SHARP RELLICH-LERAY INEQUALITY WITH A RADIAL POWER WEIGHT FOR SOLENOIDAL FIELDS

NAOKI HAMAMOTO

Abstract. In the previous work [5], following from an idea of Costin-Maz'ya [4], we computed the best constant in Rellich-Leray inequality (with a radial power weight) for axisymmetric solenoidal fields. In the present paper, we recompute the same best constant without any symmetry assumption on solenoidal fields, and compare it with that on unconstrained fields. As a result, it turns out that the two values are distinct only when the weight exponent stays within a bounded range.

1. Introduction

Throughout this paper, we use bold letters to denote vectors in an N-dimensional Euclidean space, e.g. $\boldsymbol{x} = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$, together with the notation $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{k=1}^N x_k y_k$ for the standard scalar product of two vectors. The notation $C_c^\infty(\Omega)$ denotes the set of all smooth scalar fields (i.e., smooth functions) with compact support on any open subset Ω of $\mathbb{R}^N$. By writing $\boldsymbol{u} = (u_1, u_2, \cdots, u_N) \in C_c^\infty(\Omega)^N$, we mean

$$\boldsymbol{u} : \Omega \to \mathbb{R}^N, \qquad \boldsymbol{x} \mapsto \boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), \cdots, u_N(\boldsymbol{x}))$$

is a smooth vector field with compact support on Ω.

We are concerned with functional inequalities called the Rellich-Leray inequality for vector fields, and study how its best constant can be changed if we impose differential constraints on the test vector fields. The classical Rellich-Leray (or shortly R-L) inequality

$$\Big(\frac{N(N-4)}{4}\Big)^2 \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^4} dx \le \int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 dx \tag{1.1}$$

was found for $N \ge 5$ by F. Rellich [9], where the constant factor on the left-hand side is sharp as $\boldsymbol{u}$ runs over all vector fields in $C_c^\infty(\mathbb{R}^N)^N$. This time, we mainly treat the problem whether the best constant exceeds $\big(\frac{N(N-4)}{4}\big)^2$ by assuming that $\boldsymbol{u}$ is *solenoidal*, that is, $\operatorname{div} \boldsymbol{u} = 0$ on $\mathbb{R}^N$. In a recent article [5], the new improved R-L inequality

$$\Big(\frac{N(N-4)}{4}\Big)^2 \Big(1 + \frac{16}{N(N+8)}\Big) \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^4} dx \le \int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 dx$$

was obtained by assuming that $\boldsymbol{u}$ is solenoidal and axisymmetric. The derivation of this inequality followed from an idea of Costin-Maz'ya [4] who computed the best constant in Hardy-Leray inequality for axisymmetric solenoidal fields on $\mathbb{R}^N$. In the present paper, however, we rederive the same improved R-L inequality *without any symmetry assumption* on the solenoidal fields. Moreover, in the same fashion as the

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preceding works, we treat the solenoidal improvement of sharp Rellich inequality with a radial power weight,

$$B_{N,\gamma}\int_{\mathbb{R}^N}\frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^4}|\boldsymbol{x}|^{2\gamma}dx \le \int_{\mathbb{R}^N}|\triangle\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx \qquad \forall\gamma\in\mathbb{R},$$

found for $N\ge 2$ by Caldiroli-Musina [3] as a generalization of (1.1). Here the constant number

$$B_{N,\gamma}=\min_{\nu\in\mathbb{N}\cup\{0\}}\left(\left(\nu+\tfrac{N}{2}-1\right)^2-(\gamma-1)^2\right)^2 \tag{1.2}$$

is the best possible, as $\boldsymbol{u}$ runs over all vector fields in $C_c^\infty(\mathbb{R}^N)^N$ satisfying the integrability condition $\int_{\mathbb{R}^N}|\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-4}dx<\infty$.

Now, our main result is as follows:

Theorem 1.1. *Let $N\ge 3$ and let $\boldsymbol{u}\in C_c^\infty(\mathbb{R}^N)^N$ be a solenoidal field. Then the inequality*

$$C_{N,\gamma}\int_{\mathbb{R}^N}\frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^4}|\boldsymbol{x}|^{2\gamma}dx \le \int_{\mathbb{R}^N}|\triangle\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx \tag{1.3}$$

holds if the integral on the left-hand side is finite, where the best constant $C_{N,\gamma}$ is given by

$$C_{N,\gamma}=\begin{cases}\min\left\{\dfrac{\left(\gamma-\frac{N}{2}+1\right)^2+N-1}{\left(\gamma-\frac{N}{2}-1\right)^2+N-1}\left(\dfrac{N^2}{4}-(\gamma-2)^2\right)^2,\ \left(\dfrac{N^2}{4}-(\gamma-1)^2\right)^2\right\} & \\ \qquad\qquad\text{for } |\gamma-1|<\frac{\sqrt{(N-1)^2+1}}{2}, & \\ B_{N,\gamma} \qquad \text{otherwise}, & \end{cases}$$

in terms of the same notation $B_{N,\gamma}$ as in (1.2).

For the lower-dimensional case $N=2$, the problem of the solenoidal improvement of R-L inequality can be transformed into that of the curl-free improvement, via the isometric isomorphism mapping $\boldsymbol{u}=(u_1,u_2)\mapsto(-u_2,u_1)$ from solenoidal fields onto curl-free fields. Some results involving this topic can be seen in a recent article [7] or preprint [8], where another new best constant of R-L inequality, even for general $N\ge 2$, was obtained by assuming $\boldsymbol{u}$ to be curl-free.

According to the result of Theorem 1.1, one can check by a numerical calculation that the strict inequality $C_{N,\gamma}>B_{N,\gamma}$ holds if and only if $|\gamma-1|<\frac{1}{2}\sqrt{(N-1)^2+1}$ and $\gamma\ne 2-\frac{N}{2}$. Hence the effect of the solenoidal improvement comes out only for γ staying within a bounded range. This fact may be a new remarkable phenomenon, in contrast to the previous results related to the solenoidal (or curl-free) improvement of functional inequalities with weights.

What is more, the inequality (1.3) can be further improved by adding a nonnegative term to the left-hand side; in particular, if one wish to obtain an information about the non-attainability of the equality sign, the following fact may be useful:

Remark 1.2. *In the same setting as Theorem 1.1, the further improved inequality*

$$\int_{\mathbb{R}^N}|\triangle\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx-C_{N,\gamma}\int_{\mathbb{R}^N}\frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^4}|\boldsymbol{x}|^{2\gamma}dx\ge c_{N,\gamma}\int_{\mathbb{R}^N}\left|\boldsymbol{x}\cdot\nabla\left(|\boldsymbol{x}|^{\gamma+\frac{N}{2}-2}\boldsymbol{u}\right)\right|^2|\boldsymbol{x}|^{-N}dx \tag{1.4}$$

holds for some constant number $c_{N,\gamma}>0$.

As an easy consequence of this fact, it turns out that the equality sign of (1.3) is *not* attained by any non-trivial solenoidal fields. However, we do not know the explicit expression of the best value of the constant $c_{N,\gamma}$ or that of the best possible additional term on the right-hand side of (1.4).

The proof of Theorem 1.1 (and its Remark) is based on the use of the so-called poloidal-toroidal (or shortly PT) decomposition theorem on $\mathbb{R}^N$. This PT theorem was employed in a recent paper [6] to compute the best constant of Hardy-Leray inequality for solenoidal fields, as a higher-dimensional extension of the three-dimensional PT theorem by G. Backus [1].

The remaining content of the present paper is devoted to the proof of Theorem 1.1 and organized as follows: in Section 2, we give a quick review of the standard vector calculus on $\mathbb{R}^N\backslash\{\mathbf{0}\}$ together with some notation and definitions, and provide a brief review of the N-dimensional PT theorem. In Section 3, we give the proof of Theorem 1.1 and Remark 1.2: in a similar way to [6], we compute the best constant $C_{N,\gamma}$ by separately calculating L^2 estimates for poloidal fields and for toroidal ones.

2. Preliminary: vector calculus on $\dot{\mathbb{R}}^N \cong \mathbb{R}_+ \times \mathbb{S}^{N-1}$

This section provides a quick review of the standard vector calculus on $\dot{\mathbb{R}}^N$ in terms of radial-spherical variables and lists up some elementary formulae related to PT fields, together with additional L^2 estimates including higher-order derivatives. Since a large part of the basic facts was already fully discussed in [6], we only show them here with less or without detailed proofs. We frequently use the notations

$$\dot{\mathbb{R}}^N := \mathbb{R}^N\backslash\{\mathbf{0}\} = \{\boldsymbol{x} \in \mathbb{R}^N \ ; \ \boldsymbol{x} \neq \mathbf{0}\}, \qquad \mathbb{S}^{N-1} := \big\{\boldsymbol{x} \in \mathbb{R}^N;\ |\boldsymbol{x}| = 1\big\}$$

for the subsets of $\mathbb{R}^N$.

§2.1. **Notation and definitions.** By writing $\dot{\mathbb{R}}^N \cong \mathbb{R}_+ \times \mathbb{S}^{N-1}$, we mean that $\dot{\mathbb{R}}^N$ is a smooth manifold diffeomorphic to the product of $\mathbb{S}^{N-1}$ and the half line $\mathbb{R}_+ = \{r \in \mathbb{R} \ ; \ r > 0\}$, and that every point $\boldsymbol{x}$ in $\dot{\mathbb{R}}^N$ is uniquely expressed in terms of the radius $r > 0$ and the unit vector $\boldsymbol{\sigma} \in \mathbb{S}^{N-1}$ by the formula $\boldsymbol{x} = r\boldsymbol{\sigma}$, which clearly gives the inverse transformation formulae $r = |\boldsymbol{x}|$ and $\boldsymbol{\sigma} = \boldsymbol{x}/|\boldsymbol{x}|$.

For every vector field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) : \dot{\mathbb{R}}^N \to \mathbb{R}^N$, there exists a unique pair of a scalar field u_R and a vector field $\boldsymbol{u}_S$ satisfying

$$\boldsymbol{u} = \boldsymbol{\sigma} u_R + \boldsymbol{u}_S \quad \text{and} \quad \boldsymbol{\sigma}\cdot\boldsymbol{u}_S = 0 \qquad \text{on } \dot{\mathbb{R}}^N, \tag{2.1}$$

and these fields have the explicit expressions $u_R = \boldsymbol{\sigma}\cdot\boldsymbol{u}$ and $\boldsymbol{u}_S = \boldsymbol{u} - \boldsymbol{\sigma}u_R$ which we call the radial component and the spherical part of $\boldsymbol{u}$, respectively.

The gradient operator $\nabla = \big(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_N}\big)$ and the Laplacian $\triangle = \sum_{k=1}^N (\frac{\partial}{\partial x_k})^2$ can be decomposed into radial-spherical parts as

$$\nabla = \boldsymbol{\sigma}\partial_r + \frac{1}{r}\nabla_\sigma \quad \text{and} \quad \triangle = \partial_r'\partial_r + \triangle_\sigma \tag{2.2}$$

in terms of the notations ∇_σ and $\triangle_\sigma$ denoting the spherical gradient and spherical Laplacian (or Laplace-Beltrami operator on $\mathbb{S}^{N-1}$), respectively, where

$$\partial_r := \boldsymbol{\sigma}\cdot\nabla = \sum_{k=1}^N \frac{x_k}{|\boldsymbol{x}|}\frac{\partial}{\partial x_k} \quad \text{and} \quad \partial_r' = \partial_r + \frac{N-1}{r} \tag{2.3}$$

are the radial derivative and its skew L^2 adjoint in the sense that

$$\int_{\mathbb{R}^N} f\partial_r g\,dx = -\int_{\mathbb{R}^N} (\partial_r' f) g\,dx \qquad \forall f, g \in C^\infty(\dot{\mathbb{R}}^N).$$

As a simple application of the formulae (2.2) to the scalar field $r = |\boldsymbol{x}|$ and its s-th power ($\forall s \in \mathbb{R}$), it is easy to check that $\nabla r = \boldsymbol{\sigma}$ and

$$\triangle r^s = \alpha_s r^{s-2}, \qquad \text{where} \quad \alpha_s := s(s+N-2). \tag{2.4}$$

Also applying the gradient formula in (2.2) to a vector field $\boldsymbol{u}$ in (2.1) and taking the trace part of the matrix field $\nabla\boldsymbol{u}$, one can verify that

$$\operatorname{div}\boldsymbol{u} = \partial_r' u_R + \nabla_\sigma \cdot \boldsymbol{u}_S$$

as a radial-spherical expression formula of the divergence, where $\nabla_\sigma \cdot \boldsymbol{u}_S$ is defined as the trace part of the matrix field $\nabla_\sigma \boldsymbol{u}_S$. By using this fact, we further obtain $\nabla_\sigma \cdot \nabla_\sigma f = \triangle_\sigma f$ and the spherical integration by parts formula

$$\int_{\mathbb{S}^{N-1}} \boldsymbol{u} \cdot \nabla_\sigma f \,\mathrm{d}\sigma = -\int_{\mathbb{S}^{N-1}} (\nabla_\sigma \cdot \boldsymbol{u}_S) f \,\mathrm{d}\sigma \qquad \forall f \in C^\infty(\mathbb{S}^{N-1})$$

for any fixed radius. By computing the commutation relation between $\triangle_\sigma$ and $\boldsymbol{\sigma}$ or ∇_σ, one can derive the operator identities:

$$\begin{cases} \triangle_\sigma(\boldsymbol{\sigma} f) - \boldsymbol{\sigma}\triangle_\sigma f = -(N-1)\boldsymbol{\sigma} f + 2\nabla_\sigma f, \\ \triangle_\sigma \nabla_\sigma f - \nabla_\sigma \triangle_\sigma f = -2\boldsymbol{\sigma}\triangle_\sigma f + (N-3)\nabla_\sigma f. \end{cases} \tag{2.5}$$

The proof of these identities can be seen in [7, 8] as well as in [6].

§2.2. **Poloidal-toroidal fields.** A vector field $\boldsymbol{u} : \dot{\mathbb{R}}^N \to \mathbb{R}^N$ is said to be

$$\begin{cases} \textit{pre-poloidal} \text{ if } \quad \boldsymbol{u}_S = \nabla_\sigma f \quad \text{for some } f \in C^\infty(\dot{\mathbb{R}}^N) \\ \qquad \textit{toroidal} \text{ if } \quad u_R \equiv 0 \end{cases}.$$

We denote by $\mathcal{P}(\dot{\mathbb{R}}^N)$ resp. $\mathcal{T}(\dot{\mathbb{R}}^N)$ the set of all pre-poloidal resp. toroidal fields. These sets are linear spaces and satisfies the following elementary fact:

Proposition 2.1. *The spaces of pre-poloidal and toroidal fields are $L^2(\mathbb{S}^{N-1})$ orthogonal and invariant under radial operators and Laplacians, in the sense that*

$$\int_{\mathbb{S}^{N-1}} \boldsymbol{v}(r\boldsymbol{\sigma}) \cdot \boldsymbol{w}(r\boldsymbol{\sigma})\mathrm{d}\sigma = \int_{\mathbb{S}^{N-1}} \nabla\boldsymbol{v}(r\boldsymbol{\sigma}) \cdot \nabla\boldsymbol{w}(r\boldsymbol{\sigma})\mathrm{d}\sigma = 0 \qquad \forall r > 0,$$

$$\{\zeta\boldsymbol{v}, \partial_r\boldsymbol{v}, \triangle\boldsymbol{v}, \triangle_\sigma\boldsymbol{v}\} \subset \mathcal{P}(\dot{\mathbb{R}}^N) \quad \textit{and} \quad \{\zeta\boldsymbol{w}, \partial_r\boldsymbol{w}, \triangle\boldsymbol{w}, \triangle_\sigma\boldsymbol{w}\} \subset \mathcal{T}(\dot{\mathbb{R}}^N)$$

whenever $\boldsymbol{v} \in \mathcal{P}(\dot{\mathbb{R}}^N)$ and $\boldsymbol{w} \in \mathcal{T}(\dot{\mathbb{R}}^N)$. Here $\zeta \in C^\infty(\dot{\mathbb{R}}^N)$ is any radially symmetric scalar field. Additionally, every toroidal field has zero-spherical mean:

$$\int_{\mathbb{S}^{N-1}} \boldsymbol{w}(r\boldsymbol{\sigma})\mathrm{d}\sigma = 0 \qquad \forall r > 0, \quad \forall \boldsymbol{w} \in \mathcal{T}(\dot{\mathbb{R}}^N). \tag{2.6}$$

A vector field is said to be solenoidal if it is divergence-free. Hence all toroidal fields are solenoidal; we say that a pre-poloidal field is *poloidal* whenever it is solenoidal. The second-order differential operator

$$\boldsymbol{D} := \boldsymbol{\sigma}\triangle_\sigma - r\partial_r'\nabla_\sigma, \tag{2.7}$$

which we call the *poloidal generator*, maps every scalar field to a poloidal field. The following fact serves as a fundamental tool:

Proposition 2.2. *Let $\boldsymbol{u} : \mathbb{R}^N \to \mathbb{R}^N$ be a solenoidal field smoothly defined on the entire space $\mathbb{R}^N$. Then there exists an unique pair of poloidal-toroidal fields $(\boldsymbol{u}_P, \boldsymbol{u}_T) \in \mathcal{P}(\dot{\mathbb{R}}^N) \times \mathcal{T}(\dot{\mathbb{R}}^N)$ satisfying*

$$\boldsymbol{u} = \boldsymbol{u}_P + \boldsymbol{u}_T \qquad \text{on } \dot{\mathbb{R}}^N,$$

and moreover these two fields have the expression

$$\boldsymbol{u}_P = \boldsymbol{D} f \quad \text{and} \quad \boldsymbol{u}_T = \boldsymbol{u}_S + r\partial_r'\nabla_\sigma f \qquad \text{on } \dot{\mathbb{R}}^N.$$

Here the scalar field $f = \triangle_\sigma^{-1} u_R$, which we call the poloidal potential of $\boldsymbol{u}$, is an unique solution to the Poisson-Beltrami equation $\triangle_\sigma f = u_R$ (on $\dot{\mathbb{R}}^N$) under the zero-spherical-mean constraint $\int_{\mathbb{S}^{N-1}} f(r\boldsymbol{\sigma})\mathrm{d}\sigma = 0$, $\forall r > 0$.

§2.3. **Some $L^2(\mathbb{S}^{N-1})$ estimates for poloidal-toroidal fields.** First we compute $L^2(\mathbb{S}^{N-1})$-deviation estimates of poloidal fields and their derivatives, for a perturbation of poloidal potentials by a radial scalar multiplication:

Lemma 2.3. *For any scalar field f on $\dot{\mathbb{R}}^N$, we abbreviate as $f = f(r\boldsymbol{\sigma})$ for any $(r,\boldsymbol{\sigma}) \in \mathbb{R}_+ \times \mathbb{S}^{N-1}$. Then there exists $C > 0$ depending only on N such that the inequalities*

$$\begin{aligned}
&C\int_{\mathbb{S}^{N-1}} |\boldsymbol{D}(\zeta f) - \zeta \boldsymbol{D} f|^2 \mathrm{d}\sigma \le (\partial\zeta)^2 \int_{\mathbb{S}^{N-1}} |\boldsymbol{D} f|^2 \mathrm{d}\sigma,\\
&C\int_{\mathbb{S}^{N-1}} |\nabla\boldsymbol{D}(\zeta f) - \zeta \nabla\boldsymbol{D} f|^2 \mathrm{d}\sigma \le \big((\partial\zeta)^2 + (\partial^2\zeta)^2\big) \int_{\mathbb{S}^{N-1}} \frac{|\boldsymbol{D} f|^2}{r^2} \mathrm{d}\sigma,\\
&C\int_{\mathbb{S}^{N-1}} |\triangle\boldsymbol{D}(\zeta f) - \zeta \triangle\boldsymbol{D} f|^2 \mathrm{d}\sigma \le (\partial\zeta)^2 \int_{\mathbb{S}^{N-1}} \frac{|\nabla\boldsymbol{D} f|^2}{r^2} \mathrm{d}\sigma\\
&\qquad\qquad + \Big((\partial\zeta)^2 + (\partial^2\zeta)^2 + (\partial^3\zeta)^2\Big) \int_{\mathbb{S}^{N-1}} \frac{|\boldsymbol{D} f|^2}{r^4} \mathrm{d}\sigma
\end{aligned}$$

hold for any radially symmetric scalar field ζ on $\dot{\mathbb{R}}^N$. Here the integrals are taken for any fixed radius $r > 0$, and here we use the notation $\partial = r\partial_r$ for the non-dimensional radial derivative.

Proof. First of all, notice from (2.3) and (2.7) that the poloidal generator has the expression

$$\boldsymbol{D} = \boldsymbol{\sigma}\triangle_\sigma - (\partial + N - 1)\nabla_\sigma \tag{2.8}$$

in terms of the non-dimensional derivative, and we easily get the identity

$$\boldsymbol{D}(\zeta f) - \zeta\, \boldsymbol{D} f = -(\partial\zeta)\nabla_\sigma f. \tag{2.9}$$

In order to estimate the $L^2(\mathbb{S}^{N-1})$ integral of this field, we see that the inequalities

$$(N-1)\int_{\mathbb{S}^{N-1}} |\nabla_\sigma f|^2 \mathrm{d}\sigma \le \int_{\mathbb{S}^{N-1}} (\triangle_\sigma f)^2 \mathrm{d}\sigma \le \int_{\mathbb{S}^{N-1}} |\boldsymbol{D} f|^2 \mathrm{d}\sigma \tag{2.10}$$

always hold true, where the first inequality is easy to check by using the spherical harmonics expansion of f, and the second follows clearly from $|\triangle_\sigma f| = |(\boldsymbol{D} f)_R| \le |\boldsymbol{D} f|$. Hence the $L^2(\mathbb{S}^{N-1})$ integration of (2.9) yields

$$\int_{\mathbb{S}^{N-1}} |\boldsymbol{D}(\zeta f) - \zeta \boldsymbol{D} f|^2 \mathrm{d}\sigma = (\partial\zeta)^2 \int_{\mathbb{S}^{N-1}} |\nabla_\sigma f|^2 \mathrm{d}\sigma \le \frac{(\partial\zeta)^2}{N-1} \int_{\mathbb{S}^{N-1}} |\boldsymbol{D} f|^2 \,\mathrm{d}\sigma,$$

and this proves the first inequality of the lemma. To prove the second, take the operation of $r\nabla$ on both sides of (2.9), and a direct calculation by using the Leibniz rule yields

$$\begin{aligned}
r\big(\nabla\boldsymbol{D}(\zeta f) - \zeta\nabla\boldsymbol{D} f\big) &= r\Big((\nabla\zeta)\boldsymbol{D} f - (\nabla\partial\zeta)\nabla_\sigma f - (\partial\zeta)\nabla\nabla_\sigma f\Big)\\
&= \boldsymbol{\sigma}(\partial\zeta)\boldsymbol{D} f - \boldsymbol{\sigma}(\partial^2\zeta)\nabla_\sigma f - (\partial\zeta)\big(\boldsymbol{\sigma}\partial\nabla_\sigma f + \nabla_\sigma\nabla_\sigma f\big)\\
&= \boldsymbol{\sigma}\Big((\partial\zeta)\boldsymbol{D} f - (\partial^2\zeta)\nabla_\sigma f + (\partial\zeta)\big((\boldsymbol{D} f)_S + (N-1)\nabla_\sigma f\big)\Big) - (\partial\zeta)\nabla_\sigma\nabla_\sigma f\\
&= \boldsymbol{\sigma}\Big((\partial\zeta)\big(\boldsymbol{D} f + (\boldsymbol{D} f)_S\big) + \big((N-1)\partial\zeta - \partial^2\zeta\big)\nabla_\sigma f\Big) - (\partial\zeta)\nabla_\sigma\nabla_\sigma f.
\end{aligned}$$

In order to estimate the $L^2(\mathbb{S}^{N-1})$ integral of this field, notice that the $L^2(\mathbb{S}^{N-1})$ integration by parts of the last term gives

$$\begin{aligned}\int_{\mathbb{S}^{N-1}} |\nabla_\sigma \nabla_\sigma f|^2 \mathrm{d}\sigma &= -\int_{\mathbb{S}^{N-1}} (\nabla_\sigma f)\cdot(\triangle_\sigma \nabla_\sigma f)\mathrm{d}\sigma \\ &= -\int_{\mathbb{S}^{N-1}} (\nabla_\sigma f)\cdot\Big(\nabla_\sigma\big((\triangle_\sigma + N-3)f\big)\Big)\mathrm{d}\sigma \\ &= \int_{\mathbb{S}^{N-1}} \Big((\triangle_\sigma f)^2 - (N-3)|\nabla_\sigma f|^2\Big)\mathrm{d}\sigma,\end{aligned}$$

where the second equality follows from (2.5); then we have

$$\begin{aligned} r^2 \int_{\mathbb{S}^{N-1}} &\big|\nabla \boldsymbol{D}(\zeta f) - \zeta \nabla \boldsymbol{D} f\big|^2 \mathrm{d}\sigma \\ &= \int_{\mathbb{S}^{N-1}} \Big|(\partial\zeta)\big(\boldsymbol{D}f + (\boldsymbol{D}f)_S\big) + \big((N-1)(\partial\zeta) - (\partial^2\zeta)\big)\nabla_\sigma f\Big|^2 \mathrm{d}\sigma \\ &\quad + (\partial\zeta)^2 \int_{\mathbb{S}^{N-1}} \Big((\triangle_\sigma f)^2 - (N-3)|\nabla_\sigma f|^2\Big)\mathrm{d}\sigma \\ &\lesssim (\partial\zeta)^2 \int_{\mathbb{S}^{N-1}} |\boldsymbol{D}f|^2 \mathrm{d}\sigma + \big((\partial\zeta)^2 + (\partial^2\zeta)^2\big) \int_{\mathbb{S}^{N-1}} \big(|\nabla_\sigma f|^2 + (\triangle_\sigma f)^2\big)\,\mathrm{d}\sigma \\ &\lesssim \big((\partial\zeta)^2 + (\partial^2\zeta)^2\big) \int_{\mathbb{S}^{N-1}} |\boldsymbol{D}f|^2 \mathrm{d}\sigma \end{aligned}$$

with the aid of (2.10), where and hereafter the notation "$\lesssim$" means that

$$x \lesssim y \quad :\Longleftrightarrow \quad x \le Cy \qquad \text{for some constant } C > 0 \text{ depending only on } N.$$

Therefore, we have proved the second inequality of the lemma. To prove the third, notice from the identities

$$\begin{cases} r\partial_r' = \partial + N - 1, \\ r^2 \partial_r' \partial_r = r\partial_r'(r\partial_r) - r\partial_r = \partial^2 + (N-2)\partial \end{cases}$$

that the Laplacian has the non-dimensional expression

$$r^2 \triangle = \partial^2 + (N-2)\partial + \triangle_\sigma, \tag{2.11}$$

and we start with checking the following calculation:

$$-\partial \nabla_\sigma f = (\boldsymbol{D}f)_S + (N-1)\nabla_\sigma f, \tag{2.12}$$

$$\begin{aligned} r^2 \triangle \nabla_\sigma f &= \big(\partial^2 + (N-2)\partial\big)\,\nabla_\sigma f + \triangle_\sigma \nabla_\sigma f \\ &= \partial\big((\partial + N - 1)\nabla_\sigma f\big) - \partial \nabla_\sigma f - 2\boldsymbol{\sigma}\triangle_\sigma f + \nabla_\sigma\big((\triangle_\sigma + N - 3)f\big) \\ &= -\partial(\boldsymbol{D}f)_S + (\boldsymbol{D}f)_S - 2\boldsymbol{\sigma}\triangle_\sigma f + (2N-4)\nabla_\sigma f + \nabla_\sigma \triangle_\sigma f \\ &= -(\partial \boldsymbol{D}f)_S + 3(\boldsymbol{D}f)_S - 2\boldsymbol{D}f + (2N-4)\nabla_\sigma f + \nabla_\sigma \triangle_\sigma f, \end{aligned}$$

where the third last equality holds again by using (2.5) and the second last follows from (2.12). Then, following from the Leibniz rule, we further check that

$$\begin{aligned} r^2 \triangle\big((\partial\zeta)\nabla_\sigma f\big) &= (r^2 \triangle \partial\zeta)\nabla_\sigma f + 2(\partial^2\zeta)\partial\nabla_\sigma f + (\partial\zeta) r^2 \triangle \nabla_\sigma f \\ &= \big(\partial^3\zeta + (N-2)\partial^2\zeta\big)\,\nabla_\sigma f - 2\,\big(\partial^2\zeta\big)\,\big((\boldsymbol{D}f)_S + (N-1)\nabla_\sigma f\big) \\ &\quad + (\partial\zeta)\Big(-(\partial\boldsymbol{D}f)_S + 3(\boldsymbol{D}f)_S - 2\boldsymbol{D}f + (2N-4)\nabla_\sigma f + \nabla_\sigma \triangle_\sigma f\Big) \\ &= -2(\partial\zeta)\boldsymbol{D}f - (\partial\zeta)(\partial\boldsymbol{D}f)_S + \big(3\,\partial\zeta - 2\partial^2\zeta\big)\,(\boldsymbol{D}f)_S \\ &\quad + \Big(\partial^3\zeta - N\partial^2\zeta + (2N-4)\partial\zeta\Big)\nabla_\sigma f + (\partial\zeta)\nabla_\sigma \triangle_\sigma f. \end{aligned}$$

By using this fact, we get the following calculation: the operation of $r^2\triangle =$ (2.11) on both sides of (2.9) (together with the Leibniz rule) yields

$$\begin{aligned}
&r^2\big(\triangle \boldsymbol{D}(\zeta f)-\zeta\triangle\boldsymbol{D}f\big)\\
&\quad=(r^2\triangle\zeta)\boldsymbol{D}f+2(\partial\zeta)\partial\boldsymbol{D}f-r^2\triangle((\partial\zeta)\nabla_\sigma f)\\
&\quad=\big(\partial^2\zeta+N\partial\zeta\big)\,\boldsymbol{D}f+(\partial\zeta)\big(2\,\partial\boldsymbol{D}f+(\partial\boldsymbol{D}f)_S\big)-\big(3\,\partial\zeta-2\partial^2\zeta\big)\,(\boldsymbol{D}f)_S\\
&\qquad-\Big(\partial^3\zeta-N\partial^2\zeta+(2N-4)\partial\zeta\Big)\nabla_\sigma f-(\partial\zeta)\nabla_\sigma\triangle_\sigma f.
\end{aligned}$$

Therefore, the $L^2(\mathbb{S}^{N-1})$ norm of this vector field has the estimate

$$\begin{aligned}
&r^4\int_{\mathbb{S}^{N-1}}|\triangle\boldsymbol{D}(\zeta f)-\zeta\triangle\boldsymbol{D}f|^2\mathrm{d}\sigma\\
&\qquad\lesssim\big((\partial\zeta)^2+(\partial^2\zeta)^2\big)\int_{\mathbb{S}^{N-1}}|\boldsymbol{D}f|^2\mathrm{d}\sigma+(\partial\zeta)^2\int_{\mathbb{S}^{N-1}}|\partial\boldsymbol{D}f|^2\mathrm{d}\sigma\\
&\qquad\quad+\big((\partial\zeta)^2+(\partial^2\zeta)^2+(\partial^3\zeta)^2\big)\int_{\mathbb{S}^{N-1}}|\nabla_\sigma f|^2\mathrm{d}\sigma+(\partial\zeta)^2\int_{\mathbb{S}^{N-1}}|\nabla_\sigma\triangle_\sigma f|^2\,\mathrm{d}\sigma\\
&\qquad\lesssim\big((\partial\zeta)^2+(\partial^2\zeta)^2+(\partial^3\zeta)^2\big)\int_{\mathbb{S}^{N-1}}|\boldsymbol{D}f|^2\mathrm{d}\sigma\\
&\qquad\quad+(\partial\zeta)^2r^2\int_{\mathbb{S}^{N-1}}|\nabla\boldsymbol{D}f|^2\mathrm{d}\sigma+(\partial\zeta)^2\int_{\mathbb{S}^{N-1}}|\nabla_\sigma\triangle_\sigma f|^2\,\mathrm{d}\sigma \qquad (2.13)
\end{aligned}$$

with the aid of (2.10). In order to estimate the last integral, we notice from (2.5) that

$$\begin{aligned}
\triangle_\sigma\boldsymbol{D}f&=\triangle_\sigma\left(\boldsymbol{\sigma}\triangle_\sigma f\right)-(\partial+N-1)\triangle_\sigma\nabla_\sigma f\\
&=\boldsymbol{\sigma}(\triangle_\sigma-N+1)\triangle_\sigma f+2\nabla_\sigma\triangle_\sigma f\\
&\quad+(\partial+N-1)\Big(2\boldsymbol{\sigma}\triangle_\sigma f-\nabla_\sigma\big((\triangle_\sigma+N-3)f\big)\Big)\\
&=\boldsymbol{\sigma}\big(\triangle_\sigma^2 f+(2\partial+N-1)\,\triangle_\sigma f\big)\\
&\quad+\nabla_\sigma\Big(2\triangle_\sigma f-(\partial+N-1)\big((\triangle_\sigma+N-3)f\big)\Big), \qquad (2.14)
\end{aligned}$$

and hence (from integration by parts) that

$$\begin{aligned}
&\int_{\mathbb{S}^{N-1}}|\nabla_\sigma\triangle_\sigma f|^2\mathrm{d}\sigma-\int_{\mathbb{S}^{N-1}}|\nabla_\sigma\boldsymbol{D}f|^2\,\mathrm{d}\sigma\\
&\quad=\int_{\mathbb{S}^{N-1}}\boldsymbol{D}f\cdot\triangle_\sigma\boldsymbol{D}f\,\mathrm{d}\sigma-\int_{\mathbb{S}^{N-1}}(\triangle_\sigma f)\triangle_\sigma^2 f\,\mathrm{d}\sigma\\
&\quad=\int_{\mathbb{S}^{N-1}}(\triangle_\sigma f)\big(\triangle_\sigma^2 f+(2\partial+N-1)\,\triangle_\sigma f\big)\mathrm{d}\sigma-\int_{\mathbb{S}^{N-1}}(\triangle_\sigma f)\triangle_\sigma^2 f\,\mathrm{d}\sigma\\
&\qquad-\int_{\mathbb{S}^{N-1}}\big((\partial+N-1)\nabla_\sigma f\big)\cdot\nabla_\sigma\Big(2\triangle_\sigma f-(\partial+N-1)\big((\triangle_\sigma+N-3)f\big)\Big)\mathrm{d}\sigma\\
&\quad=\int_{\mathbb{S}^{N-1}}(\triangle_\sigma f)(2\partial+N-1)\triangle_\sigma f\mathrm{d}\sigma\\
&\qquad+\int_{\mathbb{S}^{N-1}}\big((\partial+N-1)\triangle_\sigma f\big)\Big(2\triangle_\sigma f-(\partial+N-1)\big((\triangle_\sigma+N-3)f\big)\Big)\mathrm{d}\sigma\\
&\quad\lesssim\int_{\mathbb{S}^{N-1}}\Big((\partial\triangle_\sigma f)^2+|\partial\nabla_\sigma f|^2+(\triangle_\sigma f)^2+|\nabla_\sigma f|^2\Big)\mathrm{d}\sigma\\
&\quad\lesssim\int_{\mathbb{S}^{N-1}}|\partial\boldsymbol{D}f|^2\mathrm{d}\sigma+\int_{\mathbb{S}^{N-1}}|\boldsymbol{D}f|^2\mathrm{d}\sigma
\end{aligned}$$

with the aid of (2.10) and (2.12). Therefore, we get

$$\int_{\mathbb{S}^{N-1}} |\nabla_\sigma \triangle_\sigma f|^2 \mathrm{d}\sigma \lesssim r^2 \int_{\mathbb{S}^{N-1}} |\nabla \boldsymbol{D} f|^2 \mathrm{d}\sigma + \int_{\mathbb{S}^{N-1}} |\boldsymbol{D} f|^2 \mathrm{d}\sigma.$$

Finally, combine this inequality with the last term of (2.13), and we arrive at the desired last inequality of the lemma. □

The next lemma gives an $L^2(\mathbb{S}^{N-1})$-spectral estimate for toroidal fields:

Lemma 2.4. *Let $Q(\,\cdot\,)$ be a polynomial in one variable such that $\inf_{a\geq 0} Q(a) > -\infty$. We abbreviate as $\boldsymbol{u} = \boldsymbol{u}(r\boldsymbol{\sigma})$ for $r > 0$ and $\boldsymbol{\sigma} \in \mathbb{S}^{N-1}$. Then the inequality*

$$\int_{\mathbb{S}^{N-1}} \boldsymbol{u} \cdot \big(Q(-\triangle_\sigma)\boldsymbol{u}\big) \mathrm{d}\sigma \geq \min_{\nu\in\mathbb{N}} Q(\alpha_\nu) \int_{\mathbb{S}^{N-1}} |\boldsymbol{u}|^2 \mathrm{d}\sigma$$

holds for all $\boldsymbol{u} \in \mathcal{T}(\dot{\mathbb{R}}^N)$ with the same notation $\alpha_\nu = \nu(\nu + N - 2)$ in (2.4), where the integrals are taken for any fixed $r > 0$. Moreover, the equality sign of this inequality is attained if and only if $Q(-\triangle_\sigma)\boldsymbol{u} = \min_{\nu\in\mathbb{N}} Q(\alpha_\nu)\boldsymbol{u}$ on $\mathbb{S}^{N-1}$.

Proof. The key idea of the proof is the use of the spherical mean property (2.6) of toroidal fields: we find from $\int_{\mathbb{S}^{N-1}} \boldsymbol{u}\, \mathrm{d}\sigma = 0$ that the spherical harmonics decomposition of the toroidal field $\boldsymbol{u}$ has the expression

$$\boldsymbol{u} = \sum_{\nu\in\mathbb{N}} \boldsymbol{u}_\nu, \qquad -\triangle_\sigma \boldsymbol{u}_\nu = \alpha_\nu \boldsymbol{u}_\nu \quad (\nu \in \mathbb{N}).$$

Then we have

$$\begin{aligned}\int_{\mathbb{S}^{N-1}} \boldsymbol{u} \cdot Q(-\triangle_\sigma)\boldsymbol{u}\, \mathrm{d}\sigma &= \sum_{\nu\in\mathbb{N}} Q(\alpha_\nu) \int_{\mathbb{S}^{N-1}} |\boldsymbol{u}_\nu|^2 \mathrm{d}\sigma \\ &\geq \min_{\nu\in\mathbb{N}} Q(\alpha_\nu) \sum_{\nu\in\mathbb{N}} \int_{\mathbb{S}^{N-1}} |\boldsymbol{u}_\nu|^2 \mathrm{d}\sigma = Q_0 \int_{\mathbb{S}^{N-1}} |\boldsymbol{u}|^2 \mathrm{d}\sigma\end{aligned}$$

together with the abbreviation $Q_0 = \min_{\nu\in\mathbb{N}} Q(\alpha_\nu)$, where the second equality holds if and only if

$$\boldsymbol{u}_{\nu'} \equiv \boldsymbol{0} \qquad \forall \nu' \in \{\nu \in \mathbb{N}\ ;\ Q(\alpha_\nu) \neq Q_0\}$$

or equivalently $Q(-\triangle_\sigma)\boldsymbol{u} = Q_0\boldsymbol{u}$ on $\mathbb{S}^{N-1}$. Hence the proof is done. □

3. Proof of Theorem 1.1

Throughout this section, we assume $\boldsymbol{u} \not\equiv \boldsymbol{0}$, since otherwise there is nothing to prove. The integrability condition $\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-4} dx < \infty$ together with the smoothness of $\boldsymbol{u}$ implies that there exists an integer $k > -\gamma - \frac{N}{2} + 2$ satisfying

$$\boldsymbol{u}(\boldsymbol{x}) = O(|\boldsymbol{x}|^k) \quad \text{ as } \ |\boldsymbol{x}| \to 0,$$

and hence we get the additional integrability conditions:

$$\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{u}|^{2\gamma-2} dx < \infty \quad \text{ and } \quad \int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx < \infty.$$

§3.1. **Reduction to the case of PT fields with compact support on $\dot{\mathbb{R}}^N$.** Let $\boldsymbol{u} = \boldsymbol{u}_P + \boldsymbol{u}_T$ be the PT decomposition of $\boldsymbol{u}$ as shown in Proposition 2.2. Then the $L^2(\mathbb{S}^{N-1})$ orthogonality of this decomposition (in the sense of Proposition 2.1) implies that the ratio of the two integrals in R-L inequality (1.3) can be expressed as

$$\frac{\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-4} dx} = \frac{\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}_P|^2 |\boldsymbol{x}|^{2\gamma} dx + \int_{\mathbb{R}^N} |\triangle \boldsymbol{u}_T|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}_P|^2 |\boldsymbol{x}|^{2\gamma-4} dx + \int_{\mathbb{R}^N} |\boldsymbol{u}_T|^2 |\boldsymbol{x}|^{2\gamma-4} dx}.$$

Taking the infimum over all solenoidal fields $\boldsymbol{u}$ satisfying the integrability condition, we find that the best constant $C_{N,\gamma}$ of R-L inequality for solenoidal fields has the expression

$$C_{N,\gamma} = \inf_{\substack{\boldsymbol{u}\not\equiv\boldsymbol{0},\\ \operatorname{div}\boldsymbol{u}=0}} \frac{\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-4} dx} = \{C_{P,N,\gamma}, C_{T,N,\gamma}\} \tag{3.1}$$

in terms of the notation

$$C_{P,N,\gamma} := \inf_{\substack{\boldsymbol{u}_P\not\equiv\boldsymbol{0},\\ \operatorname{div}\boldsymbol{u}=0}} \frac{\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}_P|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}_P|^2 |\boldsymbol{x}|^{2\gamma-4} dx} = \inf_{\boldsymbol{u}\in\mathcal{P}} \frac{\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-4} dx}$$

$$\text{resp.}\quad C_{T,N,\gamma} := \inf_{\substack{\boldsymbol{u}_T\not\equiv\boldsymbol{0},\\ \operatorname{div}\boldsymbol{u}=0}} \frac{\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}_T|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}_T|^2 |\boldsymbol{x}|^{2\gamma-4} dx} = \inf_{\boldsymbol{u}\in\mathcal{T}} \frac{\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-4} dx}$$

denoting the best constant of the R-L inequality for poloidal resp. toroidal fields, where the abbreviation $\boldsymbol{u} \in \mathcal{P}$ resp. $\boldsymbol{u} \in \mathcal{T}$ under the inf means that $\boldsymbol{u}$ runs over all poloidal resp. toroidal fields satisfying the integrability condition as well as $\boldsymbol{u} \not\equiv \boldsymbol{0}$. Thus the evaluation of $C_{N,\gamma}$ can be reduced to that of $C_{P,N,\gamma}$ and $C_{T,N,\gamma}$.

Furthermore, we can assume that $\boldsymbol{u}$ is compactly supported on $\dot{\mathbb{R}}^N$. Indeed, by making full use of the integrability condition on $\boldsymbol{u}$, let us construct a sequence of solenoidal fields $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\dot{\mathbb{R}}^N)$ satisfying the approximation:

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\boldsymbol{u}_n|^2 |\boldsymbol{x}|^{2\gamma-4} dx \to \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-4} dx,\\
&\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}_n|^2 |\boldsymbol{x}|^{2\gamma-2} dx \to \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx,\\
&\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}_n|^2 |\boldsymbol{x}|^{2\gamma} dx \to \int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx.
\end{aligned}$$

To this end, let $f = \triangle_\sigma^{-1} u_R$ be the poloidal potential of $\boldsymbol{u}$ and recall that the PT decomposition of $\boldsymbol{u}$ has the expression

$$\boldsymbol{u} = \boldsymbol{u}_P + \boldsymbol{u}_T, \quad \boldsymbol{u}_P = \boldsymbol{D} f$$

in terms of the poloidal generator. Define $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\dot{\mathbb{R}}^N)^N$ as a sequence of solenoidal fields by the formula

$$\boldsymbol{u}_n = \boldsymbol{D}\big(\zeta_n f\big) + \zeta_n \boldsymbol{u}_T \qquad (\forall n \in \mathbb{N}),$$

where $\{\zeta_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\dot{\mathbb{R}}^N)$ are radially symmetric scalar fields given by

$$\zeta_n(\boldsymbol{x}) = \zeta_0\left(\frac{1}{n}\log|\boldsymbol{x}|\right) \qquad \forall n \in \mathbb{N}$$

together with a fixed function $\zeta_0 \in C_c^\infty(\mathbb{R})$ such that $\zeta_0(0) = 1$. In this setting, apply Lemma 2.3 to the case $\zeta = \zeta_n$; then the integral calculation by using Propositions 2.1 and 2.2 yields

$$\begin{aligned}
&C\int_{\mathbb{R}^N} |\boldsymbol{u}_n - \zeta_n \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-4} dx \le \int_{\mathbb{R}^N} (\partial\zeta_n)^2 |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-4} dx,\\
&C\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}_n - \zeta_n \nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx \le \big((\partial\zeta_n)^2 + (\partial^2\zeta_n)^2\big) \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-4} dx,\\
&C\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}_n - \zeta_n \triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \le \int_{\mathbb{R}^N} (\partial\zeta_n)^2 |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx\\
&\qquad\qquad + \int_{\mathbb{R}^N} \Big((\partial\zeta_n)^2 + (\partial^2\zeta_n)^2 + (\partial^3\zeta_n)^2\Big) |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-4} dx
\end{aligned}$$

for some constant $C > 0$ independent of n. Notice on the right-hand sides that the radial factors have the estimates

$$\begin{aligned}|\partial\zeta_n| &= \left|r\partial_r\zeta_0\left((1/n)\log r\right)\right| = \frac{1}{n}\left|\zeta_0'((1/n)\log r)\right| \le \frac{C}{n},\\ |\partial^2\zeta_n(r)| &= \frac{1}{n^2}\left|\zeta_0''\left((1/n)\log r\right)\right| \le \frac{C}{n^2},\\ |\partial^3\zeta_n(r)| &= \frac{1}{n^3}\left|\zeta_0'''\left((1/n)\log r\right)\right| \le \frac{C}{n^3}\end{aligned}$$

for some constant $C > 0$ depending only on ζ_0, and hence we have

$$\left.\begin{aligned}&\int_{\mathbb{R}^N} |\boldsymbol{u}_n - \zeta_n\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-4}dx \to 0,\\ &\int_{\mathbb{R}^N} |\nabla\boldsymbol{u}_n - \zeta_n\nabla\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-2}dx \to 0,\\ &\int_{\mathbb{R}^N} |\triangle\boldsymbol{u}_n - \zeta_n\triangle\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx \to 0\end{aligned}\right\} \quad \text{as} \ \ n\to\infty$$

by using the integrability condition. Now, combine this result together with the following fact: since $\zeta_0(0) = 1$, the L^2-dominated convergences

$$\zeta_n\boldsymbol{u} \to \boldsymbol{u}, \quad \zeta_n\nabla\boldsymbol{u} \to \nabla\boldsymbol{u} \quad \text{and} \quad \zeta_n\triangle\boldsymbol{u} \to \triangle\boldsymbol{u}$$

hold on $\mathbb{R}^N$ with respect to (in this order) the measures $|\boldsymbol{x}|^{2\gamma-4}dx$, $|\boldsymbol{x}|^{2\gamma-2}dx$ and $|\boldsymbol{x}|^{2\gamma}dx$; then we obtain

$$\left.\begin{aligned}&\int_{\mathbb{R}^N} |\boldsymbol{u}_n - \boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-4}dx \to 0,\\ &\int_{\mathbb{R}^N} |\nabla\boldsymbol{u}_n - \nabla\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-2}dx \to 0,\\ &\int_{\mathbb{R}^N} |\triangle\boldsymbol{u}_n - \triangle\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx \to 0\end{aligned}\right\} (n\to\infty)$$

through the L^2-triangle inequalities, and therefore we arrive at the desired approximation. Consequently, the evaluation of $C_{N,\gamma}$ for solenoidal fields turns out to be reduced to that of $C_{P,N,\gamma}$ and $C_{T,N,\gamma}$ for compactly supported PT fields on $\dot{\mathbb{R}}^N$.

§3.2. **Evaluation of $C_{P,N,\gamma}$.** In the present and the next subsections, we always assume that $\boldsymbol{u}$ is a poloidal field having the expression

$$\boldsymbol{u} = \boldsymbol{u}_P = \boldsymbol{D}f \qquad \text{for } f \in C_c^\infty(\dot{\mathbb{R}}^N).$$

To simplify the notation, hereafter we set

$$\lambda := \gamma - \frac{N}{2} - 1$$

as an alternative weight exponent, in place of γ.

Let us introduce a new vector field $\boldsymbol{v} \in C_c^\infty(\dot{\mathbb{R}}^N)^N$ and a scalar field $g \in C^\infty(\dot{\mathbb{R}}^N)$ by the transformation formulae

$$\begin{aligned}\boldsymbol{v}(\boldsymbol{x}) &:= |\boldsymbol{x}|^{\lambda+N-1}\boldsymbol{u}(\boldsymbol{x}),\\ g(\boldsymbol{x}) &:= |\boldsymbol{x}|^{\lambda+N-1}f(\boldsymbol{x}),\end{aligned} \tag{3.2}$$

which stems from an idea of Brezis-Vázquez [2]. We aim to express the two integrals in inequality (1.3) in terms of g, after expressing them in terms of $\boldsymbol{v}$; to this end, we start with the following differential calculation:

$$\begin{aligned}\boldsymbol{v} &= r^{\lambda+N-1}\boldsymbol{u} = r^{\lambda+N-1}\boldsymbol{D}f\\ &= r^{\lambda+N-1}\big(\boldsymbol{\sigma}\triangle_\sigma - (\partial+N-1)\nabla_\sigma\big)(r^{1-\lambda-N}g)\\ &= \boldsymbol{\sigma}\triangle_\sigma g - (\partial-\lambda)\nabla_\sigma g \end{aligned}\tag{3.3}$$

holds by using (2.8), in terms of the same notation $\partial = r\partial_r$ as in Lemma 2.3. To further simplify the notation, hereafter we introduce an alternative radial coordinate $t \in \mathbb{R}$ given by the so-called Emden transformation formula

$$t = \log|\boldsymbol{x}| \qquad \forall \boldsymbol{x} \in \dot{\mathbb{R}}^N$$

which (reproduces and) generates the non-dimensional radial differentials:

$$\partial_t = r\partial_r = \boldsymbol{x}\cdot\nabla = \partial, \quad dt = (1/r)dr \qquad \text{for} \;\; r = |\boldsymbol{x}|.$$

Then, with respect to the measure

$$|\boldsymbol{x}|^{2\gamma}dx = r^{2\gamma+N-1}dr\,\mathrm{d}\sigma = r^{2\lambda+2N+1}dr\,\mathrm{d}\sigma = r^{2(\lambda+N+1)}dt\,\mathrm{d}\sigma, \tag{3.4}$$

integration by parts of $|\boldsymbol{u}|^2/|\boldsymbol{x}|^4$ yields

$$\begin{aligned}\int_{\mathbb{R}^N}|\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-4}dx &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}|\boldsymbol{v}|^2 dt\,\mathrm{d}\sigma \qquad (3.5)\\ &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\big((\triangle_\sigma g)^2 + |(\partial-\lambda)\nabla_\sigma g|^2\big)\,dt\,\mathrm{d}\sigma\\ &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\big((\triangle_\sigma g)^2 + |\partial\nabla_\sigma g|^2 + \lambda^2|\nabla_\sigma g|^2\big)\,dt\,\mathrm{d}\sigma\\ &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}(-\triangle_\sigma g)\big(-\triangle_\sigma - \partial^2 + \lambda^2\big)\,g\,dt\,\mathrm{d}\sigma \qquad (3.6)\end{aligned}$$

by using the support compactness of $\boldsymbol{u}$ on $\dot{\mathbb{R}}^N$. On the other hand, following from (2.11), take the operation of $r^2\triangle$ on the field $\boldsymbol{u} = r^{1-N-\lambda}\boldsymbol{v}$; then we have

$$\begin{aligned}r^2\triangle\boldsymbol{u} &= \big(\partial^2 + (N-2)\partial\big)\big(r^{1-N-\lambda}\boldsymbol{v}\big) + r^{1-N-\lambda}\triangle_\sigma\boldsymbol{v}\\ &= r^{1-N-\lambda}\Big((1-N-\lambda+\partial)^2 + (N-2)(1-N-\lambda+\partial) + \triangle_\sigma\Big)\boldsymbol{v}\\ &= r^{1-N-\lambda}\big(\alpha_{1-N-\lambda} - (N+2\lambda)\partial + \partial^2 + \triangle_\sigma\big)\boldsymbol{v}\end{aligned}$$

in terms of the same notation $\alpha_{1-N-\lambda} = \alpha_{\lambda+1}$ as in $(2.4)_{s=\lambda+1}$. Therefore, integration by parts of $|\triangle\boldsymbol{u}|^2$ with respect to the measure (3.4) yields

$$\begin{aligned}\int_{\mathbb{R}^N}|\triangle\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\big|\alpha_{\lambda+1}\boldsymbol{v} - (N+2\lambda)\partial\boldsymbol{v} + \partial^2\boldsymbol{v} + \triangle_\sigma\boldsymbol{v}\big|^2\,dt\,\mathrm{d}\sigma\\ &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\Big(|\partial^2\boldsymbol{v}|^2 + \big((N-2)^2 + 2\alpha_{\lambda+1}\big)\,|\partial\boldsymbol{v}|^2 + 2|\partial\nabla_\sigma\boldsymbol{v}|^2\Big)dt\,\mathrm{d}\sigma\\ &\quad + \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}|\alpha_{\lambda+1}\boldsymbol{v} + \triangle_\sigma\boldsymbol{v}|^2 dt\,\mathrm{d}\sigma \qquad (3.7)\end{aligned}$$

with the aid of the identity $(N+2\lambda)^2 - 2\alpha_{\lambda+1} = (N-2)^2 + 2\alpha_{\lambda+1}$ and the support compactness of $\boldsymbol{v}$. In order to express in terms of g the right-hand side of the above integral equation, let us express the spherical Laplacian of $\boldsymbol{v}$ in terms of g:

a differential calculation by using (2.5) gives

$$\begin{aligned}\triangle_\sigma \boldsymbol{v} &= \triangle_\sigma(\boldsymbol{\sigma}\triangle_\sigma g) - \triangle_\sigma\nabla_\sigma\left((\partial-\lambda)g\right)\\ &= \boldsymbol{\sigma}\Big(\triangle_\sigma^2 g + \big(2\partial - (2\lambda+N-1)\big)\triangle_\sigma g\Big)\\ &\quad + \nabla_\sigma\Big(2\triangle_\sigma g - (\triangle_\sigma + N-3)\big((\partial-\lambda)g\big)\Big). \end{aligned}\tag{3.8}$$

(One may also check, by replacing λ by $1-N$, that this identity reproduces (2.14).) Then integration by parts of the scalar product of (3.3) and (3.8) gives

$$\begin{aligned}&\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} |\nabla_\sigma \boldsymbol{v}|^2 dt\,\mathrm{d}\sigma = \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} (-\triangle_\sigma \boldsymbol{v})\cdot\boldsymbol{v}\,dt\,\mathrm{d}\sigma\\ &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \left(\begin{array}{l} -\Big(\triangle_\sigma^2 g + \big(2\partial-(2\lambda+N-1)\big)\triangle_\sigma g\Big)\triangle_\sigma g\\ +\nabla_\sigma\big(2\triangle_\sigma g - (\triangle_\sigma+N-3)(\partial-\lambda)g\big)\cdot\nabla_\sigma\big((\partial-\lambda)g\big)\end{array}\right) dt\,\mathrm{d}\sigma\\ &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \left(\begin{array}{l} -\big(\triangle_\sigma^2 g - (2\lambda+N-1)\triangle_\sigma g\big)\triangle_\sigma g\\ +\Big(2\lambda\triangle_\sigma g - (\triangle_\sigma+N-3)(\partial^2-\lambda^2)g\Big)\triangle_\sigma g\end{array}\right) dt\,\mathrm{d}\sigma\\ &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} (-\triangle_\sigma g)\left(\begin{array}{l} \triangle_\sigma^2 g - (4\lambda+N-1)\triangle_\sigma g\\ +(\triangle_\sigma+N-3)\big((\partial^2-\lambda^2)g\big)\end{array}\right) dt\,\mathrm{d}\sigma.\end{aligned}\tag{3.9}$$

On the other hand, notice that the equations (3.3) and (3.8) are invariant under the replacement

$$(\boldsymbol{v}, g) \mapsto (\partial^k \boldsymbol{v}, \partial^k g) \qquad \text{for } k=1,2;$$

by applying the same replacement to the equations (3.5)=(3.6) and (3.9) and by integration by parts, we also get

$$\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} |\partial^k \boldsymbol{v}|^2 dt\,\mathrm{d}\sigma = \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} (-\triangle_\sigma g)\left(-\triangle_\sigma - \partial^2 + \lambda^2\right)(-\partial^2)^k g\,dt\,\mathrm{d}\sigma,\tag{3.10}$$

$$\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} |\partial\nabla_\sigma \boldsymbol{v}|^2 dt\,\mathrm{d}\sigma = \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} (-\triangle_\sigma g)\left(\begin{array}{l} -\triangle_\sigma^2\partial^2 g + (4\lambda+N-1)\triangle_\sigma\partial^2 g\\ -(\triangle_\sigma+N-3)\big((\partial^2-\lambda^2)\partial^2 g\big)\end{array}\right) dt\,\mathrm{d}\sigma\tag{3.11}$$

which give an expression of the second last integral in (3.7) in terms of g. In order to evaluate the last integral in (3.7), by using the expressions (3.3) and (3.8), we have

$$\begin{aligned}\triangle_\sigma\boldsymbol{v} + \alpha_{\lambda+1}\boldsymbol{v} &= \boldsymbol{\sigma}\Big(\triangle_\sigma^2 g + (2\partial - (2\lambda+N-1))\,\triangle_\sigma g\Big)\\ &\quad + \nabla_\sigma\Big(2\triangle_\sigma g - (\triangle_\sigma+N-3)\big((\partial-\lambda)g\big)\Big)\\ &\quad + \alpha_{\lambda+1}\big(\boldsymbol{\sigma}\triangle_\sigma g - (\partial-\lambda)\nabla_\sigma g\big)\\ &= \boldsymbol{\sigma}\left(2\partial + \triangle_\sigma + \alpha_\lambda\right)\triangle_\sigma g\\ &\quad + \nabla_\sigma\left(\begin{array}{l} -(\triangle_\sigma+\alpha_{\lambda+1}+N-3)\partial g\\ +(\lambda+2)\triangle_\sigma g + (\alpha_{\lambda+1}+N-3)\lambda g\end{array}\right),\end{aligned}$$

with the aid of the identity $\alpha_{\lambda+1}-(2\lambda+N-1)=\alpha_\lambda$. Then integration by parts gives the following calculation:

$$
\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} |\triangle_\sigma \boldsymbol{v}+\alpha_{\lambda+1}\boldsymbol{v}|^2 dt\,\mathrm{d}\sigma \\
&\quad=\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \big((2\partial+\triangle_\sigma+\alpha_\lambda)\triangle_\sigma g\big)^2 dt\,\mathrm{d}\sigma \\
&\qquad+\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \left|\nabla_\sigma \begin{pmatrix} -(\triangle_\sigma+\alpha_{\lambda+1}+N-3)\partial g \\ +(\lambda+2)\triangle_\sigma g+(\alpha_{\lambda+1}+N-3)\lambda g \end{pmatrix}\right|^2 dt\,\mathrm{d}\sigma \\
&\quad=\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \begin{pmatrix} 4(\partial\triangle_\sigma g)^2+\big((\triangle_\sigma+\alpha_\lambda)\triangle_\sigma g\big)^2 \\ +\big|\nabla_\sigma\big((\triangle_\sigma+\alpha_{\lambda+1}+N-3)\partial g\big)\big|^2 \\ +\big|(\lambda+2)\nabla_\sigma\triangle_\sigma g+(\alpha_{\lambda+1}+N-3)\lambda\nabla_\sigma g\big|^2 \end{pmatrix} dt\,\mathrm{d}\sigma \\
&\quad=\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} (-\triangle_\sigma g)\begin{pmatrix} \Big(-4\triangle_\sigma+(\triangle_\sigma+\alpha_{\lambda+1}+N-3)^2\Big)(-\partial^2 g) \\ +(\triangle_\sigma+\alpha_\lambda)^2(-\triangle_\sigma g) \\ +(\lambda+2)^2\triangle_\sigma^2 g+(\alpha_{\lambda+1}+N-3)^2\lambda^2 g \\ +2\lambda(\lambda+2)(\alpha_{\lambda+1}+N-3)\triangle_\sigma g \end{pmatrix} dt\,\mathrm{d}\sigma \\
&\quad=\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} (-\triangle_\sigma g)\begin{pmatrix} \Big(\big(\alpha_{\lambda+1}+N-3+\triangle_\sigma\big)^2-4\triangle_\sigma\Big)(-\partial^2 g) \\ +\big((\lambda+2)^2-\triangle_\sigma\big)\big((\alpha_\lambda+\triangle_\sigma)^2 g\big) \end{pmatrix} dt\,\mathrm{d}\sigma. \qquad (3.12)
\end{aligned}
$$

Substituting (3.10), (3.11) and (3.12) into the right-hand side of (3.7), we obtain

$$
\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} |\triangle \boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx=\begin{pmatrix} (3.10)_{k=2}+\big((N-2)^2+2\alpha_{\lambda+1}\big)\times(3.10)_{k=1} \\ +2\times(3.11)+(3.12) \end{pmatrix} \\
&=\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} (-\triangle_\sigma g)\begin{pmatrix} \Big(\partial^4+\big((N-2)^2+2\alpha_{\lambda+1}\big)(-\partial^2)\Big)\big(-\triangle_\sigma-\partial^2+\lambda^2\big) \\ +2\begin{pmatrix} \triangle_\sigma^2-(4\lambda+N-1)\triangle_\sigma \\ +(\triangle_\sigma+N-3)\big(\partial^2-\lambda^2\big) \end{pmatrix}(-\partial^2) \\ +\Big(\big(\alpha_{\lambda+1}+N-3+\triangle_\sigma\big)^2-4\triangle_\sigma\Big)(-\partial^2) \\ +\big((\lambda+2)^2-\triangle_\sigma\big)(\alpha_\lambda+\triangle_\sigma)^2 \end{pmatrix} g\,dt\,\mathrm{d}\sigma \\
&=\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} (-\triangle_\sigma g)Q(-\partial^2,-\triangle_\sigma)g\,dt\,\mathrm{d}\sigma, \qquad (3.13)
\end{aligned}
$$

where $Q(\cdot,\cdot)$ is the polynomial in two variables given by the algebraic calculation

$$
\begin{aligned}
Q(\tau,a) &:= \Big(\tau^2+\big((N-2)^2+2\alpha_{\lambda+1}\big)\tau\Big)\big(a+\tau+\lambda^2\big) \\
&\quad+2\Big(a^2+(4\lambda+N-1)a+(-a+N-3)\big(-\tau-\lambda^2\big)\Big)\tau \\
&\quad+\Big(\big(\alpha_{\lambda+1}+N-3-a\big)^2+4a\Big)\tau+\big((\lambda+2)^2+a\big)(\alpha_\lambda-a)^2 \\
&=\big(\tau+a+(\lambda+2)^2\big)\begin{pmatrix} \big(\tau+a+(\lambda+N-2)^2\big)\big(\tau+a+\lambda^2\big) \\ -(2\lambda+N-2)^2 a \end{pmatrix}. \qquad (3.14)
\end{aligned}
$$

In order to evaluate the ratio of (3.13) to (3.6), let us now apply to g the Fourier transformation in the radial direction:

$$\widehat{g}(\tau,\boldsymbol{\sigma}):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\tau t}g(e^t\boldsymbol{\sigma})dt \qquad \forall(\tau,\boldsymbol{\sigma})\in\mathbb{R}\times\mathbb{S}^{N-1},$$

where $i=\sqrt{-1}$. Then the $L^2(\mathbb{R})$ isometry of this transformation yields

$$\begin{aligned}\frac{\int_{\mathbb{R}^N}|\triangle \boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx}{\int_{\mathbb{R}^N}|\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-4}dx} &= \frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}(-\triangle_\sigma g)Q(-\partial^2,-\triangle_\sigma)g\,dt\,\mathrm{d}\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}(-\triangle_\sigma g)\left(-\triangle_\sigma-\partial^2+\lambda^2\right)g\,dt\,\mathrm{d}\sigma}\\ &= \frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}(-\triangle_\sigma\overline{\widehat{g}})Q(\tau^2,-\triangle_\sigma)\widehat{g}\,d\tau\,\mathrm{d}\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}(-\triangle_\sigma\overline{\widehat{g}})\left(-\triangle_\sigma+\tau^2+\lambda^2\right)\widehat{g}\,d\tau\,\mathrm{d}\sigma}. \end{aligned}\tag{3.15}$$

Here, let us express the spherical harmonics decomposition of $\widehat{g}$ as

$$\widehat{g}=\sum_{\nu\in\mathbb{N}}\widehat{g}_\nu,\qquad -\triangle_\sigma\widehat{g}_\nu=\alpha_\nu\widehat{g}_\nu\quad(\forall\nu\in\mathbb{N})$$

in terms of the same notation $\alpha_\nu=\nu(\nu+N-2)$ in (2.4), and we further get

$$\begin{aligned}\frac{\int_{\mathbb{R}^N}|\triangle \boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx}{\int_{\mathbb{R}^N}|\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-4}dx} &= \frac{\sum_{\nu\in\mathbb{N}}\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\alpha_\nu Q(\tau^2,\alpha_\nu)|\widehat{g}_\nu|^2dt\,\mathrm{d}\sigma}{\sum_{\nu\in\mathbb{N}}\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\alpha_\nu(\alpha_\nu+\tau^2+\lambda^2)|\widehat{g}_\nu|^2dt\,\mathrm{d}\sigma}\\ &\ge \inf_{\nu\in\mathbb{N}}\inf_{\tau\ge 0}\frac{Q(\tau,\alpha_\nu)}{\tau+\alpha_\nu+\lambda^2}=\min_{\nu\in\mathbb{N}}\frac{Q(0,\alpha_\nu)}{\alpha_\nu+\lambda^2},\end{aligned}$$

where the last equation follows from that the rational polynomial

$$\frac{Q(\tau,a)}{\tau+a+\lambda^2}=\left(\tau+a+(\lambda+2)^2\right)\left(\tau+\left(1-\frac{(2\lambda+N-2)^2}{\tau+a+\lambda^2}\right)a+(\lambda+N-2)^2\right)$$

is monotone increasing in $\tau\ge 0$ for each $a>0$. Therefore, we have obtained the R-L inequality

$$\int_{\mathbb{R}^N}|\triangle \boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx\ge C_{P,N,\gamma}\int_{\mathbb{R}^N}|\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-4}dx$$

with the constant number $C_{P,N,\gamma}$ expressed as

$$C_{P,N,\gamma}=\min_{\nu\in\mathbb{N}}\frac{Q(0,\alpha_\nu)}{\alpha_\nu+\lambda^2}=\min_{\nu\in\mathbb{N}}\frac{\alpha_\nu+(\lambda+2)^2}{\alpha_\nu+\lambda^2}(\alpha_\nu-\alpha_\lambda)^2. \tag{3.16}$$

To see that this number is the best possible, choose $\nu_0\in\mathbb{N}$ such that

$$\frac{Q(0,\alpha_{\nu_0})}{\alpha_{\nu_0}+\lambda^2}=\min_{\nu\in\mathbb{N}}\frac{Q(0,\alpha_\nu)}{\alpha_\nu+\lambda^2},$$

and define $\{g_n\}_{n\in\mathbb{N}}\subset C_c^\infty(\dot{\mathbb{R}}^N)$ as a sequence of scalar fields by

$$g_n(\boldsymbol{x})=\zeta\left(\frac{\log|\boldsymbol{x}|}{n}\right)Y_{\nu_0}(\boldsymbol{x}/|\boldsymbol{x}|)\qquad(\forall\, n\in\mathbb{N}),$$

where Y_{ν_0} is the eigenfunction of $-\triangle_\sigma$ associated with the eigenvalue α_{ν_0} and where $\zeta:\mathbb{R}\to\mathbb{R}$ is a smooth function $\not\equiv 0$ with compact support. Then, applying (3.15) to the case $g=g_n$, one can directly check that the sequence of poloidal fields

$$\left\{\boldsymbol{u}_n:=\boldsymbol{D}\big(r^{-(\lambda+N-1)}g_n\big)\right\}_{n\in\mathbb{N}}$$

satisfies

$$\begin{aligned}\frac{\int_{\mathbb{R}^N}|\triangle \boldsymbol{u}_n|^2|\boldsymbol{x}|^{2\gamma}dx}{\int_{\mathbb{R}^N}|\boldsymbol{u}_n|^2|\boldsymbol{x}|^{2\gamma-4}dx} &= \frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}(\alpha_{\nu_0}g_n)Q(-\partial^2,\alpha_{\nu_0})g_n\,dt\,\mathrm{d}\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}(\alpha_{\nu_0}g_n)\left(\alpha_{\nu_0}-\partial^2+\lambda^2\right)g_n\,dt\,\mathrm{d}\sigma}\\ &= \frac{\int_{\mathbb{R}}\zeta(t/n)Q(-\partial_t^2,\alpha_{\nu_0})\zeta(t/n)\,dt}{\int_{\mathbb{R}}\zeta(t/n)\left(\alpha_{\nu_0}-\partial_t^2+\lambda^2\right)\zeta(t/n)\,dt}\\ &= \frac{\int_{\mathbb{R}}\zeta(t)Q(-n^{-2}\partial_t^2,\alpha_{\nu_0})\zeta(t)\,dt}{\int_{\mathbb{R}}\zeta(t)\left(\alpha_{\nu_0}-n^{-2}\partial_t^2+\lambda^2\right)\zeta(t)\,dt}\\ &\longrightarrow \frac{Q(0,\alpha_{\nu_0})}{\alpha_{\nu_0}+\lambda^2}=C_{P,N,\gamma}\qquad(n\to\infty).\end{aligned}$$

Hence we get the desired best possibility of $C_{P,N,\gamma}$.

§3.3. **Additional term in the R-L inequality for poloidal fields.** Recalling from the calculation in (3.15) we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx - C_{P,N,\gamma} \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-4} dx \\
&\qquad = \iint\limits_{\mathbb{R}\times\mathbb{S}^{N-1}} (-\triangle_\sigma \overline{\widehat{g}}) \Big(Q(\tau^2, -\triangle_\sigma) - C_{P,N,\gamma} \left(-\triangle_\sigma + \tau^2 + \lambda^2 \right) \Big) \widehat{g} \, d\tau \, \mathrm{d}\sigma \\
&\qquad = \sum_{\nu\in\mathbb{N}} \alpha_\nu \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big(Q(\tau^2, \alpha_\nu) - C_{P,N,\gamma} \left(\tau^2 + \alpha_\nu + \lambda^2 \right) \Big) |\widehat{g}_\nu|^2 d\tau \, \mathrm{d}\sigma \\
&\qquad \geq c_{P,N,\gamma} \sum_{\nu\in\mathbb{N}} \alpha_\nu \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} (\tau^2 + \alpha_\nu + \lambda^2) |\widehat{g}_\nu|^2 \tau^2 d\tau \, \mathrm{d}\sigma \\
&\qquad = c_{P,N,\gamma} \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} |\partial \boldsymbol{v}|^2 dt \, \mathrm{d}\sigma \\
&\qquad = c_{P,N,\gamma} \int_{\mathbb{R}^N} \left| \boldsymbol{x} \cdot \nabla \big(|\boldsymbol{x}|^{\gamma + \frac{N}{2} - 2} \boldsymbol{u} \big) \right|^2 |\boldsymbol{x}|^{-N} dx, \qquad (3.17)
\end{aligned}$$

where we give the constant number $c_{P,N,\gamma} \geq 0$ by the formula

$$c_{P,N,\gamma} := \inf_{\tau\neq 0} \inf_{\nu\in\mathbb{N}} \frac{\dfrac{Q(\tau^2,\alpha_\nu)}{\tau^2+\alpha_\nu+\lambda^2} - C_{P,N,\gamma}}{\tau^2}. \qquad (3.18)$$

Now let us aim to show $c_{P,N,\gamma} > 0$; to this end, recall from (3.16) that

$$\frac{\dfrac{Q(\tau,\alpha_\nu)}{\tau+\alpha_\nu+\lambda^2} - C_{P,N,\gamma}}{\tau} \geq \frac{\dfrac{Q(\tau,\alpha_\nu)}{\tau+\alpha_\nu+\lambda^2} - \dfrac{Q(0,\alpha_\nu)}{\alpha_\nu+\lambda^2}}{\tau} =: R(\tau,\alpha_\nu), \qquad (3.19)$$

and evaluate the right-hand side: an elementary calculation by using (3.14) gives

$$\begin{aligned}
R(\tau, a) &= \frac{\dfrac{Q(\tau,a)}{\tau+a+\lambda^2} - \dfrac{Q(0,a)}{a+\lambda^2}}{\tau} \\
&= \frac{4(\lambda+1)(2\lambda+N-2)^2 a}{(\lambda^2+a)(\tau+\lambda^2+a)} + 2\Big(\lambda + \frac{N}{2}\Big)^2 + \frac{1}{2}(N-4)^2 + 2a + \tau
\end{aligned}$$

as a rational polynomial in (τ, a). Then we get the following two cases: when $\lambda \geq -1$, it is clear that the estimate

$$\begin{aligned}
R(\tau,\alpha_\nu) &\geq 2\Big(\lambda + \frac{N}{2}\Big)^2 + \frac{1}{2}(N-4)^2 + 2\alpha_\nu + \tau \\
&\geq 2\Big(\lambda + \frac{N}{2}\Big)^2 + \frac{1}{2}(N-4)^2 + 2(N-1) \\
&= \Big(\lambda + \frac{N}{2} - 1\Big)^2 + \Big(\lambda + \frac{N}{2} + 1\Big)^2 + \frac{1}{2}(N-2)^2 + 2
\end{aligned}$$

holds for all $\tau \geq 0$ and $\nu \in \mathbb{N}$. When $\lambda < -1$, notice that

$$\frac{4(\lambda+1)a}{(\lambda^2+a)(\tau+\lambda^2+a)} \geq \frac{4(\lambda+1)a}{(\lambda^2+a)^2} \geq \frac{4(\lambda+1)a}{4\lambda^2 a} = \frac{\lambda+1}{\lambda^2} \geq -\frac{1}{4}$$

for all $a > 0$ and $\tau \geq 0$, and hence we have

$$\begin{aligned} R(\tau,\alpha_\nu) &\geq -\frac{1}{4}(2\lambda+N-2)^2 + 2\Big(\lambda+\frac{N}{2}\Big)^2 + \frac{1}{2}(N-4)^2 + 2\alpha_\nu + \tau \\ &= \Big(\lambda+\frac{N}{2}+1\Big)^2 - 2 + \frac{1}{2}(N-4)^2 + 2\alpha_\nu + \tau \\ &\geq \Big(\lambda+\frac{N}{2}+1\Big)^2 - 2 + \frac{1}{2}(N-4)^2 + 2(N-1) \\ &= \Big(\lambda+\frac{N}{2}+1\Big)^2 + \frac{1}{2}(N-2)^2 + 2. \end{aligned}$$

Therefore, returning to (3.18) and (3.19) we obtain

$$c_{P,N,\gamma} \geq \inf_{\tau>0}\inf_{\nu\in\mathbb{N}} R(\tau,\alpha_\nu) \geq \Big(\lambda+\frac{N}{2}+1\Big)^2 + \frac{1}{2}(N-2)^2 + 2 > 0$$

for both the cases of λ.

§3.4. **Evaluation of $C_{T,N,\gamma}$.** In this subsection we always assume that $\boldsymbol{u}$ is toroidal. In the same way as (3.2), we set $\boldsymbol{v}(\boldsymbol{x}) := |\boldsymbol{x}|^{\lambda+N-1}\boldsymbol{u}(\boldsymbol{x})$. Then, applying the same calculation as (3.7) and (3.5), we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\triangle\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big(|\partial^2\boldsymbol{v}|^2 + \big((N-2)^2+2\alpha_{\lambda+1}\big)|\partial\boldsymbol{v}|^2 + 2|\partial\nabla_\sigma\boldsymbol{v}|^2\Big)dt\,\mathrm{d}\sigma \\ &\quad + \int_{\mathbb{R}^N} |\alpha_{\lambda+1}\boldsymbol{u} + \triangle_\sigma\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-4}dx \qquad (3.20) \\ &\geq \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \big((N-2)^2+2\alpha_{\lambda+1}+2\alpha_1\big)|\partial\boldsymbol{v}|^2dt\,\mathrm{d}\sigma \\ &\quad + \int_{\mathbb{R}^N} \boldsymbol{u}\cdot\Big(\left(\alpha_{\lambda+1}+\triangle_\sigma\right)^2\boldsymbol{u}\Big)|\boldsymbol{x}|^{2\gamma-4}dx \\ &\geq \big((N+\lambda)^2+\lambda^2\big)\int_{\mathbb{R}^N} \big|\boldsymbol{x}\cdot\nabla\big(|\boldsymbol{x}|^{\gamma+\frac{N}{2}-2}\boldsymbol{u}\big)\big|^2|\boldsymbol{x}|^{-N}dx \\ &\quad + \min_{\nu\in\mathbb{N}}\left(\alpha_\nu-\alpha_{\lambda+1}\right)^2\int_{\mathbb{R}^N}|\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-4}dx, \qquad (3.21) \end{aligned}$$

where the last two inequalities come from the toroidal $L^2(\mathbb{S}^{N-1})$ estimates

$$\begin{aligned} &\int_{\mathbb{S}^{N-1}} |\nabla_\sigma\partial\boldsymbol{v}|^2\mathrm{d}\sigma \geq \alpha_1\int_{\mathbb{S}^{N-1}}|\partial\boldsymbol{v}|^2\mathrm{d}\sigma, \\ &\int_{\mathbb{S}^{N-1}} \boldsymbol{u}\cdot\Big(\left(\alpha_{\lambda+1}+\triangle_\sigma\right)^2\boldsymbol{u}\Big)\mathrm{d}\sigma \geq \min_{\nu\in\mathbb{N}}\left(\alpha_\nu-\alpha_{\lambda+1}\right)^2\int_{\mathbb{S}^{N-1}}|\boldsymbol{u}|^2\mathrm{d}\sigma \end{aligned}$$

as applications of Lemma 2.4. Hence the R-L inequality

$$\int_{\mathbb{R}^N}|\triangle\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx \geq C_{T,N,\gamma}\int_{\mathbb{R}^N}|\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-4}dx$$

holds for the constant number $C_{T,N,\gamma}$ given by

$$C_{T,N,\gamma} = \min_{\nu\in\mathbb{N}}\left(\alpha_\nu-\alpha_{\lambda+1}\right)^2. \qquad (3.22)$$

To see that this number is the best possible in the R-L inequality, choose $\nu_0 \in \mathbb{N}$ to be such that

$$\left(\alpha_{\nu_0}-\alpha_{\lambda+1}\right)^2 = \min_{\nu\in\mathbb{N}}\left(\alpha_\nu-\alpha_{\lambda+1}\right)^2.$$

We set $\boldsymbol{v}_0 \in \mathcal{T}(\dot{\mathbb{R}}^N)$ as a toroidal eigenfield of $-\triangle_\sigma$ associated with the eigenvalue α_{ν_0}: more explicitly, for example, one may set

$$\boldsymbol{v}_0(\boldsymbol{x}) := \left(-x_2\frac{\partial h}{\partial x_2} - x_3\frac{\partial h}{\partial x_3},\ x_1\frac{\partial h}{\partial x_2},\ x_1\frac{\partial h}{\partial x_3}, 0, \cdots, 0\right)$$

to confirm that $\boldsymbol{x}\cdot\boldsymbol{v}_0=\mathrm{div}\boldsymbol{v}_0=0$, where $h=h(x_2,x_3)$ denotes any homogeneous harmonic polynomial of degree ν_0 in the two variables (x_2,x_3); then $\boldsymbol{v}_0$ is a toroidal field with the non-trivial components being homogeneous harmonic polynomials of degree ν_0, and hence the eigenequation $-\triangle_\sigma\boldsymbol{v}_0=\alpha_{\nu_0}\boldsymbol{v}_0$ is satisfied. In that setting, Define $\{\boldsymbol{v}_n\}_{n\in\mathbb{N}}\subset C_c^\infty(\dot{\mathbb{R}}^N)^N$ as a sequence of toroidal fields by the formula

$$\boldsymbol{v}_n(\boldsymbol{x})=\zeta\left(\frac{\log|\boldsymbol{x}|}{n}\right)\boldsymbol{v}_0(\boldsymbol{x}/|\boldsymbol{x}|)\qquad\forall\, n\in\mathbb{N},$$

where $\zeta:\mathbb{R}\to\mathbb{R}$ is a smooth function $\not\equiv 0$ with compact support. Then one can check by using (3.20) that the sequence of toroidal fields

$$\left\{\boldsymbol{u}_n:=r^{-(\lambda+N-1)}\boldsymbol{v}_n\right\}_{n\in\mathbb{N}}$$

satisfies

$$\begin{aligned}
&\frac{\int_{\mathbb{R}^N}|\triangle\boldsymbol{u}_n|^2|\boldsymbol{x}|^{2\gamma}dx}{\int_{\mathbb{R}^N}|\boldsymbol{u}_n|^2|\boldsymbol{x}|^{2\gamma-4}dx}\\
&=\frac{\displaystyle\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\left(\begin{array}{l}|\partial^2\boldsymbol{v}_n|^2+2|\partial\nabla_\sigma\boldsymbol{v}_n|^2\\ +\left((N-2)^2+2\alpha_{\lambda+1}\right)|\partial\boldsymbol{v}_n|^2\end{array}\right)dt\,\mathrm{d}\sigma}{\displaystyle\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}|\boldsymbol{v}_n|^2dt\,\mathrm{d}\sigma}+(\alpha_{\nu_0}-\alpha_{\lambda+1})^2\\
&=\frac{\displaystyle\int_{\mathbb{R}}\left(\begin{array}{l}n^{-4}(\zeta''(t/n))^2+2n^{-2}\alpha_{\nu_0}(\zeta'(t/n))^2\\ +\left((N-2)^2+2\alpha_{\lambda+1}\right)n^{-2}(\zeta'(t/n))^2\end{array}\right)dt}{\int_{\mathbb{R}}(\zeta(t/n))^2dt}+C_{T,N,\gamma}\\
&=\frac{n^{-2}\displaystyle\int_{\mathbb{R}}\left(\begin{array}{l}n^{-2}(\zeta''(t))^2+2\alpha_{\nu_0}(\zeta'(t))^2\\ +\left((N-2)^2+2\alpha_{\lambda+1}\right)(\zeta'(t))^2\end{array}\right)dt}{\int_{\mathbb{R}}(\zeta(t))^2dt}+C_{T,N,\gamma}\\
&\longrightarrow C_{T,N,\gamma}\quad(n\to\infty)
\end{aligned}$$

to obtain the desired best possibility of $C_{T,N,\gamma}$.

§3.5. **Numerical calculation of $C_{N,\gamma}$.** Recalling the equation (3.1), we see from (3.16) and (3.22) that the best constant $C_{N,\gamma}=\min\{C_{P,N,\gamma},C_{T,N,\gamma}\}$ of R-L inequality for solenoidal fields is expressed as the lesser of

$$C_{P,N,\gamma}=\min_{\nu\in\mathbb{N}}A(\alpha_\nu)\quad\text{ and }\quad C_{T,N,\gamma}=\min_{\nu\in\mathbb{N}}B(\alpha_\nu)$$

which are the best constants of R-L inequality for P-T fields. Here A and B are the rational polynomials given by

$$A(a):=\frac{(\lambda+2)^2+a}{\lambda^2+a}\left(\alpha_\lambda-a\right)^2,\qquad B(a):=(\alpha_{\lambda+1}-a)^2.$$

Our goal here is to give a more specific description of $C_{N,\gamma}$ by separating the case of $\lambda\left(=\gamma-\frac{N}{2}-1\right)$ into the following two parts:

First let us consider the case $\left|\lambda+\frac{N}{2}\right|\geq\frac{\sqrt{(N-1)^2+1}}{2}$ which is equivalent to

$$B(\alpha_0)-B(\alpha_1)=(N-1)(2\lambda^2+2N\lambda+N-1)\geq 0.$$

Following from this inequality, a direct computation gives

$$\begin{aligned} C_{T,N,\gamma} &= \min_{\nu\in\mathbb{N}} B(\alpha_\nu) = \min_{\nu\in\mathbb{N}\cup\{0\}} B(\alpha_\nu) = \min_{\nu\in\mathbb{N}\cup\{0\}} (\alpha_{\lambda+1}-\alpha_\nu)^2 \\ &= \min_{\nu\in\mathbb{N}\cup\{0\}} \left(\alpha_\nu - \alpha_{\gamma-\frac{N}{2}}\right)^2 \\ &= \min_{\nu\in\mathbb{N}\cup\{0\}} \left(\left(\nu+\tfrac{N}{2}-1\right)^2-(\gamma-1)^2\right)^2 \\ &= B_{N,\gamma} \quad \text{(the same notation as (1.2))}, \end{aligned}$$

where the second last equality can be verified by using the identity

$$\alpha_s - \alpha_t = (s+N/2-1)^2 - (t+N/2-1)^2. \tag{3.23}$$

Then one can conclude $C_{N,\gamma} = \min\{C_{P,N,\gamma}, B_{N,\gamma}\} = B_{N,\gamma}$ since the inequality $C_{P,N,\gamma} \geq B_{N,\gamma}$ is theoretically clear from the definition of $B_{N,\gamma}$ (as the best constant of R-L inequality for unconstrained fields). However, we may also check the same inequality by a direct computation as follows:

$$\begin{cases} A(\alpha_s) - B(\alpha_{s-1}) = 4s(s+\lambda+N-2)^2\,\dfrac{s^2+(N-2)s-\lambda^2-N\lambda-N+2}{s^2+(N-2)s+\lambda^2} \\ A(\alpha_s) - B(\alpha_{s+1}) = -4(s-\lambda)^2(s+N-2)\,\dfrac{s^2+(N-2)s-\lambda^2-N\lambda-N+2}{s^2+(N-2)s+\lambda^2} \end{cases},$$

which implies

$$(A(\alpha_s)-B(\alpha_{s-1}))(A(\alpha_s)-B(\alpha_{s+1})) \leq 0.$$

In particular, since either of $A(\alpha_s) \geq B(\alpha_{s-1})$ or $A(\alpha_s) \geq B(\alpha_{s+1})$ must hold true for any $s > 0$, we have

$$A(\alpha_\nu) \geq \min\{B(\alpha_{\nu-1}), B(\alpha_{\nu+1})\} \qquad \forall \nu \in \mathbb{N}.$$

Then taking the minimum on both sides yields

$$C_{P,N,\gamma} = \min_{\nu\in\mathbb{N}} A(\alpha_\nu) \geq \min_{\nu\in\mathbb{N}} \min\{B(\alpha_{\nu-1}), B(\alpha_{\nu+1})\} = \min_{\nu\in\mathbb{N}\cup\{0\}} B(\alpha_\nu) = B_{N,\gamma},$$

as desired.

Next, let us consider the remaining case $\left|\lambda+\frac{N}{2}\right| < \frac{\sqrt{(N-1)^2+1}}{2}$. Since this inequality implies $-N \leq \lambda \leq 0$, a direct computation gives

$$\alpha_2 - \alpha_\lambda = (2-\lambda)(\lambda+N) \geq 0 \tag{3.24}$$

and

$$\begin{cases} A(\alpha_2) - B(\alpha_1) = \dfrac{8(\lambda+N)^2}{\lambda^2+2N}\big(N+2-\lambda(\lambda+N)\big) \geq 0, \\ B(\alpha_{\nu+1}) - B(\alpha_1) = \nu(\nu+N)\left(\nu^2+N\nu-2\lambda(\lambda+N)\right) \geq 0 \quad \forall \nu\in\mathbb{N}. \end{cases} \tag{3.25}$$

On the other hand, a straightforward calculation of the derivative of A yields

$$\begin{aligned} \frac{dA(a)}{da} &= \frac{2\,Q(a)}{(a+\lambda^2)^2}(a-\alpha_\lambda), \\ Q(a) &:= a^2+2(\lambda^2+\lambda+1)a+2(\lambda+1)\alpha_\lambda+\lambda^2(\lambda+2)^2. \end{aligned}$$

Here we note, for all $a \geq \alpha_2$, that the quadratic function Q satisfies

$$\begin{aligned} Q(a) \geq Q(\alpha_2) &= Q(2N) \\ &= \left(\lambda^2+3\lambda+N-\frac{2}{3}\right)^2 + 3N^2 + \left(4N-\frac{17}{3}\right)\lambda^2 + \frac{16}{3}N - \frac{4}{9} \\ &\geq 0. \end{aligned}$$

This fact together with (3.24) implies that $dA(a)/da \geq 0$ for all $a \geq \alpha_2$, and hence that the function $\alpha_2 \leq a \mapsto A(a)$ is monotone increasing; in particular, we get

$$\min_{\nu\in\mathbb{N}} A(\alpha_\nu) = \min\{A(\alpha_1), A(\alpha_2)\}$$

since $2 \leq s \mapsto \alpha_s$ is monotone increasing. This result together with (3.25) gives

$$\min\left\{\min_{\nu\in\mathbb{N}} A(\alpha_\nu), \min_{\nu\in\mathbb{N}} B(\alpha_\nu)\right\} = \min\{A(\alpha_1), B(\alpha_1)\},$$

and hence we get

$$\begin{aligned} C_{N,\gamma} &= \min\{A(\alpha_1), B(\alpha_1)\} \\ &= \min\left\{\frac{(\lambda+2)^2+\alpha_1}{\lambda^2+\alpha_1}\left(\alpha_{\gamma-\frac{N}{2}-1}-\alpha_1\right)^2,\ \left(\alpha_{\gamma-\frac{N}{2}}-\alpha_1\right)^2\right\} \\ &= \min\left\{\frac{\left(\gamma-\frac{N}{2}+1\right)^2+N-1}{\left(\gamma-\frac{N}{2}-1\right)^2+N-1}\left((\gamma-2)^2-\frac{N^2}{4}\right)^2,\ \left((\gamma-1)^2-\frac{N^2}{4}\right)^2\right\} \end{aligned}$$

by recalling the notation $\lambda = \gamma - N/2 - 1$ and the identity (3.23).

Therefore, we have obtained the same desired expression of $C_{N,\gamma}$ in Theorem 1.1.

§3.6. **Proof of Remark 1.2.** Let $\boldsymbol{u} = \boldsymbol{u}_P + \boldsymbol{u}_T$ be the PT decomposition of any solenoidal field $\boldsymbol{u} \in C_c^\infty(\dot{\mathbb{R}}^N)$. It follows from $(3.17)_{\boldsymbol{u}=\boldsymbol{u}_P}$ and $(3.21)_{\boldsymbol{u}=\boldsymbol{u}_T}$ that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx - C_{N,\gamma}\int_{\mathbb{R}^N} |\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-4}dx \\ &\qquad \geq \int_{\mathbb{R}^N} |\triangle \boldsymbol{u}_P|^2|\boldsymbol{x}|^{2\gamma}dx - C_{P,N,\gamma}\int_{\mathbb{R}^N}|\boldsymbol{u}_P|^2|\boldsymbol{x}|^{2\gamma-4}dx \\ &\qquad\quad + \int_{\mathbb{R}^N} |\triangle \boldsymbol{u}_T|^2|\boldsymbol{x}|^{2\gamma}dx - C_{T,N,\gamma}\int_{\mathbb{R}^N}|\boldsymbol{u}_T|^2|\boldsymbol{x}|^{2\gamma-4}dx \\ &\qquad \geq c_{P,N,\gamma}\int_{\mathbb{R}^N}\left|\boldsymbol{x}\cdot\nabla\left(|\boldsymbol{x}|^{\gamma+\frac{N}{2}-2}\boldsymbol{u}_P\right)\right|^2|\boldsymbol{x}|^{-N}dx \\ &\qquad\quad + \left((N+\lambda)^2+\lambda^2\right)\int_{\mathbb{R}^N}\left|\boldsymbol{x}\cdot\nabla\left(|\boldsymbol{x}|^{\gamma+\frac{N}{2}-2}\boldsymbol{u}_T\right)\right|^2|\boldsymbol{x}|^{-N}dx \\ &\qquad \geq c_{N,\gamma}\int_{\mathbb{R}^N}\left|\boldsymbol{x}\cdot\nabla\left(|\boldsymbol{x}|^{\gamma+\frac{N}{2}-2}\boldsymbol{u}\right)\right|^2|\boldsymbol{x}|^{-N}dx \end{aligned}$$

holds for $c_{N,\gamma} = \min\left\{c_{P,N,\gamma}, (N+\lambda)^2+\lambda^2\right\} > 0$. This result gives the inequality (1.4) and so finishes the proof of Remark 1.2.

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Osaka City University Advanced Mathematical Institute
3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
E-mail address: yhjyoe@yahoo.co.jp (N. Hamamoto)