A SIMPLER EXPRESSION FOR COSTIN-MAZ'YA'S CONSTANT IN HARDY-LERAY INEQUALITY WITH WEIGHT

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Abstract. In this note, we obtain a simpler expression for the constant number given by Costin-Maz'ya [1] on sharp Hardy-Leray inequality for a class of solenoidal (namely divergence-free) vector fields, with respect to any radial power-weighted measure. The dependence of the constant on the weight exponent will be clear.

1. Motivation and main result

Throughout this paper, N is an integer with $N \geq 3$ and γ denotes any real number. What we call Costin-Maz'ya's constant is a real number expressed by the formula

$$C_{N,\gamma} = \left(\gamma + \tfrac{N}{2} - 1\right)^2 + \min\left\{2 + \min_{\tau \geq 0}\left(\tau + \frac{4(N-1)(\gamma-1)}{\tau + \left(\gamma - \frac{N}{2}\right)^2 + N - 1}\right),\ N-1\right\}. \tag{1.1}$$

It was found by Costin-Maz'ya [1] in the process of deriving the best constant in the weighted N-dimensional Hardy-Leray (or shortly "H-L") inequality

$$C_{N,\gamma}\int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2}|\boldsymbol{x}|^{2\gamma}dx \leq \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx \tag{1.2}$$

for solenoidal vector fields $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) : \mathbb{R}^N \to \mathbb{R}^N$ (with a suitable regularity condition). This inequality serves as a solenoidal improvement of the original sharp H-L inequality

$$\left(\gamma + \tfrac{N}{2} - 1\right)^2 \int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2}|\boldsymbol{x}|^{2\gamma}dx \leq \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2|\boldsymbol{x}|^{2\gamma}dx$$

for unconstrained fields $\boldsymbol{u} : \mathbb{R}^N \to \mathbb{R}^N$, whose prototype case $\gamma = 0$ is famous for the one-dimensional inequality by Hardy [5] and its N-dimensional extension by Leray [6].

Practically, the derivation of the expression (1.1) by Costin-Maz'ya was carried out under the axisymmetry assumption on the solenoidal fields $\boldsymbol{u}$. Strictly speaking, however, their method includes incorrect datum for $N \geq 4$ overlooking the singular behavior of axisymmetric vector fields, and in fact the inequality (1.2) together with the expression (1.1) is invalid as long as the axisymmetry of $\boldsymbol{u}$ is assumed. (For details, see [2, §2.1].) Nevertheless, the validity of the same expression can be recovered if only we remove the axisymmetry assumption; indeed, in recent papers [4, 3] the author evaluated the constant $C_{N,\gamma}$ in the inequality (1.2) without any

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symmetry assumption on the solenoidal fields $\boldsymbol{u}$, and found its best value to be the same as in (1.1), or the lesser of the two real numbers

$$\begin{cases} C_{P,N,\gamma} := \left(\gamma + \frac{N}{2} - 1\right)^2 + 2 + \min\limits_{\tau \geq 0} \left(\tau + \dfrac{4(N-1)(\gamma-1)}{\tau + \left(\gamma - \frac{N}{2}\right)^2 + N - 1} \right), \\ C_{T,N,\gamma} := \left(\gamma + \frac{N}{2} - 1\right)^2 + N - 1, \end{cases} \tag{1.3}$$

which are respectively the best constants in the H-L inequalities

$$\begin{cases} C_{P,N,\gamma} \displaystyle\int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2} |\boldsymbol{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx & \text{for poloidal fields } \boldsymbol{u}, \\ C_{T,N,\gamma} \displaystyle\int_{\mathbb{R}^N} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2} |\boldsymbol{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\nabla \boldsymbol{u}|^2 |\boldsymbol{x}|^{2\gamma} dx & \text{for toroidal fields } \boldsymbol{u}, \end{cases}$$

in the sense of the so-called poloidal-toroidal decomposition of solenoidal fields.

In view of the observation above, Costin-Maz'ya's constant

$$C_{N,\gamma} = \min\left\{C_{P,N,\gamma}, C_{T,N,\gamma}\right\}$$

can be considered as meaningful, and hence would be better if its expression could be further simplified, although it has been regarded as unwieldy. Now, our goal is to get a simpler expression of $C_{N,\gamma}$ than (1.1); the statement of our main result reads as follows:

Theorem 1.1. *Let $C_{N,\gamma}, C_{P,N,\gamma}$ and $C_{T,N,\gamma}$ be the real numbers given in* (1.1) *and* (1.3)*, and let $I_N = (\gamma_N^-, \gamma_N^+) \subset \mathbb{R}$ be the open interval between the two (extended) real numbers $\gamma_N^- < \gamma_N^+$ given by*

$$\gamma_N^- = \frac{N}{2} - \frac{N-1}{\sqrt{N+1}+2} \qquad \text{and} \qquad \gamma_N^+ = \begin{cases} \frac{N}{2} + \frac{N-1}{\sqrt{N+1}-2} & (N \geq 4) \\ \infty & (N = 3) \end{cases}.$$

Then it holds that

$$\begin{cases} C_{T,N,\gamma} < C_{P,N,\gamma} & \text{for } \gamma \in I_N, \\ C_{T,N,\gamma} = C_{P,N,\gamma} & \text{for } \gamma \in \{\gamma_N^-, \gamma_N^+\}, \\ C_{T,N,\gamma} > C_{P,N,\gamma} & \text{otherwise,} \end{cases}$$

and that $C_{P,N,\gamma} = \left(\gamma + \frac{N}{2} - 1\right)^2 \frac{\left(\gamma - \frac{N}{2}\right)^2 + N + 1}{\left(\gamma - \frac{N}{2}\right)^2 + N - 1}$ whenever $\gamma \notin I_N$. In particular,

$$\begin{aligned} C_{N,\gamma} &= \left(\gamma + \tfrac{N}{2} - 1\right)^2 + \min\left\{ 2 + \frac{4(N-1)(\gamma-1)}{\left(\gamma - \frac{N}{2}\right)^2 + N - 1},\ N - 1 \right\} \\ &= \begin{cases} \left(\gamma + \frac{N}{2} - 1\right)^2 + N - 1 & \text{for } \gamma \in I_N, \\ \left(\gamma + \frac{N}{2} - 1\right)^2 \frac{\left(\gamma - \frac{N}{2}\right)^2 + N + 1}{\left(\gamma - \frac{N}{2}\right)^2 + N - 1} & \text{otherwise.} \end{cases} \end{aligned}$$

It is clear that the expression of $C_{N,\gamma}$ in this theorem is simpler than (1.1). An advantage of such a simplification is that it helps us to verify that the equality sign in the inequality (1.2) is never achieved by any non-trivial solenoidal field $\boldsymbol{u}$. Indeed, in view of [3], one can observe that the main difficulty of this verification appears in the case where $C_{T,N,\gamma} > C_{P,N,\gamma}$ and simultaneously where the minimum in (1.3) is not achieved by $\tau = 0$; the result of Theorem 1.1 tells us that such a case is void.

In the remaining of the present paper, we will prove Theorem 1.1, after preparing a technical lemma. The proof is elementary.

2. Proof of main theorem

We start with proving the following fact, which plays a central role:

Lemma 2.1. *Let B be the quartic function given by*

$$B(\lambda) = \lambda^4 + (N-1)\left(2\lambda^2 - 4\lambda - (N-3)\right) \quad \forall \lambda \in \mathbb{R}.$$

Then it holds that $C_{T,N,\gamma} + 2 \leq C_{P,N,\gamma}$ whenever $B\left(\gamma - \frac{N}{2}\right) \leq 0$.

Proof. The key idea of the proof is an application of the intermediate value theorem to convex functions. First of all, notice that the convexity of B can be verified by the positivity of its second-order derivative:

$$B''(\lambda) = 12\lambda^2 + 4(N-1) > 0.$$

From this fact together with the datum

$$B\left(1 - \tfrac{N}{2}\right) = N^2/16 > 0 \quad \text{and} \quad B(1) = -(N-2)N < 0,$$

it follows that $\gamma > 1$ must be satisfied whenever $B\left(\gamma - \frac{N}{2}\right) \leq 0$.

Now, we set

$$F(\tau) := \tau + \frac{4(N-1)(\gamma-1)}{\tau + \left(\gamma - \frac{N}{2}\right)^2 + N - 1} - N + 3 \qquad \forall \tau \geq 0 \tag{2.1}$$

in order that

$$\min_{\tau \geq 0} F(\tau) = C_{P,N,\gamma} - C_{T,N,\gamma}. \tag{2.2}$$

To evaluate the left-hand side, notice that a direct calculation yields

$$\frac{\partial F(\tau)}{\partial \tau} = \frac{\tau^2 + 2\left(\left(\gamma - \frac{N}{2}\right)^2 + N - 1\right)\tau + B\left(\gamma - \frac{N}{2}\right)}{\left(\tau + \left(\gamma - \frac{N}{2}\right)^2 + N - 1\right)^2}. \tag{2.3}$$

Then it follows from $B\left(\gamma - \frac{N}{2}\right) \leq 0$ that the minimum of F can be achieved by the nonnegative root of the equation $\partial F/\partial \tau = 0$ which we denote by

$$\begin{aligned}
\tau_0 &:= -\left(\gamma - \tfrac{N}{2}\right)^2 - N + 1 + \sqrt{\left(\left(\gamma - \tfrac{N}{2}\right)^2 + N - 1\right)^2 - B\left(\gamma - \tfrac{N}{2}\right)} \\
&= -\left(\gamma - \tfrac{N}{2}\right)^2 - N + 1 + 2\sqrt{N-1}\sqrt{\gamma - 1}.
\end{aligned}$$

Therefore, the minimum value of $F - 2$ is given by the calculation

$$\min_{\gamma \geq 0} F(\tau) - 2 = F(\tau_0) - 2 = -\left(\gamma - \tfrac{N}{2}\right)^2 - 2N + 2 + 4\sqrt{N-1}\sqrt{\gamma - 1}.$$

Now, all that is left is to show the non-negativity of this value; it suffices to check the inequality

$$16(N-1)(\gamma - 1) \geq \left(\left(\gamma - \tfrac{N}{2}\right)^2 + 2N - 2\right)^2.$$

To this end, by putting $\lambda = \gamma - N/2$ we compute

$$\begin{aligned}
&16(N-1)\,(\gamma-1) - \left(\left(\gamma - \tfrac{N}{2}\right)^2 + 2N - 2\right)^2 + B\left(\gamma - \tfrac{N}{2}\right) \\
&\quad = 16(N-1)\left(\lambda + \tfrac{N}{2} - 1\right) - \left(\lambda^2 + 2(N-1)\right)^2 + B(\lambda) \\
&\quad = (N-1)\left(-2\lambda^2 + 12\lambda + 3N - 9\right).
\end{aligned}$$

Hence, all we have to do is to check that the quadratic function

$$E(\lambda) := -2\lambda^2 + 12\lambda + 3N - 9$$

satisfies

$$E(\lambda) \geq 0 \quad \text{whenever} \quad B(\lambda) \leq 0.$$

For this purpose, we set

$$\lambda_\pm := 3 \pm \tfrac{\sqrt{6}}{2}\sqrt{N+3}$$

as the two roots of the quadratic equation $E(\lambda) = 0$. Then, with the aid of the polynomial division identity

$$B(\lambda) = -\tfrac{1}{4}(2\lambda^2 + 12\lambda + 7N + 59)E(\lambda) + 2(13N + 77)\lambda + \tfrac{1}{4}(N-3)(17N + 181),$$

we directly compute

$$\begin{cases}
\lambda_- < 1 < \lambda_+, \\
B(1) = -(N-2)N < 0, \\
B(\lambda_+) = \tfrac{1}{4}(1305 + 442N + 17N^2) + \sqrt{6}\sqrt{N+3}\,(13N + 77) > 0, \\
B(\lambda_-) = \dfrac{1}{4}\,\dfrac{(N-3)^2(289N^2 + 538N - 503)}{17N^2 + 442N + 1305 + 4\sqrt{6}\sqrt{N+3}\,(13N+77)} \geq 0.
\end{cases}$$

In view of the concavity of E and the convexity of B, this fact tells us that

$$E(\lambda) \geq 0 \ \text{ holds whenever } \ B(\lambda) \leq 0,$$

as desired. $\square$

Proof of Theorem 1.1. First of all, in view of (2.1), notice that the two numbers $\gamma_N^\pm$ are the roots of the equation in γ:

$$F(0) = \frac{4(N-1)(\gamma-1)}{\left(\gamma - \frac{N}{2}\right)^2 + N - 1} - N + 3 = 0.$$

Hence we see from (2.1) that

$$\begin{cases}
F(0) > 0 & \text{for } \gamma \in I_N \\
F(0) = 0 & \text{for } \gamma \in \{\gamma_N^-, \gamma_N^+\}, \\
F(0) < 0 & \text{otherwise.}
\end{cases} \tag{2.4}$$

Making use of Lemma 2.1 together with (2.2), we evaluate the sign of the number $C_{P,N,\gamma} - C_{T,N,\gamma}$ in the following cases:

The case $\gamma \in I_N$. When $B\left(\gamma - \frac{N}{2}\right) \leq 0$, we have $C_{T,N,\gamma} < C_{P,N,\gamma}$ by use of Lemma 2.1. When $B\left(\gamma - \frac{N}{2}\right) > 0$, the function $F(\tau)$ is monotone increasing in τ since its derivative (2.3) is positive for all $\tau \geq 0$; then the $\min_{\tau \geq 0} F(\tau)$ is attained at $\tau = 0$. Therefore, we see from (2.2) and (2.4) that

$$C_{P,N,\gamma} - C_{T,N,\gamma} = F(0) > 0,$$

and hence again that $C_{T,N,\gamma} < C_{P,N,\gamma}$.

The case $\gamma \in \mathbb{R} \setminus I_N$. We see from (2.2) that

$$C_{P,N,\gamma} - C_{T,N,\gamma} = \min_{\tau \geq 0} F(\tau) \leq F(0) \leq 0.$$

This fact together with Lemma 2.1 implies that $B\left(\gamma - \frac{N}{2}\right)$ must be positive, whence we see in view of (2.3) that $F(\tau)$ is monotone increasing in $\tau \geq 0$. Therefore, we get $\min_{\gamma \geq 0} F(\tau) = F(0)$ and

$$C_{P,N,\gamma} = C_{T,N,\gamma} + F(0) = \left(\gamma + \tfrac{N}{2} - 1\right)^2 \frac{\left(\gamma - \frac{N}{2}\right)^2 + N + 1}{\left(\gamma - \frac{N}{2}\right)^2 + N - 1}.$$

Moreover, recalling (2.4) we obtain $\begin{cases} C_{P,N,\gamma} = C_{T,N,\gamma} & \text{for } \gamma \in \{\gamma_N^-, \gamma_N^+\}, \\ C_{P,N,\gamma} < C_{T,N,\gamma} & \text{otherwise.} \end{cases}$

The proof of Theorem 1.1 is now complete.

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