SHARP HARDY-LERAY INEQUALITY FOR CURL-FREE FIELDS WITH A REMAINDER TERM

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Abstract. In this paper, we give a new and a simpler approach to the result in [7] concerning the best constant of Hardy-Leray inequality for curl-free fields. As a by-product, we obtain an improved inequality with a remainder term. The non-attainability of the best constant is an easy consequence of the new inequality. The proof is based on a decomposition of curl-free fields into radial and spherical parts.

1. Introduction

In this paper, we concern the classical functional inequality called the Hardy-Leray inequality for smooth vector fields and its improvement.

Let $N \in \mathbb{N}$ be an integer with $N \geq 2$ and put $\boldsymbol{x} = (x_1, \cdots, x_N) \in \mathbb{R}^N$. In the following, $C_c^\infty(\Omega)^N$ denotes the set of smooth vector fields

$$\boldsymbol{u} = (u_1, u_2, \cdots, u_N) : \Omega \ni \boldsymbol{x} \mapsto \boldsymbol{u}(\boldsymbol{x}) \in \mathbb{R}^N$$

having compact supports on an open subset Ω of $\mathbb{R}^N$.

Let γ be a real number. Then it is well known that

$$\left(\gamma + \frac{N}{2} - 1\right)^2 \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2} |\boldsymbol{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx$$

holds for any vector field $\boldsymbol{u} \in C_c^\infty(\mathbb{R}^N)^N$, as far as the integral on the left-hand side is finite (or equivalently $\boldsymbol{u}(\boldsymbol{0}) = \boldsymbol{0}$ for $\gamma \leq 1 - N/2$). This was first proved by J. Leray [10] when the weight $\gamma = 0$, see also the book by Ladyzhenskaya [9]. It is also known that the constant $\left(\gamma + \frac{N}{2} - 1\right)^2$ is sharp and never attained by any non-zero vector field.

In [2], Costin and Maz'ya proved that if the smooth vector fields $\boldsymbol{u}$ are axisymmetric and subject to the divergence-free constraint $\operatorname{div} \boldsymbol{u} \equiv 0$, then the constant $\left(\gamma + \frac{N}{2} - 1\right)^2$ can be improved and replaced by a larger one. More precisely, they proved the following:

Theorem A. (Costin-Maz'ya [2]) *Let $N \geq 2$. Let $\gamma \neq 1 - N/2$ be a real number and $\boldsymbol{u} \in C_c^\infty(\mathbb{R}^N)^N$ be an axisymmetric divergence-free vector field. (If $N = 2$, the axisymmetric assumption is not needed). Assume that $\boldsymbol{u}(\boldsymbol{0}) = \boldsymbol{0}$ if $\gamma < 1 - N/2$. Then*

$$C_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2} |\boldsymbol{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx$$

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holds with the optimal constant $C_{N,\gamma}$ given by

$$C_{N,\gamma} = \begin{cases} \left(\gamma + \frac{N}{2} - 1\right)^2 \dfrac{N + 1 + \left(\gamma - \frac{N}{2}\right)^2}{N - 1 + \left(\gamma - \frac{N}{2}\right)^2} & (N \geq 3,\ \gamma \leq 1), \\ \left(\gamma + \frac{N}{2} - 1\right)^2 + 2 + \min\limits_{\kappa \geq 0} \left(\kappa + \frac{4(N-1)(\gamma-1)}{\kappa + N - 1 + \left(\gamma - \frac{N}{2}\right)^2}\right) & (N \geq 4,\ \gamma > 1), \\ \left(\gamma + \frac{1}{2}\right)^2 + 2 & (N = 3,\ \gamma > 1), \end{cases}$$

$$C_{2,\gamma} = \begin{cases} \gamma^2 \frac{3 + (\gamma-1)^2}{1 + (\gamma-1)^2} & \text{if} \quad |\gamma + 1| \leq \sqrt{3}, \\ \gamma^2 + 1 & \text{otherwise.} \end{cases}$$

Note that the expression of the best constant $C_{N,\gamma}$ is slightly different from that in [2] when $N \geq 4$, but a careful checking the proof in [2] leads to the above formula in Theorem A. (See also [3, §2.1].)

Later, the first author of this paper has succeeded in removing the axisymmetric assumption in Theorem A to obtain the best constant [6, 4]. See also [8] for another improvement of [2]. We refer to [3, 5] for the Rellich-Leray inequality for divergence-free vector fields.

For curl-free vector fields, we have recently obtained the following result.

Theorem B. ([7]) *Let $N \geq 2$. Let $\gamma \neq 1 - N/2$ be a real number and let $\boldsymbol{u} \in C_c^\infty(\mathbb{R}^N)^N$ be a curl-free vector field. We assume that $\boldsymbol{u}(\boldsymbol{0}) = \boldsymbol{0}$ if $\gamma < 1 - N/2$. Then*

$$H_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2} |\boldsymbol{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx$$

with the optimal constant $H_{N,\gamma}$ given by

$$H_{N,\gamma} = \begin{cases} \left(\gamma + \frac{N}{2} - 1\right)^2 \frac{3(N-1) + \left(\gamma + \frac{N}{2} - 2\right)^2}{N - 1 + \left(\gamma + \frac{N}{2} - 2\right)^2} & \text{if} \quad \left|\gamma + \frac{N}{2}\right| \leq \sqrt{N+1}, \\ \left(\gamma + \frac{N}{2} - 1\right)^2 + N - 1 & \text{otherwise.} \end{cases} \tag{1}$$

The method of the proof of Theorem B, which followed from that of Costin-Maz'ya [2], consists of the following items: A representation of curl-free vector fields in the spherical polar coordinates, a transformation of vector fields called Brezis-Vázquez-Maz'ya, the one-dimensional Fourier transform in the radial direction, and the eigenvector expansion for the Laplace-Beltrami operator in $L^2(\mathbb{S}^{N-1})$.

A main purpose of this paper is to give another and a simpler approach to Theorem B. We avoid the use of Fourier transform, in the hope of being helpful for the possible extension of the result to L^p-setting or to domains other than the whole space. As a by-product, we obtain the sharp Hardy-Leray inequality for curl-free vector fields with a remainder term, which is the main result of this paper:

Theorem 1. *Let $N \geq 2$. Let $H_{N,\gamma}$ be defined in (1) and let $\boldsymbol{u} \in C_c^\infty(\mathbb{R}^N)^N$ be a curl-free field such that $\boldsymbol{u}(\boldsymbol{0}) = \boldsymbol{0}$ if $\gamma \leq 1 - \frac{N}{2}$. Then the inequality*

$$\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \geq H_{N,\gamma} \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma - 2} dx + \int_{\mathbb{R}^N} \left((N-1)\mathcal{E}_{N,\gamma}[\boldsymbol{u}] + \left|\boldsymbol{x} \cdot \nabla\left(|\boldsymbol{x}|^{\gamma + \frac{N}{2} - 1} \boldsymbol{u}\right)\right|^2\right) |\boldsymbol{x}|^{-N} dx \tag{2}$$

holds with the nonnegative function $\mathcal{E}_{N,\gamma}[\boldsymbol{u}]$ *given by*

$$\mathcal{E}_{N,\gamma}[\boldsymbol{u}](\boldsymbol{x}) = \begin{cases} \left(\left(\gamma+\frac{N}{2}\right)^2-N-1\right)\varphi^2+(\boldsymbol{x}\cdot\nabla\varphi)^2 & \text{for } \left|\gamma+\frac{N}{2}\right|\geq\sqrt{N+1}, \\ \dfrac{\left(N+1-\left(\gamma+\frac{N}{2}\right)^2\right)f^2+4(1-\gamma)(\boldsymbol{x}\cdot\nabla\varphi)^2}{\left(\gamma+\frac{N}{2}-2\right)^2+N-1} & \text{for } \left|\gamma+\frac{N}{2}\right|<\sqrt{N+1}. \end{cases}$$

Here f *and* φ *are scalar fields defined by*

$$f(\boldsymbol{x}) = \omega_{N-1}^{-1}|\boldsymbol{x}|^{\gamma+\frac{N}{2}-1}\int_{\mathbb{S}^{N-1}}\boldsymbol{\sigma}\cdot\boldsymbol{u}(|\boldsymbol{x}|\boldsymbol{\sigma})\mathrm{d}\sigma,$$

$$\varphi(\boldsymbol{x}) = |\boldsymbol{x}|^{\gamma+\frac{N}{2}-2}\left(\phi(\boldsymbol{x})-\omega_{N-1}^{-1}\int_{\mathbb{S}^{N-1}}\phi(|\boldsymbol{x}|\boldsymbol{\sigma})\mathrm{d}\sigma\right),$$

in terms of the scalar potential ϕ *of* $\boldsymbol{u}$ *(that is,* $\boldsymbol{u}=\nabla\phi$*), and* ω_{N-1} *denotes the surface measure of the unit sphere* $\mathbb{S}^{N-1}$ *in* $\mathbb{R}^N$*. Moreover, the equality in* (2) *is realized if and only if the equation*

$$-\triangle_\sigma\varphi(r\boldsymbol{\sigma}) = (N-1)\varphi(r\boldsymbol{\sigma})$$

holds for all $r>0$ *and* $\boldsymbol{\sigma}\in\mathbb{S}^{N-1}$*, where* $\triangle_\sigma$ *denotes the Laplace-Beltrami operator on* $\mathbb{S}^{N-1}$*.*

Remark 2. *We directly see from* (2) *that the equation*

$$\int_{\mathbb{R}^N}|\nabla\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx = H_{N,\gamma}\int_{\mathbb{R}^N}\frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2}|\boldsymbol{x}|^{2\gamma}dx$$

does not hold for any $\boldsymbol{u}\in C_c^\infty(\mathbb{R}^N)^N\setminus\{\boldsymbol{0}\}$ *as far as the integral on the right-hand side is finite. Indeed, this equation together with* (2) *implies that the function*

$$\mathbb{R}_+\times\mathbb{S}^{N-1}\ni(r,\boldsymbol{\sigma})\mapsto r^{\gamma+\frac{N}{2}-1}\boldsymbol{u}(r\boldsymbol{\sigma})$$

is independent of r*, which violates the finiteness of the integral unless* $\boldsymbol{u}\equiv\boldsymbol{0}$*.*

The remaining content of this paper is organized as follows: In §2, we give a quick review of some differential formulae with respect to radial-spherical variables and derive an equivalent condition to the curl-free condition for vector-fields on $\mathbb{R}^N$ (the Poincaré lemma); there Proposition 4 gives a characterization of curl-free fields, which serves as a key tool for the proof of our main theorem. In §3 we prove Theorem 1 by making full use of Proposition 4. In §4 we prove the sharp Rellich-Leray inequality for curl-free vector fields with a remainder term, as another application of the method described in §2–§3.

2. Representation of curl-free fields in terms of radial-spherical variables

In this section, we recall the Poincaré lemma, which gives a scalar-potential representation of smooth curl-free fields on $\mathbb{R}^N$. By deforming this potential via Brezis-Vázquez-Maz'ya transformation, we derive another equivalent condition for test vector fields to be curl-free.

2.1. **Radial-spherical variables and the Poincaré lemma.** First of all, we introduce the transformation

$$\mathbb{R}_+\times\mathbb{S}^{N-1}\to\mathbb{R}^N\setminus\{\boldsymbol{0}\},\quad (r,\boldsymbol{\sigma})\mapsto\boldsymbol{x}=r\boldsymbol{\sigma}$$

together with its inverse

$$\mathbb{R}^N\setminus\{\boldsymbol{0}\}\to\mathbb{R}_+\times\mathbb{S}^{N-1},\quad \boldsymbol{x}\mapsto(r,\boldsymbol{\sigma})=\left(|\boldsymbol{x}|,\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\in\mathbb{R}_+\times\mathbb{S}^{N-1}.$$

Let $\boldsymbol{u} = (u_1, u_2, \cdots, u_N) : \mathbb{R}^N \setminus \{\mathbf{0}\} \to \mathbb{R}^N$ be a vector field. Then its radial scalar component $u_R = u_R(\boldsymbol{x})$ and spherical vector part $\boldsymbol{u}_S = \boldsymbol{u}_S(\boldsymbol{x})$ are defined by the formulae

$$\boldsymbol{u} = \boldsymbol{\sigma} u_R + \boldsymbol{u}_S, \quad \boldsymbol{\sigma} \cdot \boldsymbol{u}_S = 0$$

for all $\boldsymbol{x} = r\boldsymbol{\sigma} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$. In a similar way, we denote by ∂_r and ∇_σ the radial derivative and the spherical gradient, respectively:

$$\partial_r f = \boldsymbol{\sigma} \cdot \nabla f, \quad \nabla_\sigma f = r(\nabla f)_S$$

for all $f = f(\boldsymbol{x}) \in C^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})$, or equivalently

$$\nabla = \boldsymbol{\sigma}\partial_r + \frac{1}{r}\nabla_\sigma, \quad \boldsymbol{\sigma} \cdot \nabla_\sigma = 0. \tag{3}$$

The Laplace operator $\triangle = \sum_{k=1}^N \partial^2/\partial x_k^2$ is known to be represented in terms of radial-spherical variables by the formula

$$\triangle = \frac{1}{r^{N-1}}\partial_r\left(r^{N-1}\partial_r\right) + \frac{1}{r^2}\triangle_\sigma, \tag{4}$$

where $\triangle_\sigma$ denotes the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$. In the following, we use a convention with some ambiguity that for smooth scalar fields and vector fields on $\mathbb{R}^N \setminus \{\mathbf{0}\}$, we think of them as functions of $\boldsymbol{\sigma} \in \mathbb{S}^{N-1}$ for $r = |\boldsymbol{x}|$ fixed, when we apply ∇_σ or $\triangle_\sigma$ to them. As a simple example, the operation of (3) and (4) on the scalar field $r = |\boldsymbol{x}|$ or its powers gives

$$\nabla r = \boldsymbol{\sigma} \quad \text{and} \quad \triangle r^s = \alpha_s r^{s-2}, \qquad \text{where} \quad \alpha_s = s(s+N-2) \tag{5}$$

for all $s \in \mathbb{R}$.

For later use, we prove the following lemma:

Lemma 3. *For any $f \in C^\infty(\mathbb{S}^{N-1})$,*

$$\begin{cases} \triangle_\sigma(\boldsymbol{\sigma} f) - \boldsymbol{\sigma}\triangle_\sigma f = \big(2\nabla_\sigma - (N-1)\boldsymbol{\sigma}\big)f, \\ \triangle_\sigma\nabla_\sigma f - \nabla_\sigma\triangle_\sigma f = \big((N-3)\nabla_\sigma - 2\boldsymbol{\sigma}\triangle_\sigma\big)f \end{cases} \tag{6}$$

holds for all $\boldsymbol{\sigma} \in \mathbb{S}^{N-1}$.

Proof. Take any $f \in C^\infty(\mathbb{S}^{N-1})$. We identify f with $\widetilde{f} \in C^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})$ by the formula $\widetilde{f}(\boldsymbol{x}) = f(\boldsymbol{\sigma})$ where $\boldsymbol{\sigma} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \in \mathbb{S}^{N-1}$. Note that $\triangle_\sigma\boldsymbol{\sigma} = -(N-1)\boldsymbol{\sigma}$ since

$$0 = \triangle\boldsymbol{x} = \left(\partial_r^2 + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\triangle_\sigma\right)(r\boldsymbol{\sigma}) = \frac{N-1}{r}\boldsymbol{\sigma} + \frac{1}{r}\triangle_\sigma\boldsymbol{\sigma}.$$

Thus we compute

$$\begin{aligned} \triangle_\sigma(\boldsymbol{\sigma} f) &= (\triangle_\sigma\boldsymbol{\sigma})f + \boldsymbol{\sigma}(\triangle_\sigma f) + 2(\nabla_\sigma f \cdot \nabla_\sigma)\boldsymbol{\sigma} \\ &= -(N-1)\boldsymbol{\sigma} f + (\triangle_\sigma f)\boldsymbol{\sigma} + 2\nabla_\sigma f, \end{aligned}$$

where we have used $(\nabla_\sigma f \cdot \nabla_\sigma)\boldsymbol{\sigma} = (\nabla_\sigma f \cdot \nabla)\boldsymbol{x} = \nabla_\sigma f$. This proves the first identity of (6).

To prove the second identity, we note from (3) resp. (4) that $\nabla_\sigma f = r\nabla f$ resp. $\triangle_\sigma f = r^2\triangle f$, since $f = \widetilde{f}$ is independent of the radial variable r. Also recalling from $(5)_{s=1}$ the formulae $\nabla r = \boldsymbol{\sigma}$ and $\triangle r = (N-1)r^{-1}$, we have

$$\begin{aligned} (\triangle_\sigma\nabla_\sigma - \nabla_\sigma\triangle_\sigma)f &= r^2\triangle\big(r\nabla f\big) - r\nabla\big(r^2\triangle f\big) \\ &= r^2\big((\triangle r)\nabla f + 2(\nabla r \cdot \nabla)\nabla f\big) - r\big(\nabla r^2\big)\triangle f \\ &= (N-1)r\nabla f + 2r^2\partial_r r^{-1}\nabla_\sigma f - 2r^2\boldsymbol{\sigma}\triangle f \\ &= (N-3)\nabla_\sigma f - 2\boldsymbol{\sigma}\triangle_\sigma f, \end{aligned}$$

as desired. □

The *curl* of a vector field $\boldsymbol{u} = (u_1, \cdots, u_N) \in C^\infty(\mathbb{R}^N)^N$ is defined as the differential 2-form

$$\operatorname{curl} \boldsymbol{u} = d(\boldsymbol{u} \cdot d\boldsymbol{x}) = d\left(\sum_{k=1}^N u_k dx_k\right),$$

where d denotes the exterior differential. This can be expressed in terms of the standard Euclidean coordinates as

$$d(\boldsymbol{u} \cdot d\boldsymbol{x}) = \sum_{k=1}^N du_k \wedge dx_k = \sum\sum_{j<k} \left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k}\right) dx_j \wedge dx_k.$$

Thus the curl-free condition $d(\boldsymbol{u} \cdot d\boldsymbol{x}) = 0$ holds if and only if

$$\frac{\partial u_k}{\partial x_j} = \frac{\partial u_j}{\partial x_k} \quad \text{for all} \quad j, k \in \{1, \cdots, N\}. \tag{7}$$

Here we claim that any curl-free vector fields $\boldsymbol{u}$ can be represented by

$$\boldsymbol{u}(\boldsymbol{x}) = \nabla\phi(\boldsymbol{x}), \quad \phi(\boldsymbol{x}) = \int_0^{|\boldsymbol{x}|} \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \cdot \boldsymbol{u}\Big(\rho\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\Big) d\rho \quad \text{for all} \ \ \boldsymbol{x} \in \mathbb{R}^N, \tag{8}$$

which we say that $\boldsymbol{u}$ has the scalar potential $\phi \in C^\infty(\mathbb{R}^N)$. Conversely, the existence of such a potential implies $d(\boldsymbol{u} \cdot d\boldsymbol{x}) = d(\nabla\phi \cdot d\boldsymbol{x}) = dd\phi = 0$, that is, $\boldsymbol{u}$ is curl-free.

The proof of the claim (8) is standard: For every $i \in \{1, \cdots, N\}$, we have

$$\begin{aligned} u_i(\boldsymbol{x}) &= \int_0^1 \frac{d}{dt}\{t u_i(t\boldsymbol{x})\} dt = \int_0^1 \left\{ u_i(t\boldsymbol{x}) + t \sum_{j=1}^N \frac{\partial u_i(t\boldsymbol{x})}{\partial x_j} x_j \right\} dt \\ &= \int_0^1 \left\{ u_i(t\boldsymbol{x}) + t \sum_{j=1}^N \frac{\partial u_j(t\boldsymbol{x})}{\partial x_i} x_j \right\} dt \\ &= \int_0^1 \frac{\partial}{\partial x_i} \left(\sum_{j=1}^N u_j(t\boldsymbol{x}) x_j \right) dt = \frac{\partial}{\partial x_i} \int_0^1 \boldsymbol{u}(t\boldsymbol{x}) \cdot \boldsymbol{x}\, dt \qquad \forall \boldsymbol{x} \in \mathbb{R}^N, \end{aligned}$$

here we have used (7) in the third equality. Thus we see that $\phi(\boldsymbol{x}) = \int_0^1 \boldsymbol{u}(t\boldsymbol{x}) \cdot \boldsymbol{x}\, dt$ is a scalar potential of $\boldsymbol{u}$. An easy change of variables leads to (8). □

2.2. **Radial-spherical decomposition of curl-free fields.** In the following, $\lambda \in \mathbb{R}$ denotes a fixed real number. Let $\boldsymbol{u}$ be a curl-free field on $\mathbb{R}^N$, and let ϕ be its scalar potential (8). We define a new vector field $\boldsymbol{v}$ and two scalar fields f, φ on $\mathbb{R}^N \setminus \{\boldsymbol{0}\}$ by the formulae

$$\begin{cases} \boldsymbol{v}(\boldsymbol{x}) = |\boldsymbol{x}|^{1-\lambda} \boldsymbol{u}(\boldsymbol{x}), \\ f(\boldsymbol{x}) = |\boldsymbol{x}|^{1-\lambda} \omega_{N-1}^{-1} \displaystyle\int_{\mathbb{S}^{N-1}} \boldsymbol{\sigma} \cdot \boldsymbol{u}(|\boldsymbol{x}|\boldsymbol{\sigma}) \mathrm{d}\sigma, \\ \varphi(\boldsymbol{x}) = |\boldsymbol{x}|^{-\lambda} \left(\phi(\boldsymbol{x}) - \omega_{N-1}^{-1} \displaystyle\int_{\mathbb{S}^{N-1}} \phi(|\boldsymbol{x}|\boldsymbol{\sigma}) \mathrm{d}\sigma \right). \end{cases} \tag{9}$$

The transformation of the field $\boldsymbol{u} \mapsto \boldsymbol{v}$ by the multiplication of $|\boldsymbol{x}|^{1-\lambda}$ stems from an idea of Brezis-Vázquez [1] and Maz'ya [11]. Now let us denote by

$$\bar{\phi}(r) = \omega_{N-1}^{-1} \int_{\mathbb{S}^{N-1}} \phi(r\boldsymbol{\sigma}) \mathrm{d}\sigma, \quad r = |\boldsymbol{x}|$$

the spherical mean of the scalar potential ϕ in (8), together with its radial derivative

$$\frac{\partial\bar{\phi}}{\partial r} = \omega_{N-1}^{-1} \int_{\mathbb{S}^{N-1}} \frac{\partial\phi}{\partial r}(r\boldsymbol{\sigma}) \mathrm{d}\sigma = \omega_{N-1}^{-1} \int_{\mathbb{S}^{N-1}} (\boldsymbol{\sigma} \cdot \nabla\phi)(r\boldsymbol{\sigma}) \mathrm{d}\sigma.$$

Then we see that (9) can be rewritten simply in terms of ϕ as

$$\begin{cases} \boldsymbol{v}(\boldsymbol{x}) = r^{1-\lambda}\nabla\phi(\boldsymbol{x}), \\ f(r) = r^{1-\lambda}\dfrac{\partial\bar{\phi}}{\partial r}, \\ \varphi(\boldsymbol{x}) = r^{-\lambda}\left(\phi(\boldsymbol{x}) - \bar{\phi}(r)\right), \end{cases} \tag{10}$$

and that f is a spherical mean part of $r^{1-\lambda}u_R$, while φ has zero-spherical mean. Furthermore, the scalar representation of $\boldsymbol{u}(\boldsymbol{x})$ in (8) is transformed into that of $\boldsymbol{v}(\boldsymbol{x})$ by the following computation using (10):

$$\begin{aligned} \boldsymbol{v} &= r^{1-\lambda}\nabla\phi \\ &= r^{1-\lambda}\left(\nabla(\phi-\bar{\phi}) + \nabla\bar{\phi}\right) \\ &= r^{1-\lambda}\left(\nabla(r^{\lambda}\varphi) + \frac{\partial\bar{\phi}}{\partial r}\boldsymbol{\sigma}\right) \\ &= r^{1-\lambda}\nabla(r^{\lambda}\varphi) + f(r)\boldsymbol{\sigma} \\ &= r^{1-\lambda}\left(\lambda r^{\lambda-1}\varphi\boldsymbol{\sigma} + r^{\lambda}\nabla\varphi\right) + f(r)\boldsymbol{\sigma} \\ &= (\lambda\varphi + f)\,\boldsymbol{\sigma} + r\nabla\varphi \\ &= (\lambda\varphi + f + \partial_t\varphi)\,\boldsymbol{\sigma} + \nabla_\sigma\varphi. \end{aligned}$$

Here and hereafter we employ the notation $t = \log r$ which obeys the differential identities

$$\begin{cases} \partial_t = r\partial_r, \quad dt = \frac{dr}{r}, \\ r\nabla = \boldsymbol{\sigma}\partial_t + \nabla_\sigma, \\ r^2\triangle = \partial_t^2 + (N-2)\partial_t + \triangle_\sigma. \end{cases} \tag{11}$$

In view of the above computation result, we can say that f and φ are radial and spherical scalar potentials of $\boldsymbol{v}$, respectively.

In summary, we obtain the following proposition:

Proposition 4. *Let $\lambda \in \mathbb{R}$. Then a vector field $\boldsymbol{u} \in C^\infty(\mathbb{R}^N)^N$ is curl-free if and only if there exist two scalar fields $f, \varphi \in C^\infty(\mathbb{R}^N \setminus \{\boldsymbol{0}\})$ satisfying*

$$\begin{cases} f \text{ is radially symmetric and } \int_{\mathbb{S}^{N-1}} \varphi(r\boldsymbol{\sigma})\mathrm{d}\sigma = 0 \quad \forall\, r > 0, \\ \boldsymbol{v} = \boldsymbol{\sigma}\big(f + (\lambda+\partial_t)\varphi\big) + \nabla_\sigma\varphi \qquad \text{on } \mathbb{R}^N \setminus \{\boldsymbol{0}\}, \end{cases} \tag{12}$$

where $\boldsymbol{v} \in C_c^\infty(\mathbb{R}^N \setminus \{\boldsymbol{0}\})^N$ is the vector field given by the same equation $\boldsymbol{v} = r^{1-\lambda}\boldsymbol{u}$ as in (9). Moreover, such f and φ are uniquely determined and explicitly given by the equations in (9); in particular, if $\boldsymbol{u}$ has a compact support on $\mathbb{R}^N \setminus \{\boldsymbol{0}\}$, then so do f and φ.

For later use, we give an expression of the vector field $\triangle_\sigma\boldsymbol{v}$ in terms of the scalar potentials:

Lemma 5. *Let $\boldsymbol{v}$ be as in (12). Then*

$$\begin{aligned} \triangle_\sigma\boldsymbol{v} = {}& \boldsymbol{\sigma}\big(\partial_t + \lambda - 2\big)\triangle_\sigma\varphi \\ & + \nabla_\sigma\big(2\partial_t + \triangle_\sigma + 2\lambda + 2N - 4\big)\varphi - (N-1)\boldsymbol{v}. \end{aligned} \tag{13}$$

Proof. By using Lemma 3 and Proposition 4, we compute

$$\begin{aligned}
\triangle_\sigma \boldsymbol{v} &= \triangle_\sigma\Big(\boldsymbol{\sigma}\big(f+(\partial_t+\lambda)\varphi\big)\Big)+\triangle_\sigma\big(\nabla_\sigma\varphi\big)\\
&= \Big(\boldsymbol{\sigma}\triangle_\sigma+2\nabla_\sigma-(N-1)\boldsymbol{\sigma}\Big)\big(f+(\partial_t+\lambda)\varphi\big)\\
&\quad+\Big(\nabla_\sigma\triangle_\sigma+(N-3)\nabla_\sigma-2\boldsymbol{\sigma}\triangle_\sigma\Big)\varphi\\
&= \boldsymbol{\sigma}\Big((\partial_t+\lambda-2)\triangle_\sigma\varphi\Big)-(N-1)\underbrace{\boldsymbol{\sigma}\big(f+(\partial_t+\lambda)\varphi\big)}_{\boldsymbol{v}-\nabla_\sigma\varphi}\\
&\quad+\nabla_\sigma\big(2\partial_t+\triangle_\sigma+2\lambda+N-3\big)\varphi\\
&= \boldsymbol{\sigma}\big(\partial_t+\lambda-2\big)\triangle_\sigma\varphi\\
&\quad+\nabla_\sigma\big(2\partial_t+\triangle_\sigma+2\lambda+2N-4\big)\varphi-(N-1)\boldsymbol{v}.
\end{aligned}$$

□

3. Proof of Theorem 1

We assume that the left-hand side of (2) is finite, since otherwise there is nothing to prove. Then the integrability of $|\nabla\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}$ together with the smoothness of $\boldsymbol{u}$ implies the existence of an integer $m>-\frac{N}{2}-\gamma$ such that

$$\nabla\boldsymbol{u}(\boldsymbol{x})=O(|\boldsymbol{x}|^m)\quad\text{as}\quad|\boldsymbol{x}|\to 0.$$

Moreover, in view of the assumption that $\boldsymbol{u}(\boldsymbol{0})=\boldsymbol{0}$ if $\gamma\le 1-\frac{N}{2}$, we see that $\boldsymbol{u}$ satisfies

$$|\boldsymbol{x}|^{\gamma+\frac{N}{2}-1}\boldsymbol{u}(\boldsymbol{x})=O(|\boldsymbol{x}|^\varepsilon)\quad\text{as}\quad|\boldsymbol{x}|\to 0$$

for $\varepsilon>0$ given by

$$\varepsilon=\begin{cases} m+\frac{N}{2}+\gamma & \text{for}\quad \gamma\le 1-\frac{N}{2},\\ \gamma+\frac{N}{2}-1 & \text{for}\quad \gamma>1-\frac{N}{2}.\end{cases}$$

Hence also the scalar potential $\phi(\boldsymbol{x})=\int_0^{|\boldsymbol{x}|}u_R(r\boldsymbol{x}/|\boldsymbol{x}|)dr$ in (8) satisfies

$$|\boldsymbol{x}|^{\gamma+\frac{N}{2}-1}\phi(\boldsymbol{x})=O\left(|\boldsymbol{x}|^{1+\varepsilon}\right)\qquad\text{as}\quad|\boldsymbol{x}|\to 0.$$

Consequently, we have further obtained the integrability conditions

$$\int_{\mathbb{R}^N}|\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-2}dx<\infty\quad\text{and}\quad\int_{\mathbb{R}^N}\phi^2|\boldsymbol{x}|^{2\gamma-4}dx<\infty \tag{14}$$

The proof of the theorem is carried out in the following steps:

3.1. Reduction to the case of compact support distinct from the origin.

We can further assume that the curl-free field $\boldsymbol{u}=\nabla\phi$ is compactly supported on $\mathbb{R}^N\setminus\{\boldsymbol{0}\}$: Indeed, let $\{\boldsymbol{u}_n\}\subset C_c^\infty(\mathbb{R}^N\setminus\{\boldsymbol{0}\})^N$ denote the sequence of curl-free fields defined by

$$\boldsymbol{u}_n(\boldsymbol{x})=\nabla\left(\zeta\big(|\boldsymbol{x}|^{\frac{1}{n}}\big)\phi(\boldsymbol{x})\right)\quad\text{for every}\quad n\in\mathbb{N},$$

where we fix $\zeta\in C_c^\infty(\mathbb{R}_+)$ such that $\zeta(r)=\begin{cases}0 & \text{for}\quad 0<r<1/2\\ 1 & \text{for}\quad 1\le r\end{cases}$. Then we see that $\bigcup_{n=1}^\infty\operatorname{supp}\boldsymbol{u}_n$ is bounded, and that the asymptotic formulae

$$\begin{aligned}
\boldsymbol{u}_n(\boldsymbol{x})&=\zeta(|\boldsymbol{x}|^{\frac{1}{n}})\boldsymbol{u}(\boldsymbol{x})+\boldsymbol{\sigma}\,n^{-1}|\boldsymbol{x}|^{\frac{1}{n}-1}\zeta'(|\boldsymbol{x}|^{\frac{1}{n}})\phi(\boldsymbol{x})\\
&=\boldsymbol{u}(\boldsymbol{x})+o(1)\boldsymbol{u}(\boldsymbol{x})+O(1/n)|\boldsymbol{x}|^{-1}\phi(\boldsymbol{x}),\\
\nabla\boldsymbol{u}_n(\boldsymbol{x})&=\nabla\boldsymbol{u}(\boldsymbol{x})+o(1)\nabla\boldsymbol{u}(\boldsymbol{x})+O(1/n)\boldsymbol{\sigma}|\boldsymbol{x}|^{-1}\boldsymbol{u}(\boldsymbol{x})+O(1/n)\boldsymbol{\sigma}\boldsymbol{\sigma}|\boldsymbol{x}|^{-2}\phi(\boldsymbol{x})
\end{aligned}$$

hold as $n \to \infty$. Therefore, taking the L^2 integration on both sides gives

$$\int_{\mathbb{R}^N} |\boldsymbol{u}_n|^2 |\boldsymbol{x}|^{2\gamma-2} dx = \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-2} dx + o(1),$$
$$\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}_n|^2 |\boldsymbol{x}|^{2\gamma} dx = \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx + o(1),$$

thanks to the integrability conditions (14). This result shows that the integrals in the inequality (2) can be approximated by curl-free fields with compact support on $\mathbb{R}^N \setminus \{\boldsymbol{0}\}$.

3.2. **Calculation of the integrals in the Hardy-Leray inequality.** In the rest of the present section, we choose

$$\lambda = 2 - \frac{N}{2} - \gamma \tag{15}$$

in view of § 2.2. Then, with respect to the measure

$$|\boldsymbol{x}|^{2\gamma} dx = r^{2\gamma+N-1} dr \mathrm{d}\sigma = r^{4-2\lambda} \frac{dr}{r} \mathrm{d}\sigma = r^{4-2\lambda} dt\, \mathrm{d}\sigma, \tag{16}$$

the L^2 integration of $\boldsymbol{u}(\boldsymbol{x})/|\boldsymbol{x}| = r^{\lambda-2}\boldsymbol{v}$ can be expressed in terms of f and φ (in Proposition 4) as

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2} |\boldsymbol{x}|^{2\gamma} dx &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} |\boldsymbol{v}|^2 dt\, \mathrm{d}\sigma \\ &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big((f + \partial_t \varphi + \lambda\varphi)^2 + |\nabla_\sigma \varphi|^2 \Big) dt\, \mathrm{d}\sigma \\ &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big(f^2 + (\partial_t\varphi)^2 + \lambda^2\varphi^2 + |\nabla_\sigma\varphi|^2 \Big) dt\, \mathrm{d}\sigma, \end{aligned} \tag{17}$$

where the last equality follows from the integration by parts together with the support compactness and $\int_{\mathbb{S}^{N-1}} \varphi\, \mathrm{d}\sigma = 0$. On the other hand, the integration of $|\nabla \boldsymbol{u}|^2 = |\partial_r \boldsymbol{u}|^2 + r^{-2}|\nabla_\sigma \boldsymbol{u}|^2$ with respect to the measure (16) yields

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx &= \int_{\mathbb{R}^N} \Big(|\partial_r \boldsymbol{u}|^2 + r^{-2} |\nabla_\sigma \boldsymbol{u}|^2 \Big) |\boldsymbol{x}|^{2\gamma} dx \\ &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big(\big|\partial_r (r^{\lambda-1}\boldsymbol{v})\big|^2 + r^{-2} |\nabla_\sigma (r^{\lambda-1}\boldsymbol{v})|^2 \Big) r^{4-2\lambda} dt\, \mathrm{d}\sigma \\ &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big(\big|(\lambda-1)\boldsymbol{v} + \partial_t \boldsymbol{v}\big|^2 + |\nabla_\sigma \boldsymbol{v}|^2 \Big) dt\, \mathrm{d}\sigma \\ &= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big((\lambda-1)^2 |\boldsymbol{v}|^2 + |\partial_t \boldsymbol{v}|^2 \Big) dt\, \mathrm{d}\sigma + \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} |\nabla_\sigma \boldsymbol{v}|^2 dt\, \mathrm{d}\sigma. \end{aligned} \tag{18}$$

To evaluate the last integral, let us take the L^2-inner product of $\triangle_\sigma \boldsymbol{v}$ in (13) and $\boldsymbol{v} = \boldsymbol{\sigma}\big(f + (\partial_t + \lambda)\varphi\big) + \nabla_\sigma \varphi$; then integration by parts gives

(19)
$$\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} |\nabla_\sigma \boldsymbol{v}|^2 dt\, \mathrm{d}\sigma = -\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \boldsymbol{v}\cdot(\triangle_\sigma \boldsymbol{v}) dt\, \mathrm{d}\sigma$$
$$= -\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \big(f + (\partial_t+\lambda)\varphi\big)\big(\partial_t + \lambda - 2\big)\triangle_\sigma \varphi\, dt\, \mathrm{d}\sigma$$
$$+ \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big(-\nabla_\sigma\varphi\cdot\nabla_\sigma\big(2\partial_t + \triangle_\sigma + 2\lambda + 2N - 4\big)\varphi + (N-1)|\boldsymbol{v}|^2\Big) dt\, \mathrm{d}\sigma$$
$$= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \big((\triangle_\sigma\varphi)^2 + (\lambda^2 - 4\lambda - 2N + 4)|\nabla_\sigma\varphi|^2\big)\, dt\, \mathrm{d}\sigma$$
$$+ \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \big(|\partial_t \nabla_\sigma \varphi|^2 + (N-1)|\boldsymbol{v}|^2\big)\, dt\, \mathrm{d}\sigma.$$

Here we note that the spectrum of $-\triangle_\sigma$ is given by the set
$$\big\{\alpha_\nu = \nu(N+\nu-2)\ ;\ \nu \in \mathbb{N}\cup\{0\}\big\},$$
and hence the estimate
$$\frac{1}{\int_{\mathbb{S}^{N-1}} \varphi^2 \mathrm{d}\sigma} \int_{\mathbb{S}^{N-1}} \Big((\triangle_\sigma\varphi)^2 + \big(\lambda^2 - 4\lambda - 2N + 4\big)|\nabla_\sigma\varphi|^2\Big)\mathrm{d}\sigma$$
$$\geq \min_{\nu\in\mathbb{N}} \big\{\alpha_\nu^2 + (\lambda^2 - 4\lambda - 2N + 4)\alpha_\nu\ ;\ \nu \in \mathbb{N}\big\}$$
$$= \alpha_1^2 + (\lambda^2 - 4\lambda - 2N + 4)\alpha_1$$
$$= (N-1)\big((\lambda-2)^2 - N - 1\big)$$
holds for all $\varphi \in C^\infty(\mathbb{S}^{N-1}) \setminus \{0\}$ such that $\int_{\mathbb{S}^{N-1}} \varphi\, \mathrm{d}\sigma = 0$. Also by using $\int_{\mathbb{S}^{N-1}} \partial_t \varphi \mathrm{d}\sigma = 0$, we have the estimate
$$\int_{\mathbb{S}^{N-1}} |\nabla_\sigma(\partial_t\varphi)|^2 \mathrm{d}\sigma \geq (N-1)\int_{\mathbb{S}^{N-1}} |\partial_t\varphi|^2 \mathrm{d}\sigma$$
as an $L^2(\mathbb{S}^{N-1})$ version of the Poincaré inequality. Combine the above two estimates with the right-hand side of (19), and we obtain
$$\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} |\nabla_\sigma \boldsymbol{v}|^2 dt\, \mathrm{d}\sigma \geq (N-1)\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big(|\boldsymbol{v}|^2 + \big((\lambda-2)^2 - N - 1\big)\varphi^2 + (\partial_t\varphi)^2\Big) dt\, \mathrm{d}\sigma$$
to evaluate the last integral in (18); hence we get

(20)
$$\int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx$$
$$\geq \big((\lambda-1)^2 + N - 1\big)\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} |\boldsymbol{v}|^2 dt\, \mathrm{d}\sigma + \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} |\partial_t \boldsymbol{v}|^2 dt\, \mathrm{d}\sigma$$
$$+ (N-1)\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big(\big((\lambda-2)^2 - N - 1\big)\varphi^2 + (\partial_t\varphi)^2\Big) dt\, \mathrm{d}\sigma.$$

Here the equality holds if and only if $-\triangle_\sigma \varphi = \alpha_1 \varphi$; note that this equation also produces for $\partial_t\varphi$ the same equation $-\triangle_\sigma(\partial_t\varphi) = \alpha_1(\partial_t\varphi)$ since ∂_t and $\triangle_\sigma$ commutes.

To further proceed, we have the following two cases according to the sign of $(\lambda-2)^2 - N - 1$:

3.3. **The case $|\lambda-2|\ge\sqrt{N+1}$.** Discarding the last two integrals in (20) and recalling the first equation of (17), we get the Hardy–Leray inequality

$$\int_{\mathbb{R}^N}|\nabla \boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx\ge H_{N,\gamma}\int_{\mathbb{R}^N}\frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2}|\boldsymbol{x}|^{2\gamma}dx$$

for curl-free fields $\boldsymbol{u}$, with the constant

$$H_{N,\gamma}=(\lambda-1)^2+N-1=\left(\gamma+\frac{N}{2}-1\right)^2+N-1.$$

To show that this number is the best possible, let us choose the sequence of curl-free fields $\{\boldsymbol{u}_n=r^{\lambda-1}\boldsymbol{v}_n\}_{n\in\mathbb{N}}\subset C_c^\infty(\mathbb{R}^N\setminus\{\boldsymbol{0}\})^N$ by the formula

$$\boldsymbol{v}_n=\boldsymbol{\sigma}h\Big(\frac{t}{n}\Big)\qquad\Big(\text{or equivalently }\ \boldsymbol{u}_n(\boldsymbol{x})=\boldsymbol{x}|\boldsymbol{x}|^{\lambda-2}h\big(\log|\boldsymbol{x}|^{\frac{1}{n}}\big)\Big)$$

for all $(t,\boldsymbol{\sigma})\in\mathbb{R}\times\mathbb{S}^{N-1}$, where $h\in C_c^\infty(\mathbb{R})$ such that $h\not\equiv0$. Then, noticing that the triplet $(\boldsymbol{u},\boldsymbol{v},\varphi)=(\boldsymbol{u}_n,\boldsymbol{v}_n,0)$ attains the equality of the inequality (20), we get

$$\frac{\int_{\mathbb{R}^N}|\nabla\boldsymbol{u}_n|^2|\boldsymbol{x}|^{2\gamma}dx}{\int_{\mathbb{R}^N}|\boldsymbol{u}_n|^2|\boldsymbol{x}|^{2\gamma-2}dx}=H_{N,\gamma}+\frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}|\partial_t\boldsymbol{v}_n|^2dt\,\mathrm{d}\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}|\boldsymbol{v}_n|^2dt\,\mathrm{d}\sigma}=H_{N,\gamma}+\frac{1}{n^2}\frac{\int_{\mathbb{R}}(h'(t))^2dt}{\int_{\mathbb{R}}(h(t))^2dt}$$
$$\longrightarrow H_{N,\gamma}\qquad\text{as }\ n\to\infty,$$

which proves the best possibility of $H_{N,\gamma}$.

3.4. **The case $|\lambda-2|<\sqrt{N+1}$.** By using the $L^2(\mathbb{S}^{N-1})$-Poincaré inequality and equation (17), we have

$$\begin{aligned}\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\varphi^2dt\,\mathrm{d}\sigma&\le\frac{1}{\lambda^2+\alpha_1}\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\left(\lambda^2\varphi^2+|\nabla_\sigma\varphi|^2\right)dt\,\mathrm{d}\sigma\\&=\frac{1}{\lambda^2+N-1}\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\Big(|\boldsymbol{v}|^2-\big(f^2+(\partial_t\varphi)^2\big)\Big)dt\,\mathrm{d}\sigma,\end{aligned}$$

where the first equality holds if and only if $-\triangle_\sigma\varphi=\alpha_1\varphi$. Combining this estimate with (20) and noting that $(\lambda-2)^2-N-1<0$, we get

(21)
$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx\\&\quad\ge\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\Big(\big((\lambda-1)^2+N-1\big)|\boldsymbol{v}|^2+|\partial_t\boldsymbol{v}|^2+(N-1)(\partial_t\varphi)^2\Big)dt\,\mathrm{d}\sigma\\&\qquad-\frac{(N-1)\left(N+1-(\lambda-2)^2\right)}{\lambda^2+N-1}\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\Big(|\boldsymbol{v}|^2-\big(f^2+(\partial_t\varphi)^2\big)\Big)dt\,\mathrm{d}\sigma\\&\quad=\frac{(\lambda-1)^2(\lambda^2+3(N-1))}{\lambda^2+N-1}\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}|\boldsymbol{v}|^2dt\,\mathrm{d}\sigma+\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}|\partial_t\boldsymbol{v}|^2dt\,\mathrm{d}\sigma\\&\qquad+(N-1)\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\left(\frac{N+1-(\lambda-2)^2}{\lambda^2+N-1}f^2+\frac{4\big(\lambda+\frac{N}{2}-1\big)}{\lambda^2+N-1}(\partial_t\varphi)^2\right)dt\,\mathrm{d}\sigma,\end{aligned}$$

where the first equality holds if and only if $-\triangle_\sigma\varphi=\alpha_1\varphi$. In the same way as before, discard the last two integrals in (21) and recall the first equation of (17); then the Hardy-Leray inequality for curl-free fields

$$\int_{\mathbb{R}^N}|\nabla \boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx\ge H_{N,\gamma}\int_{\mathbb{R}^N}\frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2}|\boldsymbol{x}|^{2\gamma}dx$$

holds with the constant $H_{N,\gamma}$ given by

$$H_{N,\gamma} = \frac{(\lambda-1)^2(\lambda^2+3(N-1))}{\lambda^2+N-1} = \left(\gamma+\frac{N}{2}-1\right)^2 \frac{\left(\gamma+\frac{N}{2}-2\right)^2+3(N-1)}{\left(\gamma+\frac{N}{2}-2\right)^2+N-1}.$$

To show that this $H_{N,\gamma}$ is sharp, let us choose the sequence of curl-free fields $\{\boldsymbol{u}_n = r^{\lambda-1}\boldsymbol{v}_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})^N$ by the formulae

$$\begin{cases} \boldsymbol{v}_n = \boldsymbol{\sigma}(\partial_t+\lambda)\varphi_n + \nabla_\sigma\varphi_n & \Big(\text{or equivalently } \boldsymbol{u}_n(\boldsymbol{x}) = \nabla\left(|\boldsymbol{x}|^\lambda\varphi_n(\boldsymbol{x})\right)\Big) \\ \varphi_n = h\left(\frac{t}{n}\right)Y_1(\boldsymbol{\sigma}), \end{cases}$$

for all $(t,\boldsymbol{\sigma}) \in \mathbb{R}\times\mathbb{S}^{N-1}$, where $h \in C_c^\infty(\mathbb{R})\setminus\{0\}$ and where $Y_1 \in C^\infty(\mathbb{S}^{N-1})$ denotes the eigenfunction of $-\triangle_\sigma$ associated with the eigenvalue $\alpha_1 = N-1$. Then a straightforward calculation yields

$$\begin{aligned}
\frac{\int_{\mathbb{R}\times\mathbb{S}^{N-1}}(\partial_t\varphi_n)^2 dt\,d\sigma}{\int_{\mathbb{R}\times\mathbb{S}^{N-1}}|\boldsymbol{v}_n|^2 dt\,d\sigma} &= \frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}(\partial_t\varphi_n)^2 dt\,d\sigma}{\int_{\mathbb{R}\times\mathbb{S}^{N-1}}\left((\lambda^2+\alpha_1)\varphi_n^2+(\partial_t\varphi_n)^2\right)dt\,d\sigma} \\
&= \frac{\int_{\mathbb{R}} n^{-2}(h'(t))^2 dt}{\int_{\mathbb{R}}\left((\lambda^2+\alpha_1)(h(t))^2+n^{-2}(h'(t))^2\right)dt} \underset{(n\to\infty)}{\longrightarrow} 0, \\
\frac{\int_{\mathbb{R}\times\mathbb{S}^{N-1}}|\partial_t\boldsymbol{v}_n|^2 dt\,d\sigma}{\int_{\mathbb{R}\times\mathbb{S}^{N-1}}|\boldsymbol{v}_n|^2 dt\,d\sigma} &= \frac{\int_{\mathbb{R}\times\mathbb{S}^{N-1}}\left((\lambda^2+\alpha_1)(\partial_t\varphi_n)^2+(\partial_t^2\varphi_n)^2\right)dt\,d\sigma}{\int_{\mathbb{R}\times\mathbb{S}^{N-1}}\left((\lambda^2+\alpha_1)\varphi_n^2+(\partial_t\varphi_n)^2\right)dt\,d\sigma} \\
&= \frac{\int_{\mathbb{R}}\left(n^{-2}(\lambda^2+\alpha_1)(h'(t))^2+n^{-4}(h''(t))^2\right)dt}{\int_{\mathbb{R}}\left((\lambda^2+\alpha_1)(h(t))^2+n^{-2}(h'(t))^2\right)dt} \underset{(n\to\infty)}{\longrightarrow} 0.
\end{aligned}$$

Since the quadruple $(\boldsymbol{u},\boldsymbol{v},\varphi,f) = (\boldsymbol{u}_n,\boldsymbol{v}_n,\varphi_n,0)$ attains the equality in (21), the above calculation directly gives

$$\begin{aligned}
&\frac{\int_{\mathbb{R}^N}|\nabla\boldsymbol{u}_n|^2|\boldsymbol{x}|^{2\gamma}dx}{\int_{\mathbb{R}^N}|\boldsymbol{u}_n|^2|\boldsymbol{x}|^{2\gamma-2}dx} \\
&= H_{N,\gamma} + \frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\left(|\partial_t\boldsymbol{v}_n|^2 dt\,d\sigma + \frac{4(N-1)(\lambda+\frac{N}{2}-1)}{\lambda^2+N-1}(\partial_t\varphi_n)^2\right)dt\,d\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}|\boldsymbol{v}_n|^2 dt\,d\sigma} \\
&\longrightarrow H_{N,\gamma} \qquad \text{as } n\to\infty,
\end{aligned}$$

which proves the sharpness of $H_{N,\gamma}$.

3.5. **Conclusion of the proof of Theorem 1.** In view of the inequalities (20) and (21), we have already proved in §3.3 and §3.4 that every curl-free field $\boldsymbol{u} \in C_c^\infty(\mathbb{R}^N\setminus\{\mathbf{0}\})^N$ satisfies the inequality

$$\begin{aligned}
\int_{\mathbb{R}^N}|\nabla\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx &\ge H_{N,\gamma}\int_{\mathbb{R}^N}|\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-2}dx \\
&\quad + \iint_{\mathbb{R}\times\mathbb{S}^{N-1}}|\partial_t\boldsymbol{v}|^2 dt\,d\sigma + (N-1)\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}\mathcal{E}_{N,\gamma}[\boldsymbol{u}]dt\,d\sigma
\end{aligned}$$

with the constant $H_{N,\gamma}$ in Theorem B and the remainder function $\mathcal{E}_{N,\gamma}[\boldsymbol{u}]$ given by

$$\mathcal{E}_{N,\gamma}[\boldsymbol{u}](\boldsymbol{x}) = \begin{cases} \left((\lambda-2)^2-N-1\right)\varphi^2+(\partial_t\varphi)^2 & \text{for } |\lambda-2| \ge \sqrt{N+1}, \\ \dfrac{N+1-(\lambda-2)^2}{\lambda^2+N-1}f^2 + \dfrac{4(\lambda+\frac{N}{2}-1)}{\lambda^2+N-1}(\partial_t\varphi)^2 & \text{for } |\lambda-2| < \sqrt{N+1}. \end{cases}$$

Moreover, the equality in the above integral inequality holds if and only if $-\triangle_\sigma\varphi = \alpha_1\varphi$. Finally, restoring the notations

$$\lambda = 2-\frac{N}{2}-\gamma, \quad \partial_t = \boldsymbol{x}\cdot\nabla, \quad dt\,d\sigma = |\boldsymbol{x}|^{-N}dx, \quad \boldsymbol{v} = |\boldsymbol{x}|^{\gamma+\frac{N}{2}-1}\boldsymbol{u},$$

we complete the proof. □

4. A proof of the sharp Rellich-Leray inequality for curl-free fields

The same approach to prove Theorem 1 can also be applied to treat other inequalities involving higher-order derivatives. The following sharp Rellich-Leray inequality for curl-free fields was first proven in [7].

Theorem C. ([7]) *Let $\boldsymbol{u} \in C_c^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})^N$ be a curl-free vector field. Then the inequality*

$$R_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^4} |\boldsymbol{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \tag{22}$$

holds with the best constant $R_{N,\gamma}$ given by

$$R_{N,\gamma} = \min \left\{ \left(\alpha_{\gamma - \frac{N}{2}} - N + 1\right)^2,\ \min_{\nu \in \mathbb{N}} \frac{\left(\gamma + \frac{N}{2} - 1\right)^2 + \alpha_\nu}{\left(\gamma + \frac{N}{2} - 3\right)^2 + \alpha_\nu} \left(\alpha_{\gamma - \frac{N}{2} - 1} - \alpha_\nu\right)^2 \right\} \tag{23}$$

in terms of the same notation $\alpha_s = s(s + N - 2)$ as in (5).

In this section, we prove the following improvement of Theorem C.

Theorem 6. *Let $R_{N,\gamma}$ be the same as in* (23). *Then the inequality* (22) *can be further improved to be*

$$\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \geq R_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^4} |\boldsymbol{x}|^{2\gamma} dx + c_{N,\gamma} \int_{\mathbb{R}^N} \left| \boldsymbol{x} \cdot \nabla \left(|\boldsymbol{x}|^{\gamma + \frac{N}{2} - 2} \boldsymbol{u} \right) \right|^2 |\boldsymbol{x}|^{-N} dx$$

for some positive constant $c_{N,\gamma} > 0$.

As a direct consequence of this fact, the equality sign of inequality (22) is never attained by any non-zero curl-free field $\boldsymbol{u}$.

Proof. Let $\boldsymbol{u} \in C_c^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})^N$ be a curl-free field. Applying the replacement

$$\gamma \longmapsto \gamma - 1 \tag{24}$$

to equation (15), we choose

$$\lambda = 3 - N/2 - \gamma.$$

By this choice, let us calculate the integrals in inequality (22): Apply the replacement (24) to the equations in (17), and we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^4} |\boldsymbol{x}|^{2\gamma} dx &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\boldsymbol{v}|^2 dt \, d\sigma \\ &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left(f^2 + (\partial_t \varphi)^2 + \lambda^2 \varphi^2 + |\nabla_\sigma \varphi|^2 \right) dt \, d\sigma \\ &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left(f^2 + \varphi \left(\lambda^2 - \partial_t^2 - \triangle_\sigma \right) \varphi \right) dt \, d\sigma, \end{aligned}$$

where the last equality follows from integration by parts together with the support compactness. Also we notice that the condition (12) in Proposition 4 is invariant under the following replacement of the triplet:

$$(\boldsymbol{v}, f, \varphi) \longmapsto (\partial_t^k \boldsymbol{v}, \partial_t^k f, \partial_t^k \varphi) \tag{25}$$

for $k = 1, 2$. Hence we have

$$\iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\partial_t^k \boldsymbol{v}|^2 dt \, d\sigma = \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left((\partial_t^k f)^2 + \varphi \left(\lambda^2 - \partial_t^2 - \triangle_\sigma \right) (-\partial_t^2)^k \varphi \right) dt \, d\sigma. \tag{26}$$

On the other hand, with the aid of (11), we have

$$\begin{aligned}\triangle \boldsymbol{u} &= \triangle(r^{\lambda-1}\boldsymbol{v}) = (\triangle r^{\lambda-1})\boldsymbol{v} + 2\left((\nabla r^{\lambda-1})\cdot\nabla\right)\boldsymbol{v} + r^{\lambda-1}\triangle\boldsymbol{v}\\ &= \alpha_{\lambda-1}r^{\lambda-3}\boldsymbol{v} + 2(\lambda-1)r^{\lambda-3}\partial_t\boldsymbol{v} + r^{\lambda-3}\left(\partial_t^2 + (N-2)\partial_t + \triangle_\sigma\right)\boldsymbol{v}\\ &= r^{\lambda-3}\left(\alpha_{\lambda-1}\boldsymbol{v} + (2\lambda+N-4)\partial_t\boldsymbol{v} + \partial_t^2\boldsymbol{v} + \triangle_\sigma\boldsymbol{v}\right),\end{aligned}$$

where in the second line we have used the same formula $\triangle r^{\lambda-1} = \alpha_{\lambda-1}r^{\lambda-3}$ as in $(5)_{s=\lambda-1}$. Then the L^2 integration (by parts) of this result yields

(27)

$$\begin{aligned}\int\limits_{\mathbb{R}^N}|\triangle\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx &= \iint\limits_{\mathbb{R}\times\mathbb{S}^{N-1}}\left|\alpha_{\lambda-1}\boldsymbol{v} + (2\lambda+N-4)\partial_t\boldsymbol{v} + \partial_t^2\boldsymbol{v} + \triangle_\sigma\boldsymbol{v}\right|^2 dt\,\mathrm{d}\sigma\\ &= \iint\limits_{\mathbb{R}\times\mathbb{S}^{N-1}}|\partial_t^2\boldsymbol{v}|^2dt\,\mathrm{d}\sigma + \left((N-2)^2 + 2\alpha_{\lambda-1}\right)\iint\limits_{\mathbb{R}\times\mathbb{S}^{N-1}}|\partial_t\boldsymbol{v}|^2dt\,\mathrm{d}\sigma\\ &\quad + 2\iint\limits_{\mathbb{R}\times\mathbb{S}^{N-1}}|\nabla_\sigma\partial_t\boldsymbol{v}|^2dt\,\mathrm{d}\sigma + \iint\limits_{\mathbb{R}\times\mathbb{S}^{N-1}}\left|\triangle_\sigma\boldsymbol{v} + \alpha_{\lambda-1}\boldsymbol{v}\right|^2dt\,\mathrm{d}\sigma,\end{aligned}$$

where the second equality follows from the identity $(2\lambda+N-4)^2 - 2\alpha_{\lambda-1} = (N-2)^2 + 2\alpha_{\lambda-1}$. To calculate the second last integral in (27), we apply the replacement (25) to the equation in (19):

(28)

$$\begin{aligned}&\iint_{\mathbb{R}\times\mathbb{S}^{N-1}}|\partial_t\nabla_\sigma\boldsymbol{v}|^2dt\,\mathrm{d}\sigma\\ &= \iint\limits_{\mathbb{R}\times\mathbb{S}^{N-1}}\begin{pmatrix}(\triangle_\sigma\partial_t\varphi)^2 + |\nabla_\sigma\partial_t^2\varphi|^2 + \left((\lambda-2)^2-2N\right)|\nabla_\sigma\partial_t\varphi|^2\\ +(N-1)|\partial_t\boldsymbol{v}|^2\end{pmatrix}dt\,\mathrm{d}\sigma\\ &= \iint\limits_{\mathbb{R}\times\mathbb{S}^{N-1}}\begin{pmatrix}\varphi\Big(-\triangle_\sigma^2\partial_t^2 - \triangle_\sigma\partial_t^4 + \left((\lambda-2)^2-2N\right)\triangle_\sigma\partial_t^2\Big)\varphi\\ +(N-1)\Big((\partial_t f)^2 + \varphi\left(\lambda^2-\partial_t^2-\triangle_\sigma\right)(-\partial_t^2\varphi)\Big)\end{pmatrix}dt\,\mathrm{d}\sigma.\end{aligned}$$

Also to calculate the last integral in (27), let us compute from (13) and (12) that

$$\begin{aligned}&\triangle_\sigma\boldsymbol{v} + \alpha_{\lambda-1}\boldsymbol{v}\\ &\quad= \boldsymbol{\sigma}\big(\partial_t+\lambda-2\big)\triangle_\sigma\varphi + \nabla_\sigma\big(2\partial_t + \triangle_\sigma + 2(\lambda+N-2)\big)\varphi\\ &\qquad + \big(\alpha_{\lambda-1}-(N-1)\big)\big(\boldsymbol{\sigma}\big(f + (\partial_t+\lambda)\varphi\big) + \nabla_\sigma\varphi\big)\\ &\quad= \boldsymbol{\sigma}\Big((\triangle_\sigma + \alpha_{\lambda-1} - N + 1)\partial_t\varphi + (\lambda-2)\triangle_\sigma\varphi + (\alpha_{\lambda-1}-N+1)(f+\lambda\varphi)\Big)\\ &\qquad + \nabla_\sigma\left(2\partial_t + \triangle_\sigma + \alpha_\lambda\right)\varphi,\end{aligned}$$

here we have used $\alpha_{\lambda-1} - (N-1) + 2(\lambda + N - 2) = \alpha_\lambda$ in the second equality. Hence the L^2 integration by parts of this result yields

$$
\begin{aligned}
(29) \quad & \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \left|\triangle_\sigma \boldsymbol{v} + \alpha_{\lambda-1}\boldsymbol{v}\right|^2 dt\,\mathrm{d}\sigma \\
&= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \left| \begin{array}{l} (\triangle_\sigma + \alpha_{\lambda-1} - N + 1)\partial_t\varphi \\ +(\lambda-2)\triangle_\sigma\varphi + (\alpha_{\lambda-1} - N + 1)(f + \lambda\varphi) \end{array} \right|^2 dt\,\mathrm{d}\sigma \\
&\quad + \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \left|\nabla_\sigma\left(2\partial_t + \triangle_\sigma + \alpha_\lambda\right)\varphi\right|^2 dt\,\mathrm{d}\sigma \\
&= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \left(\begin{array}{l} \left((\triangle_\sigma + \alpha_{\lambda-1} - N + 1)\partial_t\varphi\right)^2 + 4\left|\partial_t\nabla_\sigma\varphi\right|^2 \\ + \left|(\lambda-2)\triangle_\sigma\varphi + (\alpha_{\lambda-1} - N + 1)\lambda\varphi\right|^2 \\ + \left|\nabla_\sigma\triangle_\sigma\varphi + \alpha_\lambda\nabla_\sigma\varphi\right|^2 + (\alpha_{\lambda-1} - N + 1)^2 f^2 \end{array} \right) dt\,\mathrm{d}\sigma \\
&= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \left(\begin{array}{l} \varphi\Big((\triangle_\sigma + \alpha_{\lambda-1} - N + 1)^2 - 4\triangle_\sigma\Big)(-\partial_t^2\varphi) \\ + \varphi\big((\lambda-2)^2 - \triangle_\sigma\big)(\triangle_\sigma + \alpha_\lambda)^2\varphi \\ + (\alpha_{\lambda-1} - N + 1)^2 f^2 \end{array} \right) dt\,\mathrm{d}\sigma.
\end{aligned}
$$

Substitute (26), (28) and (29) into (27), and after some lengthy algebraic calculations, we obtain

$$
\begin{aligned}
(30) \quad \int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx &= (26)_{k=2} + \left((N-2)^2 + 2\alpha_{\lambda-1}\right)(26)_{k=1} + 2\times(28) + (29) \\
&= \iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big(\varphi\, Q_1(-\partial_t^2, -\triangle_\sigma)\varphi + f Q_0(-\partial_t^2) f\Big) dt\,\mathrm{d}\sigma,
\end{aligned}
$$

where $Q_1(\cdot,\cdot)$ and $Q_0(\cdot)$ are the polynomials given by

$$
\begin{aligned}
(31) \quad Q_1(\tau,\alpha) &= \left(\lambda^2 + \tau + \alpha\right)\tau^2 + ((N-2)^2 + 2\alpha_{\lambda-1})\left(\lambda^2 + \tau + \alpha\right)\tau \\
&\quad + 2\Big(\alpha^2\tau + \alpha\tau^2 + \left((\lambda-2)^2 - 2N\right)\alpha\tau + (N-1)(\lambda^2 + \tau + \alpha)\tau\Big) \\
&\quad + \left((-\alpha + \alpha_{\lambda-1} - N + 1)^2 + 4\alpha\right)\tau + \left((\lambda-2)^2 + \alpha\right)(-\alpha + \alpha_\lambda)^2 \\
&= \left(\tau + \alpha + (\lambda-2)^2\right) \left(\begin{array}{l} \left(\tau + \alpha + \lambda^2\right)\left(\tau + \alpha + (\lambda + N - 2)^2\right) \\ -(2\lambda + N - 2)^2\alpha \end{array} \right), \\
(32) \quad Q_0(\tau) &= \tau^2 + \left((N-2)^2 + 2\alpha_{\lambda-1}\right)\tau + 2(N-1)\tau + (\alpha_{\lambda-1} - N + 1)^2 \\
&= \left(\tau + (\lambda-2)^2\right)\left(\tau + (\lambda + N - 2)^2\right).
\end{aligned}
$$

Therefore, we get

$$
(33) \quad \frac{\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-4} dx} = \frac{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \Big(\varphi\, Q_1(-\partial_t^2, -\triangle_\sigma)\varphi + f Q_0(-\partial_t^2) f\Big) dt\,\mathrm{d}\sigma}{\iint_{\mathbb{R}\times\mathbb{S}^{N-1}} \left(\varphi(\lambda^2 - \partial_t^2 - \triangle_\sigma)\varphi + f^2\right) dt\,\mathrm{d}\sigma}
$$

as far as $\boldsymbol{u} \not\equiv \boldsymbol{0}$.

From now on, we evaluate the right-hand side of (33). We apply to φ and f the 1-D Fourier transformation with respect to t: we set

$$
\widehat{\varphi}(\tau,\boldsymbol{\sigma}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau t}\varphi(e^t\boldsymbol{\sigma}) dt, \qquad \widehat{f}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau t} f(e^t\boldsymbol{\sigma}) dt
$$

for $(\tau,\boldsymbol{\sigma})\in\mathbb{R}\times\mathbb{S}^{N-1}$. Furthermore, we apply to $\widehat{\varphi}$ the spherical harmonics expansion:

$$\widehat{\varphi}(\tau,\boldsymbol{\sigma})=\sum_{\nu\in\mathbb{N}}\widehat{\varphi_\nu}(\tau)Y_\nu(\boldsymbol{\sigma}),\qquad \begin{cases}-\triangle_\sigma Y_\nu=\alpha_\nu Y_\nu,\\ \alpha_\nu=\nu(\nu+N-2)\quad\forall\nu\in\mathbb{N},\end{cases}$$

with the normalization $\int_{\mathbb{S}^{N-1}}|Y_\nu(\boldsymbol{\sigma})|^2d\sigma=1$. Substituting these formulae into (33) and noticing the $L^2(\mathbb{R})$ isometry of the Fourier transformation, we have

(34)

$$\begin{aligned}\frac{\int_{\mathbb{R}^N}|\triangle\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx}{\int_{\mathbb{R}^N}|\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-4}dx}&=\frac{\sum_{\nu\in\mathbb{N}}\int_{\mathbb{R}}Q_1(\tau^2,\alpha_\nu)|\widehat{\varphi_\nu}(\tau)|^2d\tau+\omega_{N-1}\int_{\mathbb{R}}Q_0(\tau^2)|\widehat{f}(\tau)|^2d\tau}{\sum_{\nu\in\mathbb{N}}\int_{\mathbb{R}}(\lambda^2+\tau^2+\alpha_\nu)|\widehat{\varphi_\nu}(\tau)|^2d\tau+\omega_{N-1}\int_{\mathbb{R}}|\widehat{f}(\tau)|^2d\tau}\\ &\geq\min\left\{\inf_{\nu\in\mathbb{N}}\inf_{\tau\in\mathbb{R}}\frac{Q_1(\tau^2,\alpha_\nu)}{\lambda^2+\tau^2+\alpha_\nu},\ \inf_{\tau\in\mathbb{R}}Q_0(\tau^2)\right\}\\ &=\min\left\{\min_{\nu\in\mathbb{N}}\frac{Q_1(0,\alpha_\nu)}{\lambda^2+\alpha_\nu},\ Q_0(0)\right\}.\end{aligned}$$

Here the last equality follows from that in view of (31) and (32) the functions

$$\frac{Q_1(\tau,\alpha_\nu)}{\lambda^2+\tau+\alpha_\nu}=\left(\tau+\alpha_\nu+(\lambda-2)^2\right)\left(\tau+\alpha_\nu\left(1-\frac{(2\lambda+N-2)^2}{\lambda^2+\tau+\alpha_\nu}\right)+(\lambda+N-2)^2\right)$$

and $Q_0(\tau)$ are monotonically increasing in $\tau\in[0,\infty)$ for each $\nu\in\mathbb{N}$. Therefore, we have proved the Rellich-Leray inequality for curl-free fields (22):

$$\int_{\mathbb{R}^N}|\triangle\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx\geq R_{N,\gamma}\int_{\mathbb{R}^N}|\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma-4}dx$$

holds with the constant $R_{N,\gamma}$ given by

$$\begin{aligned}R_{N,\gamma}&=\min\left\{\min_{\nu\in\mathbb{N}}\frac{Q_1(0,\alpha_\nu)}{\lambda^2+\alpha_\nu},\ Q_0(0)\right\}\\ &=\min\left\{\min_{\nu\in\mathbb{N}}\frac{(\lambda-2)^2+\alpha_\nu}{\lambda^2+\alpha_\nu}(\alpha_\lambda-\alpha_\nu)^2,\ (\alpha_{\lambda-1}-N+1)^2\right\}\\ &=\min\left\{\min_{\nu\in\mathbb{N}}\frac{\left(\gamma+\frac{N}{2}-1\right)^2+\alpha_\nu}{\left(\gamma+\frac{N}{2}-3\right)^2+\alpha_\nu}\left(\alpha_{\gamma-\frac{N}{2}-1}-\alpha_\nu\right)^2,\ \left(\alpha_{\gamma-\frac{N}{2}}-N+1\right)^2\right\}.\end{aligned}$$

Now, we prove the sharpness of $R_{N,\gamma}$. For this purpose, we choose $\nu_0\in\mathbb{N}\cup\{0\}$ to be such that

$$\begin{cases}\nu_0=0,\quad\text{if }\min\left\{\min_{\nu\in\mathbb{N}}\frac{Q_1(0,\alpha_\nu)}{\lambda^2+\alpha_\nu},\ Q_0(0)\right\}=Q_0(0),\\ \min_{\nu\in\mathbb{N}}\frac{Q_1(0,\alpha_\nu)}{\lambda^2+\alpha_\nu}=\frac{Q_1(0,\alpha_{\nu_0})}{\lambda^2+\alpha_{\nu_0}},\quad\text{otherwise,}\end{cases}$$

and define the sequence of vector fields $\{\boldsymbol{u}_n=r^{\lambda-1}\boldsymbol{v}_n\}_{n\in\mathbb{N}}\subset C_c^\infty(\mathbb{R}^N\setminus\{\boldsymbol{0}\})^N$ by the formulae

$$\boldsymbol{v}_n=\begin{cases}\boldsymbol{\sigma}f_n&\text{if }\ \nu_0=0,\\ \boldsymbol{\sigma}(\partial_t+\lambda)\varphi_n+\nabla_\sigma\varphi_n&\text{otherwise.}\end{cases}$$

Here

$$\begin{cases}f_n(\boldsymbol{x})=h\left(\frac{\log|\boldsymbol{x}|}{n}\right)&\text{if }\ \nu_0=0,\\ \varphi_n(\boldsymbol{x})=h\left(\frac{\log|\boldsymbol{x}|}{n}\right)Y_{\nu_0}(\boldsymbol{x}/|\boldsymbol{x}|)&\text{otherwise,}\end{cases}$$

with $h \in C_c^\infty(\mathbb{R}) \setminus \{0\}$, and $Y_{\nu_0} \in C^\infty(\mathbb{S}^{N-1})$ denotes the eigenfunction of $-\triangle_\sigma$ associated with the eigenvalue $\alpha_{\nu_0} = \nu_0(\nu_0 + N - 2)$. Notice from Proposition 4 that $\boldsymbol{u}_n$ is curl-free. Then applying the formula (33) to $(\boldsymbol{u}, f, \varphi) = (\boldsymbol{u}_n, f_n, 0)$ or $(\boldsymbol{u}, f, \varphi) = (\boldsymbol{u}_n, 0, \varphi_n)$ gives

$$\frac{\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}_n|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}_n|^2 |\boldsymbol{x}|^{2\gamma-4} dx} = \begin{cases} \dfrac{\int_{\mathbb{R}} h(\frac{t}{n}) Q_0(-\partial_t^2) h(\frac{t}{n}) dt}{\int_{\mathbb{R}} \left(h(\frac{t}{n})\right)^2 dt} & \text{if } \ \nu_0 = 0, \\ \dfrac{\int_{\mathbb{R}} h(\frac{t}{n}) Q_1(-\partial_t^2, \alpha_{\nu_0}) h(\frac{t}{n})\, dt}{\int_{\mathbb{R}} h(\frac{t}{n})(\lambda^2 - \partial_t^2 + \alpha_{\nu_0}) h(\frac{t}{n})\, dt} & \text{otherwise.} \end{cases}$$

Passing to $n \to \infty$, we get

$$\begin{aligned} \frac{\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}_n|^2 |\boldsymbol{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\boldsymbol{u}_n|^2 |\boldsymbol{x}|^{2\gamma-4} dx} &= O(1/n^2) + \begin{cases} Q_0(0) & \text{if } \ \nu_0 = 0 \\ \dfrac{Q_1(0, \alpha_{\nu_0})}{\lambda^2 + \alpha_{\nu_0}} & \text{otherwise} \end{cases} \\ &\longrightarrow R_{N,\gamma}, \end{aligned}$$

which shows the desired sharpness of $R_{N,\gamma}$.

In order to obtain further improvement, we recall that the two integrals in (22) can be expressed in terms of φ and f (in Proposition 4) as

$$\begin{aligned} \int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx &= \sum_{\nu \in \mathbb{N}} \int_{\mathbb{R}} Q_1(\tau^2, \alpha_\nu) |\widehat{\varphi_\nu}(\tau)|^2 d\tau + \omega_{N-1} \int_{\mathbb{R}} Q_0(\tau^2) |\widehat{f}(\tau)|^2 d\tau, \\ \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma-4} dx &= \sum_{\nu \in \mathbb{N}} \int_{\mathbb{R}} (\lambda^2 + \tau^2 + \alpha_\nu) |\widehat{\varphi_\nu}(\tau)|^2 d\tau + \omega_{N-1} \int_{\mathbb{R}} |\widehat{f}(\tau)|^2 d\tau \end{aligned}$$

for $\lambda = 3 - N/2 - \gamma$, together with the polynomials Q_1 and Q_0 given by (31) and (32). Also recall the expression $R_{N,\gamma} = \min\left\{ \min_{\nu \in \mathbb{N}} \frac{Q_1(0,\alpha_\nu)}{\lambda^2 + \alpha_\nu}, Q_0(0) \right\}$ of the best constant of the inequality (22) and let $\nu_1 \in \mathbb{N}$ be such that

$$\frac{Q_1(0, \alpha_{\nu_1})}{\lambda^2 + \alpha_{\nu_1}} = \min_{\nu \in \mathbb{N}} \frac{Q_1(0, \alpha_\nu)}{\lambda^2 + \alpha_\nu}.$$

Then the difference between the both sides of (22) has the following estimate:

$$\begin{aligned} &\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx - \min\left\{ \frac{Q_1(0, \alpha_{\nu_1})}{\lambda^2 + \alpha_{\nu_1}}, Q_0(0) \right\} \int_{\mathbb{R}^N} |\boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \\ &\quad \geq \int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx \\ &\qquad - \frac{Q_1(0, \alpha_{\nu_1})}{\lambda^2 + \alpha_{\nu_1}} \sum_{\nu \in \mathbb{N}} \int_{\mathbb{R}} (\tau^2 + \lambda^2 + \alpha_\nu) |\widehat{\varphi_\nu}(\tau)|^2 d\tau - Q_0(0)\, \omega_{N-1} \int_{\mathbb{R}} |\widehat{f}(\tau)|^2 d\tau \\ &\quad = \sum_{\nu \in \mathbb{N}} \int_{\mathbb{R}} \left(Q_1(\tau^2, \alpha_\nu) - \frac{Q_1(0, \alpha_{\nu_1})}{\lambda^2 + \alpha_{\nu_1}} (\tau^2 + \lambda^2 + \alpha_\nu) \right) |\widehat{\varphi_\nu}(\tau)|^2 \\ &\qquad + \omega_{N-1} \int_{\mathbb{R}} \left(Q_0(\tau^2) - Q_0(0) \right) |\widehat{f}(\tau)|^2 d\tau \\ &\quad \geq c_{1,N,\gamma} \sum_{\nu \in \mathbb{N}} \int_{\mathbb{R}} (\lambda^2 + \tau^2 + \alpha_\nu) \tau^2 |\widehat{\varphi_\nu}(\tau)|^2 d\tau + c_{0,N,\gamma}\, \omega_{N-1} \int_{\mathbb{R}} \tau^2 |\widehat{f}(\tau)|^2 d\tau \\ &\quad \geq \min\{c_{1,N,\gamma},\ c_{0,N,\gamma}\} \int_{\mathbb{R}} \left(\sum_{\nu \in \mathbb{N}} (\lambda^2 + \tau^2 + \alpha_\nu) \tau^2 |\widehat{\varphi_\nu}(\tau)|^2 + \omega_{N-1} \tau^2 |\widehat{f}(\tau)|^2 \right) d\tau \\ &\quad = \min\{c_{1,N,\gamma},\ c_{0,N,\gamma}\} \int_{\mathbb{R}^N} |\partial_t \boldsymbol{v}|^2 r^{-N} dx, \end{aligned}$$

where we have defined the two constants $c_{0,N,\gamma}$ and $c_{1,N,\gamma}$ by

$$\begin{aligned} c_{0,N,\gamma} &= \inf_{\tau\in\mathbb{R}\setminus\{0\}} \frac{Q_0(\tau^2)-Q_0(0)}{\tau^2} = Q_0'(0) = (\lambda-2)^2+(\lambda+N-2)^2 \\ &= N^2/2+2(1-\gamma)^2>0, \\ c_{1,N,\gamma} &= \inf_{\nu\in\mathbb{N}} \inf_{\tau\in\mathbb{R}\setminus\{0\}} \frac{1}{\tau^2}\left(\frac{Q_1(\tau^2,\alpha_\nu)}{\tau^2+\lambda^2+\alpha_\nu} - \frac{Q_1(0,\alpha_{\nu_1})}{\lambda^2+\alpha_{\nu_1}}\right). \end{aligned}$$

Hence it suffices to show $c_{1,N,\gamma}>0$. To this end, notice that

$$c_{1,N,\gamma} \ge \inf_{\nu\in\mathbb{N}} \inf_{\tau\in\mathbb{R}\setminus\{0\}} \frac{1}{\tau^2}\left(\frac{Q_1(\tau^2,\alpha_\nu)}{\tau^2+\lambda^2+\alpha_\nu} - \frac{Q_1(0,\alpha_\nu)}{\lambda^2+\alpha_\nu}\right) = \inf_{\nu\in\mathbb{N}} \inf_{\tau\ge 0} Q_2(\tau,\alpha_\nu)$$

for the rational polynomial $Q_2(\cdot,\cdot)$ defined by the following algebraic calculation:

$$\begin{aligned} Q_2(\tau,a) &= \frac{1}{\tau}\left(\frac{Q_1(\tau,a)}{\tau+\lambda^2+a} - \frac{Q_1(0,a)}{\lambda^2+a}\right) \\ &= \frac{4(1-\lambda)(2\lambda+N-2)^2a}{(\lambda^2+a)(\tau+\lambda^2+a)} + 2\left(\lambda+\frac{N}{2}-2\right)^2 + \frac{N^2}{2} + 2a + \tau \\ &= \frac{16(1-\lambda)(2-\gamma)^2a}{(\lambda^2+a)(\tau+\lambda^2+a)} + c_{0,N,\gamma} + 2a + \tau. \end{aligned}$$

In order to further estimate $Q_2(\tau,\alpha_\nu)$ for $\tau\ge 0$ and $\nu\in\mathbb{N}$, let us consider the following two cases: for $\lambda\le 1$, it is clear that $Q_2(\tau,\alpha_\nu)\ge c_{0,N,\gamma}+2\alpha_\nu$. For $\lambda>1$, since it is clear that $Q_2(\tau,\alpha_\nu)$ is monotone increasing in τ, we have

$$\begin{aligned} Q_2(\tau,\alpha_\nu) \ge Q_2(0,\alpha_\nu) &= -\frac{16(\lambda-1)(2-\gamma)^2\alpha_\nu}{(\lambda^2+\alpha_\nu)^2} + c_{0,N,\gamma} + 2\alpha_\nu \\ &\ge -\frac{16(\lambda-1)(2-\gamma)^2\alpha_\nu}{4\lambda^2\alpha_\nu} + c_{0,N,\gamma} + 2\alpha_\nu \\ &\ge -(2-\gamma)^2 + c_{0,N,\gamma} + 2\alpha_\nu \\ &= \gamma^2 + N^2/2 + 2(\alpha_\nu - 1) \ge N^2/2 + 2(\alpha_1-1), \end{aligned}$$

where the inequalities in the second and third lines follow from

$$(\lambda^2+\alpha_\nu)^2 \ge 4\lambda^2\alpha_\nu \quad \text{and} \quad -(\lambda-1)/\lambda^2 \ge -1/4.$$

Hence it turns out that $\inf_{\nu\in\mathbb{N}} \inf_{\tau\ge 0} Q_2(\tau,\alpha_\nu)>0$, and hence that $c_{1,N,\gamma}>0$. Therefore, we have obtained the inequality

$$\int_{\mathbb{R}^N} |\triangle \boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx - R_{N,\gamma}\int_{\mathbb{R}^N} |\boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx \ge c_{N,\gamma}\int_{\mathbb{R}^N} \left|\boldsymbol{x}\cdot\nabla\left(|\boldsymbol{x}|^{\gamma+\frac{N}{2}-1}\boldsymbol{u}\right)\right|^2|\boldsymbol{x}|^{-N}dx$$

for $c_{N,\gamma}=\min\{c_{0,N,\gamma},c_{1,N,\gamma}\}>0$. The proof of Theorem 6 is now complete, although the constant $c_{N,\gamma}$ is not ensured to be optimal. □

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