Refined construction of type II blow-up solutions for semilinear heat equations with Joseph–Lundgren supercritical nonlinearity

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Abstract

We are concerned with blow-up mechanisms in a semilinear heat equation:

$$u_t = \Delta u + |x|^{2a} u^p, \qquad x \in \mathbf{R}^N,\ t > 0,$$

where $p > 1$ and $a > -1$ are constants. As for the Fujita equation, which corresponds to $a = 0$, a well-known result due to M. A. Herrero and J. J. L. Velázquez, C. R. Acad. Sci. Paris Sér. I Math. (1994), states that if $N \geq 11$ and $p > 1 + 4/(N - 4 - 2\sqrt{N-1})$, then there exist radial blow-up solutions $u_{\ell,\mathrm{HV}}(x,t)$, $\ell \in \mathbb{N}$, such that

$$\lim_{t\to T} (T-t)^{\frac{1}{p-1}} \|u_{\ell,\mathrm{HV}}(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} = +\infty,$$

where T is the blow-up time. We revisit the idea of their construction and obtain refined estimates for such solutions by the techniques developed in recent works and elaborate estimates of the heat semigroup in backward similarity variables. Our method is naturally extended to the case $a \neq 0$. As a consequence, we obtain an example of solutions that blow up at $x = 0$, the zero point of potential $|x|^{2a}$ with $a > 0$, for $N > 10 + 8a$. This last result is contrast to backward self-similar solutions previously obtained for $N < 10 + 8a$, which blow up at $x = 0$.

Key words: non-self-similar blow-up; matched asymptotic expansions; blow-up at zero point
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1 Introduction and main results

In the present article we discuss blow-up behavior for a semilinear heat equation:

$$u_t = \Delta u + |u|^{p-1} u, \qquad x \in \mathbf{R}^N,\ t > 0, \tag{1.1}$$

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and its variant

$$u_t = \Delta u + |x|^{2a} u^p, \qquad x \in \mathbf{R}^N,\ t > 0, \tag{1.2}$$

where Δ denotes the Laplacian in $\mathbf{R}^N$, $p > 1$ and $a > -1$ are constants. Given an initial datum $u|_{t=0} = u_0 \in L^\infty(\mathbf{R}^N)$, we may uniquely obtain a local-in-time classical solution of (1.1) (resp., (1.2)). See, for instance, [38, 44].

1.1 Study of equation (1.1)

The apparently simple equation (1.1) has been widely studied by many researchers since the pioneering work [10] by H. Fujita. In particular, describing possible blow-up behavior at the blow-up time has attracted considerable attention in the past decades. We say that a solution u of (1.1) blows up in a finite time T if

$$\limsup_{t \to T} \|u(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} = +\infty. \tag{1.3}$$

A number of sufficient conditions for finite time blow-up have been obtained by many researchers. For example, if a nonnegative initial data u_0 satisfies

$$u_0(x) \geq \lambda U_\alpha(|x|), \quad x \in \mathbf{R}^N \tag{1.4}$$

for some constants $\lambda > 1$ and $\alpha > 1$, then the solution of (1.1) blows up in finite time, where $U_\alpha(|x|)$ denotes a regular stationary solution of (1.1) (see (1.9) below). In this article, we are concerned with the blow-up rate of $\|u(\cdot, t)\|_{L^\infty(\mathbf{R}^N)}$ as t approaches the blow-up time T. Local theory implies that there is a constant $C > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} \geq C(T - t)^{-1/(p-1)}, \quad 0 < t < T$$

if the maximal time of existence $T = T(u_0)$ is finite (cf. [38, Chapter II]). On the other hand, it is far from obvious whether the corresponding upper estimate holds. A blow-up is said to be of **type I** if there exists a positive constant K such that

$$\|u(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} \leq K(T - t)^{-1/(p-1)}, \quad 0 < t < T; \tag{1.5}$$

whereas the blow-up is said to be of **type II** otherwise. In the Sobolev subcritical case $p < p_{\mathrm{S}}$, where

$$p_{\mathrm{S}} := \begin{cases} +\infty, & N = 1, 2, \\ \dfrac{N+2}{N-2}, & N \geq 3, \end{cases} \tag{1.6}$$

every blow-up for (1.1) is of type I even for non-radial or sign-changing solutions [11, 12] (see also [6] for a related parabolic system). In the Sobolev supercitical case, the situation drastically changes according to whether or not p is less or greater than the Joseph–Lundgren exponent

$$p_{\mathrm{JL}} := \begin{cases} +\infty, & N \leq 10, \\ 1 + \dfrac{4}{N - 4 - 2\sqrt{N-1}}, & N \geq 11. \end{cases} \tag{1.7}$$

Indeed, if $p_{\rm S} < p < p_{\rm JL}$, only type I blow-up occurs for radial solutions under mild assumptions on initial data [26, 27, 32], whereas type II blow-up does occur for $p > p_{\rm JL}$ as we are going to recall below. To this end, let us write

$$\beta = \frac{1}{p-1}, \tag{1.8a}$$

$$\gamma = \frac{N-2-\sqrt{D}}{2}, \tag{1.8b}$$

$$D = 16\beta^2 - 8(N-4)\beta + (N-2)(N-10). \tag{1.8c}$$

All the radial regular stationary solutions, denoted by $U_\alpha(r)$, are parametrized by their values at the origin, i.e., $\alpha = U_\alpha(0) \in \mathbf{R}$. It is known (cf. Proposition 2.1 below) that, when $p > p_{\rm JL}$,

$$U_\alpha(r) = U_\infty(r) - h_\alpha r^{-\gamma} + o(r^{-\gamma}), \qquad \text{as } r \to \infty, \tag{1.9}$$

where $h_\alpha > 0$ is a constant depending on α and $U_\infty(r)$ is the singular stationary solution:

$$U_\infty(r) = c_* r^{-2\beta} \quad \text{with} \quad c_*^{p-1} = 2\beta\,(N-2-2\beta)\,. \tag{1.10}$$

Herrero and Velázquez [21, 22] proved that, as long as $N \geq 11$ and $p_{\rm JL} < p$, type II blow-up actually occurs. They constructed radial blow-up solutions $\{u_{\ell,\rm HV}\}_{\ell\in\Lambda}$, $\Lambda \subset \mathbf{N}$, (which we call *HV solutions*) satisfying $\|u_{\ell,\rm HV}(\cdot,t)\|_\infty = u_{\ell,\rm HV}(0,t)$ and

$$C_1\,(T-t)^{-\beta-2\beta\omega_\ell} \leq u_{\ell,\rm HV}(0,t) \leq C_2\,(T-t)^{-\beta-2\beta\omega_\ell} \tag{1.11a}$$

$$\text{with} \quad \omega_\ell = \frac{\lambda_\ell}{\gamma-2\beta} > 0, \quad \lambda_\ell = \ell - \frac{\gamma}{2} + \beta \tag{1.11b}$$

for some constants $C_1, C_2 > 0$. The proof requires a long argument. Though the main article [21] containing the full proof remains unpublished yet, the result as well as the idea of the proof is well explained in [22] without arguing the technical detail. A slightly shorter proof was given by [29] under the additional assumption that ℓ is even. These blow-up rates appear also for some non-radial solutions [3, 5]. The method of [21, 22] has become one of the standard tools in the study of type II singularity. Indeed, it has been applied to several nonlinear parabolic problems (cf. for instance, [1, 17–19, 40, 45]). Based on the results of [21, 22], Matano [25] and Mizoguchi [31] independently proved that, if $\lambda_n \neq 0$ for every $n \in \mathbf{N}$ and if a radial solution blows up in finite time with type II regime, then its actual blow-up rate coincides with (1.11) for some $\ell \in \Lambda$, where λ_ℓ is as in (1.11b). As to this direction, an earlier result [32] includes the same conclusion for $p > p_{\rm L}$ (so that $\lambda_0 < \lambda_1 < 0 < \lambda_2 < \cdots$), where $p_{\rm L}$ stands for the Lepin exponent:

$$p_{\rm L} = \begin{cases} +\infty, & N \leq 10, \\ 1 + \dfrac{6}{N-10}, & N \geq 11. \end{cases} \tag{1.12}$$

This was first found by [23] in the study of self-similar solutions. See [34, 37] for recent results on this topic.

For $p = p_{\rm L}$, it was proved in [42] that there exist type II blow-up solutions with exact rates much different from (1.11a) (see also [1, 2] for related results). Whether or not type

II blow-up occurs for $p = p_{\rm JL}$ had been long remained open until it was affirmatively solved in [41]. The analysis in [41, 42] is much delicate than that of [21, 22]. Our principal goal is, using the techniques developed in [41, 42] and elaborate estimates on the heat semigroup in backward similarity variables, to construct refined solutions whose blow-up mechanism is driven by a stable eigenvalue such as HV solutions. As we have already pointed out, the method originated from [21, 22] has been applied to several nonlinear parabolic problems. We expect that the refined technique developed in this article would apply to other nonlinear parabolic problems, thus obtaining completely new results or considerable improvements of the previous results.

As for the case $p = p_{\rm S}$, the existence of type II blow-up solutions have been obtained in [39] for $N = 4$ and in [7, 20] for $N = 5$. An earlier result due to [8] formally indicates that type II blow-up can occur for $N = 3, 4, 5, 6$. It was proved in [4] that type II blow-up solutions do not exist in some class of function spaces for $N \geq 7$.

1.2 Study of equation (1.2)

In the early stage of the research on (1.2), one of the main topics was to investigate the influence of decay rate of initial data at infinity for global-in-time existence of solutions. For instance, Pinsky [36] showed that the critical exponent for existence of global solutions depends on the behavior of weighted term $|x|^{2a}u^p$ as $|x| \to \infty$. Wang [44] studied sufficient conditions on initial data for global-in-time existence and the asymptotic behavior as $t \to \infty$. A comprehensive survey can be found in the introduction of [43]. In the case $a > 0$, on the other hand, the weighted term can disturb blowing up at the origin. Some recent articles discuss whether the zero point in the nonlinearity (i.e., $x = 0$) can be a blow-up point when a blow-up takes place. Several conditions which ensure non-blow-up at the zero point were obtained in [13, 14, 16]. Examples of solutions blowing up at $x = 0$, in contrast, were found in [9, 15]. Filippas and Tertikas [9] constructed self-similar solutions that blow up (in finite time) at $x = 0$ in the cases $p < p_{\rm S}(2a)$ or $p_{\rm S}(2a) < p < p_{\rm JL}(2a)$, where

$$p_{\rm S}(2a) = \begin{cases} +\infty, & N = 1, 2, \\ \dfrac{N + 2 + 4a}{N - 2}, & N \geq 3 \end{cases} \tag{1.13a}$$

and

$$p_{\rm JL}(2a) = \begin{cases} +\infty, & N \leq 10 + 8a, \\ 1 + \dfrac{4(1 + a)}{N - 2a - 4 - 2\sqrt{(N + a - 1)(a + 1)}}, & N > 10 + 8a. \end{cases} \tag{1.13b}$$

As a matter of fact, they agree with the previous notations (1.6), (1.7), respectively, when $a = 0$. Apart from the explicit examples in [9], Guo and Shimojo [15] proved the existence of a solution that blows up at the origin for $N = 3$ and $p > p_{\rm S}(2a)$. The proof of [15] is due to an argument by contradiction. To the best of the authors' knowledge, no other example of such a blow up solution has not been obtained. Our method is naturally extended to the case $a > 0$, $p > p_{\rm JL}(2a)$ with $N > 10 + 8a$ (cf. §§1.3), thereby giving a new example of solutions that blow up at the zero point. We note that our proof in fact works for $a > -1$ and thus covers the three-dimensional case. The proof is totally different from the indirect construction due to [15]. In addition, our blow-up solutions satisfy

$$\lim_{t\to T}(T-t)^{(1+a)/(p-1)}u(0,t)=+\infty. \tag{1.14}$$

Phan [35] has recently established a Liouville-type theorem for (1.2) and applied it to show the blow-up rate estimates of the form (in our notation):

$$\|u(t)\|_{L^\infty(\mathbf{R}^N)}\le C(T-t)^{-(1+a)/(p-1)},\quad t\in(0,T)$$

for $a>0$, $p<p_{\rm S}(2a)$ or for $-1<a<0$, $N\ge 2$, $p<p_{\rm S}(2a)$ and radially nonincreasing initial data. This estimate is contrast to (1.14).

1.3 The statement of the main result

Given a number $a>-1$, we re-define the constant β as follows:

$$\beta=\frac{1+a}{p-1}. \tag{1.15}$$

We keep the notations γ, D, and c_* as (1.8b), (1.8c), and (1.10), respectively, with β replaced by the one above. In the following, let us abbreviate $p_{\rm JL}(2a)$ to $p_{\rm JL}$. The family of regular stationary solutions $U_\alpha(r)$ of (1.2) has the same structure as in the case $a=0$. Let $h>0$ denote the constant in (2.8) below. Throughout this article we employ standard notations in asymptotic analysis; $\sim,\ll,\gg$, i.e., $f(\tau)\ll g(\tau)$ if $f(\tau)=o(g(\tau))$ and $f(\tau)\sim g(\tau)$ if $f(\tau)=g(\tau)(1+o(1))$ as $\tau\to\infty$.

Theorem 1.1. *Assume that $p>p_{\rm JL}$, $N>10+8a$, be in force. Let ℓ be a positive integer such that λ_ℓ in* (1.11b) *is positive and set $\omega_\ell=\lambda_\ell/(\gamma-2\beta)$. Then for every $T>0$ and $\nu>0$, there exists a positive radially decreasing solution u_ℓ of* (1.2)*, which blows up at $t=T$, $x=0$, with the following properties:*

(i) (Exact blow-up rate)

$$\lim_{t\nearrow T}(T-t)^{\beta+2\omega_\ell\beta}u_\ell(0,t)=K_T \tag{1.16}$$

with $K_T=(T/T_0)^{2\beta\omega_\ell}$, where $T_0\in(0,1)$ is a fixed small constant depending only on N,p,a,ℓ, and ν;

(ii) (Estimates in a neighborhood of the inner layer) There exists a C^∞-function $\varepsilon(\tau)$ satisfying $0<\varepsilon(\tau)<1$ and

$$\varepsilon(\tau)\sim K_T^{-\frac{1}{2\beta}}e^{-\omega_\ell\tau}\qquad\text{as } \tau\to\infty \tag{1.17}$$

such that

$$\begin{aligned}&\left|u_\ell(x,t)-\left(\frac{1}{\varepsilon(\tau)\sqrt{T-t}}\right)^{2\beta}U_1\left(\frac{|x|}{\varepsilon(\tau)\sqrt{T-t}}\right)\right|\\ &\le\left(\frac{1}{\varepsilon(\tau)\sqrt{T-t}}\right)^{2\beta}\varepsilon(\tau)^{2\theta}\Psi\left(\frac{|x|}{\varepsilon(\tau)\sqrt{T-t}}\right)\qquad\text{with}\quad \tau=-\log(T-t)\end{aligned} \tag{1.18}$$

for $|x| \le \varepsilon(\tau)^\theta\sqrt{T-t}$*,* $t<T$*, where* K_T *is the constant as in (i),* $\theta\in(0,1)$ *is a constant, and* $\Psi(\xi)$ *is a positive* C^∞*-function satisfying*

$$\Psi(\xi) = \begin{cases} O(1) & \text{as } \xi\to 0,\\ O\left(\xi^{-\gamma}\right) & \text{as } \xi\to\infty; \end{cases} \tag{1.19}$$

(iii) (Estimates in bounded regions) There holds

$$\begin{aligned}&\left| u_\ell(x,t) - U_\infty(|x|) + h\left(\varepsilon(\tau)\sqrt{T-t}\right)^{\gamma-2\beta} |x|^{-\gamma} L_\ell^{(\sqrt{D}/2)}\left(\frac{|x|^2}{4(T-t)}\right)\right| \\ &\quad < \nu\left(\varepsilon(\tau)\sqrt{T-t}\right)^{\gamma-2\beta}\left(1+\frac{|x|^2}{4(T-t)}\right)^\ell |x|^{-\gamma} \quad \text{with} \quad \tau = -\log(T-t)\end{aligned} \tag{1.20}$$

for $\varepsilon(\tau)^\theta\sqrt{T-t} < |x| \le \sqrt{T/T_0}$*,* $0<t<T$*, where* $L_\ell^{(\alpha)}(z)$ *denotes the associated Laguerre polynomial of degree* ℓ *and* T_0*,* θ *are the constants as in (i), (ii);*

(iv) (Number of intersections) There exist exactly ℓ *simple zeros* $\{r_n(t)\}_{n=1}^\ell$ *of* $u_\ell(\cdot,t) - U_\infty$ *for every* $t\in(0,T)$*, which satisfy* $r_n(t) = O(\sqrt{T-t})$ *as* $t\nearrow T$ *for* $n=1,\dots,\ell$*.*

As the blow-up rate estimate (1.16) shows, the solution u_ℓ above is from essentially the same class as of $u_{\ell,\mathrm{HV}}$ obtained by [21, 22]. Theorem 1.1 includes, however, more information about local-in-space estimates both near and away from the singularity even for $a=0$. Indeed, the proof of [21, 22] ensures an estimate of the form

$$C_1 U_{\alpha_1}\left(\frac{|x|}{(T-t)^{1/2+\omega_\ell}}\right) \le (T-t)^{\beta+2\beta\omega_\ell} u_{\ell,\mathrm{HV}}(x,t) \le C_2 U_{\alpha_2}\left(\frac{|x|}{(T-t)^{(1/2)+\omega_\ell}}\right)$$

with $\alpha_1 < 1 < \alpha_2$ for $|x| = O\left((T-t)^{1/2+\omega_\ell}\right)$. The statement (*ii*) of Theorem 1.1 shows that the leading term of u_ℓ in the region $|x|\le\varepsilon(\tau)^\theta\sqrt{T-t}$ is precisely determined as

$$u_\ell(x,t) \sim \left(\frac{1}{\varepsilon(\tau)\sqrt{T-t}}\right)^{2\beta} U_1\left(\frac{|x|}{\varepsilon(\tau)\sqrt{T-t}}\right) \qquad \text{as } t\to T$$

as well as the estimates of error terms. Counterparts for their derivatives are given in Corollary 1.2 below. Another novelty of Theorem 1.1 consists in the estimate (1.20) for bounded regions, $|x|\approx 1$, which extends the region $|x| \le (T-t)^{1/2-\sigma}$, $\sigma\in(0,1/2)$, of validity of the estimate guaranteed for $u_{\ell,\mathrm{HV}}$. Since

$$\begin{aligned}&\left(\varepsilon(\tau)\sqrt{T-t}\right)^{\gamma-2\beta}\left(1+\frac{|x|^2}{4(T-t)}\right)^\ell |x|^{-\gamma+2\beta} \\ &= \varepsilon(\tau)^{\gamma-2\beta}\left(\frac{|x|^2}{T-t}\right)^{-\gamma/2+\beta}\left(1+\frac{|x|^2}{4(T-t)}\right)^\ell \le (T-t)^{\lambda_\ell}\left(1+\frac{|x|^2}{4(T-t)}\right)^{\lambda_\ell},\end{aligned}$$

we deduce from (1.20) that

$$C'|x|^{2\lambda_\ell} \le \left|\frac{u_\ell(x,T)}{U_\infty(|x|)} - 1\right| \le C|x|^{2\lambda_\ell} \tag{1.21}$$

for every $0 < |x|$ small enough, where $u_\ell(x,T) := \lim_{t\nearrow T} u_\ell(x,t)$ denotes the blow-up profile defined outside the blow-up set. In particular, we have,

$$\lim_{|x|\to 0} \frac{u_\ell(x,T)}{U_\infty(|x|)} = 1. \tag{1.22}$$

This was established in [27, Theorem 4.1] as one of the properties characterizing (possibly sign-changing) type II blow-up (with the RHS of (1.22) replaced by ± 1) for $p > p_{\rm S}$, but no concrete example directly verifying (1.22) has been obtained so far. Our particular solutions do imply (1.22) and estimate (1.21) includes further information on the convergence. In particular, it shows optimal estimates of the error depending on each eigenvalue.

Arguing as in [41, 42], we obtain further properties on the solution.

Corollary 1.2. *Assume the same hypothesis as in Theorem 1.1 and $a \geq 0$. Let $u = u_\ell$ be the type II blow-up solution as in Theorem 1.1. Then the diffusion term $-\Delta u(x,t)$ exhibits the same growth rate as of the superlinear term $|x|^{2a}u^p(x,t)$:*

$$-\Delta u(x,t) = \left(\varepsilon(\tau)\sqrt{T-t}\right)^{-2(\beta+1)} \left[\left(\frac{|x|}{\varepsilon(\tau)\sqrt{T-t}}\right)^{2a} U_1\left(\frac{|x|}{\varepsilon(\tau)\sqrt{T-t}}\right)^p + o\left(1\right)\right], \tag{1.23}$$

$$u_t(x,t) = o\left(\left(\varepsilon(\tau)\sqrt{T-t}\right)^{-2(\beta+1)}\right) \tag{1.24}$$

as $t \nearrow T$ for every $(x,t) \in \boldsymbol{R}^N \times (0,T)$ with $|x| \leq \varepsilon(\tau)\sqrt{T-t}$, where $\tau = -\log(T-t)$ and $\varepsilon(\tau)$ is the same function as in Theorem 1.1.

Remark 1.1. Set $m(t) = \|u(\cdot,t)\|_\infty$. The following characterization of blow-up rates for any blow-up solutions of (1.1) was proved in [26, Appendix B]:

$$\text{Type I}: \quad m'(t) = O(m^p(t)) \quad \text{as } t \nearrow T, \tag{1.25}$$

$$\text{Type II}: \quad m'(t_n) = o(m^p(t_n)) \quad \text{for some sequence } t_n \nearrow T. \tag{1.26}$$

In particular, (1.26) represents the *slow nature* of type II blow-up. Corollary 1.2 shows the quantitative information about these amounts (without choosing a particular time-sequence) for the solutions. Thereby they become a prime example of this fact.

Corollary 1.3. *Assume the same hypothesis as in Theorem 1.1. Let $u = u_\ell$ be the type II blow-up solution as in Theorem 1.1. Then for every $q > q_c := N(p-1)/2(1+a)$, there exist constants $C_1, C_2 > 0$ such that*

$$C_1\left(\varepsilon(\tau)\sqrt{T-t}\right)^{N/q-2\beta} \leq \|u(\cdot,t)\|_{L^q(\boldsymbol{R}^N)} \leq C_2\left(\varepsilon(\tau)\sqrt{T-t}\right)^{N/q-2\beta} \tag{1.27}$$

for $0 < t < T$. More precisely,

$$\int_{\{|x|\leq\varepsilon(\tau)^\theta\sqrt{T-t}\}} u(x,t)^q dx = D_1\left(\varepsilon(\tau)\sqrt{T-t}\right)^{-2\beta q+N}(1+o(1)), \tag{1.28}$$

$$\int_{\{\varepsilon(\tau)^\theta\sqrt{T-t}\leq|x|\}} u(x,t)^q dx = O\left(\left(\varepsilon(\tau)^\theta\sqrt{T-t}\right)^{-2\beta q+N}\right) \tag{1.29}$$

as $t \nearrow T$, where $D_1 = \int_0^\infty U_1(\mu)^q\mu^{N-1}d\mu < \infty$.

Corollary 1.4. *Assume the same hypothesis as in Theorem 1.1. Let σ be a constant with $\sigma > 2\beta$. Then there exists an initial data u_0 satisfying*

$$u_0(x) \leq C\,(1+|x|)^{-\sigma} \qquad \text{in } \mathbf{R}^N \tag{1.30}$$

for some constant $C > 0$ such that the corresponding solution $u = u_\ell \in C\left([0,T); L^{q_c}(\mathbb{R}^N)\right)$ of (1.2) satisfies the same estimates as in (i)–(iv) of Theorem 1.1 and

$$C_3\,|\log(T-t)| \leq \|u(\cdot,t)\|_{L^{q_c}(\boldsymbol{R}^N)} \leq C_4\,|\log(T-t)| \tag{1.31}$$

for $0 < t < T$, where $C_3, C_4 > 0$ are some constants.

Remark 1.2. A recent result [33] shows that the critical L^q norm blow-up does occur for possibly non-radial solutions of (1.1) if the blow-up is of type I. The solution u_ℓ as in Theorem 1.1 exhibits type II blow-up. Nevertheless, Corollary 1.4 shows that the critical norm $\|u_\ell(\cdot,t)\|_{L^{q_c}}$ blows up as well and that, moreover, the rate is logarithmic.

The rest of this article is organized as follows. In §2 we first summarize some basic properties of stationary solutions and the linearized operator around U_∞ in the backward similarity variables. By means of matched asymptotic expansions, we then formally describe the leading terms and investigate how large the error terms can be. The last argument leads to a formulation of finite-dimensional reduction for the rigorous construction in §3. Theorem 1.1 and Corollaries 1.2–1.4 are proved therein under the assumption that a key *a priori* estimate holds. §4 and §5 are devoted to proving the *a priori* estimate.

2 Preliminary

In this section we review some known facts essentially due to [21] and discuss the formal construction. Introducing the backward similarity variables

$$\Phi(y,\tau) = (T-t)^{\beta} u(x,t), \tag{2.1a}$$

$$y = \frac{x}{\sqrt{T-t}}, \qquad \tau = -\log(T-t), \tag{2.1b}$$

we convert equation (1.1) to the rescaled equation:

$$\Phi_\tau = \Delta_y \Phi - \frac{y\cdot\nabla_y\Phi}{2} - \beta\Phi + |y|^{2a}\Phi^p \qquad \text{in } \mathbf{R}^N \times (-\log T, \infty), \tag{2.2}$$

where $\nabla_y = {}^t(\partial_{y_1}, ..., \partial_{y_N})$ and $\Delta_y = \sum_{k=1}^N \partial_{y_k}^2$. Notice that $U_\infty(r)$ as in (1.10) with $r = |y|$ is also an unbounded stationary solution of (2.2). We shall henceforth abuse notations as well such as $\Phi(r,\tau) = \Phi(y,\tau)$ for simplicity.

2.1 The description by formal asymptotics

Suppose that an inner layer near the origin appears in our sought-for solution $\Phi(r,\tau)$ of (2.2), where sharp changes in Φ arise when $\tau \to \infty$. Let $\varepsilon(\tau)$ denote the size of the inner layer, which is a priori unknown. We assume

$$\dot{\varepsilon}(\tau),\ \varepsilon(\tau) \ll 1 \qquad \text{as } \tau \to \infty. \tag{2.3}$$

To see the dynamics near the origin, we introduce inner variables $(U(\xi,\tau),\xi)$ as follows:

$$\Phi(y,\tau)=\varepsilon(\tau)^{-2\beta}U(\xi,\tau),\qquad \xi=\frac{y}{\varepsilon(\tau)}. \tag{2.4}$$

A direct computation then shows that

$$\varepsilon(\tau)^2U_\tau=\Delta_\xi U+|\xi|^{2a}U^p-\left(\varepsilon(\tau)^2-2\varepsilon(\tau)\dot{\varepsilon}(\tau)\right)\left(\frac{\xi\cdot\nabla_\xi U}{2}+\beta U\right). \tag{2.5}$$

In view of (2.3), we infer that the leading term of U as $\tau\to\infty$ would be given by a bounded stationary solution of (1.1) for $\xi=o(1/\varepsilon(\tau))$, which amounts to $|y|\ll 1$. The structure of stationary solutions of (1.1) is well understood, which we just recall here.

Proposition 2.1. *([24, Lemma 4.3]) For any $\alpha>0$, there exists a unique solution U_α of*

$$\frac{d^2U}{dr^2}+\frac{N-1}{r}\frac{dU}{dr}+r^{2a}U^p=0\quad\text{for } r>0,\qquad U(0)=\alpha,\quad U'(0)=0. \tag{2.6}$$

If $p>p_{\rm JL}$, $N>10+8a$, the family of the solutions $\{U_\alpha\}_{\alpha>0}$ has the ordered structure:

$$\alpha_1<\alpha_2\implies U_{\alpha_1}(r)<U_{\alpha_2}(r)\quad\text{for all } r>0. \tag{2.7}$$

Moreover,

$$U_1(r)=U_\infty(r)-hr^{-\gamma}+R(r), \tag{2.8}$$
$$U_1'(r)=U_\infty'(r)+h\gamma r^{-\gamma-1}+R(r)O(r^{-1}) \tag{2.9}$$

as $r\to\infty$, where $h>0$ is a constant and $R(r)=o(r^{-\gamma})$. More precisely, there holds

$$R(r)=\begin{cases} O(r^{-\gamma-\min\{\gamma-2\beta,\sqrt{D}\}}) & \text{if } \sqrt{D}\neq\gamma-2\beta,\\ O(r^{-\gamma-\sqrt{D}}\log r) & \text{if } \sqrt{D}=\gamma-2\beta.\end{cases}$$

Due to (2.4) and (2.5), it is natural to construct a solution of the form:

$$\Phi_{\rm inn}(r,\tau):=\varepsilon(\tau)^{-2\beta}U_1\left(\frac{r}{\varepsilon(\tau)}\right), \tag{2.10}$$

which describes the dynamics in the inner region $r=O\left(\varepsilon(\tau)\right)$. The asymptotic behavior (2.8) of U_α then implies

$$\Phi_{\rm inn}(r,\tau)\sim U_\infty(r)-h\varepsilon(\tau)^{\gamma-2\beta}r^{-\gamma}\quad\text{for } \varepsilon(\tau)\ll r\ll 1. \tag{2.11}$$

Hence our sought-for solution $\Phi(r,\tau)$ should behave, to the leading term, to $U_\infty(r)$ in the regions where $\varepsilon(\tau)\ll r$ as $\tau\to\infty$. It is therefore natural to linearize equation (2.2) around $U_\infty(r)$. Let us set

$$v(r,\tau)=\Phi(r,\tau)-U_\infty(r). \tag{2.12}$$

It is readily seen that v solves equation

$$v_\tau = \frac{1}{r^{N-1}\rho}\frac{\partial}{\partial r}\left(r^{N-1}\rho\frac{\partial v}{\partial r}\right) - \beta v + \frac{pc_*^{p-1}}{r^2}v + f(v) \equiv -\mathcal{A}v + f(v), \tag{2.13a}$$

$$f(\phi) := r^{2a}[(\phi + U_\infty)^p - U_\infty^p - pU_\infty^{p-1}\phi]. \tag{2.13b}$$

Let us write $\rho(r) = \exp(-r^2/4)$ and

$$L^2_{r,\rho}(\mathbf{R}^N) = \left\{ v \in L^2_{\mathrm{loc}}[0,\infty);\ \|v\|^2 \equiv \|v\|^2_{L^2_{r,\rho}(\mathbf{R}^N)} := \int_0^\infty v^2 r^{N-1}\rho dr < \infty \right\},$$

$$H^1_{r,\rho}(\mathbf{R}^N) = \left\{ v \in H^1_{\mathrm{loc}}[0,\infty);\ \|v\|^2_{H^1_{r,\rho}(\mathbf{R}^N)} := \|v\|^2 + \|v'\|^2 < \infty \right\}.$$

The linearized operator $Av \equiv \mathcal{A}v$ with $v \in \mathcal{D}(A) = C_0^\infty(0,\infty)$ is realized as a symmetric operator in $L^2_{r,\rho}(\mathbf{R}^N)$. A version of Hardy type inequality as well as integration by parts implies that, if v is smooth,

$$\begin{aligned}\langle Av, v\rangle &= \int_0^\infty \left|\frac{\partial v}{\partial r}\right|^2 r^{N-1}\rho dr - \beta\int_0^\infty v^2 r^{N-1}\rho dr + \int_0^\infty \frac{pc_*^{p-1}}{r^2}v^2 r^{N-1}\rho dr \\ &\geq \left(1 - \frac{4pc_*^{p-1}}{(N-2)^2}\right)\int_0^\infty \left|\frac{\partial v}{\partial r}\right|^2 r^{N-1}\rho dr - C\int_0^\infty v^2 r^{N-1}\rho dr\end{aligned}$$

with a certain constant $C > 0$. Consequently, if $p \geq p_{\mathrm{JL}}$, the operator A is lower bounded, i.e., $\langle A\phi, \phi\rangle \geq -C\|\phi\|^2$ for every functions $\phi \in \mathcal{D}(A)$. We still denote by A its Friedrichs extension. The following spectral result is proved by essentially the same argument as in [21, Lemma 2.3], [41, Proposition 2.3].

Proposition 2.2. *Assume that $N > 10 + 8a$ and $p \geq p_{\mathrm{JL}}$ be in force. Then the spectrum of A consists only of simple eigenvalues $\{\lambda_n\}_{n=0}^\infty$,*

$$\lambda_n = n - \frac{\gamma}{2} + \beta, \qquad n = 0, 1, 2, \ldots \tag{2.14}$$

Eigenfunctions of A associated with eigenvalues λ_n are given by

$$\phi_n(r) = c_n r^{-\gamma} M\left(-n, -\gamma + \frac{N}{2}; \frac{r^2}{4}\right), \quad n = 0, 1, 2, \ldots; \tag{2.15a}$$

$$M(a, b; z) = 1 + \sum_{j=1}^\infty \frac{(a)_j}{j!(b)_j} z^j \quad \text{with} \quad (a)_m = \prod_{j=0}^{m-1}(a + j), \tag{2.15b}$$

where $c_n > 0$ are constants such that $\|\phi_n\| = 1$. Moreover, the eigenfunctions satisfy

$$\phi_n(r) = c_n r^{-\gamma}(1 + O(r^2)) \quad \text{as } r \to 0; \tag{2.16a}$$

$$\phi_n(r) = \tilde{c}_n r^{-\gamma+2n}\left(1 + O\left(r^{-2}\right)\right) \quad \text{as } r \to \infty, \tag{2.16b}$$

where $\tilde{c}_n \in \mathbf{R}$ are constants such that $(-1)^n \tilde{c}_n > 0$ for $n = 0, 1, 2, \ldots$ Furthermore, the constants c_n and $\tilde{c}_n$ in (2.16) are represented as

$$c_n = \left(\frac{\Gamma(-\gamma + N/2 + n)}{\Gamma(-\gamma + N/2)^2\Gamma(n+1)}\right)^{1/2} 2^{\gamma - N/2 + 1/2}, \tag{2.17}$$

$$\tilde{c}_n = \frac{(-1)^n}{(-\gamma + N/2)_n}\left(\frac{\Gamma(-\gamma + N/2 + n)}{\Gamma(-\gamma + N/2)^2\Gamma(n+1)}\right)^{1/2} 2^{\gamma - N/2 + 1/2}, \tag{2.18}$$

respectively, where Γ *stands for the standard Gamma function.*

Remark 2.1. By classical results on orthogonal polynomials, the eigenfunctions are expressed by associated Laguerre polynomials $L_n^{(\nu)}(z) = (n!)^{-1}e^z z^{-\nu}(d^n/dz^n)\,(e^{-z}z^{n+\nu})$:

$$\phi_n(r) = c_n r^{-\gamma}\frac{\Gamma(\nu+1)\,n!}{\Gamma(\nu+n+1)}L_n^{(\nu)}\left(\frac{r^2}{4}\right) =: r^{-\gamma}\psi_n(r) \quad \text{with} \quad \nu = \frac{\sqrt{D}}{2} \tag{2.19}$$

We note that the polynomials $\psi_n(r)$ are uniformly bounded in every compact set of $[0,\infty)$.

We shall recall the idea of [21, 22] and then refine their argument. Due to Lemma 2.2, the solution $v \in L^2_{r,\rho}(\mathbf{R}^N)$ of (2.13) may be expanded to a Fourier series: $v(r,\tau) = \sum_{n=0}^{\infty} a_n(\tau)\phi_n(r)$, where the Fourier coefficients $a_n(\tau) = \langle v(\tau), \phi_n\rangle$ satisfy

$$\dot{a}_n(\tau) = -\lambda_n a_n(\tau) + \langle f(v(\tau)), \phi_n\rangle. \tag{2.20}$$

Consider the situation where a stable mode eventually dominates:

$$v(r,\tau) \sim a_\ell(\tau)\phi_\ell(r) \qquad \text{as } \tau \to \infty, \tag{2.21}$$

where ℓ is an integer such that $\lambda_\ell > 0$. Suppose that the term $\langle f(v(\tau)), \phi_\ell\rangle$ in (2.20) would play no role to the leading order. We then expect that the leading term of $a_\ell(\tau)$ would be determined by the homogeneous term of (2.20). Hence, as $\tau \to \infty$,

$$\Phi_{\text{out}}(r,\tau) := U_\infty(r) + v(r,\tau) \sim U_\infty(r) - d_\ell e^{-\lambda_\ell \tau}\phi_\ell(r) \tag{2.22}$$

with some constant $d_\ell > 0$. The outer expansion as $r \to 0$ then follows from (2.16a):

$$\Phi_{\text{out}}(r,\tau) \sim U_\infty(r) - c_\ell d_\ell e^{-\lambda_\ell \tau} r^{-\gamma}. \tag{2.23}$$

Matching the inner expansions (2.11) with the outer ones (2.23) in the intermediate region $\{\varepsilon(\tau) \ll |y| \ll 1\}$ in which the both expansions make sense, we obtain

$$\varepsilon(\tau)^{\gamma-2\beta} \sim C_\ell e^{-\lambda_\ell \tau} \quad \text{with} \quad C_\ell = \frac{c_\ell d_\ell}{h}. \tag{2.24}$$

Substituting (2.24) into (2.11) and returning to the original variables, we have formally obtain the asymptotic expansions of the HV solution $\{u_{\ell,\text{HV}}\}$.

While the above argument simply tells us what determines the leading terms of the outer expansions, it does not imply the possible effect of the nonlinear term $f(v)$ to $a_\ell(\tau)$ nor how large the next order corrections can be. We shall derive this result as well as expected error estimates by more careful argument.

Hypothesis 2.3. *The blow-up is driven by the stable eigenvalue* $\lambda_\ell > 0$*:*

$$|a_n(\tau)| \ll |a_\ell(\tau)| \quad (n = 0, 1, ..., \ell-1) \quad \text{as } \tau \to \infty \tag{2.25}$$

and (2.21) *holds. Moreover, the controlling factor of* $a_\ell(\tau)$ *is* $e^{-\lambda_\ell \tau}$ *and the other factors are polynomially bounded as* $\tau \to \infty$ *in the sense that*

$$\frac{C_1}{\tau^k} \le \left|e^{\lambda_\ell \tau} a_\ell(\tau)\right| \le C_2 \tau^k \tag{2.26}$$

for some constants $C_1, C_2 > 0$ *and* $k > 0$.

The rationale behind this hypothesis is the occurrence of possible behavior of $a_\ell(\tau)$, such as $a_\ell(\tau) = Ce^{-\lambda_\ell \tau}\tau^\nu$ with some $C > 0$ and $\nu \neq 0$, which actually arises in the critical case $p = p_{\mathrm{JL}}$ [41] or when $\lambda_\ell (> 0)$ is replaced by a neutral eigenvalue [42]. We will show that such behaviors cannot arise in our situation. In order Φ_{out} to be matched with the inner expansions (2.11) in the intermediate region $\{\varepsilon(\tau) \ll |y| \ll 1\}$, we must have

$$a_\ell(\tau) = -\frac{h}{c_\ell}\varepsilon(\tau)^{\gamma-2\beta} + o(\varepsilon(\tau)^{\gamma-2\beta}). \tag{2.27}$$

It then follows from (2.26) and (2.27) that

$$\varepsilon(\tau)^{\gamma-2\beta} = O\left(e^{-\lambda_\ell \tau}\tau^k\right) \qquad \text{as } \tau \to \infty. \tag{2.28}$$

For the ease of presentation, we consider only the case where N is not too large so that

$$\chi := \int_0^\infty \xi^{2a-\gamma+N-1}\left[U_1(\xi)^p - U_\infty(\xi)^p - \frac{pc_*^{p-1}}{\xi^{2a+2}}\left(U_1(\xi) - U_\infty(\xi)\right)\right] d\xi < \infty.$$

Then, arguing as in §§2.3 of [42], we obtain

$$\langle f(v(\tau)), \phi_n\rangle = \chi c_n \varepsilon(\tau)^{\gamma-2\beta+\sqrt{D}} + o\left(\varepsilon(\tau)^{\gamma-2\beta+\sqrt{D}}\right) \qquad \text{as } \tau \to \infty. \tag{2.29}$$

We now integrate the ODE (2.20) over $[\tau, \infty)$. Since $\int_1^\infty e^{\lambda_\ell s}|\langle f(v(s)), \phi_n\rangle| ds < \infty$ due to (2.28) and (2.29), it then turns out that a finite limit $A_n := \lim_{\tau_1 \to \infty} e^{\lambda_n \tau_1} a_n(\tau_1)$ exists,

$$a_n(\tau) = A_n e^{-\lambda_n \tau} - \int_\tau^\infty e^{\lambda_n(s-\tau)}\langle f(v(s)), \phi_n\rangle ds, \tag{2.30}$$

$$\int_\tau^\infty e^{\lambda_n(s-\tau)}\left|\langle f(v(s)), \phi_n\rangle\right| ds = o\left(e^{-\lambda_\ell \tau}\right) \qquad \text{as } \tau \to \infty \tag{2.31}$$

for $n = 0, 1, \ldots, \ell$. Notice that

$$A_n = 0 \quad (n = 0, \ldots, \ell - 1); \tag{2.32a}$$

$$A_\ell \neq 0. \tag{2.32b}$$

Indeed, (2.32a) is a simple consequence of (2.25), (2.30), and (2.31). If (2.32b) is false, we deduce from (2.28)–(2.30) that the controlling factor of $a_\ell(\tau)$ is not $e^{-\lambda_\ell \tau}$, a contradiction. Arguing again as above, we obtain $a_n(\tau) = O(\varepsilon(\tau)^{\gamma-2\beta+\sqrt{D}})$ for $n = 0, 1, ..., \ell - 1$ and

$$a_\ell(\tau) - A_\ell e^{-\lambda_\ell \tau} = -\int_\tau^\infty e^{\lambda_\ell(s-\tau)}\langle f(v(s)), \phi_\ell\rangle ds = O\left(\varepsilon(\tau)^{\gamma-2\beta+\sqrt{D}}\right)$$

as $\tau \to \infty$. Then (2.24) follows from (2.27). In addition, we see that A_ℓ is negative due to the matching condition (2.24). The matching condition (2.24) suggests that

$$\frac{d}{d\tau}\left(\varepsilon(\tau)^{\gamma-2\beta}\right) = -\lambda_\ell \varepsilon(\tau)^{\gamma-2\beta}\left(1 + o(1)\right) \qquad \text{as } \tau \to \infty. \tag{2.33}$$

For $n \geq \ell + 1$, we integrate the ODE (2.20) over $[\tau_0, \tau]$. By (2.29) and (2.33), we get

$$e^{\lambda_n \tau} a_n(\tau) - \widetilde{A_n} = \int_{\tau_0}^{\tau} e^{\lambda_n s} \langle f(v(s)), \phi_n \rangle ds \sim \frac{\chi c_n}{\lambda_n - (1+\kappa)\lambda_\ell} e^{\lambda_n \tau} \varepsilon(\tau)^{\gamma - 2\beta + \sqrt{D}},$$

where $\widetilde{A_n} = e^{\lambda_n \tau_0} a_n(\tau_0)$ and $\kappa = \sqrt{D}/(\gamma - 2\beta) > 0$. Due to this, we obtain the asymptotics of $a_n(\tau)$ $(n \geq \ell + 1)$ as $\tau \to \infty$. It follows that

$$\begin{aligned} Q(r,\tau) &:= v(r,\tau) - \sum_{n=0}^{\ell} a_n(\tau)\phi_n(r) \sim \chi \varepsilon(\tau)^{\gamma - 2\beta + \sqrt{D}} F_\ell(r), \\ F_\ell(r) &= \sum_{n=\ell+1}^{\infty} \frac{d_n}{n} c_n \phi_n(r), \qquad d_n = \frac{1}{1 - (\ell + \kappa \lambda_\ell)/n}, \end{aligned} \tag{2.34}$$

where the convergence is understood in an appropriate weak sense (cf. [41, 42]). We will show that $F_\ell(r) \sim w(r) := \{2(\sqrt{D}+1)\}^{-1} r^{-\gamma - \sqrt{D}}$ as $r \to 0$. Recall the identity $-2\gamma - \sqrt{D} + N - 1 = 1$ and the exact formula (2.19) of $\phi_n(r)$. Then we have

$$(\nu+1) \int_0^\infty w(r)\phi_n(r) r^{N-1} e^{-r^2/4} dr = c_n \frac{\Gamma(\nu+1) n!}{\Gamma(\nu+n+1)} \int_0^\infty L_n^{(\nu)}(z) e^{-z} dz,$$

where $\nu = \sqrt{D}$ and where the change of variable $z = r^2/4$ has been used as well. Since $n! \int_0^\infty L_n^{(\nu)}(z) e^{-z} dz = \nu(\nu+1)\cdots(\nu+n-1)$, it turns out that

$$\begin{aligned} &\int_0^\infty w(r)\phi_n(r) r^{N-1} e^{-r^2/4} dr = \frac{c_n}{\nu+n} \qquad (n = 0, 1, 2...), \\ &w(r) = \sum_{n=0}^{\ell} \frac{1}{\nu+n} c_n \phi_n(r) + \sum_{n=\ell+1}^{\infty} \frac{e_n}{n} c_n \phi_n(r) \quad \text{with} \quad e_n = \frac{1}{1 + (\nu/n)}. \end{aligned} \tag{2.35}$$

Comparing (2.34) with (2.35) and performing similar computations several times, we have

$$F_\ell(r) = \frac{1}{2(\nu+1)} r^{-\gamma - \sqrt{D}} + o\left(r^{-\gamma - \sqrt{D}}\right),$$

whence: $Q(r,\tau) \sim \frac{\chi}{2(\nu+1)} \varepsilon(\tau)^{\gamma - 2\beta + \sqrt{D}} r^{-\gamma - \sqrt{D}}$ as $r \to 0$. This is indeed much smaller than $a_\ell(\tau)\phi_\ell(r)$ as long as $\varepsilon(\tau) \ll r \ll 1$, $\tau \to \infty$.

It is also possible to refine inner expansions by computing the next order correction to (2.10). To this end, we set

$$U(\xi,\tau) = U_1(\xi) + \mu(\tau) H_1(|\xi|) + \cdots,$$

where $\mu(\tau) = \varepsilon(\tau)^2 - 2\varepsilon(\tau)\dot{\varepsilon}(\tau)$ and $\xi = y/\varepsilon(\tau)$. A standard argument then reveals that $H_1(s) = H_1(|\xi|)$ is a solution of the inhomogeneous linear ODE:

$$H'' + \frac{N-1}{s} H' + p U_1(s)^{p-1} H = \frac{s U_1'(s)}{2} + \beta U_1(s), \qquad s > 0, \tag{2.36}$$

satisfying $H(0) = H'(0) = 0$. The solution is expressed by means of variation of constants-formula. Due to Proposition 2.1 and L'Hôpital rule, we obtain

$$H_1(s) = C_1 s^{-\gamma+2} + o\left(s^{-\gamma+2}\right) \quad \text{with} \quad C_1 = \frac{h(\gamma-2\beta)}{4(2+\sqrt{D})} \tag{2.37}$$

as $s \to \infty$. Consequently, the two-term expansion for $U(\xi,\tau)$ has been obtained. In terms of the self-similar variables, this expansion reads

$$\Phi_{\rm inn}(y,\tau) = U_\infty(r) - h\varepsilon(\tau)^{\gamma-2\beta}r^{-\gamma} + \cdots + C_1\mu(\tau)\varepsilon(\tau)^{\gamma-2\beta-2}r^{-\gamma+2} + \cdots,$$

which is valid in the intermediate region $\{\varepsilon(\tau) \ll |y| \ll 1\}$.

We can observe the asymptotic matching of the outer and inner expansions even in the higher order computed above if we carefully check their coefficients in detail, but they yield no contribution to the leading terms. Hence we use them only to obtain information about a guide for rigorous construction and estimate the error to the leading terms.

2.2 Discussions toward the full construction

We have derived condition (2.32) from Hypothesis 2.3. The full proof proceeds to the opposite direction. Namely, we will find a suitable small perturbation of initial data such that (2.32a) holds and then show that Hypothesis 2.3 is true. Following [22], we shall solve this finite-dimensional problem by a topological fixed-point theorem based on mapping degree theory. To this end, we have to set an appropriate functional framework and to show *a priori* estimates for $\Phi(y,\tau)$ ensuring (2.11) and (2.22). We just mention the region where (2.22) is expected to hold. Since $v = \Phi - U_\infty(r)$ and

$$e^{-\lambda_\ell\tau}\phi_\ell(r) \approx e^{-\lambda_\ell\tau}r^{-\gamma+2\ell} = e^{-\lambda_\ell\tau}r^{2\lambda_\ell}r^{-2\beta}$$

as $r \to \infty$, the maximal region of the quadratic approximation of $f(v)$ holds is, in principle, $(\varepsilon(\tau) \ll)|y| = O(e^{\tau/2})$ as $\tau \to \infty$. This last amount is not a technical upper bound, since $e^{\tau/2}$ is a characteristic curve for the hyperbolic part of the differential operator $v_s + \mathcal{A}v \approx v_s + 2^{-1}y\cdot\nabla_y v + \beta v$, $1 \ll |y|$. Nonetheless, the authors of [22] had to restrict their *a priori* estimates to $\{|y| \le e^{\sigma\tau}\}$ with $\sigma < 1/2$. In view of the original coordinate, the set corresponds to a shrinking domain $|x| \le (T-t)^{(1/2)-\sigma}$, $t < T$. In the following sections, we show that it is possible to set a better functional framework than that of [22] and to prove an *a priori* estimate of the form:

$$\left|\Phi(r,\tau) - U_\infty(r) - e^{-\lambda_\ell\tau}\phi_\ell(r)\right| < \nu e^{-\lambda_\ell\tau}r^{-\gamma+2\ell} \qquad (1 \le r \le e^{\tau/2})$$

for every $\nu > 0$. Consequently our solution $u(x,t)$ has good estimates in the ball $\{|x| < 1\}$ uniformly in $(0,T)$.

3 Setting of initial data and functional framework

Let us set

$$\varepsilon_0(\tau) := e^{-\omega_\ell\tau} \ \text{ with } \ \omega_\ell := \frac{\lambda_\ell}{\gamma-2\beta}, \qquad a_\ell^*(\tau) := -\frac{h}{c_\ell}\varepsilon_0(\tau)^{2|\lambda_0|} = -\frac{h}{c_\ell}e^{-\lambda_\ell\tau}.$$

Let θ be a constant

$$0<\theta<\frac{\min\{2|\lambda_0|,\sqrt{D}\}}{16(2|\lambda_0|+\sqrt{D})} \tag{3.1}$$

and let τ_0, τ_1 be numbers such that $\tau_0 \le \tau_1 < \infty$. Let us write

$$\tilde{\phi}_\ell(r) := \begin{cases} a_\ell^*(\tau_0)^{-1}\left[\varepsilon_0(\tau_0)^{-2\beta}\left\{U_1\left(\frac{r}{\varepsilon_0(\tau_0)}\right)-U_\infty\left(\frac{r}{\varepsilon_0(\tau_0)}\right)\right\}-\sum_{n=0}^{\ell-1}\alpha_n\phi_n(r)\right], \\ \qquad\qquad r\le \varepsilon_0(\tau_0)^{2\theta}, \\ \phi_\ell(r), \qquad \varepsilon_0(\tau_0)^{2\theta}<r\le 2e^{\tau_0/2}, \\ a_\ell^*(\tau_0)^{-1}\left[G(r)-\sum_{n=0}^{\ell-1}\alpha_n\phi_n(r)\right], \quad 2e^{\tau_0/2}<r, \end{cases}$$

where $G(r)$ is a nonnegative continuous function satisfying

$$G(r)=o(r^{-2\beta}) \text{ as } r\to\infty. \tag{3.2}$$

We set the initial date Φ_0 as

$$\Phi_0(r;\alpha) := U_\infty(r)+a_\ell^*(\tau_0)\tilde{\phi}_\ell(r)+\sum_{n=0}^{\ell-1}\alpha_n\phi_n(r), \tag{3.3}$$

where $\alpha=(\alpha_0,\alpha_1,\ldots,\alpha_{\ell-1})\in\mathbf{R}^\ell$ is a tuple of parameters, so that

$$\Phi_0(r;\alpha)=\begin{cases} \varepsilon_0(\tau_0)^{-2\beta}U_1\left(\frac{r}{\varepsilon_0(\tau_0)}\right), & r\le\varepsilon_0(\tau_0)^{2\theta}, \\ U_\infty(r)+a_\ell^*(\tau_0)\phi_\ell(r)+\sum_{n=0}^{\ell-1}\alpha_n\phi_n(r), & \varepsilon_0(\tau_0)^{2\theta}<r\le 2e^{\tau_0/2}, \\ U_\infty(r)+G(r), & 2e^{\tau_0/2}<r. \end{cases} \tag{3.4}$$

Concerning the parameter α, we impose

$$|\alpha|<\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta} \tag{3.5}$$

(cf. (3.10)). In fact, we will convert our problem to a finite dimensional one which amounts to finding a suitable $\alpha\in\mathbf{R}^\ell$ satisfying (3.5) such that the corresponding initial data $\Phi_0(r;\alpha)$ yields a solution $\Phi(r,\tau;\alpha)$ with required estimates. To clarify the estimates, we define a functional framework for Φ in the next subsection.

3.1 The functional framework

Let M be a sufficiently large constant to be selected later (cf. Remark 4.1 below). We say that a continuous function $\Phi:\mathbf{R}_+\times[\tau_0,\tau_1]\to\mathbf{R}$ belongs to $\mathcal{A}^\nu_{\tau_0,\tau_1}$ with $\nu\in(0,1]$ if Φ fulfills the following conditions (I), $(\mathrm{II})_\mathrm{A}$, $(\mathrm{II})_\mathrm{B}$ and (III):

(I) For $\tau_0 \le \tau \le \tau_1$ and $r < \varepsilon_0(\tau)^\theta$,

$$\left|\Phi(r,\tau) - \varepsilon_0(\tau)^{-2\beta} U_1\left(\frac{r}{\varepsilon_0(\tau)}\right)\right| < \nu\varepsilon_0(\tau)^{-2\beta+\theta}\left(1+\frac{r}{\varepsilon_0(\tau)}\right)^{-\gamma}; \tag{3.6}$$

$(\mathrm{II})_{\mathrm{A}}$ For $\tau_0 \le \tau \le \tau_1$ and $\varepsilon_0(\tau)^\theta \le r \le 1$,

$$|\Phi(r,\tau) - U_\infty(r) - a_\ell^*(\tau)\phi_\ell(r)| < \nu\varepsilon_0(\tau)^{\gamma-2\beta+2\theta} r^{-\gamma}; \tag{3.7}$$

$(\mathrm{II})_{\mathrm{B}}$ For $\tau_0 \le \tau \le \tau_1$ and $1 < r \le e^{\tau/2}$,

$$|\Phi(r,\tau) - U_\infty(r) - a_\ell^*(\tau)\phi_\ell(r)| < \nu\varepsilon_0(\tau_0)^{2\theta}\varepsilon_0(\tau)^{\gamma-2\beta} r^{-\gamma+2\ell}; \tag{3.8}$$

(III) For $\tau_0 \le \tau \le \tau_1$ and $e^{\tau/2} < r$,

$$|\Phi(r,\tau) - U_\infty(r)| < \nu M r^{-2\beta}. \tag{3.9}$$

We see that $\Phi_0 \in \mathcal{A}^{1/2}_{\tau_0,\tau_0}$ for $|\alpha| < \varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}$ (see §3.1.1). We now define a subset $\mathcal{U}_{\tau_0,\tau_1} \subset \mathbf{R}^\ell$ as

$$\mathcal{U}_{\tau_0,\tau_1} := \{\alpha \in \mathbf{R}^\ell \mid \Phi(r,\tau;\alpha) \in \mathcal{A}^1_{\tau_0,\tau_1},\ |\alpha| < \varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}\}. \tag{3.10}$$

A standard continuous dependence on initial data implies that $\mathcal{U}_{\tau_0,\tau_1}$ is open with respect to the standard topology of $\mathbf{R}^\ell$. Given τ_1 with $\tau_1 \ge \tau_0$, we define a map $Q_{\tau_1} : \mathbf{R}^\ell \to \mathbf{R}^\ell$ as

$$\begin{aligned} &Q_{\tau_1} : \alpha \mapsto (q_0(\tau_1;\alpha), \dots, q_{\ell-1}(\tau_1;\alpha)) \\ &\text{with } \ q_k(\tau_1;\alpha) := \langle v(\,\cdot\,,\tau_1;\alpha), \phi_k\rangle_{L^2_\rho(\mathbf{R}^N)} \ \text{for } \ k = 0,1,\dots,\ell-1. \end{aligned}$$

Lemma 3.1. *Assume that $Q_{\tau_1}(\alpha) = 0$ for some $\alpha \in \overline{\mathcal{U}_{\tau_0,\tau_1}}$. Then:*

$$|\alpha| < \frac{1}{2}\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta} \quad \textit{and} \quad \Phi \in \mathcal{A}^{1/2}_{\tau_0,\tau_1}.$$

The proof of this lemma is postponed to §4.

3.1.1 Estimates of the initial data

Due to (3.4), the initial data $\Phi_0(r) = \Phi_0(r;\alpha)$ satisfies following estimates.

Lemma 3.2. *Let $\mu := \min\{|\lambda_0|, \sqrt{D}/2, |\lambda_0|/\lambda_\ell\}$. We then have*

$$\begin{aligned} &r^\gamma|\Phi_0(r) - U_\infty(r) - a_\ell^*(\tau_0)\phi_\ell(r)| \\ &\le \begin{cases} C\varepsilon_0(\tau_0)^{2|\lambda_0|} + Cr^{2|\lambda_0|}, & r \le \varepsilon_0(\tau_0)^{1-3\theta/\mu}, \\ C\varepsilon_0(\tau_0)^{2|\lambda_0|+\mu} r^{-\mu} + C\varepsilon_0(\tau_0)^{2|\lambda_0|} r^2, & \varepsilon_0(\tau_0)^{1-3\theta/\mu} < r \le \varepsilon_0(\tau_0)^{2\theta}, \\ C\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}(1+r^{2\ell}), & \varepsilon_0(\tau_0)^{2\theta} < r \le 2e^{\tau_0/2}, \\ C\varepsilon_0(\tau_0)^{2|\lambda_0|} r^{2\ell}, & 2e^{\tau_0/2} < r. \end{cases} \end{aligned} \tag{3.11}$$

Moreover, the initial datum Φ_0 belongs to $\mathcal{A}^{1/2}_{\tau_0,\tau_0}$.

Proof. Set $\eta := r/\varepsilon_0(\tau_0)$. Consider the region $\{r \le \varepsilon_0(\tau_0)^{2\theta}\}$. We see from (3.4) that

$$\begin{aligned}
&|\Phi_0(r) - U_\infty(r) - a_\ell^*(\tau_0)\phi_\ell(r)| \\
&\le \varepsilon_0(\tau_0)^{-2\beta}|U_1(\eta) - U_\infty(\eta) + h\eta^{-\gamma}| + (h/c_\ell)\varepsilon_0(\tau_0)^{2|\lambda_0|}|c_\ell r^{-\gamma} - \phi_\ell(r)| \\
&\le \begin{cases} C\varepsilon_0(\tau_0)^{-2\beta}\eta^{-\gamma}(1+\eta^{2|\lambda_0|}), & \eta \le \varepsilon_0(\tau_0)^{-3\theta/\mu}, \\ C\varepsilon_0(\tau_0)^{-2\beta}\eta^{-\gamma-\mu} + C\varepsilon_0(\tau_0)^{-2\beta+2}\eta^{-\gamma+2}, & \varepsilon_0(\tau_0)^{-3\theta/\mu} \le \eta \le \varepsilon_0(\tau_0)^{-1+2\theta}. \end{cases}
\end{aligned} \tag{3.12}$$

As for the region for $\varepsilon_0(\tau)^{2\theta} < r \le 2e^{\tau_0/2}$, due to (3.4) and (3.5), we have

$$|\Phi_0(r) - U_\infty(r) - a_\ell^*(\tau_0)\phi_\ell(r)| \le C\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}r^{-\gamma}(1+r^{2\ell}). \tag{3.13}$$

As for $2e^{\tau_0/2} < r$, it readily follows from (3.2) and (3.4) that

$$|\Phi_0(r) - U_\infty(r) - a_\ell^*(\tau_0)\phi_\ell(r)| \le C\varepsilon_0(\tau_0)^{2|\lambda_0|}r^{-\gamma+2\ell}. \tag{3.14}$$

Putting (3.12)–(3.14) together, we obtain (3.11).

We then verify that Φ_0 belongs to $\mathcal{A}^{1/2}_{\tau_0,\tau_0}$. A similar argument to (3.12) shows that

$$|\Phi_0(r) - \varepsilon_0(\tau_0)^{-2\beta}U_1(\eta)| \le C\varepsilon_0(\tau_0)^{-2\beta+2\theta}\eta^{-\gamma}$$

for $\varepsilon_0(\tau_0)^{2\theta} \le r \le \varepsilon_0(\tau_0)^{\theta}$ and sufficiently large τ_0, where $\eta = r/\varepsilon_0(\tau_0)$. In addition, it follows from (3.2) that $|\Phi_0(r) - U_\infty(r)| = G(r) \ll r^{-2\beta}$ for $2e^{\tau_0/2} < r$ if τ_0 is sufficiently large. These together with (3.13) give $\Phi_0 \in \mathcal{A}^{1/2}_{\tau_0,\tau_0}$. The proof is complete ☐

3.2 Proofs of Theorem 1.1 and Corollaries 1.2–1.4

Once proving the key *a priori* estimate given in Lemma 3.1, we may conclude the proof of Theorem 1.1 by the topological argument by means of mapping degree as in [21] (see also [17, 29, 40–42]). Since the argument is purely topological and independent of particular functional framework, we only write main points without discussing the detail.

Proof of Theorem 1.1. Lemma 3.1 guarantees that any root of Q_{τ_1} in $\mathcal{U}_{\tau_0,\tau_1}$ is contained in the interior of $\mathcal{U}_{\tau_0,\tau_1}$. The mapping degree of Q_τ is then preserved for $\tau_0 \le \tau \le \tau_1$ by homotopy invariance. Hence there exists $\alpha \in \mathcal{U}_{\tau_0,\tau_1}$ such that $Q_\tau(\alpha) = 0$ as long as $\mathcal{U}_{\tau_0,\tau_1} \neq \emptyset$. This last assumption is guaranteed for $|\tau_1 - \tau_0|$ small enough by standard continuous dependence results. Then, by the method of continuity, we have

$$\sup\{\tau_1 > \tau_0\,;\, \mathcal{U}_{\tau_0,\tau_1} \neq \emptyset\} = +\infty. \tag{3.15}$$

Let $\{\tau_j\} \subset (\tau_0, \infty)$ be a sequence such that $\tau_0 < \tau_1 < \cdots < \tau_j \nearrow \infty$. Due to (3.15), there exists $\alpha_j \in \mathcal{U}_{\tau_0,\tau_j}$ such that $Q_{\tau_j}(\alpha_j) = 0$. Lemma 3.1 then implies $\Phi(r,\tau;\alpha_j) \in \mathcal{A}^{1/2}_{\tau_0,\tau_j}$. By taking a subsequence, we may assume that $\{\alpha_j\}$ converges to some $\alpha^* \in \mathbf{R}^\ell$, which completely determines the initial data $\Phi_0(r;\alpha^*)$. The function $u(x,t)$ obtained by scaling back from $\Phi(y,\tau;\alpha^*)$ via (2.1) is the desired solution of (1.2) if T is small enough (denoted as T_0). The pointwise estimates stated in the theorem are obtained by those for $\Phi(y,\tau;\alpha^*)$ guaranteed by its membership to $\mathcal{A}^1_{\tau_0,\infty} := \cap_{\tau_1\in(\tau_0,\infty)}\mathcal{A}^1_{\tau_0,\tau_1}$ with $\tau_0 = -\log T_0$. The result for arbitrary blow-up time $T > 0$ is obtained by rescaling, i.e., $u_\lambda(x,t) = \lambda^{2\beta}u(\lambda x, \lambda^2 t)$ with $\lambda = \sqrt{T_0/T}$. The statement (iv) is proved by standard zero number arguments. The proof is now complete. ☐

Proof of Corollary 1.2. Let us write $V(\xi,\tau) = U(\xi,\tau) - U_1(\xi)$, $U(\xi,\tau) = \varepsilon(\tau)^{2\beta}\Phi(y,\tau)$ with $\xi = |y|/\varepsilon(\tau)$ (cf. (5.4a) below). Introducing new variables

$$s = \int_{\tau_0}^{\tau} \frac{d\tau'}{\varepsilon(\tau')^2}, \qquad W(\xi,s) = V(\xi,\tau),$$

we obtain

$$W_s = \Delta_\xi W + pU_1(\xi)^{p-1}W - \tilde{\mu}(s)\left(\frac{\xi\cdot\nabla W}{2} + \beta W\right) + f(\xi,s) \tag{3.16}$$

with

$$\begin{aligned} f(\xi,s) = |\xi|^{2a}\left\{(U_1(\xi)+W(\xi,s))^p - U_1(\xi)^p - pU_1(\xi)^{p-1}W(\xi,s)\right\} \\ - \tilde{\mu}(s)\left(\frac{\xi\cdot\nabla U_1(\xi)}{2} + \beta U_1(\xi)\right). \end{aligned} \tag{3.17}$$

Notice that the function $f(\xi,s)$ is Hölder continuous since $a \geq 0$ by assumption.

We apply standard parabolic estimates for equation (3.16) in a space-time region $Q := B_R \times (s_1+\delta,\infty)$, where $B_R = \{\xi\,;\,|\xi| < R\}$ and $\delta > 0$ is arbitrary. Due to (3.6) (with $\tau_1 = \infty$), it is readily seen that $\|f\|_{L^\alpha(Q)} \leq C\varepsilon_0(\tau^*)^{2\theta}$ holds for every $\alpha > 1$, where τ^* is the time corresponding to $s_1+\delta$ and $C > 0$ is a constant independent of s_1, δ. Let $Q'' \Subset Q' \Subset Q$ be sub-cylinders and let $W^{2,1}_\alpha(Q)$ denote the Sobolev space based on $L^\alpha(Q)$. Due to classical L^p estimate for parabolic equations, we obtain an estimate of the form $\|W\|_{W^{2,1}_\alpha(Q')} \leq C\varepsilon_0(\tau^*)^\theta$. We now choose α large enough so that $W^{2,1}_\alpha(Q')$ is embedded in the Hölder space $C^{\nu,\nu/2}(\overline{Q'})$ of order ν in $\overline{Q'}$ (with respect to parabolic distance) for some $\nu \in (0,1)$. Notice that the embedding constant does not depend on s_1, δ. Re-selecting a smaller $\nu > 0$, if necessary, we apply Schauder's interior estimate for (3.16), to get

$$\|W\|_{C^{2+\nu,1+\nu/2}(\overline{Q''})} \leq K\left(\|W\|_{L^\infty(Q')} + \|f\|_{C^{\nu,\nu/2}(\overline{Q'})}\right) \leq C'\varepsilon(\tau^*)^\theta$$

for some constant $K > 0$. Since τ^* is arbitrary, the last estimate implies

$$\sup_{\xi\in B_R}\left|\frac{\partial W}{\partial s}(\xi,s)\right| \leq C\varepsilon(\tau)^\theta,$$

Since $\frac{\partial W}{\partial s}(\xi,s) = \varepsilon(\tau)^2\frac{\partial}{\partial\tau}\left(\varepsilon(\tau)^{2\beta}\Phi(y,\tau)\right)$ and $|\dot{\varepsilon}(\tau)| \leq C\varepsilon(\tau)$, we have

$$|\Phi_\tau(y,\tau)| \leq C\varepsilon(\tau)^{-2\beta}U_1\left(\frac{|y|}{\varepsilon(\tau)}\right) + C\varepsilon(\tau)^{-2\beta-2}\left|\frac{\partial W}{\partial s}(\xi,s)\right| \leq C\varepsilon(\tau)^{-2\beta-2+\theta}$$

for $|y| \leq R\varepsilon(\tau)$, $\tau \geq \tau_0$. Returning to the original variables, we get the estimate (1.24) on u_t. Estimate (1.23) is easily obtained by (1.24), equation (1.2), and (1.18). The proof is now complete. □

Proofs of Corollaries 1.3 and 1.4. We estimate the L^q norm by splitting the region of integration defining $\|u(\cdot,t)\|_{L^q(\mathbf{R}^N)}$. The local L^q norm in $\{|x| \leq \varepsilon(\tau)^\theta\sqrt{T-t}\}$ may be readily estimated by (1.18) and the change of variable $\xi = |x|/\varepsilon(\tau)\sqrt{T-t}$, which resulted in (1.28), whereas (1.20) can be used in $\{\varepsilon(\tau)^\theta\sqrt{T-t} \leq |x| \leq 1\}$. As for $|x| \geq 1$, we use simply the decay estimate $u \leq C|x|^{-2\beta}$ when $q > q_c$, whence (1.29). When $q = q_c$, we need a faster decay $u \leq C|x|^{-d}$ for some $d > 2\beta$. Under the condition (1.30), the last estimate is guaranteed by [28, Proposition C.3]. The detail is left to the reader. □

4 Proof of a priori estimates in the outer region

In this section we prove Lemma 3.1. This task is done by showing several *a priori* estimates, which we are going to establish in the following subsections. Let us write

$$v(\,\cdot\,,\tau)=\sum_{n=0}^{\infty}a_n(\tau)\phi_n \quad \text{in } L^2_\rho(\mathbf{R}^N), \tag{4.1}$$

where $a_n(\tau)=\langle v(\,\cdot\,,\tau),\phi_n\rangle_{L^2_\rho(\mathbf{R}^N)}$. We first estimates $a_n(\tau)$ for $n=0,1,\dots,\ell-1$ in (4.1). This is accomplished in §§ 4.1 under the assumption $Q_{\tau_1}(\alpha)=0$. The reminder term $E(y,\tau)=v(y,\tau)-\sum_{n=0}^{\ell}a_n(\tau)\phi_n(y)$ yields smaller contribution than the leading mode $a_\ell(\tau)\phi_\ell(y)$ in the outer region $\{\varepsilon_0(\tau)^\theta<|y|\le e^{\tau/2}\}$. Those estimates of $a_n(\tau)$ and $E(y,\tau)$ lead to the estimates (II) and (III) with $\nu\ll 1$ in the requirement for $\mathcal{A}^{1/2}_{\tau_0,\tau_1}$. The last §§ 5 is devoted to showing the estimate (I) in the inner region. In the following, we denote by C a generic positive constant that may change from line to line.

4.1 Estimates of Fourier coefficients

Lemma 4.1. *Assume that* $\Phi\in\mathcal{A}^1_{\tau_0,\tau_1}$*. Then:*

$$|f(v(r,\tau))|\le\begin{cases} Cr^{-2\beta-2}, & r\le\varepsilon_0(\tau)^{1-\theta},\\ C\varepsilon_0(\tau)^{4|\lambda_0|}r^{-\gamma-2|\lambda_0|-2}(1+r^{4\ell}), & \varepsilon_0(\tau)^{1-\theta}<r\le e^{\Lambda_\ell\tau/2},\\ CM^pr^{-2\beta-2}, & e^{\Lambda_\ell\tau/2}<r,\end{cases} \tag{4.2}$$

for $\tau_0\le\tau\le\tau_1$*, where* $\Lambda_\ell:=1-1/2\lambda_{\ell+1}\in(1/2,1)$.

Proof. Let $\Phi_{\rm inn}(r,\tau)$ be the function as in (2.10) and set $v_{\rm inn}(r,\tau)=\Phi_{\rm inn}(r,\tau)-U_\infty(r)$,

$$|f(v(r,\tau))|\le|f(v(r,\tau))-f(v_{\rm inn}(r,\tau))|+|f(v_{\rm inn}(r,\tau))|=:r^{2a}(F_1+F_2).$$

By the condition $\Phi\in\mathcal{A}^1_{\tau_0,\tau_1}$, for $r\le\varepsilon_0(\tau)^\theta$, we know that

$$|\Phi(r,\tau)-\Phi_{\rm inn}(r,\tau)|\le\varepsilon_0(\tau)^\theta\Phi_{\rm inn}(r,\tau).$$

Let $\eta:=r/\varepsilon_0(\tau)$. For $r\le\varepsilon_0(\tau)^\theta$, we then obtain that

$$\begin{aligned}F_1&\le|\Phi(r,\tau)^p-\Phi_{\rm inn}(r,\tau)^p-p\Phi_{\rm inn}(r,\tau)^{p-1}(\Phi(r,\tau)-\Phi_{\rm inn}(r,\tau))|\\&\quad+|p(\Phi_{\rm inn}(r,\tau)^{p-1}-U_\infty(r)^{p-1})(\Phi(r,\tau)-\Phi_{\rm inn}(r,\tau))|\\&\le\begin{cases}C\varepsilon_0(\tau)^{-2\beta-2-2a+\theta}U_\infty(\eta)^p, & \eta\le\varepsilon_0(\tau)^{-\theta},\\ C\varepsilon_0(\tau)^{-2\beta-2-2a+\theta}U_\infty(\eta)^{p-2}\eta^{-2\gamma}, & \varepsilon_0(\tau)^{-\theta}<\eta\le\varepsilon_0(\tau)^{\theta-1},\end{cases}\end{aligned}$$

and

$$\begin{aligned}F_2&=|(U_\infty(r)+v_{\rm inn}(r,\tau))^p-U_\infty(r)^p-pU_\infty(r)^{p-1}v_{\rm inn}(r,\tau)|\\&\le\begin{cases}C\varepsilon_0(\tau)^{-2\beta-2-2a}U_\infty(\eta)^p, & \eta\le\varepsilon_0(\tau)^{-\theta},\\ C\varepsilon_0(\tau)^{-2\beta-2-2a}U_\infty(\eta)^{p-2}\eta^{-2\gamma}, & \varepsilon_0(\tau)^{-\theta}<\eta\le\varepsilon_0(\tau)^{\theta-1},\end{cases}\end{aligned}$$

whence:

$$|f(v(r,\tau))| \le \begin{cases} Cr^{-2\beta-2}, & r \le \varepsilon_0(\tau)^{1-\theta}, \\ C\varepsilon_0(\tau)^{4|\lambda_0|} r^{-\gamma-2|\lambda_0|-2}, & \varepsilon_0(\tau)^{1-\theta} < r \le \varepsilon_0(\tau)^{\theta}. \end{cases} \tag{4.3}$$

Since $\Phi \in \mathcal{A}^1_{\tau_0,\tau_1}$, we obtain

$$|v(r,\tau)| \le \begin{cases} Cr^{-2\beta}\varepsilon_0(\tau)^{2(1-\theta)|\lambda_0|}, & \varepsilon_0(\tau)^{\theta} < r \le 1, \\ Cr^{-2\beta}e^{-\lambda_\ell \tau/2\lambda_{\ell+1}}, & 1 < r \le e^{\Lambda_\ell \tau/2}, \\ Cr^{-2\beta}, & e^{\Lambda_\ell \tau/2} < r \le e^{\tau/2}. \end{cases}$$

Moreover, $|v(r,\tau)| < MU_\infty(r)$ for $r > e^{\tau/2}$ due to (Ⅲ) with $\nu = 1$. It then follows that

$$|f(v(r,\tau))| \le \begin{cases} C\varepsilon_0(\tau)^{4|\lambda_0|} r^{-2\gamma+2\beta-2}, & \varepsilon_0(\tau)^{\theta} < r \le 1, \\ C\varepsilon_0(\tau)^{4|\lambda_0|} r^{-2\gamma+2\beta-2+4\ell}, & 1 < r \le e^{\Lambda_\ell \tau}, \\ CM^p r^{-2\beta-2}, & e^{\Lambda_\ell \tau} < r. \end{cases} \tag{4.4}$$

Due to (4.3) and (4.4), we obtain (4.2) and the proof is complete. □

Lemma 4.2. *Assume that* $\Phi \in \mathcal{A}^1_{\tau_0,\tau_1}$. *Then*:

$$|\langle f(v(\tau)), \phi_n \rangle_{L^2_\rho(\mathbf{R}^N)}| \le Cc_n \varepsilon_0(\tau)^{2|\lambda_0|+\mu} \tag{4.5}$$

for $n = 0, 1, \dots$ *and* $\tau_0 \le \tau \le \tau_1$, *where* $\mu = \min\{|\lambda_0|, \sqrt{D}/2, |\lambda_0|/\lambda_\ell\} > 0$ *as before.*

Proof. We set

$$\begin{aligned} &H_{1;n} + H_{2;n} + H_{3;n} + H_{4;n} \\ &:= \left(\int_0^{\varepsilon_0(\tau)^{1-\theta}} + \int_{\varepsilon_0(\tau)^{1-\theta}}^1 + \int_1^{e^{\Lambda_\ell \tau/2}} + \int_{e^{\Lambda_\ell \tau/2}}^\infty \right) |f(v(r,s))||\phi_n(r)| r^{N-1}\rho(r)\, dr. \end{aligned}$$

We shall use (4.2) to estimate $|f(v(r,\tau)|$ in each subinterval. Since $N - 2 - 2\gamma = \sqrt{D}$ (cf. (1.8b), (1.15)) and $\gamma - 2\beta = 2|\lambda_0|$, we have

$$H_{1;n} \le Cc_n \int_0^{\varepsilon_0(\tau)^{1-\theta}} r^{-\gamma-2\beta+N-3}\, dr \le Cc_n \varepsilon_0(\tau)^{(1-\theta)(2|\lambda_0|+\sqrt{D})}\,; \tag{4.6}$$

$$\begin{aligned} H_{2;n} &\le Cc_n \varepsilon_0(\tau)^{4|\lambda_0|} \int_{\varepsilon_0(\tau)^{1-\theta}}^1 r^{-2\gamma-2|\lambda_0|+N-3}\, dr \\ &\le Cc_n (\varepsilon_0(\tau)^{2|\lambda_0|} + \varepsilon_0(\tau)^{\sqrt{D}-\theta(-2|\lambda_0|+\sqrt{D})}) \varepsilon_0(\tau)^{2|\lambda_0|}\,; \end{aligned} \tag{4.7}$$

$$H_{3;n} \le C|\tilde{c}_n| \varepsilon_0(\tau)^{4|\lambda_0|} \int_1^{e^{\Lambda_\ell \tau/2}} r^{-2\gamma+2\ell-2|\lambda_0|+N-3} e^{-r^2/4}\, dr \le C|\tilde{c}_n| \varepsilon_0(\tau)^{4|\lambda_0|}. \tag{4.8}$$

The contribution of $H_{4,n}$ is much smaller than those above due to the presence of exponential weight. Since $|\tilde{c}_n| \le c_n$, the estimate (4.5) follows from (3.1), (4.6)–(4.8). The proof is complete. □

Lemma 4.3. *Assume that $Q_{\tau_1}(\alpha) = 0$ for some $\alpha \in \overline{\mathcal{U}_{\tau_0,\tau_1}}$. Then:*

$$|a_n(\tau)| \le C\varepsilon_0(\tau)^{2|\lambda_0|+\mu} \tag{4.9}$$

for $\tau_0 \le \tau \le \tau_1$ and $n = 0, 1, \dots, \ell - 1$, where $\mu = \min\{|\lambda_0|, \sqrt{D}/2, |\lambda_0|/\lambda_\ell\}$ as before.

Proof. Due to $Q_{\tau_1}(\alpha) = 0$, we see that

$$\begin{aligned} |a_n(\tau)| &= \left| -\int_\tau^{\tau_1} e^{-\lambda_n(\tau-s)} \langle f(v(s)), \phi_n \rangle_{L^2_\rho(\mathbf{R}^N)}\, ds \right| \\ &\le C \int_\tau^\infty e^{-\lambda_n(\tau-s)} \varepsilon_0(s)^{2|\lambda_0|+\mu}\, ds \le C\varepsilon_0(\tau)^{2|\lambda_0|+\mu}, \end{aligned} \tag{4.10}$$

whence (4.9). The proof is complete. □

Lemma 4.4. *Assume that $Q_{\tau_1}(\alpha) = 0$ for some $\alpha \in \overline{\mathcal{U}_{\tau_0,\tau_1}}$. Then:*

$$|\alpha_n| \le C\varepsilon_0(\tau_0)^{2|\lambda_0|+2\theta(2+\sqrt{D})} \qquad \text{for } n = 0, 1, \dots, \ell - 1; \tag{4.11}$$

$$|a_n(\tau_0) - \delta_{\ell,n} a_\ell^*(\tau_0)| \le C c_n \varepsilon_0(\tau_0)^{2|\lambda_0|+2\theta(2+\sqrt{D})} \qquad \text{for } n = \ell, \ell + 1, \dots, \tag{4.12}$$

where $\delta_{\ell,n} = 1$ if $n = \ell$ and $\delta_{\ell,n} = 0$ if $n \ne \ell$.

Proof. Notice that (3.4) implies that

$$\begin{aligned} &\int_{\varepsilon_0(\tau_0)^{2\theta}}^{2e^{\tau_0/2}} v_0(r)\phi_n(r) r^{N-1} e^{-r^2/4}\, dr \\ &= \alpha_n + \left(\int_0^{\varepsilon_0(\tau_0)^{2\theta}} + \int_{2e^{\tau_0/2}}^\infty \right) \left(a_\ell^*(\tau_0)\phi_\ell(r) + \sum_{k=0}^{\ell-1} \alpha_k \phi_k(r) \right) \phi_n(r) r^{N-1} e^{-r^2/4}\, dr \end{aligned}$$

for $n = 0, 1, \dots, \ell - 1$. Then, due to $a_n(\tau_0) = \langle v_0, \phi_n \rangle_{L^2_\rho(\mathbf{R}^N)}$, we have

$$\begin{aligned} a_n(\tau_0) = \alpha_n &+ \left(\int_0^{\varepsilon_0(\tau_0)^{1-\theta}} + \int_{\varepsilon_0(\tau_0)^{1-\theta}}^{\varepsilon_0(\tau_0)^{2\theta}} + \int_{2e^{\tau_0/2}}^\infty \right) v_0(r)\phi_n(r) r^{N-1} e^{-r^2/4}\, dr \\ &+ \left(\int_0^{\varepsilon_0(\tau_0)^{2\theta}} + \int_{2e^{\tau_0/2}}^\infty \right) \left(a_\ell^*(\tau_0)\phi_\ell(r) + \sum_{n=0}^{\ell-1} \alpha_n \phi_n(r) \right) \phi_n(r) r^{N-1} e^{-r^2/4}\, dr \\ &=: \alpha_n + H'_{1;n} + H'_{2;n} + H'_{3;n} + H'_{4;n} + H'_{5;n}. \end{aligned} \tag{4.13}$$

Let $\eta := r/\varepsilon_0(\tau_0)$. We see from (3.4) that

$$|v_0(r)| \le \begin{cases} Cr^{-2\beta}, & r \le \varepsilon_0(\tau_0)^{1-\theta}, \\ C\varepsilon_0(\tau_0)^{2|\lambda_0|} r^{-\gamma}, & \varepsilon_0(\tau_0)^{1-\theta} \le r \le \varepsilon_0(\tau_0)^{2\theta}. \end{cases}$$

Due to this and identity $-2\gamma + N - 1 = \sqrt{D} + 1$, we heve

$$|H'_{1;n}| \le C c_n \int_0^{\varepsilon_0(\tau_0)^{1-\theta}} r^{2|\lambda_0|+\sqrt{D}+1}\, dr \le C c_n \varepsilon_0(\tau_0)^{(1-\theta)(2|\lambda_0|+\sqrt{D}+2)}, \tag{4.14}$$

$$|H'_{2;n}| \le C c_n \varepsilon_0(\tau_0)^{2|\lambda_0|} \int_{\varepsilon_0(\tau_0)^{1-\theta}}^{\varepsilon_0(\tau_0)^{2\theta}} r^{\sqrt{D}+1}\, dr \le C c_n \varepsilon_0(\tau_0)^{2|\lambda_0|+2\theta(\sqrt{D}+2)}. \tag{4.15}$$

Moreover, the inequality $|\alpha| \le \varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}$ ensures

$$|H'_{4;n}| \le C c_n \varepsilon_0(\tau_0)^{2|\lambda_0|} \int_0^{\varepsilon_0(\tau_0)^{2\theta}} r^{\sqrt{D}+1}\, dr \le C c_n \varepsilon_0(\tau_0)^{2|\lambda_0|+2\theta(\sqrt{D}+2)}. \tag{4.16}$$

Due to the presence of the exponential weight, the contributions of $H'_{3;n}$ and $H'_{5;n}$ are much smaller than, say $e^{-\tau_0^2/8}$, for τ_0 large enough. Due to the condition (3.1) on θ, we have $2|\lambda_0| + 2\theta(\sqrt{D}+2) < (1-\theta)(2|\lambda_0| + \sqrt{D} + 2)$. The first claim (4.11) then follows from (4.9), (4.13)–(4.16). The proof of the second claim (4.12) is similar and thus omitted. The proof is complete. □

4.2 Estimates of remainder terms

Our next goal is to estimate the higher Fourier mode: $v(r,\tau) - \sum_{n=0}^{\ell} a_n(\tau)\phi_n(r)$. To this end, it is convenient to introduce a new dependent variable

$$W(r,\tau;\alpha) := r^{\gamma} v(r,\tau;\alpha), \qquad \psi_n(r) := r^{\gamma}\phi_n(r),$$

where $v(r,\tau;\alpha) := \Phi(r,\tau;\alpha) - U_\infty(r)$. Then:

$$\begin{aligned} W_0(r;\alpha) &:= W(r,\tau_0;\alpha) \\ &= \begin{cases} \varepsilon_0(\tau_0)^{-2\beta} r^{\gamma} \left[U_1\left(\dfrac{r}{\varepsilon_0(\tau_0)}\right) - U_\infty\left(\dfrac{r}{\varepsilon_0(\tau_0)}\right)\right], & r \le \varepsilon_0(\tau_0)^{2\theta}, \\ a_\ell^*(\tau_0)\psi_\ell(r) + \displaystyle\sum_{n=0}^{\ell-1} \alpha_n \psi_n(r), & \varepsilon_0(\tau_0)^{2\theta} < r \le 2e^{\tau_0/2}, \\ r^{\gamma} G(r), & 2e^{\tau_0/2} < r, \end{cases} \end{aligned}$$

and function W satisfies $W_\tau = -\mathcal{L}W + g(W)$ where

$$-\mathcal{L}W := W'' + \left(\frac{N-2\gamma-1}{r} - \frac{r}{2}\right) W' - \lambda_0 W, \quad g(W) := r^{\gamma} f(v).$$

We set

$$m := N - 2\gamma = 2 + \sqrt{D} > 2.$$

For a while, we consider the case where m is an integer (the general case is discussed at the end of this section). Let us write

$$W(r,\tau) = \sum_{n=0}^{\ell} a_n(\tau)\psi_n(r) + E(r,\tau) \text{ where } E(r,\tau) := \sum_{n=\ell+1}^{\infty} a_n(\tau)\psi_n(r). \tag{4.17}$$

Function E satisfies

$$\langle E(\,\cdot\,,\tau), \psi_n \rangle_{L^2_\rho(\mathbf{R}^m)} = 0 \text{ for } n = 0,1,\ldots,\ell, \quad E_\tau = -\mathcal{L}E + g(W) - \sum_{n=0}^{\ell} \langle g(W), \psi_n \rangle \psi_n.$$

We denote by $S(\tau)$ the semigroup for $-\mathcal{L}$, which is expressed as

$$[S(\tau)W](y) = \sum_{n=0}^{\infty} e^{-\lambda_n \tau} \langle W, \psi_n \rangle_{L^2_\rho(\mathbf{R}^m)} \psi_n(|y|) \tag{4.18a}$$

$$= \frac{e^{\Lambda\tau}|y|^{-m/2+1}}{1-e^{-\tau}} \int_0^\infty I_\omega \left\{ \frac{|y|e^{-\tau/2} r}{2(1-e^{-\tau})} \right\} \exp\left\{ -\frac{|y|^2 e^{-\tau} + r^2}{4(1-e^{-\tau})} \right\} r^{m/2} W(r)\, dr \tag{4.18b}$$

$$= \frac{e^{|\lambda_0|\tau}}{\{4\pi(1-e^{-\tau})\}^{m/2}} \int_{\mathbf{R}^m} \exp\left\{ -\frac{|ye^{-\tau/2} - z|^2}{4(1-e^{-\tau})} \right\} W(z)\, dz \quad (m \in \mathbf{N}) \tag{4.18c}$$

with $\omega := \gamma + N/2 - 1 = (m-2)/2$ and $\Lambda := |\lambda_0| + (m-2)/4$, where I_ω denotes the modified Bessel function with order ω. The bounds for the modified Bessel function

$$|I_\omega(z)| \le \frac{C z^\omega e^z}{(1+z)^{\omega+1/2}}, \quad z \in \mathbf{R}_+, \tag{4.19}$$

yields the following estimate:

$$|[S(\tau)W](y)| \tag{4.20}$$
$$\le \frac{Ce^{|\lambda_0|\tau}}{(1-e^{-\tau})^{m/2}} \int_0^\infty \exp\left\{ -\frac{||y|e^{-\tau/2} - r|^2}{4(1-e^{-\tau})} \right\} \left(1 + \frac{|y|e^{-\tau/2} r}{2(1-e^{-\tau})} \right)^{-(m-1)/2} |W(r)| r^{m-1}\, dr.$$

The series expression (4.18a) implies $a_\ell^*(\tau)\psi_\ell(r) = S(\tau - \tau_0)\{a_\ell^*(\tau_0)\psi_\ell\}(r)$, whence:

$$W(r,\tau) - a_\ell^*(\tau)\psi_\ell(r) = S(\tau-\tau_0)\{W_0 - a_\ell^*(\tau_0)\psi_\ell\}(r) + \int_{\tau_0}^{\tau} S(\tau - s)\{g(W(s))\}(r)\, ds.$$

4.2.1 A priori estimates in the short-time case

We first discuss the short-time case, that is, $\tau_0 \le \tau \le \tau_0 + 1$. Notice that

$$e^{-1} \le e^{-(\tau-\sigma)} \le 1 \quad \text{for} \quad \tau_0 \le \sigma \le \tau \le \tau_0 + 1. \tag{4.21}$$

For simplicity, we shall abuse some notation such as $W(|y|,\tau) = W(y,\tau)$, $\psi_n(|y|) = \psi_n(y)$.

Lemma 4.5. *There holds*

$$|S(\tau-\tau_0)\{W_0 - a_\ell^*(\tau_0)\psi_\ell\}(y)| \le C\varepsilon_0(\tau)^{2|\lambda_0|+3\theta}(1+|y|^{2\ell}) \tag{4.22}$$

for $\varepsilon_0(\tau)^\theta < |y| \le e^{\tau/2}$ *and* $\tau_0 \le \tau \le \tau_0 + 1$ *with sufficiently large* τ_0.

Proof. For $i = 1, 2, 3$, we set

$$I_i := \frac{e^{|\lambda_0|(\tau-\tau_0)}}{\{4\pi(1-e^{-(\tau-\tau_0)})\}^{m/2}} \int_{D_i(\tau_0)} \exp\left\{ -\frac{|ye^{-(\tau-\tau_0)/2} - z|^2}{4(1-e^{-(\tau-\tau_0)})} \right\} |W_0(z,\tau_0) - a_\ell^*(\tau_0)\psi_\ell(z)|\, dz,$$

where

$$D_1(\sigma) := \{z \in \mathbf{R}^m \mid |z| \le \varepsilon_0(\sigma)^{2\theta}\}, \tag{4.23a}$$
$$D_2(\sigma) := \{z \in \mathbf{R}^m \mid \varepsilon_0(\sigma)^{2\theta} < |z| \le 2e^{\sigma/2}\}, \tag{4.23b}$$
$$D_3(\sigma) := \{z \in \mathbf{R}^m \mid 2e^{\sigma/2} < |z|\} \tag{4.23c}$$

for $\tau_0 \le \sigma \le \tau \le \tau_1$. Hereafter we always assume $\varepsilon_0(\tau)^\theta < |y| \le e^{\tau/2}$ and $\tau_0 \le \tau \le \tau_0 + 1$.
Estimate for I_1. Let us divide the region $D_1(\sigma)$ in (4.23a) as the disjoint union of

$$D_{1,1}(\sigma) := \{z \in \mathbf{R}^m \mid |z| \le \varepsilon_0(\sigma)^{1-3\theta/\mu}\}, \tag{4.24a}$$
$$D_{1,2}(\sigma) := \{z \in \mathbf{R}^m \mid \varepsilon_0(\sigma)^{1-3\theta/\mu} < |z| \le \varepsilon_0(\sigma)^{2\theta}\} \tag{4.24b}$$

for $\tau_0 \le \sigma \le \tau \le \tau_1$. The corresponding integrals are denoted as $I_{1,1}$ and $I_{1,2}$, respectively. For $\varepsilon_0(\tau)^\theta < |y|$ and $|z| \le \varepsilon_0(\tau_0)^{2\theta}$, there holds

$$|ye^{-(\tau-\tau_0)/2} - z| \ge e^{-1/2}|y| - |z| \ge \frac{|y|}{4} \tag{4.25}$$

if $\tau_0 \ge 1 + 2\log 2/\theta\omega_\ell$, where $\omega_\ell = \lambda_\ell/2|\lambda_0|$. By (4.21), (3.12) and (4.25), we have

$$\begin{aligned} I_{1,1} &\le \frac{C}{(1-e^{-(\tau-\tau_0)})^{m/2}} \exp\left\{-\frac{|y|^2}{64(1-e^{-(\tau-\tau_0)})}\right\} \int_{D_{1,1}(\tau_0)} (\varepsilon_0(\tau_0)^{2|\lambda_0|} + |z|^{2|\lambda_0|})\, dz \\ &\le C\left(\sup_{L\ge 0} L^m e^{-L^2}\right)|y|^{-m}\left[\varepsilon_0(\tau_0)^{2|\lambda_0|}\int_0^{\varepsilon_0(\tau_0)} r^{m-1}\,dr + \int_{\varepsilon_0(\tau_0)}^{\varepsilon_0(\tau_0)^{1-3\theta/\mu}} r^{2|\lambda_0|+m-1}\,dr\right] \\ &\le C\varepsilon_0(\tau)^{2|\lambda_0|+m-\theta(4|\lambda_0|+2m+\mu m)/\mu}. \end{aligned} \tag{4.26}$$

For $\varepsilon_0(\tau_0)^{1-3\theta/\mu} < |z| \le \varepsilon_0(\tau_0)^{2\theta}$, (3.12) implies that

$$|W_0(z) - a_\ell^*(\tau_0)\psi_\ell(z)| \le C\varepsilon_0(\tau_0)^{2|\lambda_0|}\left[\left(\frac{|z|}{\varepsilon_0(\tau_0)}\right)^{-\mu} + |z|^2\right] \le C\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}.$$

Then, due to (4.21), we obtain $I_{1,2} \le C\varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}|[S(\tau-\tau_0)1](y)| \le C\varepsilon_0(\tau)^{2|\lambda_0|+3\theta}$. It then follows from (3.1) and (4.26) that

$$I_1 \le C\varepsilon_0(\tau)^{2|\lambda_0|+3\theta}. \tag{4.27}$$

Estimate for I_2. Note that $W_0(z) - a_\ell^*(\tau_0)\psi_\ell(z) = \sum_{n=0}^{\ell-1}\alpha_n\psi_n(|z|)$ for $z \in D_2(\tau_0)$. Our fundamental assumption of $\alpha \in \overline{\mathcal{U}_{\tau_0,\tau_1}}$ ensures $|\alpha| \le \varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta}$. It then follows from (4.20) and (4.21) that

$$\begin{aligned} |I_2| &\le \frac{e^{|\lambda_0|(\tau-\tau_0)}}{\{4\pi(1-e^{-(\tau-\tau_0)})\}^{m/2}} \int_{\mathbf{R}^m} \exp\left\{-\frac{|ye^{-(\tau-\tau_0)/2} - z|^2}{4(1-e^{-(\tau-\tau_0)})}\right\} \sum_{n=0}^{\ell-1} |\alpha_n\psi_n(z)|\, dz \\ &\le C\varepsilon_0(\tau)^{2|\lambda_0|+3\theta}(1+|y|^{2\ell}). \end{aligned} \tag{4.28}$$

Estimate for I_3. We see that

$$|ye^{-(\tau-\tau_0)/2} - z| \ge |z| - |y|e^{-(\tau-\tau_0)/2} \ge \frac{|z|}{2} \tag{4.29}$$

for $|z| \ge 2|y|e^{-(\tau-\tau_0)/2}$ and

$$2|y|e^{-(\tau-\tau_0)/2} \le 2e^{\tau_0/2} \le |z| \tag{4.30}$$

for $z \in D_3(\tau_0)$. It then follows from (3.11) and (4.29) that

$$I_3 \le \frac{C\varepsilon_0(\tau_0)^{2|\lambda_0|}}{(1-e^{-(\tau-\tau_0)})^{m/2}} \int_{D_3(\tau_0)} \exp\left\{-\frac{|z|^2}{16(1-e^{-(\tau-\tau_0)})}\right\} |z|^{2\ell}\, dz \le C\varepsilon_0(\tau)^{4|\lambda_0|}. \tag{4.31}$$

Due to (4.27), (4.28) and (4.31), we obtain (4.22) and the proof is complete. □

Lemma 4.6. *There hold*

$$\left|\int_{\tau_0}^{\tau} S(\tau-s)\{g(W(s))\}(y)\, ds\right| \le C\varepsilon_0(\tau)^{\mu} \min\{\varepsilon_0(\tau)^{2|\lambda_0|}(1+|y|^{2\ell}), (1+|y|^{2|\lambda_0|})\}, \tag{4.32}$$

$$\left|\int_{\tau_0}^{\tau} S(\tau-s)\left\{\sum_{n=0}^{\ell}\langle g(W(s)),\psi_n\rangle_{L^2_\rho(\mathbf{R}^m)}\psi_n\right\}(y)\, ds\right| \le C\varepsilon_0(\tau)^{2|\lambda_0|+\mu}(1+|y|^{2\ell}) \tag{4.33}$$

for $\varepsilon_0(\tau)^{\theta} < |y|$ and $\tau_0 \le \tau \le \tau_0+1$ with sufficiently large τ_0, where $\mu > 0$ is as before.

Proof. For $i = 1, 2, 3$, we set

$$J_i := \int_{\tau_0}^{\tau} \frac{e^{|\lambda_0|(\tau-s)}}{\{4\pi(1-e^{-(\tau-s)})\}^{m/2}} \int_{E_i(s)} \exp\left\{-\frac{|ye^{-(\tau-s)/2}-z|^2}{4(1-e^{-(\tau-s)})}\right\} |g(W(z,s))|\, dzds,$$

where, for $\tau_0 \le \sigma \le \tau \le \tau_1$ and $\Lambda_\ell = 1 - 1/4\lambda_{\ell+1}$,

$$E_1(\sigma) := \{z \in \mathbf{R}^m \mid |z| \le \varepsilon_0(\sigma)^{1-\theta}\}, \tag{4.34a}$$
$$E_2(\sigma) := \{z \in \mathbf{R}^m \mid \varepsilon_0(\sigma)^{1-\theta} < |z| \le e^{\Lambda_\ell\sigma/2}\}, \tag{4.34b}$$
$$E_3(\sigma) := \{z \in \mathbf{R}^m \mid e^{\Lambda_\ell\sigma/2} < |z|\}. \tag{4.34c}$$

Estimate for J_1. We see from (4.2), (4.21) and (4.25) that

$$\begin{aligned} J_1 &\le \int_{\tau_0}^{\tau} \frac{C}{(1-e^{-(\tau-s)})^{m/2}} \exp\left\{-\frac{|y|^2}{64(1-e^{-(\tau-s)})}\right\} \int_{E_1(s)} |z|^{\gamma}|f(v(z,s))|\, dzds \\ &\le C\left(\sup_{L\ge 0} L^m e^{-L^2}\right)|y|^{-m} \int_{\tau_0}^{\tau}\int_0^{\varepsilon_0(s)^{1-\theta}} r^{\gamma-2\beta+m-3}\, drds \le C\varepsilon_0(\tau)^{-\theta m+(1-\theta)(2|\lambda_0|+\sqrt{D})}. \end{aligned} \tag{4.35}$$

Estimate for J_2. Let us set

$$B_1(y,\sigma) := \{z \in \mathbf{R}^m \mid |z| \le 2|y|e^{-(\tau-\sigma)/2}\}, \tag{4.36a}$$
$$B_2(y,\sigma) := \{z \in \mathbf{R}^m \mid 2|y|e^{-(\tau-\sigma)/2} < |z|\}. \tag{4.36b}$$

We divide the region E_2 in (4.34b) into

$$E_{2,1}(\sigma) := \{z \in \mathbf{R}^m \mid \varepsilon_0(\sigma)^{1-\theta} < |z| \le \varepsilon_0(\sigma)^{2\theta}\}, \tag{4.37a}$$
$$E_{2,2}(y,\sigma) := B_1(y,\sigma) \cap \{z \in \mathbf{R}^m \mid \varepsilon_0(\sigma)^{2\theta} < |z| \le e^{\Lambda_\ell\sigma/2}\}, \tag{4.37b}$$
$$E_{2,3}(y,\sigma) := B_2(y,\sigma) \cap \{z \in \mathbf{R}^m \mid \varepsilon_0(\sigma)^{2\theta} < |z| \le e^{\Lambda_\ell\sigma/2}\} \tag{4.37c}$$

for $y \in \mathbf{R}^m$, $\tau_0 \le \sigma \le \tau \le \tau_1$, and split J_2 as $J_2 = J_{2,1} + J_{2,2} + J_{2,3}$ accordingly. Arguing as the estimate for I_2, we see $E_{2,2}(y,s) \neq \emptyset$ for $|y| > \varepsilon_0(\tau)^\theta$ and $E_{2,3}(y,s) = \emptyset$ for $|y| > e^{-s/4\lambda_{\ell+1}} e^{\tau/2}/2$. Owing to $|\lambda_0| > 1$, (4.2), (4.21) and (4.25), we obtain

$$
\begin{aligned}
J_{2,1} &\le \int_{\tau_0}^{\tau} \frac{C}{(1-e^{-(\tau-s)})^{m/2}} \exp\left\{-\frac{|y|^2}{64(1-e^{-(\tau-s)})}\right\} \int_{E_{2,1}(s)} |z|^\gamma |f(v(z,s))|\, dzds \\
&\le C\left(\sup_{L\ge 0} e^{-L^2} L^m\right) |y|^{-m} \int_{\tau_0}^{\tau} \varepsilon_0(s)^{4|\lambda_0|} \int_{\varepsilon_0(s)^{1-\theta}}^{\varepsilon_0(s)^{2\theta}} r^{-2|\lambda_0|+m-3}\, drds \\
&\le C(\varepsilon_0(\tau)^{2\theta(-2|\lambda_0|+\sqrt{D})} + \varepsilon_0(\tau)^{(1-\theta)(-2|\lambda_0|+\sqrt{D})})\varepsilon_0(\tau)^{4|\lambda_0|-\theta(2+\sqrt{D})}.
\end{aligned}
\tag{4.38}
$$

Recall $\ell > 1$, $\lambda_\ell > 0$ and $\Lambda_\ell > 1/2$. By (4.2) and (4.21), we have

$$
\begin{aligned}
J_{2,2} &\le \int_{\tau_0}^{\tau} \frac{C\varepsilon_0(s)^{4|\lambda_0|}ds}{(1-e^{-(\tau-s)})^{m/2}} \int_{E_{2,2}(y,s)} \exp\left\{-\frac{|ye^{-(\tau-s)/2}-z|^2}{4(1-e^{-(\tau-s)})}\right\} |z|^{-2|\lambda_0|-2}(1+|z|^{4\ell})\, dz \\
&\le C\int_{\tau_0}^{\tau} \varepsilon_0(s)^{4|\lambda_0|}(\varepsilon_0(s)^{-4\theta(|\lambda_0|+1)} + (\varepsilon_0(s)^{4\theta(\lambda_\ell-1)} + e^{(\lambda_\ell-1)\Lambda_\ell s})|y|^{2\ell})\, ds \\
&\le C\varepsilon_0(\tau)^{4|\lambda_0|-4\theta(|\lambda_0|+1)} + C\varepsilon_0(\tau)^{2|\lambda_0|}(\varepsilon_0(\tau)^{2|\lambda_0|+4\theta(\lambda_\ell-1)} + e^{-\tau/2})|y|^{2\ell}
\end{aligned}
\tag{4.39}
$$

and

$$
\begin{aligned}
J_{2,2} &\le C\int_{\tau_0}^{\tau} \varepsilon_0(s)^{4|\lambda_0|}(\varepsilon_0(s)^{-4\theta(|\lambda_0|+1)} + (\varepsilon_0(s)^{4\theta(2\lambda_\ell-1)} + e^{(2\lambda_\ell-1)\Lambda_\ell s})|y|^{2|\lambda_0|})\, ds \\
&\le C\varepsilon_0(\tau)^{4|\lambda_0|-4\theta(|\lambda_0|+1)} + C(\varepsilon_0(\tau)^{4|\lambda_0|+4\theta(2\lambda_\ell-1)} + e^{-\tau/2})|y|^{2|\lambda_0|}.
\end{aligned}
\tag{4.40}
$$

On the other hand, (4.2), (4.21) and (4.29) imply that

$$
\begin{aligned}
J_{2,3} &\le \int_{\tau_0}^{\tau} \frac{C\varepsilon_0(s)^{4|\lambda_0|}ds}{(1-e^{-(\tau-s)})^{m/2}} \int_{E_{2,3}(y,s)} \exp\left\{-\frac{|ye^{-(\tau-s)/2}-z|^2}{16(1-e^{-(\tau-s)})}\right\} |z|^{-2|\lambda_0|-2}(1+|z|^{4\ell})\, dz \\
&\le \int_{\tau_0}^{\tau} \frac{C\varepsilon_0(s)^{4|\lambda_0|}ds}{(1-e^{-(\tau-s)})^{m/2}} \exp\left\{-\frac{|y|^2}{16(1-e^{-(\tau-s)})}\right\} \int_{\varepsilon_0(s)^{2\theta}}^{1} r^{-2|\lambda_0|+m-3}\, dr \\
&\quad + \int_{\tau_0}^{\tau} \frac{C\varepsilon_0(s)^{4|\lambda_0|}ds}{(1-e^{-(\tau-s)})^{m/2}} \int_0^\infty \exp\left\{-\frac{r^2}{16(1-e^{-(\tau-s)})}\right\} r^{2\lambda_\ell+2\ell+m-3}\, dr \\
&\le C\left(\sup_{L\ge 0} e^{-L^2} L^m\right) |y|^{-m}\varepsilon_0(\tau)^{4|\lambda_0|}(1+\varepsilon_0(\tau)^{2\theta(-2|\lambda_0|+\sqrt{D})}) \le \varepsilon_0(\tau)^{4|\lambda_0|-2\theta(2|\lambda_0|+1)}.
\end{aligned}
\tag{4.41}
$$

Estimate for J_3. Let us divide the region E_3 in (4.34c) with

$$
E_{3,1}(y,\sigma) := B_1(y,\sigma) \cap E_3(\sigma), \quad E_{3,2}(y,\sigma) := B_2(y,\sigma) \cap E_3(\sigma), \tag{4.42}
$$

for $y \in \mathbf{R}^m$ and $\tau_0 \le \sigma \le \tau \le \tau_1$, and split J_3 as $J_3 = J_{3,1} + J_{3,2}$, accordingly. We note that $E_{3,1}(y,s) = \emptyset$ for $\varepsilon_0(s)^\theta < |y| \le e^{-s/4\lambda_{\ell+1}} e^{\tau/2}/2$. Due to (4.2) and (4.21), we obtain

$$
\begin{aligned}
J_{3,1} &\le \int_{\tau_0}^{\tau} \frac{C}{(1-e^{-(\tau-s)})^{m/2}} \int_{E_{3,1}(y,s)} \exp\left\{-\frac{|ye^{-(\tau-s)/2}-z|^2}{4(1-e^{-(\tau-s)})}\right\} |z|^{2|\lambda_0|-2}\, dzds \\
&\le Ce^{-(\lambda_\ell+1/2)\tau}|y|^{2\ell}
\end{aligned}
\tag{4.43}
$$

and

$$J_{3,1} \le C \int_{\tau_0}^{\tau} e^{-\Lambda_\ell s} |y|^{2|\lambda_0|}\, ds \le C e^{-\tau/2} |y|^{2|\lambda_0|} \tag{4.44}$$

for $|y| > e^{-s/4\lambda_{\ell+1}} e^{\tau/2}/2$ with sufficiently large $\tau_0 > 0$. For $|y| > \varepsilon(\tau)^\theta$, it follows from $|\lambda_0| > 1$, $\Lambda_\ell > 1/2$, (4.2), (4.21), (4.29) and (4.30) that

$$\begin{aligned} J_{3,2} &\le \int_{\tau_0}^{\tau} \frac{C}{(1-e^{-(\tau-s)})^{m/2}} \int_{E_{3,2}(y,s)} \exp\left\{ -\frac{|z|^2}{16(1-e^{-(\tau-s)})} \right\} |z|^{2|\lambda_0|-2}\, dz ds \\ &\le C\varepsilon_0(\tau)^{4|\lambda_0|}. \end{aligned} \tag{4.45}$$

Because of (3.1), (4.35), (4.38)–(4.45), we obtain (4.32). To show (4.33), we use the series expression (4.18a) of $S(\tau)$ and Lemma 4.2, to get

$$\left| S(\tau-s)\{\langle g(W(s)), \psi_n\rangle_{L^2_\rho(\mathbf{R}^m)} \psi_n\}(y) \right| \le C e^{-\lambda_n(\tau-s)} \varepsilon_0(s)^{2|\lambda_0|+\mu} (1+|y|^{2n})$$

for $n = 0, ..., \ell$. Since $|\tau - s| \le 1$, we obtain (4.33) and the proof is complete. □

4.2.2 A priori estimate in the long-time case

Next, we show the estimate in the long-time $\tau_0 + 1 \le \tau \le \tau_1$. Notice that

$$1 - e^{-1} \le 1 - e^{-(\tau-\sigma)} \le 1 \quad \text{for } \tau_0 + 1 \le \sigma \le \tau \le \tau_1. \tag{4.46}$$

Let $E_0(|y|) := E(|y|, \tau_0)$ where E is as in (4.17).

Lemma 4.7. *There hold*

$$|[S(\tau-\tau_0)E_0](y)| \le \begin{cases} C\varepsilon_0(\tau)^{2|\lambda_0|+2\theta(2+\sqrt{D})} & \text{if } \varepsilon_0(\tau)^\theta < |y| \le 1, \\ C\varepsilon_0(\tau_0)^{2\theta(2+\sqrt{D})} \varepsilon_0(\tau)^{2|\lambda_0|} |y|^{2\ell} & \text{if } 1 < |y| \le e^{(\tau-\tau_0-1)/2}, \end{cases} \tag{4.47a}$$

and

$$|S(\tau-\tau_0)\{W_0 - a^*_\ell(\tau_0)\psi_\ell\}(y)| \le C\varepsilon_0(\tau_0)^{3\theta} \varepsilon_0(\tau)^{2|\lambda_0|} |y|^{2\ell} \quad \text{if } e^{(\tau-\tau_0-1)/2} < |y| \le e^{\tau/2}, \tag{4.47b}$$

for $\tau_0 + 1 \le \tau \le \tau_1$ with sufficiently large τ_0.

Proof. Since E_0 is orthogonal to the eigenfunctions ψ_n for $n = 0, 1, \dots, \ell$ in $L^2_\rho(\mathbf{R}^m)$, Lemma 4.4 (cf. (4.12)) as well as the series expansion (4.18a) imply

$$\begin{aligned} |[S(\tau-\tau_0)E_0](y)| &\le C\varepsilon_0(\tau)^{2|\lambda_0|+2\theta(2+\sqrt{D})} \sum_{n=\ell+1}^{\infty} c_n^2 e^{-(n-\ell-2\theta(2+\sqrt{D}))(\tau-\tau_0)} \\ &\le C\varepsilon_0(\tau)^{2|\lambda_0|+2\theta(2+\sqrt{D})} \end{aligned} \tag{4.48}$$

for $\varepsilon_0(\tau)^\theta < |y| \le 1$ and

$$\begin{aligned} |[S(\tau-\tau_0)E_0](y)| &\le C\varepsilon_0(\tau_0)^{2\theta(2+\sqrt{D})} \varepsilon_0(\tau)^{2|\lambda_0|} |y|^{2\ell} \sum_{n=\ell+1}^{\infty} c_n^2 e^{-(n-\ell)(\tau-\tau_0)} |y|^{2(n-\ell)} \\ &\le C\varepsilon_0(\tau_0)^{2\theta(2+\sqrt{D})} \varepsilon_0(\tau)^{2|\lambda_0|} |y|^{2\ell} \end{aligned} \tag{4.49}$$

for $1 < |y| \le e^{(\tau-\tau_0-1)/2}$, where the exact formulas of c_n and $\tilde{c}_n$ as in (2.17) and (2.18) have been used as well. We then obtain (4.47a).

Next, we prove (4.47b). We remark that $2e^{-1/2} < 2|y|e^{-(\tau-\tau_0)/2} \le 2e^{\tau_0/2}$ and $|y| \ge 1$ for $e^{(\tau-\tau_0-1)/2} < |y| \le e^{\tau/2}$ and $\tau_0 + 1 \le \tau \le \tau_1$. Due to (4.18c) and (4.46), we have

$$\begin{aligned} &|S(\tau-\tau_0)\{W_0 - a_\ell^*(\tau_0)\psi_\ell\}(y)| \\ &\le Ce^{|\lambda_0|(\tau-\tau_0)} \sum_{i=1}^{3} \int_{D_i(\tau_0)} \exp\{-|ye^{-(\tau-\tau_0)/2} - z|^2\} |W_0(z) - a_\ell^*(\tau_0)\psi_\ell(z)|\, dz \qquad (4.50) \\ &=: K_1 + K_2 + K_3, \end{aligned}$$

where $D_i(\tau_0)$ $(i = 1, 2, 3)$ are the sets as in (4.23).

Estimate for K_1. Let us write $\int_{D_1(\tau_0)} = \int_{D_{1,1}} + \int_{D_{1,2}}$, where $D_{1,1}$ and $D_{1,2}$ are the sets as in (4.24a) and (4.24b), respectively, and denote $K_1 = K_{1,1} + K_{1,2}$ accordingly. Similarly to the estimate for I_1, it follows from (3.12) that

$$\begin{aligned} K_{1,1} &\le Ce^{|\lambda_0|(\tau-\tau_0)} \int_{D_{1,1}(\tau_0)} (\varepsilon_0(\tau_0)^{2|\lambda_0|} + |z|^{2|\lambda_0|})\, dz \\ &\le Ce^{|\lambda_0|(\tau-\tau_0)} (\varepsilon_0(\tau_0)^{2|\lambda_0|+(1-3\theta/\mu)m} + \varepsilon_0(\tau_0)^{(1-3\theta/\mu)(2|\lambda_0|+m)}). \end{aligned}$$

Notice that we have $|y|^{-2\ell} \le e^\ell e^{-\ell(\tau-\tau_0)}$ for $e^{(\tau-\tau_0-1)/2} < |y|$ and $|\lambda_0| - \ell = -\lambda_\ell$, whence:

$$K_{1,1} \le C(\varepsilon_0(\tau_0)^{(1-3\theta/\mu)m} + \varepsilon_0(\tau_0)^{m-(1-3\theta/\mu)(2|\lambda_0|+m)}) e^{-\lambda_\ell \tau} |y|^{2\ell}. \qquad (4.51)$$

A similar argument shows

$$\begin{aligned} K_{1,2} &\le Ce^{|\lambda_0|(\tau-\tau_0)} \int_{D_{1,2}(\tau_0)} (\varepsilon_0(\tau_0)^{2|\lambda_0|+\mu} |z|^{-\mu} + C\varepsilon_0(\tau_0)^{2|\lambda_0|} |z|^2)\, dz \\ &\le C(\varepsilon_0(\tau_0)^{\mu+2\theta(\mu+m)} + \varepsilon_0(\tau_0)^{2\theta(m+2)}) e^{-\lambda_\ell \tau} |y|^{2\ell}. \end{aligned} \qquad (4.52)$$

Estimate for K_2. Let us write $\int_{D_2(\tau_0)} = \int_{D_{2,1}(y,\tau_0)} + \int_{D_{2,2}(y,\tau_0)}$ with the sets $D_{2,1}(y,\sigma) := B_1(y,\sigma) \cap D_2(\sigma)$ and $D_{2,2}(y,\sigma) := B_2(y,\sigma) \cap D_2(\sigma)$, where $B_1(y,\sigma)$ and $B_2(y,\sigma)$ are as in (4.36a) and (4.36b), respectively, and denote $K_2 = K_{2,1} + K_{2,2}$ accordingly. We note that $D_{2,1}, D_{2,2} \ne \emptyset$ for $e^{(\tau-\tau_0-1)/2} < |y| \le e^{\tau/2}$. Because of (3.11), we have

$$\begin{aligned} K_{2,1} &\le Ce^{-\lambda_0(\tau-\tau_0)} \varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta} \int_{D_{2,1}(y,\tau_0)} \exp\{-|ye^{-(\tau-\tau_0)/2} - z|^2\} (1 + |z|^{2\ell})\, dz \qquad (4.53) \\ &\le Ce^{-\lambda_0(\tau-\tau_0)} \varepsilon_0(\tau_0)^{2|\lambda_0|+3\theta} \left[(|y|e^{-(\tau-\tau_0-1)/2})^{2\ell} + (2|y|e^{-(\tau-\tau_0)/2})^{2\ell} \right] \le C\varepsilon_0(\tau_0)^{3\theta} e^{-\lambda_\ell \tau} |y|^{2\ell}. \end{aligned}$$

In addition, we know from (3.11) and (4.29) that

$$K_{2,2} \le Ce^{-\lambda_0(\tau-\tau_0)} \varepsilon_0(\tau_0)^{2|\lambda_0|+\theta} \int_{D_{2,2}(y,\tau_0)} e^{-|z|^2/4} (1 + |z|^{2\ell})\, dz \le C\varepsilon_0(\tau_0)^{3\theta} e^{-\lambda_\ell \tau} |y|^{2\ell}. \quad (4.54)$$

Estimate for K_3. Inequalities (3.11), (4.29) and (4.30) imply

$$K_3 \le Ce^{-\lambda_0(\tau-\tau_0)} \varepsilon_0(\tau_0)^{2|\lambda_0|} \int_{D_{2,2}(y,\tau_0)} (|z|/2e^{\tau_0/2})^2 e^{-|z|^2/4} |z|^{2\ell}\, dz \le Ce^{-\tau_0} e^{-\lambda_\ell \tau} |y|^{2\ell}. \quad (4.55)$$

Because of (3.1), (4.51)-(4.55), we obtain (4.47b) and the proof is complete. □

Lemma 4.8. *There hold*

$$\left|S(\tau-s)\left\{g(W(s))-\sum_{n=0}^{\ell}\langle g(W(s)),\psi_n\rangle_{L^2_\rho(\mathbf{R}^m)}\psi_n\right\}(y)\right| \le \begin{cases} C\varepsilon_0(s)^{\mu-2\theta(2+\sqrt{D})}\varepsilon_0(\tau)^{2|\lambda_0|+2\theta(2+\sqrt{D})}, & \varepsilon_0(\tau)^\theta\le|y|<1,\\ C\varepsilon_0(s)^\mu\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}, & 1\le|y|,\end{cases} \tag{4.56}$$

and

$$|S(\tau-s)g(W(s))(y)|\le C\varepsilon_0(s)^\mu\min\{\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell},|y|^{2|\lambda_0|}\},\quad |y|\ge e^{(\tau-s-1)/2}, \tag{4.57}$$

as long as $\tau_0+1\le\tau\le\tau_1$ *and* $\tau_0\le s\le\tau-1$ *with sufficiently large* τ_0*, where* $\mu=\min\{|\lambda_0|,\sqrt{D}/2,|\lambda_0|/\lambda_\ell\}>0$ *as before.*

Proof. Recall that $\langle f(v),\phi_n\rangle_{L^2_\rho(\mathbf{R}^N)}=\langle g(W),\psi_n\rangle_{L^2_\rho(\mathbf{R}^m)}$. Arguing as in the proof of Lemma 4.7, we obtain the first and second inequalities of (4.56) for $|y|<e^{(\tau-s-1)/2}$ from (4.5) and (4.18a).

We next prove (4.57) and the second inequality of (4.56) for $|y|\ge e^{(\tau-s-1)/2}$. To this end, we assume $|y|\ge e^{(\tau-s-1)/2}$ hereafter without stating explicitly. Due to (4.18c) and (4.46), we have

$$|S(\tau-s)g(W(s))(y)|\le Ce^{|\lambda_0|(\tau-s)}\int_{\mathbf{R}^m}\exp\{-|ye^{-(\tau-s)/2}-z|^2\}|g(W(z,s))|\,dz. \tag{4.58}$$

We split $\mathbf{R}^m$ to the subregions E_i $(i=1,2,3)$ defined in (4.34) and denote the corresponding integrals as L_i $(i=1,2,3)$ accordingly. We remark $1<2e^{-1/2}<2|y|e^{-(\tau-s)/2}$, $|y|\ge1$ and $r^\gamma f(v)=g(W)$.

Estimate for L_1. By (4.2), we see that

$$\begin{aligned} L_1&\le Ce^{|\lambda_0|(\tau-s)}\int_{E_1(s)}|z|^\gamma|f(v(z,s))|\,dz\\ &\le Ce^{|\lambda_0|(\tau-s)}(e^{-(\tau-s-1)/2}|y|)^{2|\lambda_0|}\int_0^{\varepsilon_0(s)^{1-\theta}}r^{\gamma-2\beta+m-3}\,dr\\ &\le C\varepsilon_0(s)^{\sqrt{D}-\theta(2|\lambda_0|+\sqrt{D})}e^{-\lambda_\ell\tau}|y|^{2\ell}.\end{aligned} \tag{4.59}$$

Estimate for L_2. Let us divide further the region E_2 as

$$\begin{aligned} &E'_{2,1}(s):=\{z\in\mathbf{R}^m\mid\varepsilon_0(s)^{1-\theta}<|z|\le1\},\\ &E'_{2,2}(y,s):=B_1(y,s)\cap\{z\in\mathbf{R}^m\mid1<|z|\le e^{\Lambda_\ell s/2}\},\\ &E'_{2,3}(y,s):=B_2(y,s)\cap\{z\in\mathbf{R}^m\mid1<|z|\le e^{\Lambda_\ell s/2}\},\end{aligned}$$

and denote the corresponding integrals by $L_{2,j}$ $(j=1,2,3)$ accordingly. We note that $E'_{2,2}(y,s)\ne\emptyset$ for $|y|>e^{(\tau-s-1)/2}$ and $E'_{2,3}(y,s)=\emptyset$ for $|y|>e^{-s/4\lambda_{\ell+1}}e^{\tau/2}/2$. The estimate

(4.2) of $f(v(z,s))$ in $E'_{2,1}(s)$ implies that

$$\begin{aligned} L_{2,1} &\le Ce^{|\lambda_0|(\tau-s)} \int_{E'_{2,1}(s)} |z|^\gamma |f(v(z,s))|\, dz \\ &\le Ce^{|\lambda_0|(\tau-s)} (e^{-(\tau-s-1)/2}|y|)^{2|\lambda_0|} e^{-2\lambda_\ell s} \int_{\varepsilon_0(s)^{1-\theta}}^1 r^{-2|\lambda_0|+m-3}\, dr \\ &\le C(e^{-\lambda_\ell s} + \varepsilon_0(s)^{(1-\theta)\sqrt{D}}) e^{-\lambda_\ell s}|y|^{2|\lambda_0|} \le C(e^{-\lambda_\ell s} + \varepsilon_0(s)^{(1-\theta)\sqrt{D}}) e^{-\lambda_\ell \tau}|y|^{2\ell} \end{aligned} \tag{4.60}$$

for $|y| > e^{(\tau-s-1)/2}$. Due to (4.2), we have

$$\begin{aligned} L_{2,2} &\le Ce^{-\lambda_0(\tau-s)} e^{-2\lambda_\ell s} \int_{E'_{2,2}(y,s)} \exp\{-|ye^{-(\tau-s)/2} - z|^2\} |z|^{2\ell+2\lambda_\ell-2}\, dz \\ &\le C(e^{-\lambda_\ell s} + e^{-s/2}) e^{-\lambda_\ell \tau}|y|^{2\ell} \end{aligned} \tag{4.61}$$

and

$$L_{2,2} \le Ce^{|\lambda_0|(\tau-s)} e^{-2\lambda_\ell s} (2|y|e^{-(\tau-s)/2})^{2|\lambda_0|} (1+e^{(2\lambda_\ell-1)\Lambda_\ell s}) \le C(e^{-2\lambda_\ell s}+e^{-s/2})|y|^{2|\lambda_0|}. \tag{4.62}$$

A similar estimate to (4.29) shows

$$L_{2,3} \le Ce^{|\lambda_0|(\tau-s)} e^{-2\lambda_\ell s} \int_{E'_{2,3}(y,s)} e^{-|z|^2/4} |z|^{2\ell+2\lambda_\ell-2}\, dz \le Ce^{-\lambda_\ell s} e^{-\lambda_\ell \tau}|y|^{2\ell}. \tag{4.63}$$

Estimate for L_3. Let us split E_3 to the disjoint union of $E_{3,1}, E_{3,2}$ as in (4.42) and denote the corresponding integrals as $L_{3,j}$ $(j = 1, 2)$ accordingly. We note that $E_{3,1} = \emptyset$ for $e^{(\tau-s-1)/2} < |y| \le 2^{-1}e^{-s/4\lambda_{\ell+1}} e^{\tau/2}$. The fact of $|\lambda_0| > 1$ and (4.2) imply that

$$\begin{aligned} L_{3,1} &\le Ce^{|\lambda_0|(\tau-s)} \int_{E_{3,1}(y,s)} \exp\{-|ye^{-(\tau-s)/2} - z|^2\} |z|^{2|\lambda_0|-2}\, dz \\ &\le Ce^{|\lambda_0|(\tau-s)} e^{-\lambda_{\ell+1}\Lambda_\ell s} (2|y|e^{-(\tau-s)/2})^{2\ell} \le Ce^{-s/2} e^{-\lambda_\ell \tau}|y|^{2\ell} \end{aligned} \tag{4.64}$$

and

$$L_{3,1} \le Ce^{|\lambda_0|(\tau-s)} e^{-\Lambda_\ell s} (2|y|e^{-(\tau-s)/2})^{2|\lambda_0|} \le Ce^{-s/2}|y|^{2|\lambda_0|} \tag{4.65}$$

for $|y| > 2^{-1}e^{-s/4\lambda_{\ell+1}} e^{\tau/2}$. Similarly to the argument in (4.63), we see that

$$\begin{aligned} L_{3,2} &\le Ce^{|\lambda_0|(\tau-s)} \int_{E_{3,2}(y,s)} e^{-|z|^2/4} |z|^{2|\lambda_0|-2}\, dz \\ &\le Ce^{|\lambda_0|(\tau-s)} (2|y|e^{-(\tau-s)/2})^{2|\lambda_0|} \int_{e^{s/4}}^\infty \left(\frac{r}{e^{-s/4}}\right)^{8\lambda_\ell} e^{-r^2/4} r^{2|\lambda_0|+m-3}\, dz \\ &\le Ce^{-2\lambda_\ell s}|y|^{2|\lambda_0|} \le Ce^{-\lambda_\ell s} e^{-\lambda_\ell \tau}|y|^{2\ell}. \end{aligned} \tag{4.66}$$

Due to (3.1), (4.59)–(4.66), we have (4.57).

To obtain the second inequality of (4.56), it is sufficient to show

$$\left| S(\tau - s) \left\{ \sum_{n=0}^{\ell} \langle g(W(s)), \psi_n \rangle_{L^2_\rho(\mathbf{R}^m)} \psi_n \right\} (y) \right| \le C\varepsilon_0(s)^\mu \varepsilon_0(\tau)^{2|\lambda_0|} |y|^{2\ell} \tag{4.67}$$

for $e^{(\tau-s-1)/2} \le |y|$. Notice that $|y| \ge 1$. Recalling Lemma 4.2 and the series expression (4.18a) of $S(\tau)$, we easily obtain

$$\begin{aligned}&\left|S(\tau-s)\{\langle g(W(s)),\psi_n\rangle_{L^2_\rho(\mathbf{R}^m)}\psi_n\}(y)\right|\\&\le C\varepsilon_0(s)^\mu e^{-\lambda_\ell s}e^{-\lambda_n(\tau-s)}|y|^{2n} = C\varepsilon_0(s)^\mu e^{-\lambda_\ell\tau}|y|^{2\ell}\left(e^{-(\tau-s)/2}|y|\right)^{2(n-\ell)}\end{aligned}$$

for $n = 0, 1, \ldots, \ell$. Estimate (4.67) then follows at once since $e^{-(\tau-s)/2}|y| \ge e^{-1/2}$. The proof is complete □

Lemma 4.9. *There hold*

$$\begin{aligned}&\left|\int_{\tau_0}^{\tau} S(\tau-s)\left\{g(W(s)) - \sum_{n=0}^{\ell}\langle g(W(s)),\psi_n\rangle_{L^2_\rho(\mathbf{R}^m)}\psi_n\right\}(y)ds\right|\\&\le \begin{cases} C\varepsilon_0(\tau_0)^{\mu-2\theta(2+\sqrt{D})}\varepsilon_0(\tau)^{2|\lambda_0|+2\theta(2+\sqrt{D})}, & \varepsilon_0(\tau)^\theta \le |y| \le 1,\\ C\varepsilon_0(\tau_0)^\mu\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}, & 1 \le |y|,\end{cases}\end{aligned} \tag{4.68}$$

and

$$\left|\int_{\tau_0}^{\tau} S(\tau-s)g(W(s))(y)ds\right| \le C\varepsilon_0(\tau_0)^\mu \min\{\varepsilon_0(\tau)^{2|\lambda_0|}|y|^{2\ell}, |y|^{2|\lambda_0|}\}, \quad |y| \ge e^{\tau/2} \tag{4.69}$$

as long as $\tau_0 + 1 < \tau \le \tau_1$ with sufficiently large τ_0, where $\mu > 0$ is as before.

Proof. We first divide integral in time as $\int_{\tau_0}^{\tau} = \int_{\tau_0}^{\tau-1} + \int_{\tau-1}^{\tau}$. Clearly the later integral may be estimated as in (4.32) and (4.33). It thus suffices to consider the former integral. Fix now y with $1 < |y| < e^{(\tau-\tau_0-1)/2}$. Then there is a unique number $\bar{s} = \bar{s}(y) \in (\tau_0, \tau-1)$ such that $|y| = e^{(\tau-\bar{s}-1)/2}$. We then use the second inequality of (4.56) to estimate the integrand over $[\tau_0, \tau-1]$. As a result, we obtain

$$\begin{aligned}&\left|\int_{\tau_0}^{\tau-1} S(\tau-s)\left\{g(W(s)) - \sum_{n=0}^{\ell}\langle g(W(s)),\psi_n\rangle_{L^2_\rho(\mathbf{R}^m)}\psi_n\right\}(y)\,ds\right|\\&\le \varepsilon_0(\tau_0)^\mu\varepsilon_0(\tau)^{2|\lambda_0|}(1+|y|^{2\ell})\end{aligned} \tag{4.70}$$

Similar estimates may be derived for $\varepsilon_0(\tau)^\theta \le |y| \le 1$ and $e^{(\tau-\tau_0-1)/2} \le |y|$, respectively, without splitting the integral in s, whence (4.68). Integrating the both sides of (4.57) in s over $[\tau_0, \tau-1]$, we readily obtain (4.69). The proof is complete. □

4.2.3 A priori estimates for $|y| > e^{\tau/2}$

Lemma 4.10. *There exists a constant $C > 0$ depending only on N, λ_ℓ and h such that*

$$|W(y,\tau)| \le C|y|^{2|\lambda_0|}, \quad |y| > e^{\tau/2} \tag{4.71}$$

for $\tau_0 \le \tau \le \tau_1$ with sufficiently large τ_0.

Proof. Similarly to (3.11), there exists a positive constant C such that

$$|W_0(r)| \le Cr^{2|\lambda_0|}, \quad r > 0. \tag{4.72}$$

A careful check of our setting of initial data shows that the constant C in (4.72) depends only on λ_ℓ, N and h. We define functions W_1 and W_2 as

$$W(y,\tau;\alpha) = [S(\tau-\tau_0)W_0](y) + \int_{\tau_0}^{\tau} [S(\tau-s)g(W(s))](y)\,ds =: W_1 + W_2. \tag{4.73}$$

By (4.72), we have

$$\begin{aligned}|W_1| &\le \frac{Ce^{|\lambda_0|(\tau-\tau_0)}}{(1-e^{-(\tau-\tau_0)})^{m/2}} \left(\int_{B_1(y,\tau_0)} + \int_{B_2(y,\tau_0)}\right) \exp\left\{-\frac{|ye^{-(\tau-\tau_0)/2}-z|^2}{4(1-e^{-(\tau-\tau_0)})}\right\} |z|^{2|\lambda_0|}\,dz \\ &\le Ce^{|\lambda_0|(\tau-\tau_0)}(2|y|e^{-(\tau-\tau_0)/2})^{2|\lambda_0|} + Ce^{|\lambda_0|(\tau-\tau_0)}(|y|e^{-\tau/2})^{2|\lambda_0|} \le C|y|^{2|\lambda_0|}\end{aligned}$$

for $|y| > e^{\tau/2}$. It follows from Lemmas 4.6 and 4.9 that $|W_2| \le C\varepsilon_0(\tau_0)^{\mu}|y|^{2|\lambda_0|}$ for $|y| > e^{\tau/2}$. The proof is complete. □

Remark 4.1. Let us comment on the choice of the constant M from the definition of functional set $\mathcal{A}^{\nu}_{\tau_0,\tau_1}$ in §3.1, as we haven't clarified how large it should be. We now choose M as in (3.9) so that $M > 2C$ holds, where C is the constant appearing in Lemma 4.10.

4.3 Completion of the key a priori estimate

We now prove Lemma 3.1. Due to Lemmas 4.3, 4.10 and 5.1 below, it suffices to show

$$|W(r,\tau) - a^*_\ell(\tau)\psi_\ell(r)| < \frac{1}{4}\varepsilon_0(\tau)^{\gamma-2\beta+2\theta}, \quad \varepsilon(\tau)^{\theta} \le r \le 1, \tag{4.74a}$$

$$|W(r,\tau) - a^*_\ell(\tau)\psi_\ell(r)| < \frac{1}{4}\varepsilon_0(\tau_0)^{2\theta}\varepsilon_0(\tau)^{\gamma-2\beta}r^{2\ell}, \quad 1 \le r \le e^{\tau/2} \tag{4.74b}$$

as long as $\tau_0 \le \tau \le \tau_1$ with τ_0 large enough. When $\tau_0 \le \tau_1 \le \tau_0 + 1$, the estimates (4.74) follow from Lemmas 4.5 and 4.6. When $\tau_0 + 1 < \tau_1$, we obtain (4.74) from Lemmas 4.7 and 4.9 as well as the estimate (4.9) of Fourier coefficients. The proof of Lemma 3.1 is now complete.

4.4 The general case $m \in \mathbf{R}_+$

In the case where $m \in \mathbf{R}_+$ is not an integer, the representation (4.18c) is no longer available. Instead, we may use (4.18b), which have the estimate (4.20). As in §§§ 4.2.1 and §§§4.2.2, we split the region of integrations to various subintervals to obtain suitable estimates. See [42, §§§4.3.2] for a similar argument.

5 Proof of a priori estimates in the inner region

In this subsection we will prove a priori estimates in the region where $|y| \le \varepsilon_0(\tau)^\theta$ using the idea of [41, 42]. This together with the lemmas in §4 complete the proof of Lemma 3.1.

Lemma 5.1. *Assume that $p > p_{\rm JL}$, $N > 10 + 8a$. Assume also that*

$$|\Phi_0(r) - \Phi_{\rm out}(r, \tau_0)| \le C\varepsilon_0(\tau_0)^{\gamma-2\beta+2\theta} r^{-\gamma} \tag{5.1}$$

for $r \le \varepsilon_0(\tau_0)^\theta$. If there exists a constant $M_0 > 0$ such that

$$\left| \Phi(r,\tau) - U_\infty(r) + \frac{h}{c_\ell}\varepsilon_0(\tau)^{\gamma-2\beta}\phi_\ell(r) \right| \le M_0 \varepsilon_0(\tau)^{\gamma-2\beta+2\theta} r^{-\gamma} \tag{5.2}$$

for $r = \varepsilon_0(\tau)^\theta$, $\tau_0 \le \tau \le \tau_1$, then there exists a positive smooth function $H(s)$ with

$$H(s) = \begin{cases} O(s^{-\gamma}) & \text{as } s \to \infty, \\ O(1) & \text{as } s \to 0 \end{cases}$$

such that

$$|\Phi(r,\tau) - \Phi_{\rm inn}(r,\tau)| \le \varepsilon_0(\tau)^{-2\beta+2\theta} H\left(\frac{r}{\varepsilon_0(\tau)}\right) \tag{5.3}$$

for $r \le \varepsilon_0(\tau)^\theta$, $\tau_0 \le \tau \le \tau_1$ with τ_0 large enough.

Proof. We extend the idea of the proof of [41, Proposition 2.1] for the neutral eigenvalues to that of stable ones. Let us abbreviate $\varepsilon_0(\tau)$ to $\varepsilon(\tau)$. We shall recall equation (2.5):

$$\varepsilon(\tau)^2 U_\tau = \Delta_\xi U + |\xi|^{2a} U^p - \mu(\tau)\left(\frac{\xi\cdot\nabla_\xi U}{2} + \beta U\right),$$

where $\xi = y/\varepsilon(\tau)$, $U(\xi,\tau) = \varepsilon(\tau)^{2\beta}\Phi(y,\tau)$, and $\mu(\tau) := \varepsilon(\tau)^2 - 2\varepsilon(\tau)\dot\varepsilon(\tau)$. Consider the new dependent variable:

$$V(\xi,\tau) = U(\xi,\tau) - U_1(\xi).$$

Equation (2.5) for $U(\xi,\tau)$ is then converted to the one for $V(\xi,\tau)$ as follows:

$$\begin{aligned} \mathcal{N}V := -\varepsilon(\tau)^2\frac{\partial V}{\partial\tau} + \Delta_\xi V + |\xi|^{2a}\Big[pU_2(\xi)^{p-1}V + (U_1(\xi)+V)^p - U_1(\xi)^p \qquad \text{(5.4a)}\\ - pU_1(\xi)^{p-1}V + p(U_1(\xi)^{p-1} - U_2(\xi)^{p-1})V\Big] - \mu(\tau)\left[T_0(\xi) + \left(\frac{\xi\cdot\nabla_\xi V}{2} + \beta V\right)\right] = 0 \end{aligned}$$

$$\text{with } T_0(s) = \frac{sU_1'(s)}{2} + \beta U_1(s). \tag{5.4b}$$

Here and henceforth, we shall abuse the notation such as $U_1(\xi) = U_1(|\xi|)$ for simplicity. The ordered structure of the family $\{U_\alpha\}_{\alpha>0}$ implies that

$$T_0(s) = \beta\frac{\partial}{\partial\alpha}U_\alpha(s)\Big|_{\alpha=1} > 0 \quad \text{for any } s > 0; \quad T_0(0) = \beta.$$

Let us write

$$H_0(\xi) := \frac{\partial}{\partial \alpha} U_\alpha(\xi)\bigg|_{\alpha=2},$$

which solves

$$\begin{cases} 0 = H_0'' + \dfrac{N-1}{s} H_0' + p s^{2a} U_2^{p-1}(s) H_0, \quad s > 0 \text{ with } s = |\xi|, \\ H_0(0) = 0, \ H_0'(0) = 0. \end{cases}$$

Taking advantage of the asymptotics of $U_1(s)$ and $U_1'(s)$ as in Proposition 2.1, we obtain

$$T_0(s) = \frac{h(\gamma - 2\beta)}{2} s^{-\gamma} + o\left(s^{-\gamma}\right), \tag{5.5}$$

$$H_0(s) = C_0 s^{-\gamma} + o\left(s^{-\gamma}\right) \quad \text{with} \quad C_0 = \frac{h(\gamma-2\beta)(p-1)}{2 \cdot 2^{\gamma(p-1)/2}} \tag{5.6}$$

as $s \to \infty$. Let $H_1(\xi)$ be a solution of the inhomogeneous ODE:

$$H'' + \frac{N-1}{s} H' + p s^{2a} U_1(s)^{p-1} H = T_0(s), \tag{5.7}$$

satisfying $H(0) = H'(0) = 0$. A standard computation then shows that

$$\begin{aligned} H_1(s) =& H_0(s) \int_0^s \frac{1}{\{H_0(t)\}^2 t^{N-1}} \int_0^t \eta^{N-1} H_0(\eta) T_0(\eta)\, d\eta dt \\ =& C_1 s^{-\gamma+2} + o\left(s^{-\gamma+2}\right) \quad \text{with} \quad C_1 = \frac{h(\gamma-2\beta)}{4(2+\sqrt{D})} \end{aligned} \tag{5.8}$$

as $s \to \infty$. Let $M_1 > 0$ be a constant to be chosen later. We will construct sub- and supersolutions of (5.4a), using auxiliary functions

$$V_{1,\pm}(\xi,\tau) := \mu_\pm(\tau) H_1(\xi) \pm M_1 \varepsilon(\tau)^{2\theta} H_0(\xi) \tag{5.9a}$$

$$\text{with } \mu_\pm(\tau) = \mu(\tau) \mp \sqrt{M_1} \varepsilon(\tau)^{2+2\theta}. \tag{5.9b}$$

The functions $V_\pm(\xi,\tau) := V_{1,\pm}(\xi,\tau)$ satisfy

$$\begin{aligned} \mathcal{N} V_{1,\pm} =& |\xi|^{2a} \Big[(U_1(\xi) + V_{1,\pm})^p - U_1(\xi)^p - p U_1(\xi)^{p-1} V_{1,\pm} + \\ &+ p(U_1(\xi)^{p-1} - U_2(\xi)^{p-1}) V_{1,\pm} \Big] + (\mu_\pm(\tau) - \mu(\tau)) T_0(\xi) - \\ &- \mu(\tau)\mu_\pm(\tau) \left(\frac{\xi \cdot \nabla_\xi H_1(\xi)}{2} + \beta H_1(\xi) \right) - \varepsilon(\tau)^2 \frac{d}{d\tau} \left(\mu_\pm(\tau)\right) H_1(\xi) \mp \\ &\mp M_1 \varepsilon(\tau)^{2\theta} \mu(\tau) \left(\frac{\xi \cdot \nabla_\xi H_0(\xi)}{2} + \beta H_0(\xi) \right) \mp M_1 \varepsilon(\tau)^2 \frac{d}{d\tau} \left(\varepsilon(\tau)^{2\theta}\right) H_0(\xi). \end{aligned} \tag{5.10}$$

Notice that the last two terms in (5.10) are roughly of oder $\varepsilon(\tau)^{2+2\theta}$ as $\tau \to \infty$, which is the same as of $(\mu_\pm(\tau) - \mu(\tau)) T_0(\xi)$. To cancel out the terms proportional to $T_1(\xi) :=$

$\xi \cdot \nabla_\xi H_0(\xi)/2 + \beta H_0(\xi)$ and $M_1 H_0(\xi)$ in (5.10), respectively, we introduce the functions

$$J_1(\xi) = H_0(\xi) \int_0^\xi \frac{dt}{\{H_0(t)\}^2 t^{N-1}} \int_0^t \eta^{N-1} H_0(\eta) T_1(\eta) d\eta,$$
$$\tilde{J}_1(\xi) = H_0(\xi) \int_0^\xi \frac{dt}{\{H_0(t)\}^2 t^{N-1}} \int_0^t \eta^{N-1} H_0(\eta)^2 d\eta.$$

Notice that they solve ODEs

$$J_1'' + \frac{N-1}{s} J_1' + p s^{2a} U_2(s)^{p-1} J_1 = T_1(\xi),$$
$$\tilde{J}_1'' + \frac{N-1}{s} \tilde{J}_1' + p s^{2a} U_2(s)^{p-1} \tilde{J}_1 = H_0(\xi)$$

with boundary conditions $J_1(0) = J_1'(0) = 0$, $\tilde{J}(0) = \tilde{J}'(0) = 0$. We now set

$$V_{2,\pm}(\xi,\tau) = \pm M_1 \mu(\tau) \varepsilon_0(\tau)^{2\theta} J_1(\xi) \pm M_1 \varepsilon(\tau)^2 \frac{d}{d\tau} \left(\varepsilon_0(\tau)^{2\theta}\right) \tilde{J}_1(\xi).$$

Using H'Lôpital rule, we readily obtain

$$T_1(\xi) \approx \left(-\frac{\gamma}{2} + \beta\right) C_0 \xi^{-\gamma} + o\left(\xi^{-\gamma}\right), \tag{5.11a}$$
$$J_1(\xi) \approx \left(-\frac{\gamma}{2} + \beta\right) C_0 \xi^{-\gamma+2} + o\left(\xi^{-\gamma+2}\right), \tag{5.11b}$$
$$T_2(\xi) := \frac{\xi \cdot \nabla_\xi J_1(\xi)}{2} + \beta J_1(\xi) \approx \left(-\frac{\gamma}{2} + \beta\right)\left(1 - \frac{\gamma}{2} + \beta\right) C_0 \xi^{-\gamma+2} + o\left(\xi^{-\gamma+2}\right) \tag{5.11c}$$

as $\xi \to \infty$. Notice that $(-\gamma/2 + \beta)(1 - \gamma/2 + \beta) = \lambda_0 \lambda_1 > 0$ for $p > p_{\rm JL}$. The redefined function

$$V_\pm(\xi,\tau) := V_{1,\pm}(\xi,\tau) + V_{2,\pm}(\xi,\tau)$$

satisfies

$$\begin{aligned} \mathcal{N} V_\pm =& |\xi|^{2a} \Big[(U_1 + V_\pm)^p - U_1^p - p U_1^{p-1} V_\pm + p \left\{U_1^{p-1} - U_2^{p-1}\right\} V_\pm \Big] \\ &+ \{\mu_\pm(\tau) - \mu(\tau)\} T_0(\xi) - \mu(\tau)\mu_\pm(\tau) \left(\frac{\xi \cdot \nabla_\xi H_1(\xi)}{2} + \beta H_1(\xi)\right) \\ &- \varepsilon_0(\tau)^2 \frac{d}{d\tau} \left(\mu_\pm(\tau)\right) H_1(\xi) + \mathcal{E}(\xi,\tau), \end{aligned} \tag{5.12}$$

where

$$\begin{aligned} \mathcal{E}(\xi,\tau) =& \pm M_1 \mu(\tau)^2 \varepsilon(\tau)^{2\theta} T_2(\xi) \mp M_1 \varepsilon(\tau)^2 \frac{d}{d\tau}\left(\mu(\tau)\varepsilon(\tau)^{2\theta}\right) J_1(\xi) \\ &\pm M_1 \mu(\tau) \varepsilon(\tau)^2 \frac{d}{d\tau}\left(\varepsilon(\tau)^{2\theta}\right) \left(\frac{\xi \cdot \nabla_\xi \tilde{J}_1(\xi)}{2} + \beta \tilde{J}_1(\xi)\right) \\ &\mp \varepsilon(\tau)^2 \frac{d}{d\tau}\left(M_1 \varepsilon(\tau)^2 \frac{d}{d\tau}\left(\varepsilon(\tau)^{2\theta}\right)\right) \tilde{J}_1(\xi). \end{aligned}$$

As is readily seen, there is a constant $C_{M_1} > 0$ such that

$$|\mathcal{E}(\xi,\tau)| \le C_{M_1}\varepsilon(\tau)^{4+2\theta}(1+\xi)^{-\gamma+2}.$$

This is smaller than $\{\mu_\pm(\tau)-\mu(\tau)\}T_0(\xi) = \mp\sqrt{M_1}\varepsilon(\tau)^{2+2\theta}T_0(\xi)$ in its modulus. Due to (5.8), we have

$$\begin{aligned}&\left|-\mu(\tau)\mu_\pm(\tau)\left(\frac{\xi\cdot\nabla_\xi H_1(\xi)}{2}+\beta H_1(\xi)\right)-\varepsilon_0(\tau)^2\frac{d}{d\tau}\left(\mu_\pm(\tau)\right)H_1(\xi)\right|\\&\le C\varepsilon(\tau)^4(1+\xi)^{-\gamma+2}\le C'\varepsilon(\tau)^{2+2\theta}(1+\xi)^{-\gamma}\end{aligned}\tag{5.13}$$

for $\xi \le \varepsilon(\tau)^{\theta-1}$. We now choose M_1 large enough, so that the last quantity of (5.13) is dominated by $\{\mu_\pm(\tau)-\mu(\tau)\}T_0(\xi)$ as well.

Consider the case where the plus sign of $V_\pm$ is selected. Since $T_0(\xi)$ is positive, it follows from (5.5), (5.9b), (5.12), and (5.13) that

$$\mathcal{N}V_+ \le |\xi|^{2a}\Big[(U_1+V_+)^p - U_1^p - pU_1^{p-1}V_+ + p\left\{U_1^{p-1}-U_2^{p-1}\right\}V_+\Big] - \frac{1}{3}\varepsilon(\tau)^{2+2\theta}T_0(\xi)$$

holds for $\xi \le \varepsilon(\tau)^{\theta-1}$. Moreover, it is easily seen that this last term dominates for $1 \ll \xi \le \varepsilon(\tau)^{\theta-1}$, whereas the negative term $p\left\{U_1^{p-1}-U_2^{p-1}\right\}V_+$ dominates for $\xi = O(1)$. Therefore the function V_+ is a supersolution. The case where the negative sign of $V_\pm$ is selected is similar. In this case the both $(U_1+V_-)^p - U_1^p - pU_1^{p-1}V_-$ and $p\left\{U_1^{p-1}-U_2^{p-1}\right\}V_-$ are positive. Consequently, the function V_- is a subsolution.

Next, we verify

$$V_-(\xi,\tau) \le U(\xi,\tau)-U_1(\xi) \le V_+(\xi,\tau) \text{ for } |\xi| = \varepsilon(\tau)^{\theta-1},\ \tau_0\le\tau\le\tau_1. \tag{5.14}$$

To this end, we recall

$$\Phi_{\text{out}}(r,\tau) = U_\infty(r) + a_\ell^*(\tau)\phi_\ell(r)\,;\quad a_\ell^*(\tau) = -\frac{h}{c_\ell}\varepsilon(\tau)^{\gamma-2\beta}. \tag{5.15}$$

Due to (2.15) and (2.8), we obtain, as $r = \varepsilon(\tau)^\theta \to 0$,

$$\begin{aligned}&\Phi_{\text{out}}(r,\tau)-\Phi_{\text{inn}}(r,\tau)\\&=\widetilde{C_1}\varepsilon(\tau)^{\gamma-2\beta}r^{-\gamma+2}(1+O(r^2))+\varepsilon(\tau)^{\gamma-2\beta+\min\{2|\lambda_0|,\sqrt{D}\}}O(r^{-\gamma-\min\{2|\lambda_0|,\sqrt{D}\}}),\end{aligned}\tag{5.16}$$

where $\widetilde{C_1} = h\ell/2(2+\sqrt{D}) = C_1(1+2\omega_\ell)$ (cf. (5.8)). Combining (5.2) with (5.16), we get

$$|\Phi(r,\tau)-\Phi_{\text{inn}}(r,\tau)-\widetilde{C_1}\varepsilon(\tau)^{\gamma-2\beta}r^{-\gamma+2}| \le 2M_0\varepsilon(\tau)^{\gamma-2\beta+2\theta}r^{-\gamma} \tag{5.17}$$

for $r = \varepsilon(\tau)^\theta$, $\tau_0 \le \tau \le \tau_1$. Rewriting this estimate by the inner variables, we obtain $|U(\xi,\tau)-U_1(\xi)-\mu(\tau)H_1(\xi)| \le 3M_0\varepsilon(\tau)^{2\theta}\xi^{-\gamma}$. It then follows that

$$|U(\xi,\tau)-U_1(\xi)| \le \mu(\tau)H_1(\xi)+3M_0\varepsilon(\tau)^{2\theta}\xi^{-\gamma} \le V_{1,\pm}(\xi,\tau)+M_1\varepsilon(\tau)^{2\theta}H_0(\xi).$$

We thus obtain (5.14) with $M_1 = 5M_0 C_0^{-\gamma}$ if τ_0 is large enough (cf. (5.6)). We finally verify if the bound corresponding to (5.14) at $\tau = \tau_0$ is true for $|\xi| \le \varepsilon(\tau_0)^{\theta-1}$, which amounts to asking if there holds

$$\left|\Phi_0(r) - \Phi_{\rm inn}(r,\tau_0) - \varepsilon(\tau_0)^{-2\beta}\mu(\tau_0) H_1\left(\frac{r}{\varepsilon(\tau_0)}\right)\right| \le M_1 \varepsilon(\tau_0)^{-2\beta+2\theta} H_0\left(\frac{r}{\varepsilon(\tau_0)}\right) \qquad (5.18)$$

for $r \le \varepsilon(\tau_0)^{\theta}$. This is clearly satisfied for $r \le \varepsilon(\tau_0)^{2\theta}$, since $\Phi_0(r) = \Phi_{\rm inn}(r,\tau_0)$ there and $\varepsilon(\tau_0)^{2(1-\theta)} H_1(\xi) \ll H_0(\xi)$ with $\xi = r/\varepsilon(\tau_0) \le 2\varepsilon(\tau_0)^{2\theta-1}$. As for the region $\{\varepsilon(\tau_0)^{2\theta} < r \le \varepsilon(\tau_0)^{\theta}\}$, an estimate similar to (5.16) with $\tau = \tau_0$ implies

$$\left|\Phi_{\rm out}(r,\tau_0) - \Phi_{\rm inn}(r,\tau_0) - \widetilde{C_1}\varepsilon(\tau_0)^{\gamma-2\beta} r^{-\gamma+2}\right| \le C\varepsilon(\tau_0)^{\gamma-2\beta+4\theta} r^{-\gamma}.$$

Combining this with the assumption (5.1), we readily obtain (5.18). Comparison principle completes the proof. □

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