DECAY ESTIMATE OF SOLUTIONS TO DISSIPATIVE NONLINEAR SCHRÖDINGER EQUATIONS

NAOYASU KITA AND CHUNHUA LI

Abstract

This paper presents the decay estimate of solutions to the initial value problem of 1D Schrödinger equations containing a sub-critical dissipative nonlinearity $\lambda|u|^{p-1}u$, where $2.468 \approx p_0 \leq p < 3$, $\text{Im}\lambda < 0$ and $(p-1)|\text{Re}\lambda| \leq 2\sqrt{p}|\text{Im}\lambda|$. Our aim is to obtain the decay estimate of the solutions, without assuming size restriction on the initial data and under the extended lower bound of nonlinear power.

1 Introduction and main results

We consider the Cauchy problem of 1D-nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t v + \frac{1}{2}\partial_x^2 v = \lambda|v|^{p-1}v, \\ v(0,x) = \phi(x), \end{cases} \tag{1.1}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$, $1 < p < 3$ and $\lambda = \lambda_1 + i\lambda_2$ $(\lambda_1, \lambda_2 \in \mathbb{R})$ satisfying

$$\lambda_2 < 0, \quad |\lambda_2| \geq \frac{p-1}{2\sqrt{p}}|\lambda_1|. \tag{1.2}$$

It is well known that the asymptotic behavior of solutions to (1.1) with $1 < p \leq 3$ is different from that of solutions to the corresponding free equation (see [1]). There is some research on the small initial problem (1.1)(see e.g., [2] and [6]). The large initial problem (1.1) with (1.2) was first considered in [5], where the $\mathbf{L}^\infty$-decay estimates of the global solutions for $2.686\cdots \approx \frac{5+\sqrt{33}}{4} < p \leq 3$ was investigated. Jin-Jin-Li also considered the large initial problem (1.1) with (1.2) and improved the previous result in [5]. They proved the

---

$\mathbf{L}^\infty$-decay estimate of the global solutions to (1.1) with(1.2), i.e., $\|v(t)\|_{\mathbf{L}^\infty} \leq C(1+t)^{-1/(p-1)}$ for $t > 0$, if $2.586 \cdots \approx \frac{19+\sqrt{145}}{12} < p < 3$ holds; $\|v(t)\|_{\mathbf{L}^\infty} \leq C(1+t)^{-1/2}\left(\log(1+t)\right)^{-1/2}$ for $t > 0$, if $p = 3$. Since $\frac{19+\sqrt{145}}{12} < \frac{5+\sqrt{33}}{4}$, Jin-Jin-Li extended the lower bound of p by using the operator $J^2(t) = U(t)x^2 U(-t)$. Our purpose is to investigate $\mathbf{L}^2$- decay estimates of solutions and achieve a better $\mathbf{L}^\infty$-decay estimates of solutions to (1.1) with (1.2) for arbitrarily large initial data.

Let $\mathbf{L}^q(\mathbb{R})$ denote the usual Lebesgue space with the norm

$$\|\phi\|_{\mathbf{L}^q(\mathbb{R})} = \left(\int_{\mathbb{R}} |\phi(x)|^q\, dx\right)^{\frac{1}{q}}$$

if $1 \leq q < \infty$ and

$$\|\phi\|_{\mathbf{L}^\infty(\mathbb{R})} = \operatorname*{ess.sup}_{x \in \mathbb{R}} |\phi(x)|.$$

For $m, s \in \mathbb{R}$ and $1 \leq q \leq \infty$, weighted Sobolev space $\mathbf{H}_q^{m,s}(\mathbb{R})$ is defined by

$$\mathbf{H}_q^{m,s}(\mathbb{R}) = \left\{ f \in \mathbf{L}^q(\mathbb{R})\,;\, \|f\|_{\mathbf{H}_q^{m,s}(\mathbb{R})} = \left\|(1 - \partial_x^2)^{\frac{m}{2}}(1 + |x|^2)^{\frac{s}{2}} f\right\|_{\mathbf{L}^q(\mathbb{R})} < \infty \right\}.$$

We write $\mathbf{H}_2^{m,s}(\mathbb{R}) = \mathbf{H}^{m,s}$ and $\mathbf{H}^{m,0}(\mathbb{R}) = \mathbf{H}^m$ for simplicity.

Let us introduce some more notations. We define the dilation operator by

$$(D_t \phi)(x) = \frac{1}{(it)^{\frac{1}{2}}} \phi\left(\frac{x}{t}\right)$$

and define $M = e^{\frac{i}{2t}x^2}$ for $t \neq 0$. Evolution operator $U(t)$ is written as

$$U(t) = M D_t \mathcal{F} M,$$

where the Fourier transform of f is

$$(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

We also have

$$U(-t) = M^{-1}\mathcal{F}^{-1} D_t^{-1} M^{-1},$$

where the inverse Fourier transform of f is

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} f(\xi)d\xi.$$

We denote by the same letter C various positive constants.

The standard generator of the Galilei transformations is given as

$$J(t) = U(t)\, x\, U(-t) = x + it\partial_x.$$

We have

$$J^2(t) = U(t)\, x^2\, U(-t) = M(-t^2\partial_x^2)M^{-1}.$$

We also have commutation relations with J^α and $L = i\partial_t + \frac{1}{2}\partial_x^2$ such that

$$[L, J^\alpha] = 0,$$

where $\alpha = 1, 2$.

Before stating our main theorem, we introduce the function space

$$X_{2,T} = \{y \ ; \ U(-t)y \in C([0,T]; \mathbf{H}^1 \cap \mathbf{H}^{0,2}), \|y\|_{X_{2,T}} < \infty\},$$

where $\|y\|_{X_{2,T}} = \sup_{0 \le t < T} \|U(-t)y\|_{\mathbf{H}^1 \cap \mathbf{H}^{0,2}}$. Our main result is

Theorem 1.1. *Let* $p_0 < p \le \frac{19+\sqrt{145}}{12}$, *where* $p_0 \approx 2.486\cdots$ *is a unique real root of*

$$12p^3 - 41p^2 + 35p - 18 = 0.$$

We assume that $\phi \in \mathbf{H}^{0,2} \cap \mathbf{H}^1$ *(without size restriction) and the strong dissipative condition* (1.2) *holds. Then* (1.1) *has a unique global solution* $v \in X_{2,\infty}$ *satisfying the following time-decay estimates :*

$$\|v(t)\|_{\mathbf{L}^2} \le C(1+t)^{-(2/3)d}, \tag{1.3}$$
$$\|v(t)\|_{\mathbf{L}^\infty} \le C(1+t)^{-1/(p-1)}, \tag{1.4}$$

for $t > 0$, *where* $d = 1/(p-1) - 1/2$.

Combining Theorem 1.1 and theorems in [3], we have

Corollary 1.2. *Let* $p_0 < p < 3$, *where* $p_0 \approx 2.486\cdots$ *is a unique real root of*

$$12p^3 - 41p^2 + 35p - 18 = 0.$$

We assume that $\phi \in \mathbf{H}^{0,2} \cap \mathbf{H}^1$ *(without size restriction) and the strong dissipative condition* (1.2) *holds. Then* (1.1) *has a unique global solution* $v \in X_{2,\infty}$ *satisfying the following time-decay estimates :*

$$\|v(t)\|_{\mathbf{L}^\infty} \le C(1+t)^{-1/(p-1)}, \tag{1.5}$$

for $t > 0$.

Remark 1.1. In fact, we have L^2- decay (1.3) under the same assumptions as in Corollary 1.2. This will be proved in Section 2 (see Proposition 2.1).

Remark 1.2. The decay rate of $\|v(t)\|_{\mathbf{L}^\infty}$ is similar to that of the solution to the ODE: $i\partial_t v = \lambda|v|^{p-1}v$. This suggests that the nonlinear effect is dominant to the dispersion emerging from $\frac{1}{2}\partial_x^2$ for large $t > 0$.

In our strategy for the proof, we are going to be engaged in the refinement of the estimate of $\|J^2 v(t)\|_{\mathbf{L}^2}$, where $J = x + it\partial_x$. In [3], they derived, from (1.1) and (1.2),

$$
\begin{aligned}
\frac{d}{dt}\|J^2 v\|_{\mathbf{L}^2} &\leq C\|v\|_{\mathbf{L}^\infty}^{p-2}\|Jv\|_{\mathbf{L}^\infty}\|Jv\|_{\mathbf{L}^2} \\
&\leq Ct^{-1/2}\|v\|_{\mathbf{L}^\infty}^{p-2}\|J^2 v\|_{\mathbf{L}^2}^{1/2}\|Jv\|_{\mathbf{L}^2}^{3/2},
\end{aligned}
$$

and used $\|v\|_{\mathbf{L}^\infty} \leq Ct^{-1/2}\|v\|_{\mathbf{L}^2}^{1/2}\|Jv\|_{\mathbf{L}^2}^{1/2}$ together with $\|v\|_{\mathbf{L}^2} + \|Jv\|_{\mathbf{L}^2} \leq C$ and $\phi \in \mathbf{H}^{0,2}$. Let $2 < p < 3$. Solving the above differential inequality, they obtained

$$\|J^2 v\|_{\mathbf{L}^2} \leq C(1+t)^{3-p} \tag{1.6}$$

for $t > 0$, which played an important role to control the remainder terms appearing in the process to determine the decay estimate of $v(t)$. On the other hand, we will, more carefully than Jin-Jin-Li did, treat $\|v\|_{\mathbf{L}^\infty}$ (refer to Proposition 2.3) after deriving some decay estimate of $\|v(t)\|_{\mathbf{L}^2}$ (refer to Proposition 2.1), and obtain

$$\|J^2 v\|_{\mathbf{L}^2} \leq C(1+t)^{p(11-4p)/6} \tag{1.7}$$

for $t > 0$, if $p_0 < p \leq \frac{19+\sqrt{145}}{12}$. We remark here that the growth order in (1.7) is better than that of (1.6), since $p(11-4p)/6 < 3-p$ holds sufficiently if $p_0 < p \leq \frac{19+\sqrt{145}}{12}$. Finally, making use of (1.7), we will obtain the sharp decay rate of $\|v(t)\|_{\mathbf{L}^\infty}$. The number p_0 will arise in the estimate of (3.5) below. In (3.5), we will regard $t^{-1/(p-1)+1/2}$ as a dominant to $t^{-(12p^2-29p)/12}$ for $t > 1$. This observation is valid if $p_0 < p$.

2   Preliminaries

We first consider the $\mathbf{L}^2$-decay estimate of $v(t)$. This will be used later for a rough $\mathbf{L}^\infty$-decay estimate (see Proposition 2.3).

Proposition 2.1 (L^2-Decay)**.** *Assume the same assumptions as in Corollary 1.2. Let $v \in X_{2,\infty}$ be the global solution to (1.1). Then it satisfies*

$$\|v(t)\|_{\mathbf{L}^2} \leq C(1+t)^{-(2/3)d}, \tag{2.1}$$

for $t > 0$, where $d = 1/(p-1) - 1/2$.

When we prove Proposition 2.1, the lemma below will be often taken into account, the proof of which follows from [3] and [5]. We note that, to derive Lemma 2.2 without size restriction on ϕ, the strong dissipative condition (1.2) is useful.

Lemma 2.2. *Let $v \in X_{2,\infty}$ be the global solution to (1.1). Then it satisfies*

$$\|\partial_x v(t)\|_{\mathbf{L}^2} \leq \|\partial_x \phi\|_{\mathbf{L}^2}, \tag{2.2}$$

$$\|Jv(t)\|_{\mathbf{L}^2} \leq \|x\phi\|_{\mathbf{L}^2}. \tag{2.3}$$

Proof of Proposition 2.1. We first note that $\mathcal{F}U(-t)v$ satisfies

$$\partial_t(\mathcal{F}U(-t)v) = -i\lambda t^{-(p-1)/2}|\mathcal{F}U(-t)v|^{p-1}\mathcal{F}U(-t)v + R(t), \tag{2.4}$$

where $U(t) = \exp(it\partial_x^2/2)$ and

$$R(t) = -i\lambda t^{-(p-1)/2} \left(\mathcal{F}M^{-1}\mathcal{F}^{-1}|\mathcal{F}MU(-t)v|^{p-1}\mathcal{F}MU(-t)v \right.$$
$$\left. -|\mathcal{F}U(-t)v|^{p-1}\mathcal{F}U(-t)v \right) \tag{2.5}$$

with $M = \exp(ix^2/2t)$. By (2.4), we have

$$\partial_t\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^2$$
$$= -2|\lambda_2|t^{-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^{p+1}}^{p+1} + 2\mathrm{Re}(\mathcal{F}U(-t)v, R(t)). \tag{2.6}$$

To estimate $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^{p+1}}^{p+1}$ on the RHS of (2.6) from the below, we firstly use Hölder's inequality : $\|f\|_{\mathbf{L}^2} \leq \|f\|_{\mathbf{L}^1}^{(p-1)/2p}\|f\|_{\mathbf{L}^{p+1}}^{(p+1)/2p}$ and secondly use $\|f\|_{\mathbf{L}^1} \leq C\|f\|_{\mathbf{L}^2}^{1/2}\|\xi f\|_{\mathbf{L}^2}^{1/2}$. Considering also

$$\begin{aligned}
\|R(t)\|_{\mathbf{L}^2} &\leq Ct^{-(p-1)/2-1/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1}\|Jv\|_{\mathbf{L}^2} \\
&\leq Ct^{-p/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(p-1)/2}\|\partial_\xi\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(p-1)/2}\|Jv\|_{\mathbf{L}^2} \\
&\leq Ct^{-p/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(p-1)/2}\|Jv\|_{\mathbf{L}^2}^{(p+1)/2},
\end{aligned}$$

in which the following Sobolev embedding

$$\|f\|_{\mathbf{L}^\infty} \leq C\|f\|_{\mathbf{L}^2}^{\frac{1}{2}}\|\partial_x f\|_{\mathbf{L}^2}^{\frac{1}{2}}$$

is used, we see that

$$
\begin{aligned}
\partial_t \|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} \ \leq \ & -Ct^{-(p-1)/2}\frac{\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2}}{\|\partial_x v\|_{\mathbf{L}^2}^{(p-1)/2}} \\
& +Ct^{-p/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(p-1)/2}\|Jv\|_{\mathbf{L}^2}^{(p+1)/2}.
\end{aligned}
$$

By (2.2) and (2.3),

$$
\begin{aligned}
\partial_t \|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} \ \leq \ & -Ct^{-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} \\
& +Ct^{-p/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(p-1)/2}. \quad\quad (2.7)
\end{aligned}
$$

Young's inequality : $ab \leq h\theta a^{1/\theta} + h^{-\theta/(1-\theta)}(1-\theta)b^{1/(1-\theta)}$, where $0 < \theta < 1$ and $h > 0$, gives

$$
\begin{aligned}
& t^{-p/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(p-1)/2} \\
& \leq \ \varepsilon t^{-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} + C_\varepsilon t^{-(2p-1)(p+1)/4p}
\end{aligned}
$$

for any $\varepsilon = \frac{h(p-1)}{3p-1} > 0$. Applying this inequality to (2.7) and using $\|v\|_{\mathbf{L}^2} \leq C$, we have

$$
\begin{aligned}
\partial_t \|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} \ \leq \ & -Ct^{-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} \\
& + \ Ct^{-(2p-1)(p+1)/4p}. \quad\quad (2.8)
\end{aligned}
$$

To solve (2.8), we invoke Sunagawa's idea [4]. The quantity $t^\gamma\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}$ with $\gamma > 0$ sufficiently large satisfies the following differential inequality:

$$
\begin{aligned}
& \partial_t(t^\gamma\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}) \\
& = \ \gamma t^{\gamma-1}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} + t^\gamma\partial_t\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} \\
& \leq \ \gamma t^{\gamma-1}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} - Ct^{\gamma-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} \\
& \quad +Ct^{\gamma-(2p-1)(p+1)/4p}. \quad\quad (2.9)
\end{aligned}
$$

Young's inequality gives

$$
t^{\gamma-1}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} \leq \varepsilon t^{\gamma-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} + C_\varepsilon t^{\gamma-2p/3(p-1)},
$$

where $\varepsilon > 0$. Let $\varepsilon > 0$ be so small that $\gamma\varepsilon < C$. Then, applying this inequality to (2.9), we see that

$$
\begin{aligned}
\partial_t(t^\gamma\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}) \ \leq \ & \gamma\varepsilon t^{\gamma-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} \\
& -Ct^{\gamma-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} \\
& +\gamma C_\varepsilon t^{\gamma-2p/3(p-1)} + Ct^{\gamma-(2p-1)(p+1)/4p} \\
\leq \ & \gamma C_\varepsilon t^{\gamma-2p/3(p-1)} + Ct^{\gamma-(2p-1)(p+1)/4p} \quad\quad (2.10)
\end{aligned}
$$

Integrating (2.10) over $[1, t]$, we have

$$\|\mathcal{F}U(-t)v(t)\|_{\mathbf{L}^2}$$
$$\leq \ t^{-\gamma}\|\mathcal{F}U(-1)v(1)\|_{\mathbf{L}^2} + Ct^{1-2p/3(p-1)} + Ct^{1-(2p-1)(p+1)/4p}. \quad (2.11)$$

Since the second term of (2.11) is dominant if $p > p_0$, taking $\gamma > 0$ sufficiently large and noting that $1 - 2p/3(p-1) = -(2/3)d$ yield

$$\|\mathcal{F}U(-t)v(t)\|_{\mathbf{L}^2} \ \leq \ Ct^{-(2/3)d}$$

for $t > 1$. By Plancherell's identity, $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} = \|v\|_{\mathbf{L}^2}$. This completes the proof of Proposition 2.1. $\square$

We next derive a rough decay estimate of $\|v(t)\|_{\mathbf{L}^\infty}$ by making use of Proposition 2.1.

Proposition 2.3 (Rough L^∞-Decay). *Assume the same assumption as in Theorem 1.1. Let $v \in X_{2,\infty}$ be the global solution to (1.1). Then it satisfies*

$$\|v(t)\|_{\mathbf{L}^\infty} \leq C(1+t)^{-(4p-3)/12}, \quad (2.12)$$

for $t > 0$.

Proof of Proposition 2.3. Since

$$
\begin{aligned}
\|v\|_{\mathbf{L}^\infty} \ &= \ t^{-1/2}\|\mathcal{F}MU(-t)v\|_{\mathbf{L}^\infty} \\
&\leq \ t^{-1/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty} + t^{-1/2}\|\mathcal{F}(M-1)U(-t)v\|_{\mathbf{L}^\infty} \\
&\leq \ t^{-1/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty} \\
&\quad + Ct^{-1/2}\|\mathcal{F}(M-1)U(-t)v\|_{\mathbf{L}^2}^{1/2}\|\partial_\xi \mathcal{F}(M-1)U(-t)v\|_{\mathbf{L}^2}^{1/2} \\
&\leq \ t^{-1/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty} + Ct^{-3/4}\|Jv\|_{\mathbf{L}^2} \quad (2.13)
\end{aligned}
$$

in which the Sobolev embedding

$$\|f\|_{\mathbf{L}^\infty} \leq C\|f\|_{\mathbf{L}^2}^{\frac{1}{2}}\|\partial_x f\|_{\mathbf{L}^2}^{\frac{1}{2}}$$

is used, it is enough to observe the decay of $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}$. From (2.4) and

$$\|R(t)\|_{\mathbf{L}^\infty} \leq Ct^{-(p-1)/2-1/4}(\|\mathcal{F}MU(-t)v\|_{\mathbf{L}^\infty}^{p-1} + \|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1})\|Jv\|_{\mathbf{L}^2},$$

it follows that

$$
\begin{aligned}
\partial_t |\mathcal{F}U(-t)v| \\
\leq \ -|\lambda_2|t^{-(p-1)/2}|\mathcal{F}U(-t)v|^p \\
+ Ct^{-(p-1)/2-1/4}(\|\mathcal{F}MU(-t)v\|_{\mathbf{L}^\infty}^{p-1} + \|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1})\|Jv\|_{\mathbf{L}^2} (2.14)
\end{aligned}
$$

Since

$$\|\mathcal{F}MU(-t)v\|_{\mathbf{L}^\infty} \leq \|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty} + Ct^{-1/4}\|Jv\|_{\mathbf{L}^2}$$

holds from (2.13) and $\|Jv\|_{\mathbf{L}^2} \leq C$ due to Lemma 2.2, we have

$$\partial_t|\mathcal{F}U(-t)v| \leq -|\lambda_2|t^{-(p-1)/2}|\mathcal{F}U(-t)v|^p$$
$$+Ct^{-(p-1)/2-1/4}(\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1} + t^{-(p-1)/4}). \quad (2.15)$$

To estimate $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1}$ in (2.15), we use the Sobolev embedding :

$$\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty} \leq C\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{1/2}\|\partial_\xi\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{1/2}$$
$$= C\|v\|_{\mathbf{L}^2}^{1/2}\|Jv\|_{\mathbf{L}^2}^{1/2}.$$

Apply Proposition 2.1 to $\|v\|_{\mathbf{L}^2}^{1/2}$ and Lemma 2.2 to $\|Jv\|_{\mathbf{L}^2}^{1/2}$. Then we see that

$$\partial_t|\mathcal{F}U(-t)v| \leq -|\lambda_2|t^{-(p-1)/2}|\mathcal{F}U(-t)v|^p$$
$$+Ct^{-(p-1)/2-1/4}(t^{-(p-1)d/3} + t^{-(p-1)/4}). \quad (2.16)$$

Since $(p-1)d/3 < (p-1)/4$ and $-(p-1)/2-1/4-(p-1)d/3 = -(4p+3)/12$, (2.16) yields

$$\partial_t|\mathcal{F}U(-t)v| \leq -|\lambda_2|t^{-(p-1)/2}|\mathcal{F}U(-t)v|^p + Ct^{-(4p+3)/12}. \quad (2.17)$$

We now invoke Sunagawa's idea again. By (2.17), we see that

$$\partial_t(t^\gamma|\mathcal{F}U(-t)v|)$$
$$\leq \gamma t^{\gamma-1}|\mathcal{F}U(-t)v| - |\lambda_2|t^{\gamma-(p-1)/2}|\mathcal{F}U(-t)v|^p + Ct^{\gamma-(4p+3)/12}.(2.18)$$

Young's inequality : $t^{\gamma-1}f \leq \varepsilon t^{\gamma-(p-1)/2}f^p + C_\varepsilon t^{\gamma-1/(p-1)-1/2}$ with $\varepsilon > 0$ and $\gamma\varepsilon<|\lambda_2|$ gives

$$\partial_t(t^\gamma|\mathcal{F}U(-t)v|) \leq Ct^{\gamma-1/(p-1)-1/2} + Ct^{\gamma-(4p+3)/12}. \quad (2.19)$$

Since $p_0 < p \leq \frac{19+\sqrt{145}}{12}$, then the second term on the RHS of (2.19) is dominant for large $t > 0$. Integrating (2.19) over $[1,t]$, we have

$$\|\mathcal{F}U(-t)v(t)\|_{\mathbf{L}^\infty} \leq Ct^{-\gamma} + Ct^{-(4p-9)/12}$$
$$\leq Ct^{-(4p-9)/12}, \quad (2.20)$$

if $\gamma > 0$ is taken large enough. Combining (2.13) and (2.20), we obtain Proposition 2.3. □

In virtue of Proposition 2.3, we achieve the estimate of $\|J^2v\|_{\mathbf{L}^2}$, which is a refinement of Jin-Jin-Li's estimate.

Proposition 2.4 (Estimate of J^2v). *Assume that $p_0 < p \leq \frac{19+\sqrt{145}}{12}$. Let $v \in X_{2,\infty}$ be the global solution to (1.1). Then it satisfies*

$$\|J^2 v(t)\|_{\mathbf{L}^2} \leq C(1+t)^{p(11-4p)/6}, \tag{2.21}$$

for $t > 0$.

Proof of Proposition 2.4. By (1.2), we see that

$$
\begin{aligned}
\partial_t \|J^2 v\|_{\mathbf{L}^2} &\leq C\|v\|_{\mathbf{L}^\infty}^{p-2}\|JvJv\|_{\mathbf{L}^2} \\
&\leq C\|v\|_{\mathbf{L}^\infty}^{p-2}\|Jv\|_{\mathbf{L}^\infty}\|Jv\|_{\mathbf{L}^2} \\
&\leq Ct^{-1/2}\|v\|_{\mathbf{L}^\infty}^{p-2}\|J^2 v\|_{\mathbf{L}^2}^{1/2}\|Jv\|_{\mathbf{L}^2}^{3/2}.
\end{aligned}
\tag{2.22}
$$

Apply Proposition 2.3 to $\|v\|_{\mathbf{L}^\infty}^{p-2}$ and Lemma 2.2 to $\|Jv\|_{\mathbf{L}^2}^{3/2}$. Then we have

$$\partial_t \|J^2 v\|_{\mathbf{L}^2}^{1/2} \leq Ct^{-1/2-(p-2)(4p-3)/12}. \tag{2.23}$$

Integrating (2.23) over $[1,t]$, we obtain Proposition 2.4. □

3 Proof of Theorem 1.1

Since Proposition 2.1 proves (1.3), we are going to concentrate ourselves into the proof of (1.4).

Proof of Theorem 1.1. By (2.4) and

$$\|R(t)\|_\infty \leq Ct^{-(p-1)/2-3/4}(\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1} + t^{-(p-1)/4})\|J^2 v\|_{\mathbf{L}^2}, \tag{3.1}$$

we have

$$
\begin{aligned}
\partial_t &|\mathcal{F}U(-t)v| \\
&\leq -|\lambda_2|t^{-(p-1)/2}|\mathcal{F}U(-t)v|^p \\
&\quad +Ct^{-(p-1)/2-3/4}(\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1} + t^{-(p-1)/4})\|J^2 v\|_{\mathbf{L}^2}.
\end{aligned}
\tag{3.2}
$$

Apply (2.20) to $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1}$ and Proposition 2.4 to $\|J^2 v\|_{\mathbf{L}^2}$. Also notice that $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1}$ is dominant to $t^{-(p-1)/4}$ in (3.2). Then, after some careful computation to determine the decay order of the remainder term, we have

$$\partial_t |\mathcal{F}U(-t)v| \leq -|\lambda_2|t^{-(p-1)/2}|\mathcal{F}U(-t)v|^p + Ct^{-(12p^2-29p+12)/12}. \tag{3.3}$$

Similarly to the derivation of (2.19),

$$\partial_t(t^\gamma|\mathcal{F}U(-t)v|) \leq Ct^{\gamma-1/(p-1)-1/2} + Ct^{\gamma-(12p^2-29p+12)/12}. \quad (3.4)$$

Integrating (3.4) over $[1,t]$, we see that

$$|\mathcal{F}U(-t)v| \leq Ct^{-\gamma} + Ct^{-1/(p-1)+1/2} + Ct^{-(12p^2-29p)/12}. \quad (3.5)$$

We here want to regard $t^{-1/(p-1)+1/2}$ as a dominant term and $t^{-(12p^2-29p)/12}$ as a remainder one. Such observation becomes true if

$$-\frac{1}{p-1} + \frac{1}{2} > -\frac{12p^2-29p}{12},$$

which is equivalent to

$$12p^3 - 41p^2 + 35p - 18 > 0.$$

There is only one real root of the polynomial $12p^3 - 41p^2 + 35p - 18$, which is numerically described as $p = p_0 \approx 2.486 \cdots$. This is the main reason why the weird number p_0 is included in the assumption of Theorem 1.1. Anyway, if $p_0 < p$, (3.5) yields

$$\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty} \leq Ct^{-1/(p-1)+1/2}$$

for $t > 1$. Applying the above inequality to $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}$, we get

$$\begin{aligned}
\|v(t)\|_{\mathbf{L}^\infty} &\leq Ct^{-\frac{1}{2}}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty} + Ct^{-\frac{1}{2}}\|\mathcal{F}(M-1)U(-t)v(t)\|_{\mathbf{L}^\infty} \\
&\leq Ct^{-\frac{1}{p-1}} + Ct^{-\frac{3}{4}}\|xU(-t)v(t)\| \\
&\leq Ct^{-\frac{1}{p-1}}
\end{aligned}$$

for $t > 1$. $\square$

References

[1] J. Barab, *Nonexistence of asymptotically free solutions for a nonlinear Schrödinger equation*, J. Math. Phy., **25** (1984), 3270-3273.

[2] N. Hayashi and P. I. Naumkin, *Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations*, American Journal of Mathematics, **120** (1998), 369-389.

[3] G.Jin, Y.Jin and C.Li, *The initial value problem for nonlinear Schrödinger equations with a dissipative nonlinearity in one space dimension*,Journal of Evolution Equations, **16** (2016),no.4, 983-995.

[4] S. Katayama, C. Li and H. Sunagawa, *A remark on decay rates of solutions for a system of quadratic nonlinear Schrödinger equations in 2D*, Differential Integral Equations, **27** (2014), 301-312.

[5] N. Kita and A. Shimomura, *Large time behavior of solutions to Schrödinger equations with a dissipative nonlinearity for arbitrarily large initial data*, J. Math. Soc. Japan, **61** (2009), no.1, 39-64.

[6] A. Shimomura, *Asymptotic behavior of solutions to Schrödinger equations with dissipative nonlinearities*, Comm. Partial Differential Equations, **31** (2006), no.1, 1407-1423.

FACULTY OF ADVANCED SCIENCE AND TECHNOLOGY, KUMAMOTO UNIVERSITY
KUROKAMI 2-39-1, KUMAMOTO, 860-8555 JAPAN
    E-mail address: `nkita@kumamoto-u.ac.jp`

DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, YANBIAN UNIVERSITY
NO.977 GONGYUAN ROAD, YANJI CITY, JILIN PROVINCE,133002, CHINA
    E-mail address: `sxlch@ybu.edu.cn`