\begin{aligned}
|B^\sharp(s)| \;&=\; Ce^{\beta s}\times\exp\left(|\mathrm{Im}\eta_2|\int_{s}^{\infty}|A^\sharp(\sigma)|^{p_2-1}d\sigma\right)\\
&=\; Ce^{\beta s} + CR(s), \qquad\qquad (3.5)
\end{aligned}$$

where the remainder is given by

$$R(s) = e^{\beta s}\left\{\exp\left(|\mathrm{Im}\eta_2|\int_{s}^{\infty}|A^\sharp(\sigma)|^{p_2-1}d\sigma\right) - 1\right\}.$$

By (3.4), we see that

$$\begin{aligned}
R(s) \;&\leq\; Ce^{\beta s}\int_{s}^{\infty}|A^\sharp(\sigma)|^{p_2-1}d\sigma\\
&\leq\; CC_2^{p_2-1}e^{\beta s}\int_{s}^{\infty}\exp(-C''e^{C'\sigma})d\sigma. \qquad\qquad (3.6)
\end{aligned}$$

By l'Hôpital's rule, we have

$$\begin{aligned}
&\lim_{s\to\infty}\frac{\int_s^\infty \exp(-C''e^{C'\sigma})d\sigma}{\exp(-C''e^{C's})\times e^{-C's}}\\
&= \lim_{s\to\infty}\frac{-\exp(-C''e^{C's})}{-C''C'\exp(-C''e^{C's})-C'\exp(-C''e^{C's})\times e^{-C's}}\\
&= \frac{1}{C''C'}.
\end{aligned}$$

Hence, from (3.6), it follows that $R(s)=O(\exp(-C'e^{Cs}))$ as $s\to\infty$, and so we obtain the asymptotic profile of $|B^\sharp(s)|$, i.e.,

$$|B^\sharp(s)| = Ce^{\beta s}+O(\exp(-C'''e^{C's})) \tag{3.7}$$

as $s\to\infty$.

(Step 4) We will show the sharp decay estimate of $|A^\sharp(s)|$ as $s\to\infty$. By Lemma 2.1, we have

$$\begin{aligned}
&|A^\sharp(s)|\\
&= |\mu|\times\frac{|B^\sharp(s)|^{\alpha/\beta}}{|\nu|^{\alpha/\beta}}\exp\left(\frac{\mathrm{Im}\eta_1}{(p_1-1)\beta}|B^\sharp(s)|^{p_1-1}-\frac{\mathrm{Im}\eta_2}{(p_2-1)\beta}|A^\sharp(s)|^{p_2-1}\right).
\end{aligned}$$

Applying (3.4) and (3.7), we see that

$$\begin{aligned}
|A^\sharp(s)| &= |\mu|\times\frac{C^{\alpha/\beta}e^{\alpha s}}{|\nu|^{\alpha/\beta}}\exp\left(\frac{\mathrm{Im}\eta_1 C^{p_1-1}}{(p_1-1)\beta}e^{(p_1-1)\beta s}\right)\\
&\quad\times\left(1+O(\exp(-C'''e^{C's}))\right)
\end{aligned} \tag{3.8}$$

as $s\to\infty$. Recall the deformation of $A(t)$ and $B(t)$ in §2. Then we see that $|A(t)|=t^{-\alpha}|A^\sharp(\log t)|$ and $|B(t)|=t^{-\beta}|B^\sharp(\log t)|$. By (3.7) and (3.8), we obtain

$$\begin{aligned}
|A(t)| &= |\mu|\times\frac{C^{\alpha/\beta}}{|\nu|^{\alpha/\beta}}\exp\left(\frac{\mathrm{Im}\eta_1 C^{p_1-1}}{(p_1-1)\beta}t^{(p_1-1)\beta}\right)\\
&\quad\times\left(1+O(\exp(-C'''t^{C'}))\right)
\end{aligned} \tag{3.9}$$

and

$$|B(t)| = C+O(\exp(-C'''t^{C'})) \tag{3.10}$$

as $t\to\infty$. Since $\|u(t)\|_{L^\infty}=|A(t)U_{m_1}(t)\delta_a|$ and $\|v(t)\|_{L^\infty}=|B(t)U_{m_2}(t)\delta_b|$ together with $\|U_m(t)\delta_c\|_{L^\infty}=(m/2\pi t)^{n/2}$, (3.9) and (3.10) yield Theorem 1.1 ($i$).

($ii$) By exchanging the roles of $|A^\sharp|$ and $|B^\sharp|$, the proof follows analogously in the proof of ($i$).

($iii$) The assumption in the statement ($iii$) suggests that $f(a_s)=g(b_s)$. Both the solutions $|A^\sharp(s)|$ and $|B^\sharp(s)|$ satisfying (2.2) are monotonically increasing while they do not exceed

$a_s$ and $b_s$ respectively. The $|A^\sharp(s)|$ never reaches $a_s$ for finite $s$, and $|B^\sharp(s)|$ never reaches $b_s$ either. In fact, if there exists some $s_0$ for which $|A^\sharp(s_0)| = a_s$, then, for the same $s_0$, $|B^\sharp(s_0)| = b_s$. Note that $(a_s, b_s)$ is the stationary solution to (2.2), and the uniqueness of the solution yields $(|A^\sharp(s)|, |B^\sharp(s)|) = (a_s, b_s)$ for $s \in (-\infty, s_0]$. But it contradicts the fact that $\lim\limits_{s\to-\infty} |A^\sharp(s)| = 0$ and $\lim\limits_{s\to-\infty} |B^\sharp(s)| = 0$. Hence we see that $\lim\limits_{s\to\infty} |A^\sharp(s)| \le a_s$ and $\lim\limits_{s\to\infty} |B^\sharp(s)| \le b_s$.

Suppose that $\lim\limits_{s\to\infty} |A^\sharp(s)| = a^*(< a_s)$ and $\lim\limits_{s\to\infty} |B^\sharp(s)| = b^*(< b_s)$. Then we will have contradiction. In fact, from (2.2), it follows that

$$\begin{aligned}
|A^\sharp(s)| - |A^\sharp(s_0)| &= \int_{s_0}^{s} (\alpha + \mathrm{Im}\eta_1 |B^\sharp(\sigma)|^{p_1-1})|A^\sharp(\sigma)|d\sigma \\
&> \int_{s_0}^{s} (\alpha + \mathrm{Im}\eta_1 |b^*|^{p_1-1})|A^\sharp(s_0)|d\sigma \\
&= (\alpha + \mathrm{Im}\eta_1 |b^*|^{p_1-1})|A^\sharp(s_0)|(s - s_0).
\end{aligned}$$

Taking $s \to \infty$, we see that this inequality causes a contradiction. Therefore we have $\lim\limits_{s\to\infty} |A^\sharp(s)| = a_s$. Since $f(a_s) = g(b_s)$, we also have $\lim\limits_{s\to\infty} |B^\sharp(s)| = b_s$, which implies that $|A(t)| \sim a_s t^{-\alpha}$ and $|B(t)| \sim b_s t^{-\beta}$ as $t \to \infty$. Hence it follows that

$$\begin{aligned}
\|u(t)\|_{L^\infty} &= \|A(t)U_{m_1}(t)\delta_a\|_{L^\infty} \sim \left(\frac{m_1}{2\pi}\right)^{n/2} a_s t^{-1/(p_2-1)}, \\
\|v(t)\|_{L^\infty} &= \|B(t)U_{m_2}(t)\delta_b\|_{L^\infty} \sim \left(\frac{m_2}{2\pi}\right)^{n/2} b_s t^{-1/(p_1-1)}
\end{aligned}$$

as $t \to \infty$. Now the proof of $(iii)$ is complete. □

# 4 Proof of Theorem 1.2 and 1.3

We will prove only Theorem 1.2 $(i)$ and Theorem 1.3 $(i)$.

**Proof of Theorem 1.2 $(i)$.**
(Step1) We first show that $|B^\sharp(s)| \to \infty$ as $s \to \infty$. Let $f(\xi)$ and $g(\xi)$ as defined in (3.1) and (3.2). By Lemma 2.1, the solution $(A^\sharp(s), B^\sharp(s))$ is subject to $f(|A^\sharp(s)|) = g(|B^\sharp(s)|)$, and $|A^\sharp(s)|$, $|B^\sharp(s)|$ are monotone increasing as long as $|B^\sharp(s)| < b_s$ where $b_s$ is defined at the beginning of §3. Let $\xi^*$ be the uniquely determined value such that $f(\xi^*) = g(b_s)$. Then, by Lemma 2.1, $|A^\sharp(s)| \le \xi^*$ always holds, which may be easily understood by referring to Figure 4.1. From the second equation of (2.2), it follows that

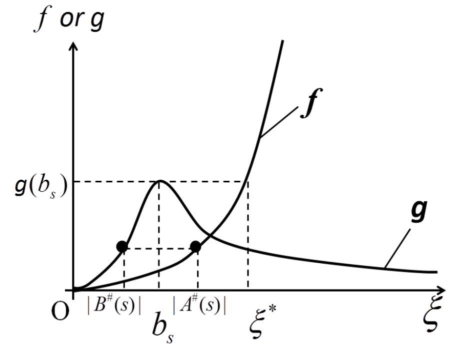


Figure 4.1: Transition of $|A^\sharp(s)|$ and $|B^\sharp(s)|$

$$\frac{d|B^\sharp(s)|}{ds} \le (\beta + \mathrm{Im}\eta_2(\xi^*)^{p_2-1})|B^\sharp(s)|,$$

which yields

$$|B^\sharp(s)| \leq |B^\sharp(s_0)| \exp\left\{(\beta + \mathrm{Im}\eta_2(\xi^*)^{p_2-1})(s-s_0)\right\} < \infty$$

for $s > s_0$. Hence the solution $(A^\sharp(s), B^\sharp(s))$ exists globally in time. By the second equation in (2.2), we see that

$$\frac{d|B^\sharp(s)|}{ds} > \beta|B^\sharp(s)|,$$

and so we have

$$|B^\sharp(s)| \;\; > \;\; |B^\sharp(s_0)|e^{\beta(s-s_0)} \tag{4.1}$$

for any $s > s_0$. Hence it follows that $|B^\sharp(s)| \to \infty$ as $s \to \infty$.
(Step 2) We will show that $|A^\sharp(s)| \to 0$ as $s \to \infty$. In fact, by the first equation of (2.2), we have, for $s > s_0$,

$$|A^\sharp(s)| \;\; = \;\; |A^\sharp(s_0)| \exp\left(\int_{s_0}^{s} (\alpha + \mathrm{Im}\eta_1 |B^\sharp(\sigma)|^{p_1-1}) d\sigma\right).$$

Applying (4.1), we see that

$$\begin{aligned} |A^\sharp(s)| \;\; &\leq \;\; |A^\sharp(s_0)| \exp\left(\int_{s_0}^{s} (\alpha - Ce^{\beta(p_1-1)(\sigma-s_0)}) d\sigma\right) \\ &\leq \;\; C_1 \exp\left(\alpha(s-s_0) - C' e^{\beta(p_1-1)(s-s_0)}\right) \\ &\leq \;\; C_2 \exp(-C_3 e^{\beta(p_1-1)s}) \qquad (4.2) \\ &\to \;\; 0 \quad (\text{as } s \to \infty). \end{aligned}$$

(Step 3) We will show that $|B^\sharp(s)| = O(e^{\beta s})$. In fact, by the second equation of (2.2), we have, for $s > s_0$,

$$|B^\sharp(s)| \;\; = \;\; |B^\sharp(s_0)| \exp\left(\int_{s_0}^{s} (\beta + \mathrm{Im}\eta_2 |A^\sharp(\sigma)|^{p_2-1}) d\sigma\right).$$

Since (4.2) yields $\int_{s_0}^{\infty} |A^\sharp(\sigma)|^{p_2-1} d\sigma < \infty$, it follows that

$$\begin{aligned} |B^\sharp(s)| \;\; &= \;\; Ce^{\beta s} \times \exp\left(-\mathrm{Im}\eta_2 \int_{s}^{\infty} |A^\sharp(\sigma)|^{p_2-1} d\sigma\right) \\ &= \;\; Ce^{\beta s} + CR(s), \qquad (4.3) \end{aligned}$$

where the remainder is given by

$$R(s) = e^{\beta s}\left\{\exp\left(-\mathrm{Im}\eta_2 \int_{s}^{\infty} |A^\sharp(\sigma)|^{p_2-1} d\sigma\right) - 1\right\}.$$

By (4.2), we see that

$$\begin{aligned} R(s) \;\; &\leq \;\; Ce^{\beta s} \int_{s}^{\infty} |A^\sharp(\sigma)|^{p_2-1} d\sigma \\ &\leq \;\; Ce^{\beta s} \int_{s}^{\infty} \exp(-C' e^{\beta(p_1-1)\sigma}) d\sigma. \qquad (4.4) \end{aligned}$$

By l'Hôpital's rule, we have

$$\begin{aligned}
&\lim_{s\to\infty}\frac{\int_s^\infty \exp(-C'e^{\beta(p_1-1)\sigma})d\sigma}{\exp(-C'e^{\beta(p_1-1)s})\times e^{-\beta(p_1-1)s}}\\
&= \lim_{s\to\infty}\frac{-\exp(-C'e^{\beta(p_1-1)s})}{-\beta(p_1-1)\exp(-C'e^{\beta(p_1-1)s})(C'+e^{-\beta(p_1-1)s})}\\
&= \frac{1}{C'\beta(p_1-1)}.
\end{aligned}$$

Hence, from (4.4), it follows that $R(s)=O(\exp(-C''e^{\beta(p_1-1)s}))$ as $s\to\infty$, and so we obtain the asymptotic profile of $|B^\sharp(s)|$, i.e.,

$$|B^\sharp(s)| = Ce^{\beta s}+O(\exp(-C''e^{\beta(p_1-1)s})) \tag{4.5}$$

as $s\to\infty$.

(Step 4) We will show the sharp decay estimate of $|A^\sharp(s)|$ as $s\to\infty$. By Lemma 2.1, we have

$$\begin{aligned}
&|A^\sharp(s)|\\
&= |\mu|\times\frac{|B^\sharp(s)|^{\alpha/\beta}}{|\nu|^{\alpha/\beta}}\exp\left(\frac{\mathrm{Im}\eta_1}{(p_1-1)\beta}|B^\sharp(s)|^{p_1-1}-\frac{\mathrm{Im}\eta_2}{(p_2-1)\beta}|A^\sharp(s)|^{p_2-1}\right).
\end{aligned}$$

Applying (4.2) and (4.5), we see that

$$\begin{aligned}
|A^\sharp(s)| &= |\mu|\times\frac{C^{\alpha/\beta}e^{\alpha s}}{|\nu|^{\alpha/\beta}}\exp\left(\frac{\mathrm{Im}\eta_1 C^{p_1-1}}{(p_1-1)\beta}e^{(p_1-1)\beta s}\right)\\
&\quad\times\left(1+O(\exp(-C'e^{Cs}))\right)
\end{aligned}\tag{4.6}$$

as $s\to\infty$. Recall the deformation of $A(t)$ and $B(t)$ in §2. Then we see that $|A(t)|=t^{-\alpha}|A^\sharp(\log t)|$ and $|B(t)|=t^{-\beta}|B^\sharp(\log t)|$. By (4.5) and (4.6), we obtain

$$\begin{aligned}
|A(t)| &= |\mu|\times\frac{C^{\alpha/\beta}}{|\nu|^{\alpha/\beta}}\exp\left(\frac{\mathrm{Im}\eta_1 C^{p_1-1}}{(p_1-1)\beta}t^{(p_1-1)\beta}\right)\\
&\quad\times\left(1+O(\exp(-C't^C))\right)
\end{aligned}\tag{4.7}$$

and

$$|B(t)| = C+O(\exp(-C't^C)) \tag{4.8}$$

as $t\to\infty$. Since $\|u(t)\|_{L^\infty}=|A(t)U_{m_1}(t)\delta_a|$ and $\|v(t)\|_{L^\infty}=|B(t)U_{m_2}(t)\delta_b|$ together with $\|U_m(t)\delta_C\|_{L^\infty}=(m/2\pi t)^{-n/2}$, (4.7) and (4.8) yield Theorem 1.2($i$). The proof of the statement ($ii$) follows in similar way. □

The proof of Theorem 1.3 is easy.

**Proof of Theorem 1.3 ($i$).** By the first equation of (1.2), we see that $|A(t)|=|\mu|$. Substitute it into the second equation, we have

$$\frac{dB(t)}{dt}=-i\eta_2|\mu|^{p_2-1}t^{-d_2}B(t).$$

It is easy to solve this equation, and we obtain

$$B(t) = \nu \exp\left(-i\frac{2\eta_2|\mu|^{p_2-1}}{2-n(p_2-1)}t^{1-n(p_2-1)/2}\right).$$

This completes the proof. The proof of $(ii)$ similarly follows. □

# 5 Proof of Theorem 1.4

In this final section, we will prove the blowing-up result by making use of Lemma 2.1.

**Proof of Theorem 1.4.** We only consider the case of $p_1 < p_2$.

(Step 1) We first show that $|A^\sharp(s)|$ and $|B^\sharp(s)|$ blow up in finite time by the contradiction argument. Suppose that the solution $(A^\sharp(s), B^\sharp(s))$ exists globally in time. By the equations in (2.2), $\frac{d}{ds}|A^\sharp(s)| > \alpha|A^\sharp(s)|$ and $\frac{d}{ds}|B^\sharp(s)| > \beta|B^\sharp(s)|$ hold. Then we have $|A^\sharp(s)| > |A^\sharp(s_0)|e^{\alpha(s-s_0)}$ and $|B^\sharp(s)| > |B^\sharp(s_0)|e^{\beta(s-s_0)}$ for any $s > s_0$, which implies that $\lim_{s\to\infty}|A^\sharp(s)| = \infty$ and $\lim_{s\to\infty}|B^\sharp(s)| = \infty$. Note that Lemma 2.1 yields $f(|A^\sharp(s)|) = g(|B^\sharp(s)|)$, where $f$ and $g$ were defined at the beginning of §3. Since $p_1 < p_2$ is assumed, there exists some $\xi_0 > 0$ such that $f(\xi) > g(\xi)$ holds for any $\xi > \xi_0$. This means that $|A^\sharp(s)| < |B^\sharp(s)|$ holds for sufficiently large $s > 0$ as in Figure 5.1. Then, from (2.2), it follows that

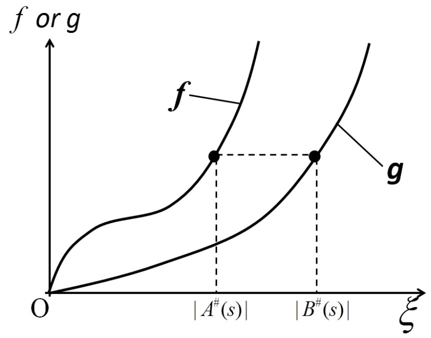

Figure 5.1: Transition of $|A^\sharp(s)|$ and $|B^\sharp(s)|$

$$\begin{aligned}\frac{d|A^\sharp(s)|}{ds} &> (\alpha + \mathrm{Im}\eta_1|A^\sharp(s)|^{p_1-1})|A^\sharp(s)| \\ &> \mathrm{Im}\eta_1|A^\sharp(s)|^{p_1}.\end{aligned}$$

Solving this differential inequality, we have

$$|A^\sharp(s)|^{-(p_1-1)} < |A^\sharp(s_0)|^{-(p_1-1)} - (p_1-1)\mathrm{Im}\eta_1(s-s_0).$$

But this inequality fails by taking $s$ sufficiently large. Thus there exists some $s^* \in \mathbb{R}$ such that $\lim_{s\uparrow s^*}|A^\sharp(s)| = \infty$. Since $f(|A^\sharp(s)|) = g(|B^\sharp(s)|)$, we also have $\lim_{s\uparrow s^*}|B^\sharp(s)| = \infty$.

(Step 2) We will determine the blowing-up rates of $|A^\sharp(s)|$ and $|B^\sharp(s)|$. When $s$ is closely lower than $s^*$, both $|A^\sharp(s)|$ and $|B^\sharp(s)|$ take large values. Applying Lemma 2.1 and noting that $\exp\left(\frac{\mathrm{Im}\eta_2}{p_2-1}|A^\sharp(s)|^{p_2-1}\right)$ is remarkably lager than $|A^\sharp(s)|^\beta$ etc., we see that, for any $\varepsilon > 0$, there exists some $s' \in \mathbb{R}$ such that, if $s \in (s', s^*)$, then

$$\exp\left\{(1-\varepsilon)\frac{\mathrm{Im}\eta_2}{p_2-1}|A^\sharp(s)|^{p_2-1}\right\} < \exp\left\{(1+\varepsilon)\frac{\mathrm{Im}\eta_1}{p_1-1}|B^\sharp(s)|^{p_1-1}\right\}$$

and

$$\exp\left\{(1+\varepsilon)\frac{\mathrm{Im}\eta_2}{p_2-1}|A^\sharp(s)|^{p_2-1}\right\} > \exp\left\{(1-\varepsilon)\frac{\mathrm{Im}\eta_1}{p_1-1}|B^\sharp(s)|^{p_1-1}\right\}.$$

Obviously, these inequalities are equivalent to

$$(1-\varepsilon)\frac{\mathrm{Im}\eta_2}{p_2-1}|A^\sharp(s)|^{p_2-1} < (1+\varepsilon)\frac{\mathrm{Im}\eta_1}{p_1-1}|B^\sharp(s)|^{p_1-1} \tag{5.1}$$

and

$$(1+\varepsilon)\frac{\mathrm{Im}\eta_2}{p_2-1}|A^\sharp(s)|^{p_2-1} > (1-\varepsilon)\frac{\mathrm{Im}\eta_1}{p_1-1}|B^\sharp(s)|^{p_1-1}. \tag{5.2}$$

Apply (5.1) and (5.2) to the first equation of (2.2). Then we see that

$$\begin{aligned}
&\alpha|A^\sharp(s)| + \frac{1-\varepsilon}{1+\varepsilon}\times\frac{p_1-1}{p_2-1}\mathrm{Im}\eta_2|A^\sharp(s)|^{p_2} < \frac{d|A^\sharp(s)|}{ds},\\
&\frac{d|A^\sharp(s)|}{ds} < \alpha|A^\sharp(s)| + \frac{1+\varepsilon}{1-\varepsilon}\times\frac{p_1-1}{p_2-1}\mathrm{Im}\eta_2|A^\sharp(s)|^{p_2}.
\end{aligned}$$

It is written in the way that

$$\begin{aligned}
&\frac{1-\varepsilon}{1+\varepsilon}\times\frac{p_1-1}{p_2-1}\mathrm{Im}\eta_2 e^{(p_2-1)\alpha s}\left(e^{-\alpha s}|A^\sharp(s)|\right)^{p_2} < \frac{d}{ds}\left(e^{-\alpha s}|A^\sharp(s)|\right),\\
&\frac{d}{ds}\left(e^{-\alpha s}|A^\sharp(s)|\right) < \frac{1+\varepsilon}{1-\varepsilon}\times\frac{p_1-1}{p_2-1}\mathrm{Im}\eta_2 e^{(p_2-1)\alpha s}\left(e^{-\alpha s}|A^\sharp(s)|\right)^{p_2}.
\end{aligned}$$

By taking the integration from $s$ to $s^*$, it turns out to be

$$\begin{aligned}
&\frac{1-\varepsilon}{1+\varepsilon}\times\frac{p_1-1}{p_2-1}\mathrm{Im}\eta_2(e^{(p_2-1)\alpha(s^*-s)}-1) < \alpha|A^\sharp(s)|^{-(p_2-1)},\\
&\alpha|A^\sharp(s)|^{-(p_2-1)} < \frac{1+\varepsilon}{1-\varepsilon}\times\frac{p_1-1}{p_2-1}\mathrm{Im}\eta_2(e^{(p_2-1)\alpha(s^*-s)}-1).
\end{aligned}$$

Multiply $(s^*-s)^{-1}$ and taking the $\liminf_{s\uparrow s^*}$ and $\limsup_{s\uparrow s^*}$, we have

$$\begin{aligned}
&\frac{1-\varepsilon}{1+\varepsilon}(p_1-1)\mathrm{Im}\eta_2\\
&\quad\le \liminf_{s\uparrow s^*}(s^*-s)^{-1}|A^\sharp(s)|^{-(p_2-1)}\\
&\quad\le \limsup_{s\uparrow s^*}(s^*-s)^{-1}|A^\sharp(s)|^{-(p_2-1)}\\
&\quad\le \frac{1+\varepsilon}{1-\varepsilon}(p_1-1)\mathrm{Im}\eta_2.
\end{aligned}$$

Taking $\varepsilon\downarrow 0$, we see that

$$\lim_{s\uparrow s^*}(s^*-s)^{-1}|A^\sharp(s)|^{-(p_2-1)} = (p_1-1)\mathrm{Im}\eta_2. \tag{5.3}$$

Let $T^* = e^{s^*}$ and $t = e^s$. Recall $|A(t)| = t^{-\alpha}|A^\sharp(\log t)|$. Then, from (5.3), it follows that

$$\lim_{t\uparrow T^*}(T^*-t)|A(t)|^{p_2-1} = \frac{(T^*)^{(p_2-1)n/2}}{(p_1-1)\mathrm{Im}\eta_2}.$$

The proof of (1.18) is complete. The proof of (1.19) similarly follows. □

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