HIGHER LEVEL q-OSCILLATOR REPRESENTATIONS FOR $U_q(C_n^{(1)}), U_q(C^{(2)}(n+1))$ AND $U_q(B^{(1)}(0,n))$

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ABSTRACT. We introduce higher level q-oscillator representations for the quantum affine (super)algebras of type $C_n^{(1)}, C^{(2)}(n+1)$ and $B^{(1)}(0,n)$. They are constructed from the fusion procedure from the fundamental q-oscillator representations obtained through the studies of the tetrahedron equation. We prove that they are irreducible for type $C_n^{(1)}$ and $C^{(2)}(n+1)$, and give their characters.

1. INTRODUCTION

Let $\mathfrak{g}$ be an affine Lie algebra and $U_q(\mathfrak{g})$ the Drinfeld-Jimbo quantum group (without derivation) associated to it. For a node r of the Dynkin diagram of $\mathfrak{g}$ except 0 and a positive integer s there exists a family of finite-dimensional $U_q(\mathfrak{g})$-modules $W^{r,s}$ called Kirillov-Reshetikhin modules. They have distinguished properties. One of them is the existence of crystal bases in Kashiwara's sense (see [1, 5, 18] and references therein).

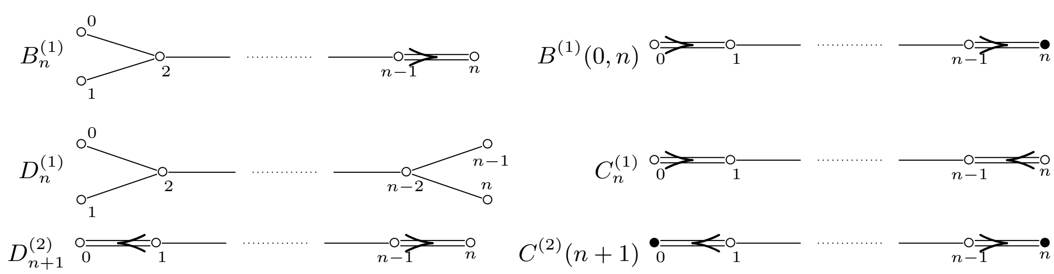

TABLE 1. Dynkin diagrams of $(\mathfrak{g}, \overline{\mathfrak{g}})$

Consider the affine Lie algebras $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}$, whose Dynkin diagrams are given in the left side of Table 1. The Kirillov-Reshetikhin modules corresponding to the node n and the integer 1 have a simple structure. Let V be a two dimensional vector space. The action of $U_q(\mathfrak{g})$ on $W^{n,1}$ has an easy description on $V^{\otimes n}$. It is irreducible when $\mathfrak{g} = B_n^{(1)}, D_{n+1}^{(2)}$, but for $\mathfrak{g} = D_n^{(1)}$ it decomposes into two components; $V^{\otimes n} = W^{n,1} \oplus W^{n-1,1}$. For a quantum

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group $U_q(\mathfrak{g})$ we can consider the quantum R matrix. We introduce a spectral parameter x to the representation $W^{n,1}$, and denote the associated representation by $W^{n,1}(x)$. Let Δ be the coproduct and $\Delta^{\rm op}$ its opposite. Then the quantum R matrix $R(x/y)$ is defined as an intertwiner of Δ and $\Delta^{\rm op}$, namely, linear operator satisfying $R(x/y)\Delta(u) = \Delta^{\rm op}(u)R(x/y)$ for any $u \in U_q(\mathfrak{g})$ on $W^{n,1}(x) \otimes W^{n,1}(y)$. ($R$ is found to depend only on x/y.)

In [15], Kuniba and Sergeev initiated an attempt to obtain quantum R matrices from the solution to the tetrahedron equation, three dimensional analogue of the Yang-Baxter equation. Let $\mathcal{L}$ be a solution of the tetrahedron equation. It is a linear operator on $F\otimes V\otimes V$ where F is an infinite-dimensional vector space spanned by $\{\, |m\rangle \,|\, m \in \mathbb{Z}_{\geq 0} \,\}$. By composing this $\mathcal{L}$ n times and applying suitable boundary vectors in F and F^*, they obtained linear operators on $(V^{\otimes n})\otimes(V^{\otimes n})$ satisfying the Yang-Baxter equation. The commuting symmetry algebras were found to be $U_q(B_n^{(1)}), U_q(D_n^{(1)})$ or $U_q(D_{n+1}^{(2)})$. The reason they had variations was that there were two choices of boundary vectors in each F and F^* corresponding to the shapes of the Dynkin diagrams at each end.

To the tetrahedron equation, there is yet another solution $\mathcal{R}$, which is a linear operator on $F^{\otimes 3}$. In [13], Kuniba and the second author performed the same scheme to $\mathcal{R}$ and constructed linear operators on $(F^{\otimes n}) \otimes (F^{\otimes n})$. For the symmetry algebra this time, they found $U_q(C_n^{(1)}), U_q(D_{n+1}^{(2)})$ and $U_q(A_{2n}^{(2)})$. They called these representations on $\mathcal{W} = F^{\otimes n}$ q-oscillator ones. To be precise, for type $C_n^{(1)}$ there are two irreducible components $\mathcal{W}_+, \mathcal{W}_-$, so one can think $\mathcal{W}$ of either $\mathcal{W}_+$ or $\mathcal{W}_-$. By construction, the q-oscillator representation $\mathcal{W}$ is a bosonic analogue of $W^{n,1}$, and it is natural to ask whether we have a higher level q-oscillator representation corresponding to $W^{n,s}$ for $s \geq 1$. However, there is a difficulty in understanding $\mathcal{W}$ since they do not have a suitable classical limit ($q \to 1$) for type $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}$.

In this paper, we first resolve this difficulty by considering $\mathcal{W}$ for these two types as q-oscillator representations over quantum affine superalgebras $\overline{\mathfrak{g}}$ given in the right side of Table 1 by using the twistor on quantum covering groups [4]. The filled nodes in the Dynkin diagrams signify anisotropic odd simple roots. If they were not filled, the third Dynkin diagram would be $D_{n+1}^{(2)}$ and the first one $A_{2n}^{(2)\dagger}$, where the latter is the same diagram as $A_{2n}^{(2)}$ but the opposite labeling of nodes. We then investigate the quantum R matrices for $\mathcal{W}(x) \otimes \mathcal{W}(y)$ and apply the fusion construction. As a result, we obtain a higher level representation $\mathcal{W}^{(s)}$ for any $s \in \mathbb{Z}_{>0}$ and each $U_q(C_n^{(1)}), U_q(C^{(2)}(n+1))$ and $U_q(B^{(1)}(0,n))$.

Our main purpose in this paper is to prove the irreducibility of $\mathcal{W}^{(s)}$ and compute its character for $U_q(C_n^{(1)})$ and $U_q(C^{(2)}(n+1))$. We investigate the crystal base of $\mathcal{W}^{(s)}$ in detail to show this. We further prove that $\mathcal{W}^{(s)}$ is classically irreducible, that is, irreducible as a module over the subalgebra generated by e_i, f_i, k_i for $i \neq 0$. Rather surprisingly, this coincides with the fact that the corresponding $W^{n,s}$ is classically irreducible. We also give conjectures on the irreducibility of $\mathcal{W}^{(s)}$ and its character formula for $B^{(1)}(0,n)$.

We would like to remark that the correspondence between $W^{n,s}$ and $\mathcal{W}^{(s)}$ as representations of finite-dimensional simple Lie (super)algebras after a classical limit, appears in the context of super duality [3]. The theory of super duality is an equivalence between certain parabolic Bernstein-Gelfand-Gelfand categories of classical Lie (super)algebras of infinite-rank. As a special case, this yields an equivalence between the categories for $\mathcal{G}_\infty$ and $\overline{\mathcal{G}}_\infty$, where $(\mathcal{G}_\infty, \overline{\mathcal{G}}_\infty) = (B_\infty, B(0,\infty)), (D_\infty, C_\infty)$. Their Dynkin diagrams are given in Table 2. Let $\mathcal{G}_n$ and $\overline{\mathcal{G}}_n$ denote the subalgebras of $\mathcal{G}_\infty$ and $\overline{\mathcal{G}}_\infty$ of finite rank n, respectively. Let V_∞ be a given integrable highest weight $\mathcal{G}_\infty$-module. Under this equivalence, it corresponds to an irreducible highest weight $\overline{\mathcal{G}}_\infty$-module, say W_∞, called an oscillator representation. By applying a truncation functor to V_∞ and W_∞, we also obtain irreducible modules V_n and W_n of $\mathcal{G}_n$ and $\overline{\mathcal{G}}_n$, respectively.

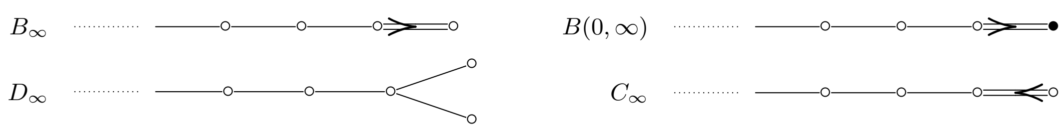

TABLE 2. Dynkin diagrams of $(\mathcal{G}_\infty, \overline{\mathcal{G}}_\infty)$

Let $(\mathfrak{g}, \overline{\mathfrak{g}})$ be one of the pairs of affine Lie (super)algebras $(B_n^{(1)}, B^{(1)}(0,n))$, $(D_n^{(1)}, C_n^{(1)})$, $(D_{n+1}^{(2)}, C^{(2)}(n+1))$ in Table 1. Let $\mathcal{G}_n$ and $\overline{\mathcal{G}}_n$ be the subalgebra of $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ corresponding to $I \setminus \{0\}$, respectively. Assume that $\mathfrak{g} = D_n^{(1)}$, $D_{n+1}^{(2)}$. Now we see that if V_n is the classical limit of a classically irreducible Kirillov-Reshetikhin $U_q(\mathfrak{g})$-module, then W_n corresponds to the classical limit of a higher level q-oscillator $U_q(\overline{\mathfrak{g}})$-module in Theorems 5.1 and 5.20. The character formula in Conjecture 5.22 is based on this observation in case of $(\mathfrak{g}, \overline{\mathfrak{g}}) = (B_n^{(1)}, B^{(1)}(0,n))$, which is true for $s = 2$. We strongly expect that there is a quantum affine analogue of super duality which relates the category of finite-dimensional $U_q(\mathfrak{g})$-modules and a suitable category of infinite-dimensional $U_q(\overline{\mathfrak{g}})$-modules including the q-oscillator modules, and hence explains the correspondence in this paper.

The paper is organized as follows: In Section 2, we briefly review the notion of quantum superalgebras. In Section 3, we construct a level one q-oscillator representation $\mathcal{W}$ of $U_q(\overline{\mathfrak{g}})$ and study some of its properties including the crystal base. In Section 4, we introduce the quantum R matrix on $\mathcal{W}(x) \otimes \mathcal{W}(y)$ and apply fusion construction to define $\mathcal{W}^{(s)}$. In Section 5, we prove the irreducibility of $\mathcal{W}^{(s)}$ and give its character formula when $\overline{\mathfrak{g}} = C_n^{(1)}, C^{(2)}(n+1)$. A conjecture when $\overline{\mathfrak{g}} = B^{(1)}(0,n)$ is also given. In Appendix A, we explain how to construct a level one q-oscillator representation of $U_q(\overline{\mathfrak{g}})$ when $\overline{\mathfrak{g}} = C^{(2)}(n+1)$ and $B^{(1)}(0,n)$ from the one for $D_{n+1}^{(2)}$ and $A_{2n}^{(2)\dagger}$ in [13], respectively, by using the quantum covering groups and twistor [4]. In Appendices B and C, we construct the quantum R matrix on $\mathcal{W}(x) \otimes \mathcal{W}(y)$ for $U_q(\overline{\mathfrak{g}})$ from the one in [13].

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2. Quantum superalgebras

2.1. **Variant of q-integer.** Throughout the paper, we let q be an indeterminate. Following [4], we introduce variants of q-integer, q-factorial and q-binomial coefficient. Let $\epsilon = \pm 1$. For $m \in \mathbb{Z}_{\geq 0}$, we set

$$[m]_{q,\epsilon} = \frac{(\epsilon q)^m - q^{-m}}{\epsilon q - q^{-1}}.$$

For $m \in \mathbb{Z}_{\geq 0}$, set

$$[m]_{q,\epsilon}! = [m]_{q,\epsilon}[m-1]_{q,\epsilon} \cdots [1]_{q,\epsilon} \quad (m \geq 1), \quad [0]_{q,\epsilon}! = 1.$$

For integers m, n such that $0 \leq n \leq m$, we define

$$\begin{bmatrix} m \\ n \end{bmatrix}_{q,\epsilon} = \frac{[m]_{q,\epsilon}!}{[n]_{q,\epsilon}![m-n]_{q,\epsilon}!}.$$

They all belong to $\mathbb{Z}[q, q^{-1}]$. Let A_0 be the subring of $\mathbb{Q}(q)$ consisting of rational functions without a pole at $q = 0$. Then we have

$$[m]_{q,\epsilon} \in q^{1-m}(1 + qA_0), \quad [m]_{q,\epsilon}! \in q^{-m(m-1)/2}(1 + A_0), \quad \begin{bmatrix} m \\ n \end{bmatrix}_{q,\epsilon} \in q^{-n(m-n)}(1 + qA_0).$$

We simply write $[m] = [m]_{q,1}$, $[m]! = [m]_{q,1}!$ and $\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}_{q,1}$.

2.2. **Quantum (super)algebra $U_q(sl_2)$ and $U_q(osp_{1|2})$.** The quantum (super)algebras $U_q(sl_2)$ $(\epsilon = 1)$ and $U_q(osp_{1|2})$ $(\epsilon = -1)$ are defined as a $\mathbb{Q}(q)$-algebra generated by $e, f, k^{\pm 1}$ satisfying the following relations:

$$kk^{-1} = k^{-1}k = 1, \quad kek^{-1} = q^2 e, \quad kfk^{-1} = q^{-2} f, \quad ef - \epsilon fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

Set $e^{(m)} = e^m / [m]_{q,\epsilon}!$ and $f^{(m)} = f^m / [m]_{q,\epsilon}!$. We will use the following formula.

Proposition 2.1.

$$e^{(m)} f^{(n)} = \sum_{j \geq 0} \frac{\epsilon^{mn - j(j+1)/2}}{[j]_{q,\epsilon}!} f^{(n-j)} \left(\prod_{l=0}^{j-1} \frac{(\epsilon q)^{2j-m-n-l} k - q^{-2j+m+n+l} k^{-1}}{q - q^{-1}} \right) e^{(m-j)}.$$

Proof. The $U_q(sl_2)$ $(\epsilon = 1)$ case is derived easily from (1.1.23) of [12]. The $U_q(osp_{1|2})$ $(\epsilon = -1)$ case can be shown by induction. □

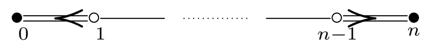
2.3. **Quantum affine (super)algebras $U_q(C_n^{(1)}), U_q(C^{(2)}(n+1)), U_q(B^{(1)}(0,n))$.** Set $I = \{0, 1, \ldots, n\}$. In this paper, we consider the following three Cartan data $(a_{ij})_{i,j \in I}$, or Dynkin diagrams (cf. [9]), and $(d_i)_{i \in I}$ such that $d_i a_{ij} = d_j a_{ji}$ for $i, j \in I$.

- $C_n^{(1)}$:

$$(a_{ij})_{i,j\in I} = \begin{pmatrix} 2 & -1 & & & \\ -2 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -2 \\ & & & -1 & 2 \end{pmatrix}$$

$$(d_i)_{i\in I} = (2, 1, \ldots, 1, 2)$$

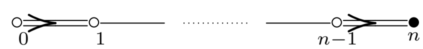
- $C^{(2)}(n+1)$:



$$(a_{ij})_{i,j\in I} = \begin{pmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}$$

$$(d_i)_{i\in I} = \left(\tfrac{1}{2}, 1, \ldots, 1, \tfrac{1}{2}\right)$$

- $B^{(1)}(0,n)$:



$$(a_{ij})_{i,j\in I} = \begin{pmatrix} 2 & -1 & & & \\ -2 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}$$

$$(d_i)_{i\in I} = (2, 1, \ldots, 1, \tfrac{1}{2}).$$

Let $d = \min\{\, d_i \mid i \in I \,\}$. For $i \in I$, let $q_i = q^{d_i}$, and let $p(i) = 0, 1$ such that $p(i) \equiv 2d_i \pmod 2$. Set

$$[m]_i = [m]_{q_i,(-1)^{p(i)}}, \quad [m]_i! = [m]_{q_i,(-1)^{p(i)}}!, \quad \begin{bmatrix} m \\ k \end{bmatrix}_i = \begin{bmatrix} m \\ k \end{bmatrix}_{q_i,(-1)^{p(i)}},$$

for $0 \le k \le m$ and $i \in I$.

For a Cartan datum $X = C_n^{(1)}, C^{(2)}(n+1), B^{(1)}(0,n)$, the quantum affine (super)algebra $U_q(X)$ is defined to be the $\mathbb{Q}(q^d)$-algebra generated by $k_i^{\pm 1}, e_i, f_i$ $(i \in I)$ with the following relations:

$$k_i k_j = k_j k_i, \quad k_i e_j k_i^{-1} = q_i^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j,$$
$$e_i f_j - (-1)^{p(i)p(j)} f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{m=0}^{1-a_{ij}}(-1)^{m+p(i)m(m-1)/2+mp(i)p(j)}e_i^{(1-a_{ij}-m)}e_je_i^{(m)}=0\quad(i\neq j),$$
$$\sum_{m=0}^{1-a_{ij}}(-1)^{m+p(i)m(m-1)/2+mp(i)p(j)}f_i^{(1-a_{ij}-m)}f_jf_i^{(m)}=0\quad(i\neq j),$$

where

$$e_i^{(m)}=\frac{e_i^m}{[m]_i!},\quad f_i^{(m)}=\frac{f_i^m}{[m]_i!}.$$

We define the automorphism τ of $U_q(X)$ for $X=C_n^{(1)},C^{(2)}(n+1)$ by

$$\tau(k_i)=k_{n-i}^{-1},\quad \tau(e_i)=f_{n-i},\quad \tau(f_i)=e_{n-i},\quad \text{if } X=C_n^{(1)}, \tag{2.1}$$

$$\tau(k_i)=k_{n-i}^{-1},\ \tau(e_i)=(-1)^{\delta_{in}}f_{n-i},\ \tau(f_i)=(-1)^{\delta_{i0}}e_{n-i},\ \text{if } X=C^{(2)}(n+1) \tag{2.2}$$

for $i\in I$ and the anti-automorphism η of $U_q(X)$ by

$$\begin{cases}\eta(k_i)=k_i\\ \eta(e_i)=(-1)^{\delta_{i0}+\delta_{in}}q_i^{-1}k_i^{-1}f_i & \text{if } X=C_n^{(1)},B^{(1)}(0,n),\\ \eta(f_i)=(-1)^{\delta_{i0}+\delta_{in}}q_i^{-1}k_ie_i\end{cases}$$

$$\begin{cases}\eta(k_i)=k_i\\ \eta(e_i)=(-1)^{\delta_{in}}q_i^{-1}k_i^{-1}f_i & \text{if } X=C^{(2)}(n+1)\\ \eta(f_i)=(-1)^{\delta_{in}}q_i^{-1}k_ie_i\end{cases}$$

for $i\in I$. Both τ and η are involutions.

When $X=C^{(2)}(n+1),B^{(1)}(0,n)$, let

$$U_q(X)^\sigma=U_q(X)\oplus U_q(X)\sigma$$

be the semidirect product of $U_q(X)$ and the group algebra generated by σ, where

$$\sigma^2=1,\quad \sigma k_i=k_i\sigma,\quad \sigma e_i=(-1)^{p(i)}e_i\sigma,\quad \sigma f_i=(-1)^{p(i)}f_i\sigma\quad(i\in I). \tag{2.3}$$

τ and η are extended to $U_q(X)^\sigma$ by $\tau(\sigma)=\eta(\sigma)=\sigma$.

The algebras $U_q(C_n^{(1)}),U_q(C^{(2)}(n+1))^\sigma,U_q(B^{(1)}(0,n))^\sigma$ have a Hopf algebra structure. In particular, the coproduct Δ is given by

$$\begin{aligned}&\Delta(k_i)=k_i\otimes k_i,\quad \Delta(\sigma)=\sigma\otimes\sigma,\\ &\Delta(e_i)=e_i\otimes\sigma^{p(i)\delta_{i0}}k_i^{-1}+\sigma^{p(i)\delta_{in}}\otimes e_i,\\ &\Delta(f_i)=f_i\otimes\sigma^{p(i)\delta_{i0}}+\sigma^{p(i)\delta_{in}}k_i\otimes f_i\end{aligned} \tag{2.4}$$

for $i\in I$.

3. Level one q-oscillator representation

Let $\mathcal{W}$ be an infinite-dimensional vector space over $\mathbb{Q}(q^d)$ defined by

$$\mathcal{W}=\bigoplus_{\mathbf{m}}\mathbb{Q}(q^d)|\mathbf{m}\rangle,$$

where $|\mathbf{m}\rangle$ is a basis vector parametrized by $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n$. Let $|\mathbf{m}| = \sum_{j=1}^n m_j$, and let $\mathbf{e}_j$ be the j-th standard vector in $\mathbb{Z}^n$ for $1 \leq j \leq n$. In this section, we introduce the so-called q-oscillator representation of level one for each algebra.

3.1. **Type $C_n^{(1)}$.**

3.1.1. *$U_q(C_n^{(1)})$-module $\mathcal{W}_\pm$.* Consider the quantum affine algebra $U_q(C_n^{(1)})$. Let $U_q(C_n)$ and $U_q(A_{n-1})$ be the subalgebras generated by k_i, e_i, f_i for $i \in I \setminus \{0\}$ and $i \in I \setminus \{0, n\}$, respectively.

Proposition 3.1. *For a non-zero $x \in \mathbb{Q}(q)$, the space $\mathcal{W}$ admits a $U_q(C_n^{(1)})$-module structure given as follows:*

$$
\begin{aligned}
e_0|\mathbf{m}\rangle &= xq^{-1}\frac{[m_1+1][m_1+2]}{[2]}|\mathbf{m}+2\mathbf{e}_1\rangle,\\
f_0|\mathbf{m}\rangle &= -x^{-1}\frac{q}{[2]}|\mathbf{m}-2\mathbf{e}_1\rangle,\\
k_0|\mathbf{m}\rangle &= q^{2m_1+1}|\mathbf{m}\rangle,\\
e_j|\mathbf{m}\rangle &= [m_{j+1}+1]|\mathbf{m}-\mathbf{e}_j+\mathbf{e}_{j+1}\rangle,\\
f_j|\mathbf{m}\rangle &= [m_j+1]|\mathbf{m}+\mathbf{e}_j-\mathbf{e}_{j+1}\rangle,\\
k_j|\mathbf{m}\rangle &= q^{-m_j+m_{j+1}}|\mathbf{m}\rangle,\\
e_n|\mathbf{m}\rangle &= -\frac{q}{[2]}|\mathbf{m}-2\mathbf{e}_n\rangle,\\
f_n|\mathbf{m}\rangle &= q^{-1}\frac{[m_n+1][m_n+2]}{[2]}|\mathbf{m}+2\mathbf{e}_n\rangle,\\
k_n|\mathbf{m}\rangle &= q^{-2m_n-1}|\mathbf{m}\rangle,
\end{aligned}
$$

where $1 \leq j \leq n-1$. Here we understand the vector on the right-hand side is zero when any of its components does not belong to $\mathbb{Z}_{\geq 0}$.

Remark 3.2. For $|\mathbf{m}\rangle \in \mathcal{W}$, set $\tau(|\mathbf{m}\rangle) = |m_n, \ldots, m_1\rangle$, and extend linearly to any vector of $\mathcal{W}$. Then, when $x = 1$ we have the following symmetry

$$\tau(u|\mathbf{m}\rangle) = \tau(u)\tau(|\mathbf{m}\rangle),$$

for $u \in U_q(C_n^{(1)})$. Here the automorphism τ on $U_q(C_n^{(1)})$ is given in (2.1).

Remark 3.3. This representation originally appeared in [13, Proposition 3]. The presentation above is obtained from the one in [13] by applying the basis change $|\mathbf{m}\rangle^{\text{new}} = \frac{(q[2])^{|\mathbf{m}|/2}}{\prod_{i=1}^n [m_i]!}|\mathbf{m}\rangle^{\text{old}}$ and the automorphism of $U_q(C_n^{(1)})$ sending $f_0 \mapsto -f_0, e_n \mapsto -e_n, k_i \mapsto -k_i$ for $i = 0, n$ with the other generators fixed.

We assume that ε denotes $+$ or $-$. Set $\varsigma(\varepsilon) = 0$ and 1, when $\varepsilon = +$ and $-$, respectively. For $m \in \mathbb{Z}_{\geq 0}$, let $\operatorname{sgn}(m)$ be $+$ and $-$ if m is even and odd, respectively.

Define the subspace $\mathcal{W}_\varepsilon$ of $\mathcal{W}$ by

$$\mathcal{W}_\varepsilon = \bigoplus_{\operatorname{sgn}(|\mathbf{m}|)=\varepsilon} \mathbb{Q}(q)|\mathbf{m}\rangle.$$

Proposition 3.4. *For a non-zero $x \in \mathbb{Q}(q)$, $\mathcal{W}_\varepsilon$ is an irreducible $U_q(C_n^{(1)})$-module.*

We denote this module by $\mathcal{W}_\varepsilon(x)$, and call it a (level one) q-oscillator representation. We simply write $\mathcal{W}_\varepsilon = \mathcal{W}_\varepsilon(1)$ as a $U_q(C_n^{(1)})$-module.

Let $s_\lambda(x_1,\dots,x_n)$ denote the Schur polynomial in $x_1,\dots,x_n$ corresponding to a partition λ. Then as a $U_q(A_{n-1})$-module, we have

$$\begin{aligned}\mathrm{ch}\mathcal{W}_+ &= \sum_{l\in 2\mathbb{Z}_{\geq 0}} s_{(l)}(x_1,\dots,x_n) = \frac{1}{\prod_{i=1}^n(1-x_i^2)},\\ \mathrm{ch}\mathcal{W}_- &= \sum_{l\in 1+2\mathbb{Z}_{\geq 0}} s_{(l)}(x_1,\dots,x_n) = \frac{\prod_{i=1}^n(1+x_i)-1}{\prod_{i=1}^n(1-x_i^2)}.\end{aligned}$$

Here the weight lattice of $U_q(C_n^{(1)})$ is identified with the $\mathbb{Z}$-lattice spanned by $\mathbf{e}_i$ for $1 \leq i \leq n$, and hence the variable x_i corresponds to the weight of $\mathbf{e}_i$.

3.1.2. *Classical limit.* Let A be the localization of $\mathbb{Z}[q,q^{-1}]$ at $[2] = q+q^{-1}$. Let

$$\mathcal{W}_\varepsilon(x)_A = \sum_{\mathrm{sgn}(|\mathbf{m}|)=\varepsilon} A|\mathbf{m}\rangle.$$

Then $\mathcal{W}_\varepsilon(x)_A$ is invariant under e_i, f_i, k_i and $\{k_i\} := \frac{k_i-k_i^{-1}}{q_i-q_i^{-1}}$ for $i \in I\setminus\{0\}$. Let

$$\overline{\mathcal{W}_\varepsilon(x)} = \mathcal{W}_\varepsilon(x)_A \otimes_A \mathbb{C},$$

where $\mathbb{C}$ is an A-module such that $f(q)\cdot c = f(1)c$ for $f(q) \in A$ and $c \in \mathbb{C}$.

Let E_i, F_i and H_i be the $\mathbb{C}$-linear endomorphisms on $\overline{\mathcal{W}_\varepsilon(x)}$ induced from e_i, f_i and $\{k_i\}$ for $i \in I\setminus\{0\}$. We can check that they satisfy the defining relations for the universal enveloping algebra $U(C_n)$ of type C_n (cf. [7, Chapter 5]). Hence $\overline{\mathcal{W}_\varepsilon(x)}$ becomes a $U(C_n)$-module.

Lemma 3.5. *The space $\overline{\mathcal{W}_\varepsilon(x)}$ is isomorphic to the irreducible highest weight $U(C_n)$-module with highest weight $-(\frac{1}{2}+\varsigma(\varepsilon))\varpi_n$, where ϖ_n is the n-th fundamental weight for C_n.*

Proof. It is clear that $E_i(|\mathbf{0}\rangle\otimes 1) = 0$ for all $i \in I\setminus\{0\}$. Since

$$H_n(|\mathbf{0}\rangle\otimes 1) = \left(\frac{k_n-k_n^{-1}}{q_n-q_n^{-1}}|\mathbf{0}\rangle\right)\otimes 1 = \left(-\frac{1}{q+q^{-1}}|\mathbf{0}\rangle\right)\otimes 1 = -\frac{1}{2}|\mathbf{0}\rangle\otimes 1,$$

and $H_i(|\mathbf{0}\rangle\otimes 1) = 0$ for $1 \leq i \leq n-1$, $\overline{\mathcal{W}_+(x)}$ is a highest weight $U(C_n)$-module with highest weight $-\frac{1}{2}\varpi_n$. It follows from the actions of E_i for $i \in I\setminus\{0\}$ that any submodule of $\overline{\mathcal{W}_+(x)}$ contains $|\mathbf{0}\rangle\otimes 1$. This implies that $\overline{\mathcal{W}_+(x)}$ is irreducible. The proof for $\mathcal{W}_-(x)$ is similar. □

3.1.3. *Polarization.* Define a symmetric bilinear form on $\mathcal{W}_\varepsilon$ by

$$(|\mathbf{m}\rangle, |\mathbf{m}'\rangle) = \delta_{\mathbf{m},\mathbf{m}'}\frac{q^{-\frac{1}{2}\sum_{i=1}^n m_i(m_i-1)}}{\prod_{i=1}^n [m_i]!}, \tag{3.1}$$

for $|\mathbf{m}\rangle$, $|\mathbf{m}'\rangle$ with $\mathbf{m} = (m_1,\dots,m_n)$. Note that $(|\mathbf{m}\rangle,|\mathbf{m}\rangle) \in 1+qA_0$.

Lemma 3.6. *The bilinear form in* (3.1) *is a polarization on* $\mathcal{W}_\varepsilon$*, that is,*

$$(uv, v') = (v, \eta(u)v'),$$

for $u \in U_q(C_n^{(1)})$ *and* $v, v' \in \mathcal{W}_\varepsilon$*.*

Proof. It suffices to show when u is one of the generators. If $u = k_i$, it is trivial. Let us show that

$$(e_i|\mathbf{m}\rangle, |\mathbf{m}'\rangle) = (|\mathbf{m}\rangle, \eta(e_i)|\mathbf{m}'\rangle), \tag{3.2}$$

for $i \in I$ and $|\mathbf{m}\rangle, |\mathbf{m}'\rangle \in \mathcal{W}_\varepsilon$. The proof for f_i is almost identical since (3.1) is symmetric.

Case 1. Suppose that $1 \leq i \leq n-1$. We may assume $\mathbf{m}' = \mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}$. The right-hand side is

$$(|\mathbf{m}\rangle, \eta(e_i)|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle) = (|\mathbf{m}\rangle, q_i^{-1}k_i^{-1}f_i|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle) = [m_i]q^{-1+m_i-m_{i+1}}(|\mathbf{m}\rangle, |\mathbf{m}\rangle),$$

and the left-hand side is

$$\begin{aligned}(e_i|\mathbf{m}\rangle, |\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle) &= [m_{i+1}+1](|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle, |\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle) \\ &= \frac{q^A[m_{i+1}+1]}{[m_i-1]![m_{i+1}+1]!\prod_{j\neq i,i+1}[m_j]!} \\ &= q^{m_i-m_{i+1}-1}[m_i](|\mathbf{m}\rangle, |\mathbf{m}\rangle),\end{aligned}$$

since

$$\begin{aligned}A &= -\frac{1}{2}\sum_{j\neq i,i+1} m_j(m_j-1) - \frac{1}{2}(m_i-1)(m_i-2) - \frac{1}{2}(m_{i+1}+1)m_{i+1} \\ &= -\frac{1}{2}\sum_{1\leq j\leq n} m_j(m_j-1) + m_i - m_{i+1} - 1.\end{aligned}$$

Hence (3.2) holds.

Case 2. Suppose that $i = n$. We may assume $\mathbf{m}' = \mathbf{m} - 2\mathbf{e}_n$. The right-hand side is

$$(|\mathbf{m}\rangle, \eta(e_n)|\mathbf{m} - 2\mathbf{e}_n\rangle) = (|\mathbf{m}\rangle, -q_n^{-1}k_n^{-1}f_n|\mathbf{m} - 2\mathbf{e}_n\rangle) = -q^{2m_n-2}\frac{[m_n-1][m_n]}{[2]}(|\mathbf{m}\rangle, |\mathbf{m}\rangle),$$

and the left-hand side is

$$\begin{aligned}(e_n|\mathbf{m}\rangle, |\mathbf{m} - 2\mathbf{e}_n\rangle) &= -\frac{q}{[2]}(|\mathbf{m} - 2\mathbf{e}_n\rangle, |\mathbf{m} - 2\mathbf{e}_n\rangle) = -\frac{q}{[2]}\frac{q^B}{[m_n-2]!\prod_{j\neq n}[m_j]!}(|\mathbf{m}\rangle, |\mathbf{m}\rangle) \\ &= -q^{2m_n-2}\frac{[m_n-1][m_n]}{[2]}(|\mathbf{m}\rangle, |\mathbf{m}\rangle),\end{aligned}$$

since

$$B = -\frac{1}{2}\sum_{j\neq n} m_j(m_j-1) - \frac{1}{2}(m_n-2)(m_n-3) = -\frac{1}{2}\sum_{1\leq j\leq n} m_j(m_j-1) + 2m_n - 3.$$

Hence (3.2) holds.

Case 3. Suppose that $i = 0$. We have to show $(e_0v, v') = (v, -q^{-2}k_0^{-1}f_0v')$. By Remark 3.2 and the property $(\tau(|\mathbf{m}\rangle), \tau(|\mathbf{m}'\rangle)) = (|\mathbf{m}\rangle, |\mathbf{m}'\rangle)$, it is equivalent to $(f_n\tau(v), \tau(v')) = (\tau(v), -q^{-2}k_ne_n\tau(v'))$. However, it is equivalent to the one proved in *Case 1.* □

3.1.4. *Crystal base.* Let M be a $U_q(C_n^{(1)})$-module. For $1 \leq j \leq n-1$, we assume that e_j and f_j are locally nilpotent on M, and define $\widetilde{e}_j$, $\widetilde{f}_j$ to be the usual lower crystal operators [12]. For $i = 0, n$, we introduce new operators $\widetilde{e}_i$ and $\widetilde{f}_i$ as follows:

Case 1. Let $u \in M$ be a weight vector such that $e_n u = 0$ and $k_n u = q_n^{-l} u$ for some $l > 0$. Put

$$u_k := q_n^{\frac{k(k+2l-1)}{2}} f_n^{(k)} u \quad (k \geq 0). \tag{3.3}$$

Then we define

$$\widetilde{f}_n u_k = u_{k+1}, \quad \widetilde{e}_n u_{k+1} = u_k \quad (k \geq 0). \tag{3.4}$$

Case 2. Let $u \in M$ be a weight vector such that $f_0 u = 0$ and $k_0 u = q_0^{l} u$ for some $l > 0$. Put

$$u_k := q_0^{\frac{k(k+2l-1)}{2}} e_0^{(k)} u \quad (k \geq 0). \tag{3.5}$$

Then we define

$$\widetilde{e}_0 u_k = u_{k+1}, \quad \widetilde{f}_0 u_{k+1} = u_k \quad (k \geq 0). \tag{3.6}$$

Remark 3.7. The definitions of $\widetilde{e}_i$ and $\widetilde{f}_i$ $(i = 0, n)$ are based on the idea that

$$(\widetilde{f}_n^k u, \widetilde{f}_n^k u) \in 1 + qA_0 \quad (\widetilde{e}_0^k u', \widetilde{e}_0^k u') \in 1 + qA_0 \quad (k \geq 0), \tag{3.7}$$

for $u, u' \in \mathcal{W}_\varepsilon$ such that $e_n u = 0$ and $f_0 u' = 0$ (use Proposition 2.1).

Let A_0 be the subring of $\mathbb{Q}(q)$ consisting of functions which are regular at $q = 0$. We define A_0-lattice $\mathcal{L}_\varepsilon$ of $\mathcal{W}_\varepsilon$ and a $\mathbb{Q}$-basis $\mathcal{B}_\varepsilon$ of $\mathcal{L}_\varepsilon / q\mathcal{L}_\varepsilon$ by

$$\mathcal{L}_\varepsilon = \bigoplus_{\mathrm{sgn}(\mathbf{m})=\varepsilon} A_0 |\mathbf{m}\rangle, \quad \mathcal{B}_\varepsilon = \{\, |\mathbf{m}\rangle \pmod{q\mathcal{L}} \,|\, \mathrm{sgn}(\mathbf{m}) = \varepsilon \,\}.$$

It is clear from (3.1) that $(\mathcal{L}_\varepsilon, \mathcal{L}_\varepsilon) \subset A_0$, and $\mathcal{B}_\varepsilon$ is an orthonormal basis of $\mathcal{L}_\varepsilon / q\mathcal{L}_\varepsilon$ with respect to $(\ ,\ )|_{q=0}$.

Proposition 3.8. *The pair $(\mathcal{L}_\varepsilon, \mathcal{B}_\varepsilon)$ is a crystal base of $\mathcal{W}_\varepsilon$, that is,*

(1) *$\mathcal{L}_\varepsilon$ is invariant under $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$,*
(2) *$\widetilde{e}_i \mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon \cup \{0\}$ and $\widetilde{f}_i \mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon \cup \{0\}$ for $i \in I$, where we have*

$$\widetilde{f}_i |\mathbf{m}\rangle \equiv \begin{cases} |\mathbf{m} + 2\mathbf{e}_n\rangle & \text{if } i = n, \\ |\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle & \text{if } m_{i+1} \geq 1 \text{ and } 1 \leq i \leq n-1, \\ |\mathbf{m} - 2\mathbf{e}_1\rangle & \text{if } m_1 \geq 2 \text{ and } i = 0, \\ 0 & \text{otherwise}, \end{cases} \pmod{q\mathcal{L}_\varepsilon}.$$

Proof. It is enough to prove (2).

Case 1. Suppose that $1 \leq i \leq n-1$. Let $|\mathbf{m}\rangle = |m_1, \ldots, m_n\rangle \in \mathcal{L}_\varepsilon$ be given with $m_{i+1} \geq 1$. Since $e_i |\mathbf{m} - m_i \mathbf{e}_i + m_i \mathbf{e}_{i+1}\rangle = 0$, we have

$$\widetilde{f}_i^{m_i} |\mathbf{m} - m_i \mathbf{e}_i + m_i \mathbf{e}_{i+1}\rangle = \frac{f_i^{m_i}}{[m_i]!} |\mathbf{m} - m_i \mathbf{e}_i + m_i \mathbf{e}_{i+1}\rangle = |\mathbf{m}\rangle,$$

and hence $\widetilde{f}_i|\mathbf{m}\rangle = \widetilde{f}_i^{m_i+1}|\mathbf{m} - m_i\mathbf{e}_i + m_i\mathbf{e}_{i+1}\rangle = |\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle$.

Case 2. Suppose that $i = n$. First, suppose that m_n is even. Since $e_n|\mathbf{m} - m_n\mathbf{e}_n\rangle = 0$ and $k_n|\mathbf{m} - m_n\mathbf{e}_n\rangle = q^{-1}|\mathbf{m} - m_n\mathbf{e}_n\rangle$, we have

$$\widetilde{f}_n^{\frac{m_n}{2}}|\mathbf{m} - m_n\mathbf{e}_n\rangle = q^{\left(\frac{m_n}{2}\right)^2} \frac{f_n^{\frac{m_n}{2}}}{\left[\frac{m_n}{2}\right]_n!}|\mathbf{m} - m_n\mathbf{e}_n\rangle = (1+q^2)^{-\frac{m_n}{2}} q^{\left(\frac{m_n}{2}\right)^2} \frac{[m_n]!}{\left[\frac{m_n}{2}\right]_n!}|\mathbf{m}\rangle,$$

and hence

$$\begin{aligned}
\widetilde{f}_n|\mathbf{m}\rangle &= (1+q^2)^{\frac{m_n}{2}} q^{-\left(\frac{m_n}{2}\right)^2} \frac{\left[\frac{m_n}{2}\right]_n!}{[m_n]!}\ \widetilde{f}_n^{\frac{m_n}{2}+1}|\mathbf{m} - m_n\mathbf{e}_n\rangle \\
&= (1+q^2)^{\frac{m_n}{2}} q^{-\left(\frac{m_n}{2}\right)^2} \frac{\left[\frac{m_n}{2}\right]_n!}{[m_n]!}\ (1+q^2)^{-\frac{m_n}{2}-1} q^{\left(\frac{m_n}{2}+1\right)^2} \frac{[m_n+2]!}{\left[\frac{m_n}{2}+1\right]_n!}|\mathbf{m} + 2\mathbf{e}_n\rangle \\
&= (1+q^2)^{-1} q^{\left(\frac{m_n}{2}+1\right)^2 - \left(\frac{m_n}{2}\right)^2} \frac{[m_n+2][m_n+1]}{\left[\frac{m_n}{2}+1\right]_n}|\mathbf{m} + 2\mathbf{e}_n\rangle \\
&\equiv |\mathbf{m} + 2\mathbf{e}_n\rangle \qquad (\mathrm{mod}\ q\mathcal{L}_\varepsilon),
\end{aligned}$$

since

$$q^{\left(\frac{m_n}{2}+1\right)^2 - \left(\frac{m_n}{2}\right)^2} \frac{[m_n+2][m_n+1]}{\left[\frac{m_n}{2}+1\right]_n} = q^{m_n+1} \frac{[m_n+2][m_n+1]}{\left[\frac{m_n}{2}+1\right]_n} \in (1 + qA_0).$$

Next, suppose that m_n is odd. Since $e_n|\mathbf{m} - (m_n-1)\mathbf{e}_n\rangle = 0$ and $k_n|\mathbf{m} - (m_n-1)\mathbf{e}_n\rangle = q^{-3}|\mathbf{m} - (m_n-1)\mathbf{e}_n\rangle$, we have

$$\begin{aligned}
\widetilde{f}_n^{\frac{m_n-1}{2}}|\mathbf{m} - (m_n-1)\mathbf{e}_n\rangle &= q^{\left(\frac{m_n-1}{2}\right)\left(\frac{m_n+3}{2}\right)} \frac{f_n^{\frac{m_n-1}{2}}}{\left[\frac{m_n-1}{2}\right]_n!}|\mathbf{m} - (m_n-1)\mathbf{e}_n\rangle \\
&= (1+q^2)^{-\frac{m_n-1}{2}} q^{\left(\frac{m_n-1}{2}\right)\left(\frac{m_n+3}{2}\right)} \frac{[m_n]!}{\left[\frac{m_n-1}{2}\right]_n!}|\mathbf{m}\rangle,
\end{aligned}$$

and hence

$$\begin{aligned}
&\widetilde{f}_n|\mathbf{m}\rangle \\
&= (1+q^2)^{\frac{m_n-1}{2}} q^{-\left(\frac{m_n-1}{2}\right)\left(\frac{m_n+3}{2}\right)} \frac{\left[\frac{m_n-1}{2}\right]_n!}{[m_n]!}\ \widetilde{f}_n^{\frac{m_n+1}{2}}|\mathbf{m} - (m_n-1)\mathbf{e}_n\rangle \\
&= (1+q^2)^{\frac{m_n-1}{2}} q^{-\left(\frac{m_n-1}{2}\right)\left(\frac{m_n+3}{2}\right)} \frac{\left[\frac{m_n-1}{2}\right]_n!}{[m_n]!}\ (1+q^2)^{-\frac{m_n+1}{2}} q^{\left(\frac{m_n+1}{2}\right)\left(\frac{m_n+5}{2}\right)} \frac{[m_n+2]!}{\left[\frac{m_n+1}{2}\right]_n!}|\mathbf{m} + 2\mathbf{e}_n\rangle \\
&= (1+q^2)^{-1} q^{\left(\frac{m_n+1}{2}\right)\left(\frac{m_n+5}{2}\right) - \left(\frac{m_n-1}{2}\right)\left(\frac{m_n+3}{2}\right)} \frac{[m_n+2][m_n+1]}{\left[\frac{m_n+1}{2}\right]_n}|\mathbf{m} + 2\mathbf{e}_n\rangle \\
&\equiv |\mathbf{m} + 2\mathbf{e}_n\rangle \qquad (\mathrm{mod}\ q\mathcal{L}_\varepsilon),
\end{aligned}$$

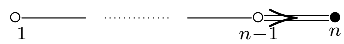
since

$$q^{\left(\frac{m_n+1}{2}\right)\left(\frac{m_n+5}{2}\right) - \left(\frac{m_n-1}{2}\right)\left(\frac{m_n+3}{2}\right)} \frac{[m_n+2][m_n+1]}{\left[\frac{m_n+1}{2}\right]_n} = q^{m_n+1} \frac{[m_n+2][m_n+1]}{\left[\frac{m_n+1}{2}\right]_n} \in (1 + qA_0).$$

Case 3. Suppose that $i = 0$. We can prove this case by the same arguments as in *Case 2* by using the automorphism τ (2.1). □

3.2. **Type $C^{(2)}(n+1)$.**

3.2.1. *$U_q(C^{(2)}(n{+}1)$-module $\mathcal{W}$.* Consider the quantum affine superalgebra of type $C^{(2)}(n+1)$. Let $U_q(B(0,n))$ and $U_q(A_{n-1})$ be the subalgebras of $U_q(C^{(2)}(n{+}1))$ generated by k_i, e_i, f_i for $i \in I \setminus \{0\}$ and $i \in I \setminus \{0, n\}$, respectively. We also write $U_q(B(0,n)) = U_q(osp_{1|2n})$, where $osp_{1|2n}$ is the orthosymplectic Lie superalgebra corresponding to the Dynkin diagram:



Proposition 3.9. *For a non-zero $x \in \mathbb{Q}(q^{\frac{1}{2}})$, the space $\mathcal{W}$ admits an irreducible $U_q(C^{(2)}(n+1))^\sigma$-module structure given as follows:*

$$\begin{aligned}
e_0|\mathbf{m}\rangle &= xq^{-\frac{1}{2}}[m_1+1]|\mathbf{m}+\mathbf{e}_1\rangle, \\
f_0|\mathbf{m}\rangle &= x^{-1}q^{\frac{1}{2}}|\mathbf{m}-\mathbf{e}_1\rangle, \\
k_0|\mathbf{m}\rangle &= q^{m_1+\frac{1}{2}}|\mathbf{m}\rangle, \\
e_j|\mathbf{m}\rangle &= [m_{j+1}+1]|\mathbf{m}-\mathbf{e}_j+\mathbf{e}_{j+1}\rangle, \\
f_j|\mathbf{m}\rangle &= [m_j+1]|\mathbf{m}+\mathbf{e}_j-\mathbf{e}_{j+1}\rangle, \\
k_j|\mathbf{m}\rangle &= q^{-m_j+m_{j+1}}|\mathbf{m}\rangle, \\
e_n|\mathbf{m}\rangle &= -q^{\frac{1}{2}}|\mathbf{m}-\mathbf{e}_n\rangle, \\
f_n|\mathbf{m}\rangle &= q^{-\frac{1}{2}}[m_n+1]|\mathbf{m}+\mathbf{e}_n\rangle, \\
k_n|\mathbf{m}\rangle &= q^{-m_n-\frac{1}{2}}|\mathbf{m}\rangle, \\
\sigma|\mathbf{m}\rangle &= (-1)^{|\mathbf{m}|}|\mathbf{m}\rangle,
\end{aligned}$$

where $1 \leq j \leq n-1$.

We denote this module by $\mathcal{W}(x)$ and call it a (level one) q-oscillator representation. We simply write $\mathcal{W} = \mathcal{W}(1)$ as a $U_q(C^{(2)}(n+1))$-module. Note that as a $U_q(A_{n-1})$-module, we have

$$\mathrm{ch}\mathcal{W} = \sum_{l \in \mathbb{Z}_{\geq 0}} s_{(l)}(x_1, \ldots, x_n) = \frac{1}{\prod_{i=1}^n (1-x_i)}.$$

Remark 3.10. When $x = 1$ we also have the following symmetry

$$\tau(u|\mathbf{m}\rangle) = \tau(u)\tau(|\mathbf{m}\rangle),$$

for $u \in U_q(C^{(2)}(n+1))$ (cf. Remark 3.2). Here the automorphism τ on $U_q(C^{(2)}(n+1))$ is given in (2.2).

3.2.2. *Classical limit.* Let

$$\mathcal{W}(x)_A = \sum_{\mathbf{m}} A|\mathbf{m}\rangle, \quad \overline{\mathcal{W}(x)} = \mathcal{W}(x)_A \otimes_A \mathbb{C}, \tag{3.8}$$

where $A = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ and $\mathbb{C}$ is an A-module such that $f(q^{\frac{1}{2}}) \cdot c = f(1)c$ for $f(q^{\frac{1}{2}}) \in A$ and $c \in \mathbb{C}$.

One can check directly that $\mathcal{W}(x)_A$ is invariant under e_i, f_i, and $\{k_i\}$ for $i \in I \setminus \{0\}$, and the induced operators E_i, F_i, and H_i on $\overline{\mathcal{W}(x)}$, respectively, satisfy the defining relations of $U(osp_{1|2n})$.

Lemma 3.11. *The space $\overline{\mathcal{W}(x)}$ is isomorphic to the irreducible highest weight $U(osp_{1|2n})$-module with highest weight $-\varpi_n$, where ϖ_n is the n-th fundamental weight for $osp_{1|2n}$.*

Proof. We have

$$H_n(|\mathbf{0}\rangle \otimes 1) = \left(\frac{k_n - k_n^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}|\mathbf{0}\rangle\right) \otimes 1 = \left(\frac{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}|\mathbf{0}\rangle\right) \otimes 1 = -|\mathbf{0}\rangle \otimes 1,$$

and $H_i(|\mathbf{0}\rangle \otimes 1) = 0$ for $1 \leq i \leq n-1$. By the same argument as in Lemma 3.5, $\overline{\mathcal{W}(x)}$ is an irreducible highest weight $U_q(osp_{1|2n})$-module with highest weight $-\varpi_n$. □

3.2.3. *Polarization.* Define a symmetric bilinear form on $\mathcal{W}$ by (3.1).

Lemma 3.12. *The bilinear form in* (3.1) *is a polarization on $\mathcal{W}$, that is,*

$$(uv, v') = (v, \eta(u)v'),$$

for $u \in U_q(C^{(2)}(n+1))$ and $v, v' \in \mathcal{W}$.

Proof. Let us show $(e_n|\mathbf{m}\rangle, |\mathbf{m}'\rangle) = (|\mathbf{m}\rangle, \eta(e_n)|\mathbf{m}'\rangle)$ for $|\mathbf{m}\rangle, |\mathbf{m}'\rangle \in \mathcal{W}$ only. The proof for e_i $(1 \leq i \leq n-1)$ is identical to Lemma 3.1, and the proof for e_0 is obtained by using τ. We may assume $\mathbf{m}' = \mathbf{m} - \mathbf{e}_n$. The right-hand side is

$$(|\mathbf{m}\rangle, \eta(e_n)|\mathbf{m} - \mathbf{e}_n\rangle) = (|\mathbf{m}\rangle, -q_n^{-1}k_n^{-1}f_n|\mathbf{m} - \mathbf{e}_n\rangle) = -q_n^{2m_n-1}[m_n](|\mathbf{m}\rangle, |\mathbf{m}\rangle),$$

and the left-hand side is

$$\begin{aligned}(e_n|\mathbf{m}\rangle, |\mathbf{m} - \mathbf{e}_n\rangle) &= -q^{\frac{1}{2}}(|\mathbf{m} - \mathbf{e}_n\rangle, |\mathbf{m} - \mathbf{e}_n\rangle) \\ &= -q^{\frac{1}{2}}\frac{q^{-\frac{1}{2}\sum m_i(m_i-1)}}{\prod_{i=1}^n [m_i]!}[m_n]\, q^{m_n-1} = -q^{m_n-\frac{1}{2}}[m_n](|\mathbf{m}\rangle, |\mathbf{m}\rangle).\end{aligned}$$

Hence the equality holds. □

3.2.4. *Crystal base.* Let M be a $U_q(C^{(2)}(n+1))$-module. For $1 \leq j \leq n-1$, we assume that e_j and f_j are locally nilpotent on M, and define $\widetilde{e}_j$, $\widetilde{f}_j$ to be the usual lower crystal operators. For $i = 0, n$, we consider the operators $\widetilde{e}_i$ and $\widetilde{f}_i$ defined in the same way as in $U_q(C_n^{(1)})$ (3.3)–(3.6), which also satisfy (3.7).

Let A_0 be the subring of $\mathbb{Q}(q^{\frac{1}{2}})$ consisting of functions which are regular at $q^{\frac{1}{2}} = 0$. We define the A_0-lattice $\mathcal{L}$ of $\mathcal{W}$ and a $\mathbb{Q}$-basis $\mathcal{B}$ of $\mathcal{L}/q^{\frac{1}{2}}\mathcal{L}$ by

$$\mathcal{L} = \bigoplus_{\mathbf{m}} A_0|\mathbf{m}\rangle, \quad \mathcal{B} = \{\, |\mathbf{m}\rangle \pmod{q^{\frac{1}{2}}\mathcal{L}} \,\}. \tag{3.9}$$

It is clear from (3.1) that $(\mathcal{L}, \mathcal{L}) \subset A_0$, and $\mathcal{B}$ is an orthonormal basis of $\mathcal{L}/q^{\frac{1}{2}}\mathcal{L}$ with respect to $(\ ,\ )|_{q^{\frac{1}{2}}=0}$.

Proposition 3.13. *The pair $(\mathcal{L},\mathcal{B})$ is a crystal base of $\mathcal{W}$ in the sense of Proposition 3.8, where*

$$\widetilde{f}_i|\mathbf{m}\rangle \equiv \begin{cases} |\mathbf{m}+\mathbf{e}_n\rangle & \text{if } i=n,\\ |\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}\rangle & \text{if } m_{i+1}\geq 1 \text{ and } 1\leq i\leq n-1,\\ |\mathbf{m}-\mathbf{e}_1\rangle & \text{if } m_1\geq 1 \text{ and } i=0,\\ 0 & \text{otherwise,}\end{cases} \qquad (\text{mod } q^{\frac{1}{2}}\mathcal{L}).$$

Proof. It suffices to prove (2) when $i=0,n$ since the other cases are proved in Proposition 3.8. Let us prove the case of $\widetilde{f}_n$ only. Recall that $[m]_n=[m]_{q^{\frac{1}{2}},-1}$ for $m\in\mathbb{Z}_{\geq 0}$.

Let $|\mathbf{m}\rangle$ be given. Since $e_n|\mathbf{m}-m_n\mathbf{e}_n\rangle=0$ and $k_n|\mathbf{m}-m_n\mathbf{e}_n\rangle=q^{-\frac{1}{2}}|\mathbf{m}-m_n\mathbf{e}_n\rangle$, we have

$$\widetilde{f}_n^{m_n}|\mathbf{m}-m_n\mathbf{e}_n\rangle = q_n^{\frac{m_n(m_n+1)}{2}}\frac{f_n^{m_n}}{[m_n]_{q_n,-1}!}|\mathbf{m}-m_n\mathbf{e}_n\rangle = q_n^{\frac{m_n(m_n-1)}{2}}\frac{[m_n]!}{[m_n]_{q_n,-1}!}|\mathbf{m}\rangle,$$

and hence

$$\begin{aligned}\widetilde{f}_n|\mathbf{m}\rangle &= q_n^{-\frac{m_n(m_n-1)}{2}}\frac{[m_n]_{q_n,-1}!}{[m_n]!}\ \widetilde{f}_n^{m_n+1}|\mathbf{m}-m_n\mathbf{e}_n\rangle\\ &= q_n^{-\frac{m_n(m_n-1)}{2}}\frac{[m_n]_{q_n,-1}!}{[m_n]!}\ q_n^{\frac{m_n(m_n+1)}{2}}\frac{[m_n+1]!}{[m_n+1]_{q_n,-1}!}|\mathbf{m}+\mathbf{e}_n\rangle\\ &\equiv q_n^{m_n}\frac{[m_n+1]}{[m_n+1]_{q_n,-1}}|\mathbf{m}+\mathbf{e}_n\rangle = |\mathbf{m}+\mathbf{e}_n\rangle \qquad (\text{mod } q^{\frac{1}{2}}\mathcal{L}).\end{aligned}$$

□

3.3. Type $B^{(1)}(0,n)$.

3.3.1. *$U_q(B^{(1)}(0,n))$-module $\mathcal{W}$.* Consider the quantum affine superalgebra of type $B^{(1)}(0,n)$. Let $U_q(B(0,n))$ (or $U_q(osp_{1|2n})$) and $U_q(A_{n-1})$ be the subalgebras of $U_q(B^{(1)}(0,n))$ generated by k_i,e_i,f_i for $i\in I\setminus\{0\}$ and $i\in I\setminus\{0,n\}$, respectively.

Proposition 3.14. *For a non-zero $x\in\mathbb{Q}(q^{\frac{1}{2}})$, the space $\mathcal{W}$ admits an irreducible $U_q(B^{(1)}(0,n))^\sigma$-module structure given as follows:*

$$\begin{aligned}
e_0|\mathbf{m}\rangle &= xq^{-1}\frac{[m_1+1][m_1+2]}{[2]}|\mathbf{m}+2\mathbf{e}_1\rangle,\\
f_0|\mathbf{m}\rangle &= -x^{-1}\frac{q}{[2]}|\mathbf{m}-2\mathbf{e}_1\rangle,\\
k_0|\mathbf{m}\rangle &= q^{2m_1+1}|\mathbf{m}\rangle,\\
e_j|\mathbf{m}\rangle &= [m_{j+1}+1]|\mathbf{m}-\mathbf{e}_j+\mathbf{e}_{j+1}\rangle,\\
f_j|\mathbf{m}\rangle &= [m_j+1]|\mathbf{m}+\mathbf{e}_j-\mathbf{e}_{j+1}\rangle,\\
k_j|\mathbf{m}\rangle &= q^{-m_j+m_{j+1}}|\mathbf{m}\rangle,\\
e_n|\mathbf{m}\rangle &= -q^{\frac{1}{2}}|\mathbf{m}-\mathbf{e}_n\rangle,\\
f_n|\mathbf{m}\rangle &= q^{-\frac{1}{2}}[m_n+1]|\mathbf{m}+\mathbf{e}_n\rangle,
\end{aligned}$$

$$k_n|\mathbf{m}\rangle = q^{-m_n-\frac{1}{2}}|\mathbf{m}\rangle,$$
$$\sigma|\mathbf{m}\rangle = (-1)^{|\mathbf{m}|}|\mathbf{m}\rangle,$$

where $1 \leq j \leq n-1$.

We also denote this module by $\mathcal{W}(x)$ and call it a (level one) q-oscillator representation. Note that the classical limit of $\mathcal{W}(x)$ as a $U_q(osp_{1|2n})$-module is the same as in Lemma 3.11.

3.3.2. *Polarization and crystal base.*

Lemma 3.15. *The bilinear form in* (3.1) *is a polarization on* $\mathcal{W}$*, that is,*

$$(uv, v') = (v, \eta(u)v'),$$

for $u \in U_q(B^{(1)}(0,n))$ *and* $v, v' \in \mathcal{W}$.

Proof. All the cases are already shown in Lemmas 3.6 and 3.12 since the action of e_i for $0 \leq i < n$ (resp. $i = n$) is the same as the one for $C_n^{(1)}$ (resp. $C^{(2)}(n+1)$). □

We define the A_0-lattice $\mathcal{L}$ of $\mathcal{W}$ and a $\mathbb{Q}$-basis $\mathcal{B}$ of $\mathcal{L}/q^{\frac{1}{2}}\mathcal{L}$ as in (3.9). We also define the operators $\widetilde{e}_i$ and $\widetilde{f}_i$ in the same way as in $U_q(C_n^{(1)})$ and $U_q(C^{(2)}(n+1))$.

Proposition 3.16. *The pair* $(\mathcal{L}, \mathcal{B})$ *is a crystal base of* $\mathcal{W}$ *in the sense of Proposition 3.8, where*

$$\widetilde{f}_i|\mathbf{m}\rangle \equiv \begin{cases} |\mathbf{m}+\mathbf{e}_n\rangle & \text{if } i = n, \\ |\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}\rangle & \text{if } m_{i+1} \geq 1 \text{ and } 1 \leq i \leq n-1, \\ |\mathbf{m}-2\mathbf{e}_1\rangle & \text{if } m_1 \geq 2 \text{ and } i = 0, \\ 0 & \text{otherwise}, \end{cases} \pmod{q^{\frac{1}{2}}\mathcal{L}}.$$

Proof. It follows from Propositions 3.8 and 3.13. □

4. Quantum R-matrix and fusion construction

In this section, we review the quantum R-matrix and its spectral decomposition for each quantum affine (super)algebra and explain how to construct higher level q-oscillator representations by so-called fusion construction.

Let $x, y \in \mathbb{Q}(q^d)$ be generic, and let $\mathcal{W}$ be a level one q-oscillator representation of $U_q(X)$ including $\mathcal{W}_\varepsilon$ ($\varepsilon = \pm$) for type $C_n^{(1)}$. The quantum R-matrix $R(x,y)$ on $\mathcal{W}(x) \otimes \mathcal{W}(y)$ is defined as a linear operator satisfying

$$R(x,y)\Delta(a) = \Delta^{\mathrm{op}}(a)R(x,y)$$

for $a \in U_q(X)$, where Δ^{op} denotes the opposite coproduct, namely, the coproduct obtained by interchanging the first and second components in Δ. If $\mathcal{W}(x) \otimes \mathcal{W}(y)$ is irreducible, then $R(x,y)$ is unique up to a scalar function of x, y and depends only on $z = x/y$. Let P be the linear operator on $\mathcal{W}(x) \otimes \mathcal{W}(y)$ such that $P(u \otimes v) = v \otimes u$ and set $\check{R}(x,y) = PR(x,y)$. Then $\check{R}(x,y)$ maps $\mathcal{W}(x) \otimes \mathcal{W}(y)$ to $\mathcal{W}(y) \otimes \mathcal{W}(x)$.

We also need to care about the difference between the coproduct (2.4) and that of [13] and Appendix A. Let $\bar{\Delta}$ be the coproduct of the latter. Let ς be the automorphism given by $\varsigma(e_i) = e_i k_i^{-1}, \varsigma(f_i) = k_i f_i, \varsigma(k_i) = k_i$. Then we have $\Delta(a) = \bar{\Delta}^{\rm op}(\varsigma(a))$. Hence, to translate the results in [13] and Appendix A, we replace $\check{R}(x,y)$ with $\check{R}(y,x)$. The component V_l or V_l^{ε} appearing in the spectral decomposition should be replaced with PV_l. Thus, we obtain the spectral decomposition of $\check{R}(x,y)$ as follows. Note that $z = x/y$.

For type $C_n^{(1)}$, we have

$$\check{R}_{\varepsilon}(x,y) = \sum_{l\in 2\mathbb{Z}_{\geq 0}} \prod_{j=1}^{l/2} \frac{1-q^{4j-2}z}{z-q^{4j-2}} P_l^{\varepsilon} \tag{4.1}$$

where $\check{R}_{\varepsilon}(x,y) : \mathcal{W}_{\varepsilon}(x)\otimes\mathcal{W}_{\varepsilon}(y) \to \mathcal{W}_{\varepsilon}(y)\otimes\mathcal{W}_{\varepsilon}(x)$ for $\varepsilon = +,-$ and P_l^{ε} is the projection onto V_l^{ε}.

For $C^{(2)}(n+1)$, from Proposition C.4 and the spectral decomposition for $U_q(D_{n+1}^{(2)})$ in [13, Proposition 7], we have

$$\check{R}(x,y) = \sum_{l\in \mathbb{Z}_{\geq 0}} \prod_{j=1}^{l} \frac{1+(-q)^j z}{z+(-q)^j} P_l, \tag{4.2}$$

where P_l is the projection onto V_l.

Finally for $B^{(1)}(0,n)$, from Proposition C.4 and the spectral decomposition for $U_q(A_{2n}^{(2)\dagger})$ in Appendix B, we have

$$\check{R}(x,y) = \sum_{l\in 2\mathbb{Z}_{\geq 0}} \prod_{j=1}^{l/2} \frac{1-q^{4j-1}z}{z-q^{4j-1}} P_l + \sum_{l\in 1+2\mathbb{Z}_{\geq 0}} \prod_{j=0}^{(l-1)/2} \frac{1-q^{4j+1}z}{z-q^{4j+1}} P_l. \tag{4.3}$$

Next, we explain the fusion construction. For $s\geq 2$, let $\mathfrak{S}_s$ denote the group of permutations on s letters generated by $s_i = (i\ i+1)$ for $1\leq i\leq s-1$. We have $U_q(X)$-linear maps

$$\check{R}_w(x_1,\ldots,x_s) : \mathcal{W}(x_1)\otimes\cdots\otimes\mathcal{W}(x_s) \longrightarrow \mathcal{W}(x_{w(1)})\otimes\cdots\otimes\mathcal{W}(x_{w(s)}),$$

for $w\in\mathfrak{S}_s$ and generic $x_1,\ldots,x_s\in\mathbb{Q}(q)$ satisfying

$$\begin{aligned}
&\check{R}_1(x_1,\ldots,x_s) = {\rm id}_{\mathcal{W}(x_1)\otimes\cdots\otimes\mathcal{W}(x_s)},\\
&\check{R}_{s_i}(x_1,\ldots,x_s) = \left(\otimes_{j<i}{\rm id}_{\mathcal{W}(x_j)}\right)\otimes\check{R}(x_i/x_{i+1})\otimes\left(\otimes_{j>i+1}{\rm id}_{\mathcal{W}(x_j)}\right),\\
&\check{R}_{ww'}(x_1,\ldots,x_s) = \check{R}_{w'}(x_{w(1)},\ldots,x_{w(s)})\check{R}_w(x_1,\ldots,x_s),
\end{aligned}$$

for $w,w'\in\mathfrak{S}_s$ with $\ell(ww') = \ell(w)+\ell(w')$ where $\ell(w)$ denotes the length of w. Hence we have a $U_q(X)$-linear map $\check{R}_s = \check{R}_{w_0}(x_1,\ldots,x_s)$ with $x_i = q^{d(2i-s-1)}$:

$$\check{R}_s : \mathcal{W}(q^{d(1-s)})\otimes\ldots\otimes\mathcal{W}(q^{d(s-1)}) \longrightarrow \mathcal{W}(q^{d(s-1)})\otimes\ldots\otimes\mathcal{W}(q^{d(1-s)}).$$

Here w_0 is the longest element in $\mathfrak{S}_s$ and $d = \min\{\, d_i \,|\, i\in I\,\}$. Now we define a $U_q(X)$-module

$$\mathcal{W}^{(s)} = {\rm Im}\check{R}_s. \tag{4.4}$$

Remark 4.1. Let R^{univ} be the universal R matrix for the quantum affine (super)algebra $U_q(X)$ [6]. Suppose that $\mathcal{W}$ is a finite-dimensional irreducible $U_q(X)$-module. Then R^{univ} is rationally renormalizable in the sense of [11], that is, there exists $c \in \mathbb{Q}(q^d)((y/x))$ such that we have a well-defined map

$$cR^{\text{univ}} : \mathcal{W}(x) \otimes \mathcal{W}(y) \longrightarrow \mathcal{W}(y) \otimes \mathcal{W}(x), \tag{4.5}$$

for x, y. Then we may apply [11, Theorem 3.12] to prove that $\mathcal{W}^{(s)}$ is irreducible. However, the q-oscillator module $\mathcal{W}$ is infinite dimensional and R^{univ} on $\mathcal{W}(x)\otimes\mathcal{W}(y)$ is not rationally renormalizable. We expect that (4.5) still has a meaning, but do not know how to justify it.

5. Higher level q-oscillator representation

5.1. **Type $C_n^{(1)}$.** For $s \geq 2$ and $\varepsilon = \pm$, let $\mathcal{W}_\varepsilon^{(s)}$ denote the higher level q-oscillator module in (4.4) corresponding to $\mathcal{W}_\varepsilon$. The following is the main result in this section.

Theorem 5.1. *For $s \geq 2$, $\mathcal{W}_\varepsilon^{(s)}$ is an irreducible $U_q(C_n^{(1)})$-module, which is also irreducible as a $U_q(C_n)$-module. Moreover, its character is given by*

$$\mathrm{ch}\mathcal{W}_\varepsilon^{(s)} = \sum_{\substack{\lambda \in \mathscr{P}_\varepsilon \\ \ell(\lambda) \leq s}} s_\lambda(x_1, \dots, x_n),$$

where $\mathscr{P}_\varepsilon$ is the set of partitions $\lambda = (\lambda_i)_{i \geq 1}$ with $\mathrm{sgn}(\lambda_i) = \varepsilon$ for all i with $\lambda_i \neq 0$, and $\ell(\lambda)$ denotes the length of λ.

Corollary 5.2. *The character of $\mathcal{W}_\varepsilon^{(s)}$ has a stable limit for $s \geq n$ as follows:*

$$\begin{aligned}\mathrm{ch}\mathcal{W}_\varepsilon^{(s)} &= \sum_{\substack{\lambda \in \mathscr{P}_\varepsilon \\ \ell(\lambda) \leq n}} s_\lambda(x_1, \dots, x_n) \\ &= \frac{1}{\prod_{1 \leq i \leq j \leq n}(1 - x_i x_j)} \quad (\varepsilon = +).\end{aligned}$$

Let us construct a certain $\mathbb{Q}(q)$-basis of $\mathcal{W}_\varepsilon^{(2)}$, which is compatible with the action of $\check{R}(z)$, and plays an important role in the proof of Theorem 5.1. We note from (4.1) that

$$\mathcal{W}_\varepsilon^{(2)} = V_0^\varepsilon = U_q(C_n)(|\varsigma(\varepsilon)\mathbf{e}_n\rangle \otimes |\varsigma(\varepsilon)\mathbf{e}_n\rangle),$$

and hence it is irreducible. Moreover, we have the following character formula for $\mathcal{W}_\varepsilon^{(2)}$.

Proposition 5.3. *We have*

$$\mathrm{ch}\mathcal{W}_\varepsilon^{(2)} = \mathrm{ch}V_0^\varepsilon = \sum_{\substack{\lambda \in \mathscr{P}_\varepsilon \\ \ell(\lambda) \leq 2}} s_\lambda(x_1, \dots, x_n).$$

Proof. Write $\mathcal{W}_\varepsilon = \mathcal{W}_\varepsilon(q^{\pm 1})$ for short since we may consider the action of $U_q(C_n)$ only. Let $(\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon)_A$ be the A-span of $|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle$ in $\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon$. Then $(\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon)_A$ is also invariant under e_i, f_i, k_i and $\{k_i\}$ for $i \in I \setminus \{0\}$. This yields its classical limit $\overline{\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon} := (\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon)_A \otimes_A \mathbb{C}$, which is a $U(C_n)$-module. Also, we have as a $U(C_n)$-module

$$\overline{\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon} \cong \overline{\mathcal{W}_\varepsilon} \otimes \overline{\mathcal{W}_\varepsilon}.$$

By Lemma 3.5, $\overline{\mathcal{W}_\varepsilon}$ is an irreducible highest weight module. By the theory of super duality [3], it belongs to a semisimple category of $U(C_n)$-module which is closed under tensor product (see [16, Section 5.4] for more details, where we put $m=0$ there). Hence $\overline{\mathcal{W}_\varepsilon}\otimes\overline{\mathcal{W}_\varepsilon}$ is semisimple, and the classical limit $\overline{V_0^\varepsilon}$, the submodule generated by $(|\varsigma(\varepsilon)\mathbf{e}_n\rangle\otimes|\varsigma(\varepsilon)\mathbf{e}_n\rangle)\otimes 1$, is an irreducible highest weight $U(C_n)$-module with highest weight $-(1+2\varsigma(\varepsilon))\varpi_n$. The character of $\overline{V_0^\varepsilon}$ and hence V_0^ε follows from [17, Theorem 6.1]. □

We construct a $\mathbb{Q}(q)$-basis of $\mathcal{W}_\varepsilon^{(2)}$ which is compatible with its $U_q(A_{n-1})$-crystal base. For this, we find all the $U_q(A_{n-1})$-highest weight vectors in $\mathcal{W}_\varepsilon^{(2)}$.

For $l\in\mathbb{Z}_{\geq 0}$, let

$$\mathbf{v}_l=\sum_{k=0}^{l}(-1)^k q^{k(k-l+1)}\begin{bmatrix} l\\ k\end{bmatrix}^{-1}|k\mathbf{e}_{n-1}+(l-k)\mathbf{e}_n\rangle\otimes|(l-k)\mathbf{e}_{n-1}+k\mathbf{e}_n\rangle. \tag{5.1}$$

Lemma 5.4. *For $l\in\mathbb{Z}_{\geq 0}$, $\mathbf{v}_l$ is a $U_q(A_{n-1})$-highest weight vector in $\mathcal{W}_\varepsilon^{(2)}$, and*

$$\mathbf{v}_l\equiv|l\mathbf{e}_n\rangle\otimes|l\mathbf{e}_{n-1}\rangle\pmod{q\mathcal{L}_\varepsilon^{\otimes 2}},$$

where $\mathrm{sgn}(l)=\varepsilon$.

Proof. It is straightforward to check that $e_i\mathbf{v}_l=0$ for $1\leq i\leq n-1$. Next we claim that $\mathbf{v}_l\in\mathcal{W}_\varepsilon^{(2)}$. Note that

$$\mathrm{ch}\mathcal{W}_\varepsilon=\sum_{l\in\varsigma(\varepsilon)+2\mathbb{Z}_{>0}}s_{(l)}(x_1,\dots,x_n),$$

and hence

$$\mathrm{ch}\mathcal{W}_\varepsilon^{\otimes 2}=(\mathrm{ch}\mathcal{W}_\varepsilon)^2=\sum_{\substack{\mathrm{sgn}(|\lambda|)=+\\ \ell(\lambda)\leq 2}}m_\lambda s_\lambda(x_1,\dots,x_n), \tag{5.2}$$

where for $\lambda=(\lambda_1,\lambda_2)$,

$$m_\lambda=\begin{cases}\frac{\lambda_1-\lambda_2}{2}-\varsigma(\varepsilon) & \text{if } \lambda_1>\lambda_2,\\ 1 & \text{if } \lambda_1=\lambda_2.\end{cases}$$

Let S_l be the $U_q(A_{n-1})$-submodule of $\mathcal{W}_\varepsilon^{\otimes 2}$ generated by $\mathbf{v}_l$. Since the character of S_l is $s_{(l^2)}(x_1,\dots,x_n)$, and the multiplicity of $s_{(l^2)}(x_1,\dots,x_n)$ in (5.2) is one, it follows from Proposition 5.3 that $S_l\subset\mathcal{W}_\varepsilon^{(2)}$. This shows that $\mathbf{v}_l\in\mathcal{W}_\varepsilon^{(2)}$. The lemma follows from $q^{k(k-l+1)}\begin{bmatrix} l\\ k\end{bmatrix}^{-1}\in q^k(1+qA_0)$. □

One can prove more directly that $\mathbf{v}_l\in\mathcal{W}_\epsilon^{(2)}$ using the following lemma.

Lemma 5.5. *Set $\mathcal{E}=e_{n-2}^{(2)}\cdots e_1^{(2)}e_0$, where it should be understood as e_0 when $n=2$. Then for $l\in\mathbb{Z}_{\geq 0}$ we have*

$$(\mathcal{E}e_1^{(2)}\mathcal{E}-\frac{1}{[3]!}(e_1\mathcal{E})^2)\mathbf{v}_l=q^{-2}\frac{[2]}{[3]}([l+1][l+2])^2\mathbf{v}_{l+2}.$$

Proof. Denote the module $\mathcal{W}_\varepsilon$ by $\mathcal{W}_{\varepsilon,n}$ to signify the rank n and let $\mathcal{W}^0_{\epsilon,n}$ be a linear subspace of $\mathcal{W}_{\varepsilon,n}$ spanned by the vectors $|0^{n-2}, m_{n-1}, m_n\rangle$. Let $\pi : \mathcal{W}_{\varepsilon,n} \to \mathcal{W}_{\varepsilon,2}$ be a linear map defined by $\pi(|\mathbf{m}\rangle) = |m_{n-1}, m_n\rangle$, where $\mathbf{m} = (m_1, \ldots, m_n)$. Then we can show by direct calculation that the following diagram commutes.

$$\begin{array}{ccc}
(\mathcal{W}^0_{\varepsilon,n})^{\otimes 2} & \xrightarrow{\pi^{\otimes 2}} & \mathcal{W}^{\otimes 2}_{\varepsilon,2} \\
\varepsilon \downarrow & & \downarrow e_0 \\
(\mathcal{W}^0_{\varepsilon,n})^{\otimes 2} & \xrightarrow[\pi^{\otimes 2}]{} & \mathcal{W}^{\otimes 2}_{\varepsilon,2}
\end{array}$$

This fact reduces the proof of the lemma to the case of $n = 2$.

When $n = 2$, one calculates

$$\begin{aligned}
e_0 e_1^{(2)} e_0 \mathbf{v}_l =& \sum_k c_k [l-k+1][l-k+2] \\
&\times \{q^{-2l+2k-4}[k+1][k+2]|k+2, l-k+2\rangle \otimes |l-k, k\rangle \\
&+ (q^{-1}[l-k+1][l-k+2] + q^{-2l-7}[k-1][k])|k, l-k+2\rangle \otimes |l-k+2, k\rangle \\
&+ q^{-2k}[l-k+3][l-k+4]|k-2, l-k+2\rangle \otimes |l-k+4, k\rangle\}.
\end{aligned}$$

Here $c_k = (-1)^k q^{k(k-l+1)} \begin{bmatrix} l \\ k \end{bmatrix}^{-1}$ and we have used the relation $q^{l-2k-2}[l-k]c_{k+1} + [k+1]c_k = 0$. On the other hand, we also get

$$\begin{aligned}
(e_1 e_0)^2 \mathbf{v}_l =& [2] \sum_k c_k [l-k+1][l-k+2] \\
&\times \{[3] q^{-2l+2k-4}[k+1][k+2]|k+2, l-k+2\rangle \otimes |l-k, k\rangle \\
&+ A_k |k, l-k+2\rangle \otimes |l-k+2, k\rangle \\
&+ [3] q^{-2k}[l-k+3][l-k+4]|k-2, l-k+2\rangle \otimes |l-k+4, k\rangle\},
\end{aligned}$$

where

$$\begin{aligned}
A_k = \frac{q^{l-2k}}{q-q^{-1}} &\{(1+q^{-2l-6})(q^2[k+1][l-k+2] - [k][l-k+3]) \\
&- q^{-2l+2k}(1+q^{-4})([k+1][l-k+2] - q^{-4}[k][l-k+3])\}.
\end{aligned}$$

Combining these results, we obtain

$$\begin{aligned}
(e_0 e_1^{(2)} e_0 - &\frac{1}{[3]!}(e_1 e_0)^2)\mathbf{v}_l \\
&= \frac{[2]}{[3]}[l+1][l+2] \sum_k c_k q^{-2k-2}[l-k+1][l-k+2]|k, l-k+2\rangle \otimes |l-k+2, k\rangle \\
&= q^{-2}\frac{[2]}{[3]}([l+1][l+2])^2 \mathbf{v}_{l+2}.
\end{aligned}$$

□

For $l \in \mathbb{Z}_{\geq 0}$ and $l' = 2m \in 2\mathbb{Z}_{\geq 0}$, set

$$\mathbf{v}_{l,l'} = q_n^{\frac{m(m+2l+1)}{2}} f_n^{(m)} \mathbf{v}_l.$$

Note that $\mathbf{v}_{l,l'}$ may not be equal to $\widetilde{f}_n^m \mathbf{v}_l$ in the sense of (3.4) since $e_n \mathbf{v}_{l,l'} \neq 0$ in general.

Lemma 5.6. *For $l \in \mathbb{Z}_{\geq 0}$ and $l' \in 2\mathbb{Z}_{\geq 0}$ with $\mathrm{sgn}(l) = \varepsilon$, $\mathbf{v}_{l,l'}$ is a $U_q(A_{n-1})$-highest weight vector in $\mathcal{W}_\varepsilon^{(2)}$, and*

$$\mathbf{v}_{l,l'} \equiv |l\mathbf{e}_n\rangle \otimes |l\mathbf{e}_{n-1} + l'\mathbf{e}_n\rangle \pmod{q\mathcal{L}_\varepsilon^{\otimes 2}}.$$

Proof. Let us assume that l is even, and hence $\varepsilon = +$, since the proof for odd l is almost identical. Since e_j $(1 \leq j \leq n-1)$ commutes with f_n, it is clear that $\mathbf{v}_{l,l'}$ is a $U_q(A_{n-1})$-highest weight vector in $\mathcal{W}_\varepsilon^{(2)}$.

Let $l' = 2m$. For $0 \leq c \leq l$, we have

$$|c\mathbf{e}_{n-1} + (l-c)\mathbf{e}_n\rangle \equiv \begin{cases} \widetilde{f}_n^{\lfloor \frac{l-c}{2} \rfloor} |c\mathbf{e}_{n-1}\rangle & \text{if } c \text{ is even,} \\ \widetilde{f}_n^{\lfloor \frac{l-c}{2} \rfloor} |c\mathbf{e}_{n-1} + \mathbf{e}_n\rangle & \text{if } c \text{ is odd,} \end{cases} \qquad \pmod{q\mathcal{L}_+}.$$

Put $a = \lfloor \frac{l-c}{2} \rfloor$ and $b = \lfloor \frac{c}{2} \rfloor$.

Case 1. Suppose that c is even. Let

$$u_1 = |c\mathbf{e}_{n-1}\rangle, \quad u_2 = |(l-c)\mathbf{e}_{n-1}\rangle.$$

We have

$$\begin{aligned}
&\Delta(f_n^{(m)})(\widetilde{f}_n^a u_1 \otimes \widetilde{f}_n^b u_2) \\
&= \sum_{k=0}^m q_n^{-k(m-k)} f_n^{(m-k)} k_n^k (\widetilde{f}_n^a u_1) \otimes f_n^{(k)} (\widetilde{f}_n^b u_2) \\
&= \sum_{k=0}^m q_n^{-k(m-k)-(\frac{1}{2}+2a)k} f_n^{(m-k)} \widetilde{f}_n^a u_1 \otimes f_n^{(k)} \widetilde{f}_n^b u_2 \\
&= \sum_{k=0}^m q_n^{-k(m-k)-(\frac{1}{2}+2a)k+\frac{a^2}{2}+\frac{b^2}{2}} f_n^{(m-k)} f_n^{(a)} u_1 \otimes f_n^{(k)} f_n^{(b)} u_2 \\
&= \sum_{k=0}^m q_n^{-k(m-k)-(\frac{1}{2}+2a)k+\frac{a^2}{2}+\frac{b^2}{2}} \begin{bmatrix} m-k+a \\ a \end{bmatrix}_n \begin{bmatrix} k+b \\ b \end{bmatrix}_n f_n^{(m-k+a)} u_1 \otimes f_n^{(k+b)} u_2 \\
&= \sum_{k=0}^m q_n^{-k(m-k)-(\frac{1}{2}+2a)k+\frac{a^2}{2}+\frac{b^2}{2}-\frac{(m-k+a)^2}{2}-\frac{(k+b)^2}{2}} \\
&\qquad\qquad \begin{bmatrix} m-k+a \\ a \end{bmatrix}_n \begin{bmatrix} k+b \\ b \end{bmatrix}_n \widetilde{f}_n^{m-k+a} u_1 \otimes \widetilde{f}_n^{k+b} u_2 \\
&= \sum_{k=0}^m f_{a,b}(q) \widetilde{f}_n^{m-k+a} u_1 \otimes \widetilde{f}_n^{k+b} u_2.
\end{aligned}$$

Multiplying $q_n^{\frac{m(m+2l+1)}{2}}$ on both sides, we have $q_n^{\frac{m(m+2l+1)}{2}} f_{a,b}(q) \in q^d(1+qA_0)$, where

$$\begin{aligned} d &= m(m+2l+1) - 2k(m-k) - 2\left(\frac{1}{2}+2a\right)k \\ &\qquad + a^2 + b^2 - (m-k+a)^2 - (k+b)^2 - 2(m-k)a - 2kb \\ &= 2lm + (m-k) - 4ma - 4kb = 2lm + (m-k) - 4m\left(\frac{l-c}{2}\right) - 4k\left(\frac{c}{2}\right) \\ &= (m-k) + 2c(m-k) = (2c+1)(m-k) \end{aligned} \tag{5.3}$$

since $a = \frac{l-c}{2}$ and $b = \frac{c}{2}$.

Case 2. Suppose that c is odd. Let

$$u_1 = |c\mathbf{e}_{n-1} + \mathbf{e}_n\rangle, \quad u_2 = |(l-c)\mathbf{e}_{n-1} + \mathbf{e}_n\rangle.$$

We have

$$\begin{aligned} &\Delta(f_n^{(m)})(\widetilde{f}_n^a u_1 \otimes \widetilde{f}_n^b u_2) \\ &= \sum_{k=0}^m q_n^{-k(m-k)} f_n^{(m-k)} k_n^k (\widetilde{f}_n^a u_1) \otimes f_n^{(k)} (\widetilde{f}_n^b u_2) \\ &= \sum_{k=0}^m q_n^{-k(m-k)-(\frac{3}{2}+2a)} f_n^{(m-k)} \widetilde{f}_n^{(a)} u_1 \otimes f_n^{(k)} \widetilde{f}_n^{(b)} u_2 \\ &= \sum_{k=0}^m q_n^{-k(m-k)-(\frac{3}{2}+2a)k+\frac{a(a+2)}{2}+\frac{b(b+2)}{2}} f_n^{(m-k)} f_n^{(a)} u_1 \otimes f_n^{(k)} f_n^{(b)} u_2 \\ &= \sum_{k=0}^m q_n^{-k(m-k)-(\frac{3}{2}+2a)k+\frac{a(a+2)}{2}+\frac{b(b+2)}{2}} \begin{bmatrix} m-k+a \\ a \end{bmatrix}_n \begin{bmatrix} k+b \\ b \end{bmatrix}_n f_n^{(m-k+a)} u_1 \otimes f_n^{(k+b)} u_2 \\ &= \sum_{k=0}^m q_n^{-k(m-k)-(\frac{3}{2}+2a)k+\frac{a(a+2)}{2}+\frac{b(b+2)}{2}-\frac{(m-k+a)(m-k+a+2)}{2}-\frac{(k+b)(k+b+2)}{2}} \\ &\qquad\qquad\qquad \begin{bmatrix} m-k+a \\ a \end{bmatrix}_n \begin{bmatrix} k+b \\ b \end{bmatrix}_n \widetilde{f}_n^{m-k+a} u_1 \otimes \widetilde{f}_n^{k+b} u_2 \\ &= \sum_{k=0}^m g_{a,b}(q) \widetilde{f}_n^{m-k+a} u_1 \otimes \widetilde{f}_n^{k+b} u_2. \end{aligned}$$

Multiplying $q_n^{\frac{m(m+2l+1)}{2}}$ on both sides, we have $q_n^{\frac{m(m+2l+1)}{2}} g_{a,b}(q) \in q^{d'}(1+qA_0)$, where

$$\begin{aligned} d' &= d - 2k + 2a + 2b - 2(m-k+a) - 2(k+b) \\ &= d - 2k - 2m \\ &= 2lm + (m-k) - 4ma - 4kb - 2k - 2m \\ &= 2lm + (m-k) - 4m\left(\frac{l-c-1}{2}\right) - 4k\left(\frac{c-1}{2}\right) - 2k - 2m \\ &= (m-k) + 2c(m-k) = (2c+1)(m-k) \end{aligned} \tag{5.4}$$

by putting $a = \frac{l-c-1}{2}$ and $b = \frac{c-1}{2}$. By (5.3), (5.4), and Lemma 5.4, we have

$$q_n^{\frac{m(m+2l+1)}{2}} f_n^{(m)} \mathbf{v}_l \equiv |l\mathbf{e}_n\rangle \otimes |l\mathbf{e}_{n-1} + 2m\mathbf{e}_n\rangle \pmod{q\mathcal{L}_+^{\otimes 2}}.$$

□

Corollary 5.7. *The set* $\{\, \mathbf{v}_{l,l'} \,|\, l \in \mathbb{Z}_{\geq 0},\ l' \in 2\mathbb{Z}_{\geq 0},\ \mathrm{sgn}(l) = \varepsilon \,\}$ *is the set of* $U_q(A_{n-1})$*-highest weight vectors in* $\mathcal{W}_\varepsilon^{(2)}$.

Proof. The character of the $U_q(A_{n-1})$-submodule of $\mathcal{W}_\varepsilon$ generated by $\mathbf{v}_{l,l'}$ is $s_\lambda(x_1, \ldots, x_n)$ where $\lambda = (l' + l, l)$. Hence it follows from Proposition 5.3 that there is no other $U_q(A_{n-1})$-highest weight vectors in $\mathcal{W}_\varepsilon^{(2)}$. □

Now we define the pair $(\mathcal{L}_\varepsilon^{(2)}, \mathcal{B}_\varepsilon^{(2)})$ by

$$\mathcal{L}_\epsilon^{(2)} = \sum_{\substack{l_1 \in \mathbb{Z}_{\geq 0} \\ \mathrm{sgn}(l_1) = \varepsilon}} \sum_{l_2 \in 2\mathbb{Z}_{\geq 0}} \sum_{\substack{r \geq 0 \\ 1 \leq i_1, \ldots, i_r \leq n-1}} A_0 \widetilde{f}_{i_1} \ldots \widetilde{f}_{i_r} \mathbf{v}_{l_1, l_2},$$

$$\mathcal{B}_\epsilon^{(2)} = \Big\{ \widetilde{f}_{i_1} \ldots \widetilde{f}_{i_r} \mathbf{v}_{l_1,l_2} \pmod{q\mathcal{L}_\varepsilon^{(2)}} \,\Big|\, l_1 \in \mathbb{Z}_{\geq 0},\ \mathrm{sgn}(l_1) = \varepsilon,\ l_2 \in 2\mathbb{Z}_{\geq 0},\ r \geq 0,\ 1 \leq i_1, \ldots, i_r \leq n-1 \Big\} \setminus \{0\}.$$

Proposition 5.8. *We have*

(1) $\mathcal{L}_\varepsilon^{(2)} \subset \mathcal{L}_\varepsilon^{\otimes 2}$ *and* $\mathcal{B}_\varepsilon^{(2)} \subset \mathcal{B}_\varepsilon^{\otimes 2}$,

(2) $(\mathcal{L}_\varepsilon^{(2)}, \mathcal{B}_\varepsilon^{(2)})$ *is a* $U_q(A_{n-1})$*-crystal base of* $\mathcal{W}_\varepsilon^{(2)}$.

Proof. (1) By Proposition 3.8, $\mathcal{L}_\varepsilon^{\otimes 2}$ is a crystal base of $\mathcal{W}_\varepsilon^{\otimes 2}$ as a $U_q(A_{n-1})$-module, hence it is invariant under $\widetilde{f}_i$ for $1 \leq i \leq n-1$. By Lemma 5.6, we have $\widetilde{f}_{i_1} \ldots \widetilde{f}_{i_r} \mathbf{v}_{l_1,l_2} \in \mathcal{L}_\varepsilon^{\otimes 2}$ and hence $\widetilde{f}_{i_1} \ldots \widetilde{f}_{i_r} \mathbf{v}_{l_1,l_2} \in \mathcal{B}_\varepsilon^{\otimes 2} \pmod{q\mathcal{L}_\varepsilon^{\otimes 2}}$.

(2) By definition of $(\mathcal{L}_\varepsilon^{(2)}, \mathcal{B}_\varepsilon^{(2)})$ and Lemma 5.6, $(\mathcal{L}_\varepsilon^{(2)}, \mathcal{B}_\varepsilon^{(2)})$ is a $U_q(A_{n-1})$-crystal base of the submodule V of $\mathcal{W}_\epsilon^{(2)}$ generated by $\mathbf{v}_{l_1,l_2}$ for l_1, l_2. On the other hand, we have $V = \mathcal{W}_\varepsilon^{(2)}$ by Proposition 5.3. Hence $(\mathcal{L}_\varepsilon^{(2)}, \mathcal{B}_\varepsilon^{(2)})$ is a $U_q(A_{n-1})$-crystal base of $\mathcal{W}_\varepsilon^{(2)}$. □

For $|\mathbf{m}\rangle = |m_1, \ldots, m_n\rangle \in \mathcal{W}$, let $T(\mathbf{m})$ denote the semistandard tableau of shape $(|\mathbf{m}|)$, a single row of length $|\mathbf{m}|$, with letters in $\{\overline{n} < \cdots < \overline{1}\}$ such that the number of occurrences of $\overline{i}$ is m_i for $1 \leq i \leq n$.

Suppose that $|\mathbf{m}_1\rangle, \ldots, |\mathbf{m}_s\rangle$ are given such that $|\mathbf{m}_1| \leq \cdots \leq |\mathbf{m}_s|$. Let $\lambda = (|\mathbf{m}_s| \geq \cdots \geq |\mathbf{m}_1|)$, which is a partition or its Young diagram, and λ^π denote the Young diagram obtained by 180°-rotation of λ. We denote by $T(\mathbf{m}_1, \ldots, \mathbf{m}_s)$ the row-semistandard tableau of shape λ^π, whose j-th row from the top is equal to $T(\mathbf{m}_j)$ for $1 \leq j \leq s$.

Example 5.9. Suppose that $n = 5$. If $|\mathbf{m}_1\rangle = |2, 1, 0, 0, 2\rangle$ and $|\mathbf{m}_2\rangle = |0, 1, 2, 3, 1\rangle$, then

$$T(\mathbf{m}_1, \mathbf{m}_2) = \begin{array}{ccccccc} & & \overline{5} & \overline{5} & \overline{2} & \overline{1} & \overline{1} \\ \overline{5} & \overline{4} & \overline{4} & \overline{4} & \overline{3} & \overline{3} & \overline{2} \end{array}.$$

Proposition 5.10. *We have*

$$\mathcal{B}_\varepsilon^{(2)} = \Big\{ |\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle \pmod{q\mathcal{L}_\varepsilon^{(2)}} \,\Big|\, |\mathbf{m}_1| \leq |\mathbf{m}_2|,\ T(\mathbf{m}_1, \mathbf{m}_2) \text{ is semistandard} \Big\}.$$

Proof. For $l_1 \in \mathbb{Z}_{\geq 0}$ and $l_2 \in 2\mathbb{Z}_{\geq 0}$ with $\mathrm{sgn}(l_1) = \varepsilon$, let us identify $\mathbf{v}_{l_1,l_2} = |l_1\mathbf{e}_n\rangle \otimes |l_1\mathbf{e}_{n-1} + l_2\mathbf{e}_n\rangle$ in $\mathcal{B}_\varepsilon^{\otimes 2}$ with the pair $(l_1\mathbf{e}_n, l_1\mathbf{e}_{n-1} + l_2\mathbf{e}_n)$ and the connected component of $\mathbf{v}_{l_1,l_2}$ as a $U_q(A_{n-1})$-crystal with the set of corresponding set of pairs $(\mathbf{m}_1, \mathbf{m}_2)'s$. Then $T(\mathbf{v}_{l_1,l_2})$ is the semistandard tableau of shape $(l_1 + l_2, l_1)^\pi$. Since $\widetilde{e}_j\mathbf{v}_{l_1,l_2} = 0$ for $1 \leq j \leq n-1$, $T(\mathbf{v}_{l_1,l_2})$ is the tableau of highest weight and the set

$$\left\{ T\left(\widetilde{f}_{i_1}\dots\widetilde{f}_{i_r}\mathbf{v}_{l_1,l_2}\right) \,\middle|\, r \geq 0, 1 \leq i_1, \dots, i_r \leq n-1 \right\} \setminus \{0\} \tag{5.5}$$

is equal to the set of semistandard tableau of shape $(l_1 + l_2, l_1)^\pi$ with letters $\{\overline{n} < \cdots < \overline{1}\}$. □

Let $|\mathbf{m}_1\rangle, |\mathbf{m}_2\rangle \in \mathcal{B}_\varepsilon$ be given with $|\mathbf{m}_1| = d_1$ and $|\mathbf{m}_2| = d_2$, let $P(\mathbf{m}_1, \mathbf{m}_2)$ denote a unique semistandard tableau of shape μ^π for some partition μ, which is equivalent to $|\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle$ as an element of $U_q(A_{n-1})$-crystals. Indeed, if we read the row word of $T(\mathbf{m}_1)$ from left to right, and then apply the Schensted's column insertion to $T(\mathbf{m}_2)$ in a reverse way starting from the right-most column, then the resulting tableau is $P(\mathbf{m}_1, \mathbf{m}_2)$. So $P(\mathbf{m}_1, \mathbf{m}_2)$ is of shape $(d_2', d_1')^\pi$ for some $d_1' \leq d_2'$ with $d_1' \leq d_1$, $d_2' \geq d_2$, and $d_1' + d_2' = d_1 + d_2$. In particular, $P(\mathbf{m}_1, \mathbf{m}_2) = T(\mathbf{m}_1, \mathbf{m}_2)$ if $d_1 \leq d_2$ and $|\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle \in \mathcal{B}_\varepsilon^{(2)}$.

Example 5.11. Let $|\mathbf{m}_1\rangle, |\mathbf{m}_2\rangle$ be as in Example 5.9. Then

$$P(\mathbf{m}_1, \mathbf{m}_2) = \begin{array}{cccccccccc} & & & & & & & & \overline{5} & \overline{5} \\ \overline{5} & \overline{4} & \overline{4} & \overline{4} & \overline{3} & \overline{3} & \overline{2} & \overline{2} & \overline{1} & \overline{1} \end{array}.$$

Let $l_1 \in \mathbb{Z}_{\geq 0}$ and $l_2 \in 2\mathbb{Z}_{\geq 0}$ be given with $\mathrm{sgn}(l_1) = \varepsilon$. Put $\lambda = (\lambda_1, \lambda_2) = (l_1 + l_2, l_1)$. Let $SST(\lambda^\pi)$ be the set of semistandard tableaux of shape λ^π with letters in $\{\,\overline{n} < \cdots < \overline{1}\,\}$. For each $T \in SST(\lambda^\pi)$, we choose $i_1, \dots, i_r \in I \setminus \{0, n\}$ such that $T = T\left(\widetilde{f}_{i_1}\dots\widetilde{f}_{i_r}\mathbf{v}_{l_1,l_2}\right)$ (see (5.5)), and define

$$\mathbf{v}_T = \widetilde{f}_{i_1}\dots\widetilde{f}_{i_r}\mathbf{v}_{l_1,l_2} \in \mathcal{L}_\varepsilon^{(2)}. \tag{5.6}$$

By Proposition 5.10, we have a $\mathbb{Q}(q)$-basis of $\mathcal{W}_\varepsilon^{(2)}$

$$\bigsqcup_{\substack{\lambda \in \mathscr{P}_\varepsilon \\ \ell(\lambda) \leq 2}} \{\, \mathbf{v}_T \,|\, T \in SST(\lambda^\pi)\,\}\,. \tag{5.7}$$

Lemma 5.12. *For $T \in SST(\lambda^\pi)$, we have*

$$\mathbf{v}_T = |\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle + \sum_{\mathbf{m}_1', \mathbf{m}_2'} c_{\mathbf{m}_1', \mathbf{m}_2'} |\mathbf{m}_1'\rangle \otimes |\mathbf{m}_2'\rangle,$$

where $P(\mathbf{m}_1, \mathbf{m}_2) = T$, $P(\mathbf{m}_1', \mathbf{m}_2')$ is of shape μ^π with $\mu \rhd \lambda$ and $\mu \neq \lambda$, and $c_{\mathbf{m}_1', \mathbf{m}_2'} \in qA_0$. Here $\rhd$ denotes a dominance order on partitions, that is, $\mu_1 > \lambda_1$, and $\mu_1 + \mu_2 = \lambda_1 + \lambda_2$.

Proof. By Lemmas 5.4 and 5.6 (see also their proofs), we observe that

$$\mathbf{v}_{l_1,l_2} = |l_1\mathbf{e}_n\rangle \otimes |l_1\mathbf{e}_{n-1} + l_2\mathbf{e}_n\rangle + \sum c_{x,y,z,w} |x\mathbf{e}_{n-1} + y\mathbf{e}_n\rangle \otimes |z\mathbf{e}_{n-1} + w\mathbf{e}_n\rangle, \tag{5.8}$$

where the sum is over (x, y, z, w) such that

(1) $0 < x \leq l_1$ with $x + z = l_1$,

(2) $y \geq z$, $w \geq x$ with $y + w = l_1 + l_2$,

(3) $c_{x,y,z,w} \in qA_0$.

We may regard $|l_1\mathbf{e}_n\rangle \otimes |l_1\mathbf{e}_{n-1} + l_2\mathbf{e}_n\rangle$ as the case when $(x, y, z, w) = (0, l_1, l_1, l_2)$. Then it is not difficult to see that if the shape of $P(x\mathbf{e}_{n-1} + y\mathbf{e}_n, z\mathbf{e}_{n-1} + w\mathbf{e}_n)$ is $\mu^\pi = (\mu_1, \mu_2)^\pi$, then $\mu_2 = z = l_1 - x \leq l_1$ and hence $\mu \rhd \lambda$, and $\mu \neq \lambda$ when $x > 0$.

Let $i_1, \ldots, i_r \in I \setminus \{0, n\}$ be the sequence in (5.6). By the tensor product rule of crystals, we have

$$\widetilde{f}_{i_1} \ldots \widetilde{f}_{i_r}(|x\mathbf{e}_{n-1} + y\mathbf{e}_n\rangle \otimes |z\mathbf{e}_{n-1} + w\mathbf{e}_n\rangle) = \sum_{\mathbf{m}_1, \mathbf{m}_2} c_{\mathbf{m}_1,\mathbf{m}_2} |\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle, \tag{5.9}$$

where the sum is over $\mathbf{m}_1, \mathbf{m}_2$ such that

(1) $c_{\mathbf{m}_1,\mathbf{m}_2}(q) \in A_0$ such that

$$c_{\mathbf{m}_1,\mathbf{m}_2}(0) = \begin{cases} 1 & \text{if } |\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle = \widetilde{f}_{i_1} \ldots \widetilde{f}_{i_r}(|x\mathbf{e}_{n-1} + y\mathbf{e}_n\rangle \otimes |z\mathbf{e}_{n-1} + w\mathbf{e}_n\rangle), \\ 0 & \text{otherwise.} \end{cases}$$

(2) $\nu \rhd \lambda$ and $\nu \neq \lambda$, where ν^π is the shape of $P(\mathbf{m}_1, \mathbf{m}_2)$.

Therefore, we obtain the result by (5.8) and (5.9). □

Corollary 5.13. *We have* $\mathcal{L}_\varepsilon^{(2)} = \mathcal{L}_\varepsilon^{\otimes 2} \cap \mathcal{W}_\varepsilon^{(2)}$.

Proof. It is clear that $\mathcal{L}_\varepsilon^{(2)} \subset \mathcal{L}_\varepsilon^{\otimes 2} \cap \mathcal{W}_\varepsilon^{(2)}$ by Proposition 5.8. Conversely, suppose that $v \in \mathcal{L}_\varepsilon^{\otimes 2} \cap \mathcal{W}_\varepsilon^{(2)}$ is given. By (5.7), we have

$$v = \sum_T c_T \mathbf{v}_T, \tag{5.10}$$

for some $c_T \in \mathbb{Q}(q)$. We may assume that all the shape of T in (5.10) is the same. Fix T with $c_T \neq 0$. Let $|\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle$ be such that $|\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle$ appears in (5.10) with non-zero coefficient, and $P(\mathbf{m}_1, \mathbf{m}_2) = T$. By Lemma 5.12, the coefficient of $|\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle$ is c_T. Hence $c_T \in A_0$, and $v \in \mathcal{L}_\varepsilon^{(2)}$. □

Proof of Theorem 5.1. Let $\mathcal{W}_\varepsilon^{\otimes 2} = \mathcal{W}_\varepsilon^{(2)} \oplus W$, where W is the complement of $\mathcal{W}_\varepsilon^{(2)}$ in $\mathcal{W}_\varepsilon^{\otimes 2}$ as a $U_q(A_{n-1})$-module since it is completely reducible. By Corollary 5.13, we have

$$\mathcal{L}_\varepsilon^{\otimes 2} = \mathcal{L}_\varepsilon^{(2)} \oplus \mathcal{M}^{(2)}, \tag{5.11}$$

where $\mathcal{M}^{(2)} = \mathcal{L}_\varepsilon^{\otimes 2} \cap W$ is the crystal lattice of W as a $U_q(A_{n-1})$-module. Then we have

$$\check{R}_2(\mathcal{L}_\varepsilon^{\otimes 2}) \subset \mathcal{L}_\varepsilon^{(2)}, \quad \check{R}_2|_{q=0}(\mathcal{B}_\varepsilon^{\otimes 2}) \subset \mathcal{B}_\varepsilon^{(2)}. \tag{5.12}$$

More generally, by (4.1) and (5.11), we have for $a \in \mathbb{Z}_{>0}$

$$\check{R}(q^{-2a})(\mathcal{L}_\varepsilon^{\otimes 2}) \subset \mathcal{L}_\varepsilon^{\otimes 2}. \tag{5.13}$$

For each $1 \leq i \leq s-1$, we have

$$\check{R}_s = \check{R}_{s_i}(\cdots, \underbrace{q^{s-2i-1}}_{i}, \underbrace{q^{s-2i+1}}_{i+1}, \cdots) \check{R}_{w_0 s_i}(q^{1-s}, \cdots, q^{s-1}).$$

We have $\check{R}_{w_0 s_i}(q^{1-s},\cdots,q^{s-1})(\mathcal{L}_\varepsilon^{\otimes s})\subset\mathcal{L}_\varepsilon^{\otimes s}$ by (5.13), and hence by (5.12)

$$\check{R}_s(\mathcal{L}_\varepsilon^{\otimes s})\subset\mathcal{L}_\varepsilon^{\otimes i-1}\otimes\mathcal{L}_\varepsilon^{(2)}\otimes\mathcal{L}_\varepsilon^{\otimes s-i-1},$$
$$\check{R}_s(\mathcal{B}_\varepsilon^{\otimes s})\subset\mathcal{B}_\varepsilon^{\otimes i-1}\otimes\mathcal{B}_\varepsilon^{(2)}\otimes\mathcal{B}_\varepsilon^{\otimes s-i-1}.$$

Therefore $\check{R}_s(\mathcal{B}_\varepsilon^{\otimes s})$ is spanned by $\mathcal{B}_\varepsilon^{(s)}$, where

$$\mathcal{B}_\varepsilon^{(s)}=\left\{\,|\mathbf{m}_1\rangle\otimes\ldots\otimes|\mathbf{m}_s\rangle\pmod{q\mathcal{L}_\varepsilon^{\otimes s}}\,\middle|\,|\mathbf{m}_j\rangle\otimes|\mathbf{m}_{j+1}\rangle\in\mathcal{B}_\varepsilon^{(2)}\ (1\le j\le s-2)\,\right\}.$$

By Proposition 5.10, the set

$$\left\{\,T(\mathbf{m}_1,\ldots,\mathbf{m}_s)\,\middle|\,|\mathbf{m}_1\rangle\otimes\ldots\otimes|\mathbf{m}_s\rangle\in\mathcal{B}_\varepsilon^{(s)}\,\right\}$$

is equal to the set of semistandard tableau of shape λ^π where $\lambda=(|\mathbf{m}_s|\geq\cdots\geq|\mathbf{m}_1|)$. Hence

$$\mathrm{ch}\mathcal{W}_\varepsilon^{(s)}=\sum_{\substack{\lambda\in\mathscr{P}_\varepsilon\\ \ell(\lambda)\le s}}s_\lambda(x_1,\ldots,x_n).\tag{5.14}$$

Let $V_0^{(s)}$ be the $U_q(C_n)$-submodule of $\mathcal{W}_\varepsilon^{(s)}$ generated $|\varsigma(\varepsilon)\mathbf{e}_n\rangle^{\otimes s}$. The classical limit $\overline{V_0^{(s)}}$ of $V_0^{(s)}$ is a highest weight $U(C_n)$-module with highest weight

$$\Lambda^{(s)}:=-s(\frac{1}{2}+\varsigma(\varepsilon))\varpi_n.$$

On the other hand, by [17, Theorem 6.1] the character of the irreducible highest weight $U(C_n)$-module with highest weight $\Lambda^{(s)}$, say $V(\Lambda^{(s)})$, is also equal to (5.14). Since $V(\Lambda^{(s)})$ is a quotient of $\overline{V_0^{(s)}}$, we conclude that

$$\mathrm{ch}\mathcal{W}_\varepsilon^{(s)}=\mathrm{ch}V_0^{(s)}=\mathrm{ch}\overline{V_0^{(s)}}=\mathrm{ch}V(\Lambda^{(s)}).$$

In particular, $V_0^{(s)}$ is an irreducible $U_q(C_n)$-module and hence $\mathcal{W}_\varepsilon^{(s)}=V_0^{(s)}$ is an irreducible $U_q(C_n^{(1)})$-module. This completes the proof. □

5.2. **Type** $C^{(2)}(n+1)$**.** Let us prove that $\mathcal{W}^{(s)}$ is an irreducible $U_q(C^{(2)}(n+1))$-module. The proof is similar to that of Theorem 5.1 for $U_q(C_n^{(1)})$. So we give a sketch of the proof and leave the details to the reader.

We first consider $\mathcal{W}^{(2)}$. By (4.2), we have

$$\mathcal{W}^{(2)}=V_0=U_q(osp_{1|2n})|\mathbf{0}\rangle\otimes|\mathbf{0}\rangle,\tag{5.15}$$

which is an irreducible representation of $U_q(osp_{1|2n})$ and hence of $U_q(C^{(1)}(n+1))$. By similar arguments as in Proposition 5.3, we have the following.

Proposition 5.14. *We have*

$$\mathrm{ch}\mathcal{W}^{(2)}=\mathrm{ch}V_0=\sum_{\substack{\lambda\in\mathscr{P}\\ \ell(\lambda)\le 2}}s_\lambda(x_1,\ldots,x_n).$$

Lemma 5.15. *For $l\in\mathbb{Z}_{\geq 0}$, let $\mathbf{v}_l$ be the vector in* (5.1)*. Then $\mathbf{v}_l$ is a $U_q(A_{n-1})$-highest weight vector in $\mathcal{W}^{(2)}$, and $\mathbf{v}_l\equiv|l\mathbf{e}_n\rangle\otimes|l\mathbf{e}_{n-1}\rangle\pmod{q^{\frac{1}{2}}\mathcal{L}^{\otimes 2}}$.*

Proof. Since the actions of Chevalley generators for $1 \le i \le n-1$ is the same as in the case of $C_n^{(1)}$, it follows from Lemma 5.4 that $\mathbf{v}_l$ is a $U_q(A_{n-1})$-highest weight vector. Note that

$$\mathrm{ch}\mathcal{W}^{\otimes 2} = (\mathrm{ch}\mathcal{W})^2 = \sum_{\ell(\lambda)\le 2} m_\lambda s_\lambda(x_1,\dots,x_n), \tag{5.16}$$

where $m_\lambda = \lambda_1 - \lambda_2$. Then we have $\mathbf{v}_l \in \mathcal{W}^{(2)}$ by the same argument as in Lemma 5.4. □

We have an analogue of Lemma 5.5, which also proves that $\mathbf{v}_l \in \mathcal{W}^{(2)}$.

Lemma 5.16. *Set $\mathcal{E} = e_{n-2}\cdots e_1e_0$, where it is understood as e_0 when $n=2$. Then for $l \ge 0$ we have*

$$(\mathcal{E}e_{n-1}\mathcal{E} - \frac{1}{[2]}e_{n-1}\mathcal{E}^2)\mathbf{v}_l = (-1)^l q^{-5/2}\frac{(1+q)}{[2]}[l+1]^2\mathbf{v}_{l+1}.$$

Lemma 5.17. *For $l,m \in \mathbb{Z}_{\ge 0}$, let*

$$\mathbf{v}_{l,m} = q_n^{\frac{m(m+4l+3)}{2}} f_n^{(m)}\mathbf{v}_l.$$

Then $\mathbf{v}_{l,m}$ is a $U_q(A_{n-1})$-highest weight vector in $\mathcal{W}^{(2)}$, and

$$\mathbf{v}_{l,m} \equiv |l\mathbf{e}_n\rangle \otimes |l\mathbf{e}_{n-1} + m\mathbf{e}_n\rangle \pmod{q^{\frac12}\mathcal{L}^{\otimes 2}}.$$

Proof. Since e_j for $1 \le j \le n-1$ commutes with f_n, $\mathbf{v}_{l,m}$ is a $U_q(A_{n-1})$-highest weight vector in $\mathcal{W}_\epsilon^{(2)}$.

For $0 \le c \le l$, put $a = l-c$ and $b = c$. Let

$$u_1 = |c\mathbf{e}_{n-1}\rangle, \quad u_2 = |(l-c)\mathbf{e}_{n-1}\rangle.$$

By (2.4), we have

$$\begin{aligned}
&\Delta(f_n^{(m)})(\widetilde{f}_n^a u_1 \otimes \widetilde{f}_n^b u_2)\\
&= \sum_{k=0}^m \sigma^k q_n^{-k(m-k)} f_n^{(m-k)}k_n^k(\widetilde{f}_n^a u_1)\otimes f_n^{(k)}(\widetilde{f}_n^b u_2)\\
&= \sum_{k=0}^m \sigma^k q_n^{-k(m-k)-(1+2a)k} f_n^{(m-k)}\widetilde{f}_n^a u_1\otimes f_n^{(k)}\widetilde{f}_n^b u_2\\
&= \sum_{k=0}^m \sigma^k q_n^{-k(m-k)-(1+2a)k+\frac{a(a+1)}{2}+\frac{b(b+1)}{2}} f_n^{(m-k)}f_n^{(a)} u_1\otimes f_n^{(k)}f_n^{(b)} u_2\\
&= \sum_{k=0}^m \sigma^k q_n^{-k(m-k)-(1+2a)k+\frac{a(a+1)}{2}+\frac{b(b+1)}{2}} \begin{bmatrix} m-k+a \\ a\end{bmatrix}_n \begin{bmatrix} k+b \\ b\end{bmatrix}_n f_n^{(m-k+a)} u_1\otimes f_n^{(k+b)} u_2\\
&= \sum_{k=0}^m \sigma^k q_n^{-k(m-k)-(1+2a)k+\frac{a(a+1)}{2}+\frac{b(b+1)}{2}-\frac{(m-k+a)(m-k+a+1)}{2}-\frac{(k+b)(k+b+1)}{2}}\\
&\qquad\qquad \begin{bmatrix} m-k+a \\ a\end{bmatrix}_n \begin{bmatrix} k+b \\ b\end{bmatrix}_n \widetilde{f}_n^{m-k+a} u_1\otimes \widetilde{f}_n^{k+b} u_2\\
&= \sum_{k=0}^m \sigma^k f_{a,b}(q)\widetilde{f}_n^{m-k+a} u_1\otimes \widetilde{f}_n^{k+b} u_2.
\end{aligned}$$

Multiplying $q_n^{\frac{m(m+4l+3)}{2}}$ on both sides, it is straightforward to see that

$$q_n^{\frac{m(m+4l+3)}{2}} f_{a,b}(q) \in q_n^{(2c+1)(m-k)}(1 + q^{\frac{1}{2}} A_0).$$

This implies that $\mathbf{v}_{l,m} \equiv |l\mathbf{e}_n\rangle \otimes |l\mathbf{e}_{n-1} + m\mathbf{e}_n\rangle \pmod{q^{\frac{1}{2}}\mathcal{L}^{\otimes 2}}$. □

Now we define the pair $(\mathcal{L}^{(2)}, \mathcal{B}^{(2)})$ by

$$\mathcal{L}^{(2)} = \sum_{l_1,l_2\in\mathbb{Z}_{\geq 0}} \sum_{\substack{r\geq 0\\ 1\leq i_1,\dots,i_r\leq n-1}} A_0 \widetilde{f}_{i_1}\dots\widetilde{f}_{i_r}\mathbf{v}_{l_1,l_2},$$

$$\mathcal{B}^{(2)} = \left\{ \widetilde{f}_{i_1}\dots\widetilde{f}_{i_r}\mathbf{v}_{l_1,l_2} \pmod{q^{\frac{1}{2}}\mathcal{L}^{(2)}} \,\middle|\, l_1,l_2\in\mathbb{Z}_{\geq 0},\ r\geq 0,\ 1\leq i_1,\dots,i_r\leq n-1 \right\} \setminus \{0\}.$$

Proposition 5.18. *We have*

(1) $\mathcal{L}^{(2)} \subset \mathcal{L}^{\otimes 2}$ *and* $\mathcal{B}^{(2)} \subset \mathcal{B}^{\otimes 2}$,

(2) $(\mathcal{L}^{(2)}, \mathcal{B}^{(2)})$ *is a* $U_q(A_{n-1})$*-crystal base of* $\mathcal{W}^{(2)}$*, where*

$$\mathcal{B}^{(2)} = \left\{ |\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle \pmod{q^{\frac{1}{2}}\mathcal{L}^{(2)}} \,\middle|\, |\mathbf{m}_1| \leq |\mathbf{m}_2|,\ T(\mathbf{m}_1,\mathbf{m}_2) \text{ is semistandard} \right\}.$$

Proof. It follows from the same arguments as in Propositions 5.8 and 5.10. □

Corollary 5.19. *We have* $\mathcal{L}^{(2)} = \mathcal{L}^{\otimes 2} \cap \mathcal{W}^{(2)}$.

Proof. By Proposition 5.18, one can check that Lemma 5.12 also holds for $\mathcal{W}^{(2)}$, which implies $\mathcal{L}^{(2)} = \mathcal{L}^{\otimes 2} \cap \mathcal{W}^{(2)}$. □

Theorem 5.20. *For* $s \geq 2$, $\mathcal{W}^{(s)}$ *is an irreducible* $U_q(C^{(2)}(n+1))$*-module, which is also irreducible as a* $U_q(osp_{1|2n})$*-module. Moreover, its character is given by*

$$\mathrm{ch}\mathcal{W}^{(s)} = \sum_{\substack{\lambda\in\mathscr{P}\\ \ell(\lambda)\leq s}} s_\lambda(x_1,\dots,x_n).$$

Proof of Theorem 5.20. We may apply the same arguments as in Theorem 5.1 and the result in [17, Theorem 6.1] by using Proposition 5.18 and Corollary 5.19. □

Corollary 5.21. *The character of* $\mathcal{W}_\varepsilon^{(s)}$ *has a stable limit for* $s \geq n$ *as follows:*

$$\mathrm{ch}\mathcal{W}^{(s)} = \sum_{\substack{\lambda\in\mathscr{P}\\ \ell(\lambda)\leq n}} s_\lambda(x_1,\dots,x_n) = \frac{1}{\prod_{1\leq i\leq n}(1-x_i)\prod_{1\leq i<j\leq n}(1-x_ix_j)}.$$

5.3. **Type** $B^{(1)}(0,n)$**.** As usual, we identify the weight lattice for $U_q(B(0,n))$ with $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}\mathbf{e}_i$ equipped with the standard symmetric bilinear form such that $(\mathbf{e}_i,\mathbf{e}_j) = \delta_{ij}$. Then the simple roots α_i $(i \in I\setminus\{0\})$ are given by $\boldsymbol{\alpha}_i = \mathbf{e}_{i+1} - \mathbf{e}_i$ for $1 \leq i \leq n-1$ and $\boldsymbol{\alpha}_n = -\mathbf{e}_n$, and $\varpi_n = -\frac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n)$.

For $\lambda \in \mathscr{P}_+$ with $\ell(\lambda) \leq \min\{n, s/2\}$, we put

$$\Lambda_\lambda^{(s)} = -s\varpi_n + \sum_{i=1}^n \lambda_i \mathbf{e}_{n-i+1}.$$

Let $V(\Lambda_\lambda^{(s)})$ be the irreducible highest weight $U(osp_{1|2n})$-module with highest weight $\Lambda_\lambda^{(s)}$. Note that $\Lambda_{(l)}^{(2)}$ is the weight of the maximal vector v_l and $V(\Lambda_{(l)}^{(2)}) = V_l$ for $l \geq 0$. Generalizing the decomposition of $\mathcal{W}^{(2)}$ into $U_q(osp_{1|2n})$-modules, we have the following conjecture on $\mathcal{W}^{(s)}$.

Conjecture 5.22. *For $s \geq 2$, $\mathcal{W}^{(s)}$ is an irreducible $U_q(B^{(1)}(0,n))$-module and its character is given by*

$$\mathrm{ch}\mathcal{W}^{(s)} = \sum_{\substack{\lambda \in \mathscr{P}_+ \\ \ell(\lambda) \leq \min\{n, s/2\}}} \mathrm{ch}V(\Lambda_\lambda^{(s)}).$$

Remark 5.23. The family of infinite-dimensional $U(osp_{1|2n})$-modules $V(\Lambda_\lambda^{(s)})$ have been introduced in [2] in connection with Howe duality. They are unitarizable and form a semisimple tensor category. The Weyl-Kac type character formula for $V(\Lambda_\lambda^{(s)})$ can be found in [2, Theorem 6.13].

Corollary 5.24. *For $s \geq 2n$, we have*

$$\begin{aligned}\mathrm{ch}\mathcal{W}^{(s)} &= \frac{\sum_{\lambda \in \mathscr{P}_+} s_\lambda(x_1, \ldots, x_n)}{\prod_{1 \leq i \leq n}(1 - x_i)\prod_{1 \leq i < j \leq n}(1 - x_i x_j)} \\ &= \frac{1}{\prod_{1 \leq i \leq n}(1 - x_i)(1 - x_i^2)\prod_{1 \leq i < j \leq n}(1 - x_i x_j)^2}.\end{aligned}$$

Proof. The first equation follows from the fact [17, Corollary 6.6] that if $\lambda \in \mathscr{P}_+$ with $\ell(\lambda) \leq n$, then

$$\mathrm{ch}V(\Lambda_\lambda^{(s)}) = \frac{s_\lambda(x_1, \ldots, x_n)}{\prod_{1 \leq i \leq n}(1 - x_i)\prod_{1 \leq i < j \leq n}(1 - x_i x_j)}.$$

The second one follows from the well-known Littlewood identity. □

Appendix A. Twistor

In this appendix, we review the twistor introduced in [4] that relate quantum groups to quantum supergroups. We use it to relate the q-oscillator representation of $U_q(D_{n+1}^{(2)})$ in [13] to a representation of $U_q(C^{(2)}(n+1))$. An advantage to do so is that in the latter we can take a classical limit $q \to 1$. We also obtain a representation of $U_q(B^{(1)}(0,n))$ from the q-oscillator representation of $U_q(A_{2n}^{(2)\dagger})$, where $A_{2n}^{(2)\dagger}$ is the same Dynkin diagram as $A_{2n}^{(2)}$ in [8] but the labeling of nodes are opposite.

A.1. **The twistor of the covering quantum group.** We review the covering quantum group and the twistor map introduced in [4]. Our notations for a Cartan datum is closer to Kac's book [8]. Let I be the index set of the Dynkin diagram, $\{\alpha_i\}_{i \in I}$ the set of simple roots, $(a_{ij})_{i,j \in I}$ the Cartan matrix. The symmetric bilinear form $(\cdot, \cdot)$ on the weight lattice is normalized so that it satisfies $d_i = (\alpha_i, \alpha_i)/2 \in \mathbb{Z}$ for any $i \in I$. It is also assumed that $a_{ij} \in 2\mathbb{Z}$ if $d_i \equiv 1 \pmod 2$ and $j \in I$. The parity function $p(i)$ taking values in $\{0, 1\}$ is consistent with d_i, namely, $p(i) \equiv d_i \pmod 2$. We set $q_i = q^{d_i}, \pi_i = \pi^{d_i}$.

Let q, π be indeterminates and $\mathbf{i} = \sqrt{-1}$. For a ring R with 1, we set $R^\pi = R[\pi]/(\pi^2 - 1)$. The covering quantum group $\mathbf{U}$ associated to a Cartan datum is the $\mathbb{Q}^\pi(q, \mathbf{i})$-algebra with generators $E_i, F_i, K_i^{\pm 1}, J_i^{\pm 1}$ for $i \in I$ subject to the following relations.

$$\begin{aligned}
&J_iJ_j = J_jJ_i, \quad K_iK_j = K_jK_i, \quad J_iK_j = K_jJ_i,\\
&J_iE_j = \pi^{a_{ij}}E_jJ_i, \quad J_iF_j = \pi^{a_{ij}}F_jJ_i,\\
&K_iE_j = q^{a_{ij}}E_jK_i, \quad K_iF_j = q^{a_{ij}}F_jK_i,\\
&E_iF_j - \pi^{p(i)p(j)}F_jE_i = \delta_{ij}\frac{J_iK_i - K_i^{-1}}{\pi_iq_i - q_i^{-1}},\\
&\sum_{l=0}^{1-a_{ij}} (-1)^l \pi^{l(l-1)p(i)/2+lp(i)p(j)} \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i,\pi_i} E_i^{1-a_{ij}-l}E_jE_i^l = 0 \quad (i \neq j),\\
&\sum_{l=0}^{1-a_{ij}} (-1)^l \pi^{l(l-1)p(i)/2+lp(i)p(j)} \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i,\pi_i} F_i^{1-a_{ij}-l}F_jF_i^l = 0 \quad (i \neq j).
\end{aligned}$$

Remark A.1. We changed the notations from [4]. We replaced v with q, $\mathbf{t}$ with $\mathbf{i}$, J_{d_ii} and K_{d_ii} with J_i and K_i.

We extend $\mathbf{U}$ by introducing generators T_i, Υ_i for $i \in I$. They commute with each other and with J_i, K_i. They also have the commutation relations with E_i, F_i as

$$T_iE_j = \mathbf{i}^{a_{ij}}E_jT_i, \quad T_iF_j = \mathbf{i}^{-a_{ij}}F_jT_i, \quad \Upsilon_iE_j = \mathbf{i}^{\phi_{ij}}E_j\Upsilon_i, \quad \Upsilon_iF_j = \mathbf{i}^{-\phi_{ij}}F_j\Upsilon_i$$

where

$$\phi_{ij} = \begin{cases} d_ia_{ij} & \text{if } i > j,\\ d_i & \text{if } i = j,\\ -2p(i)p(j) & \text{if } i < j.\end{cases}$$

We denote this extended algebra by $\widehat{\mathbf{U}}$.

Theorem A.2 ([4])**.** *There is a $\mathbb{Q}(\mathbf{i})$-algebra automorphism $\widehat{\Psi}$ on $\widehat{\mathbf{U}}$ such that*

$$\begin{aligned}
&E_i \mapsto \mathbf{i}^{-d_i}\Upsilon_i^{-1}T_iE_i, \quad && F_i \mapsto F_i\Upsilon_i, \quad && K_i \mapsto T_iK_i,\\
&J_i \mapsto T_i^2J_i, && T_i \mapsto T_i, && \Upsilon_i \mapsto \Upsilon_i,\\
&q \mapsto \mathbf{i}^{-1}q, && \pi \mapsto -\pi.
\end{aligned}$$

A.2. **Image of the twistor $\widehat{\Psi}$.** We apply the twistor $\widehat{\Psi}$ given in the previous subsection for the Cartan datum corresponding to B_n, namely, $I = \{1, 2, \dots, n\}$ and the Cartan matrix is given by

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & & & \\ & & \ddots & & \\ & & & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}.$$

Through it, we are to regard the q-oscillator representation $\mathcal{W} = \bigoplus_{\mathbf{m}} \mathbb{Q}(q^{\frac{1}{2}})|\mathbf{m}\rangle$ of $U_q(B_n)$ (subalgebra of $U_q(D_{n+1}^{(2)})$ generated by e_i, f_i, k_i for $1 \leq i \leq n$) given in Proposition 1 of [13] as a representation of $U_q(osp_{1|2n})$. Although we normalized the symmetric bilinear form on the weight lattice so that $(\alpha_i,\alpha_i)\in\mathbb{Z}$ for any $i\in I$ in the previous subsection, we renormalize it so that $(\alpha_n,\alpha_n)=\frac{1}{2}$ to adjust it to the notations in [13]. The generators T_i,Υ_i are represented on $\mathcal{W}$ as

$$T_i|\mathbf{m}\rangle=\begin{cases}\mathbf{i}^{m_{i+1}-m_i}|\mathbf{m}\rangle & (1\le i<n)\\ \mathbf{i}^{-2m_n}|\mathbf{m}\rangle & (i=n)\end{cases},\quad \Upsilon_i|\mathbf{m}\rangle=\begin{cases}\mathbf{i}^{-2m_i}|\mathbf{m}\rangle & (1\le i<n)\\ \mathbf{i}^{|\mathbf{m}|-2m_n}|\mathbf{m}\rangle & (i=n)\end{cases}.$$

Let u_i ($i\in I$, $u=e,f,k$) be the generators of $U_q(B_n)$ ($\pi=1$) and $\bar{u}_i=\hat{\Psi}(u_i)$ be the image ($\pi=-1$) of the twistor $\hat{\Psi}$. Then $\bar{u}_i$ satisfy the relations for $U_{\bar{q}}(osp_{1|2n})$ where $\bar{q}^{\frac{1}{2}}=\mathbf{i}^{-1}q^{\frac{1}{2}}$. On the space $\mathcal{W}$, they act as follows.

$$\begin{aligned}\bar{e}_i|\mathbf{m}\rangle&=\mathbf{i}^{2m_{i+1}}[m_i]|\mathbf{m}-\mathbf{e}_i+\mathbf{e}_{i+1}\rangle,\\ \bar{f}_i|\mathbf{m}\rangle&=\mathbf{i}^{-2m_i}[m_{i+1}]|\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}\rangle,\\ \bar{k}_i|\mathbf{m}\rangle&=\mathbf{i}^{2m_i-2m_{i+1}}q^{-m_i+m_{i+1}}|\mathbf{m}\rangle,\\ \bar{e}_n|\mathbf{m}\rangle&=\kappa\,\mathbf{i}^{1-|\mathbf{m}|}[m_n]|\mathbf{m}-\mathbf{e}_n\rangle,\\ \bar{f}_n|\mathbf{m}\rangle&=\mathbf{i}^{|\mathbf{m}|-2m_n}|\mathbf{m}+\mathbf{e}_n\rangle,\\ \bar{k}_n|\mathbf{m}\rangle&=\mathbf{i}^{2m_n+1}q^{-m_n-\frac{1}{2}}|\mathbf{m}\rangle,\end{aligned}$$

where $1\le i<n$, $\kappa=(q+1)/(q-1)$.

By introducing the actions of $\bar{e}_0,\bar{f}_0,\bar{k}_0$, we want to make $\mathcal{W}$ a quantum group module in Section A.1 associated to the affine Dynkin datum $C^{(2)}(n+1)$ or $B^{(1)}(0,n)$. For the former we set

$$\begin{aligned}\bar{e}_0|\mathbf{m}\rangle&=x\,\mathbf{i}^{2m_1-|\mathbf{m}|}|\mathbf{m}+\mathbf{e}_1\rangle,\\ \bar{f}_0|\mathbf{m}\rangle&=x^{-1}\kappa\,\mathbf{i}^{|\mathbf{m}|+1}[m_1]|\mathbf{m}-\mathbf{e}_1\rangle,\\ \bar{k}_0|\mathbf{m}\rangle&=\mathbf{i}^{-2m_1-1}q^{m_1+\frac{1}{2}}|\mathbf{m}\rangle,\end{aligned}$$

and for the latter

$$\begin{aligned}\bar{e}_0|\mathbf{m}\rangle&=x(-1)^{|\mathbf{m}|}|\mathbf{m}+2\mathbf{e}_1\rangle,\\ \bar{f}_0|\mathbf{m}\rangle&=x^{-1}(-1)^{|\mathbf{m}|}\frac{[m_1][m_1-1]}{[2]^2}|\mathbf{m}-2\mathbf{e}_1\rangle,\\ \bar{k}_0|\mathbf{m}\rangle&=-q^{2m_1+1}|\mathbf{m}\rangle,\end{aligned}$$

where x is the so-called spectral parameter. We also note that the quantum parameter is still $\bar{q}=\mathbf{i}^{-1}q^{\frac{1}{2}}$.

To obtain the representation for the quantum parameter q, we need to we switch $q^{\frac{1}{2}}$ to $\mathbf{i}q^{\frac{1}{2}}$ ($\bar{q}^{\frac{1}{2}}$ to $q^{\frac{1}{2}}$). Also, the relations in Section A.1 and those in Section 2.3 are different. For the node i that is signified as $\bullet$ in the Dynkin diagram, there is a relation

$$e_if_i+f_ie_i=\frac{k_i-k_i^{-1}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$$

in Section 2.3 rather than

$$e_if_i+f_ie_i=\frac{k_i-k_i^{-1}}{-q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$$

in Section A.1. The former relation is realized by deleting κ from the action of $\bar{e}_i$ or $\bar{f}_i$ in the formulas of the q-oscillator representation above. By doing so, we obtain

$$\bar{e}_0|\mathbf{m}\rangle = \begin{cases} x\,\mathbf{i}^{2m_1-|\mathbf{m}|}|\mathbf{m}+\mathbf{e}_1\rangle & \text{for } U_q(C^{(2)}(n+1)) \\ x(-1)^{|\mathbf{m}|}|\mathbf{m}+2\mathbf{e}_1\rangle & \text{for } U_q(B^{(1)}(1,n)) \end{cases},$$

$$\bar{f}_0|\mathbf{m}\rangle = \begin{cases} x^{-1}\mathbf{i}^{|\mathbf{m}|+2m_1+1}[m_1]|\mathbf{m}-\mathbf{e}_1\rangle & \text{for } U_q(C^{(2)}(n+1)) \\ x^{-1}(-1)^{|\mathbf{m}|+1}\frac{[m_1][m_1-1]}{[2]^2}|\mathbf{m}-2\mathbf{e}_1\rangle & \text{for } U_q(B^{(1)}(0,n)) \end{cases},$$

$$\bar{k}_0|\mathbf{m}\rangle = \begin{cases} q^{m_1+\frac{1}{2}}|\mathbf{m}\rangle & \text{for } U_q(C^{(2)}(n+1)) \\ q^{2m_1+1}|\mathbf{m}\rangle & \text{for } U_q(B^{(1)}(0,n)) \end{cases},$$

$$\bar{e}_i|\mathbf{m}\rangle = (-1)^{-m_i+m_{i+1}+1}[m_i]|\mathbf{m}-\mathbf{e}_i+\mathbf{e}_{i+1}\rangle,$$
$$\bar{f}_i|\mathbf{m}\rangle = (-1)^{-m_i+m_{i+1}+1}[m_{i+1}]|\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}\rangle,$$
$$\bar{k}_i|\mathbf{m}\rangle = q^{-m_i+m_{i+1}}|\mathbf{m}\rangle,$$
$$\bar{e}_n|\mathbf{m}\rangle = \mathbf{i}^{1-|\mathbf{m}|+2m_n}[m_n]|\mathbf{m}-\mathbf{e}_n\rangle,$$
$$\bar{f}_n|\mathbf{m}\rangle = \mathbf{i}^{|\mathbf{m}|-2m_n}|\mathbf{m}+\mathbf{e}_n\rangle,$$
$$\bar{k}_n|\mathbf{m}\rangle = q^{-m_n-\frac{1}{2}}|\mathbf{m}\rangle,$$

for $1 \le i \le n-1$.

To obtain the actions of $U_q(C^{(2)}(n+1))$ (resp. $U_q(B^{(1)}(0,n))$) in Proposition 3.9 (resp. 3.14), we perform the basis change $|\mathbf{m}\rangle$ to $\mathbf{i}^{s(\mathbf{m})}q^{-|\mathbf{m}|/2}\prod_{j=1}^{n}[m_j]!|\mathbf{m}\rangle$ where $s(\mathbf{m}) = -|\mathbf{m}|(|\mathbf{m}|+1)/2 - \sum_j m_j^2$. Next we apply the algebra automorphism sending $e_n \mapsto -e_n, f_n \mapsto -f_n$ and the other generators fixed. For $U_q(C^{(2)}(n+1))^{\sigma}$, we also apply $e_0 \mapsto \sigma e_0$, $f_0 \mapsto f_0\sigma$. Accordingly, the coproduct also changes. For $U_q(B^{(1)}(0,n))$, we alternatively apply $e_0 \mapsto \mathbf{i}[2]e_0, f_0 \mapsto \frac{1}{\mathbf{i}[2]}f_0$.

Appendix B. Quantum R matrix for $U_q(A_{2n}^{(2)\dagger})$

In this appendix, we consider the quantum R matrix for the q-oscillator representation of $U_q(A_{2n}^{(2)\dagger})$ where $A_{2n}^{(2)\dagger}$ is the Dynkin diagram whose nodes have the opposite labelings to $A_{2n}^{(2)}$. Next we identify it as the one for $U_q(B^{(1)}(0,n))$.

B.1. **q-oscillator representation for $U_q(A_{2n}^{(2)\dagger})$.** By $A_{2n}^{(2)\dagger}$ we denote the following Dynkin diagram.

Although we did not deal with the q-oscillator representation for $U_q(A_{2n}^{(2)\dagger})$ in [13], it is easy to guess from other cases given there. On the space $\mathcal{W}$, the actions are given as follows.

$$
\begin{aligned}
e_0|\mathbf{m}\rangle &= x|\mathbf{m}+2\mathbf{e}_1\rangle,\\
f_0|\mathbf{m}\rangle &= x^{-1}\frac{[m_1][m_1-1]}{[2]^2}|\mathbf{m}-2\mathbf{e}_1\rangle,\\
k_0|\mathbf{m}\rangle &= -q^{2m_1+1}|\mathbf{m}\rangle,\\
e_i|\mathbf{m}\rangle &= [m_i]|\mathbf{m}-\mathbf{e}_i+\mathbf{e}_{i+1}\rangle,\\
f_i|\mathbf{m}\rangle &= [m_{i+1}]|\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}\rangle,\\
k_i|\mathbf{m}\rangle &= q^{-2m_i+2m_{i+1}}|\mathbf{m}\rangle,\\
e_n|\mathbf{m}\rangle &= \mathbf{i}\kappa[m_n]|\mathbf{m}-\mathbf{e}_n\rangle,\\
f_n|\mathbf{m}\rangle &= |\mathbf{m}+\mathbf{e}_n\rangle,\\
k_n|\mathbf{m}\rangle &= \mathbf{i}q^{-m_n-1/2}|\mathbf{m}\rangle,
\end{aligned}
$$

where $0<i<n$, $\kappa=(q+1)/(q-1)$. Denote this representation map by π_x.

$U_q(B_n)$-highest weight vectors $\{v_l \mid l\in\mathbb{Z}_{\geq 0}\}$ are calculated in [13, Prop 4]. We take the coproduct (C.1) with $\pi=1$.

Lemma B.1. *For $x,y\in\mathbb{Q}(q)$ we have*

(1) $(\pi_x\otimes\pi_y)\Delta(f_0f_1^{(2)}\cdots f_{n-1}^{(2)})v_l=-\frac{[l][l-1]}{[2]^2}(q^{2l-2}x^{-1}+q^{-1}y^{-1})v_{l-2}\quad(l\geq 2)$,

(2) $(\pi_x\otimes\pi_y)\Delta(e_ne_{n-1}^{(2)}\cdots e_1^{(2)}e_0)v_0=\frac{\mathbf{i}\kappa[2]}{1-q}((y+qx)v_1-q(y+x)\Delta(f_n)v_0)$.

Define $\check{R}_{KO}(z,q)$ as in Proposition C.4 for $U_q(B_n^{(1)})$. The existence of such $\check{R}_{KO}(z,q)$ is essentially given in [13, Theorem 13]. Namely, although $A_{2n}^{(2)\dagger}$ is not listed there, the corresponding gause transformed quantum R matrix is $S^{2,1}(z)$ and the proof has been done as the cases (i),(iv) and (v).

Proposition B.2. *We have the following spectral decomposition*

$$
\check{R}_{KO}(z)=\sum_{l\in2\mathbb{Z}_+}\prod_{j=1}^{l/2}\frac{z+q^{4j-1}}{1+q^{4j-1}z}P_l+\sum_{l\in1+2\mathbb{Z}_+}\prod_{j=0}^{(l-1)/2}\frac{z+q^{4j+1}}{1+q^{4j+1}z}P_l,
$$

where P_l is the projector on the subspace generated by the $U_q(B_n)$-highest weight vector v_l $(l\geq 0)$.

Appendix C. Quantum R matrix for $U_q(C^{(2)}(n+1))$ and $U_q(B^{(1)}(0,n))$

In this appendix, we compare the quantum R matrix for the q-oscillator representation for $U_q(C^{(2)}(n+1))$ with the one for $U_q(D_{n+1}^{(2)})$ given in [13]. We also consider the quantum R matrix for $U_q(B^{(1)}(0,n))$ based on the results in [13].

C.1. **Gauge transformation.** We take the following coproduct

$$
\begin{aligned}
\Delta(k_i) &= k_i\otimes k_i,\\
\Delta(e_i) &= 1\otimes e_i+e_i\otimes\sigma^{\frac{1-\pi}{2}p(i)}k_i,\\
\Delta(f_i) &= f_i\otimes\sigma^{\frac{1-\pi}{2}p(i)}+k_i^{-1}\otimes f_i,
\end{aligned}
\tag{C.1}
$$

for $i \in I$, where σ satisfies (2.3). We also take the same coproduct (C.1) for $\overline{u}_i$. Let Γ be an operator acting on $\mathcal{W}^{\otimes 2}$ by

$$\Gamma|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle = \mathbf{i}^{\sum_{k,l}\varphi_{kl}m_k m'_l}|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle, \tag{C.2}$$

for $\mathbf{m} = (m_1, \dots, m_n)$ and $\mathbf{m}' = (m'_1, \dots, m'_n)$. Here we have the constraint $\varphi_{kl} + \varphi_{lk} = 0$. Then by [20] (see also [19]),

$$\Delta^{\Gamma}(u) = \Gamma^{-1}\Delta(u)\Gamma$$

gives another coproduct of $U_q(B_n)$ acting on $\mathcal{W}^{\otimes 2}$. Take φ_{kl} to be 1 for $k < l$. We also set

$$K|\mathbf{m}\rangle = \mathbf{i}^{c(\mathbf{m})}|\mathbf{m}\rangle, \tag{C.3}$$

where

$$c(\mathbf{m}) = -\frac{1}{2}\sum_k m_k^2 + \sum_k \left(k - n - \frac{1}{2}\right) m_k.$$

Set

$$\gamma_i(\mathbf{m}) = \begin{cases} -|\mathbf{m}| + m_1 & (i = 0 \text{ and for } U_q(C^{(2)}(n+1))) \\ -2|\mathbf{m}| + 2m_1 & (i = 0 \text{ and for } U_q(B^{(1)}(0,n))) \\ m_i + m_{i+1} & (0 < i < n) \\ -|\mathbf{m}| + m_n & (i = n) \end{cases},$$

$$\beta_i(\mathbf{m}) = \begin{cases} m_1 + n & (i = 0 \text{ and } U_q(C^{(2)}(n+1))) \\ 2m_1 + 2n + 1 & (i = 0 \text{ and } U_q(B^{(1)}(0,n))) \\ -m_i + m_{i+1} & (0 < i < n) \\ -m_n & (i = n) \end{cases}.$$

Let $\boldsymbol{\alpha}_0 = \mathbf{e}_1$ for $U_q(C^{(2)}(n+1))), 2\mathbf{e}_1$ for $U_q(B^{(1)}(0,n)))$, $\boldsymbol{\alpha}_i = -\mathbf{e}_i + \mathbf{e}_{i+1}$ $(0 < i < n)$, and $\boldsymbol{\alpha}_n = -\mathbf{e}_n$.

Lemma C.1. *The following formulas hold for* $\mathbf{m}$, $\mathbf{m}'$, *and* $i \in I$;

(1) $\Gamma^{-1}(1 \otimes e_i)\Gamma\, |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle = \mathbf{i}^{-\gamma_i(\mathbf{m})}|\mathbf{m}\rangle \otimes e_i|\mathbf{m}'\rangle$,
(2) $\Gamma^{-1}(e_i \otimes 1)\Gamma\, |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle = \mathbf{i}^{\gamma_i(\mathbf{m}')}e_i|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle$,
(3) $\Gamma^{-1}(1 \otimes f_i)\Gamma\, |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle = \mathbf{i}^{\gamma_i(\mathbf{m}-\boldsymbol{\alpha}_i)}|\mathbf{m}\rangle \otimes f_i|\mathbf{m}'\rangle$,
(4) $\Gamma^{-1}(f_i \otimes 1)\Gamma\, |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle = \mathbf{i}^{-\gamma_i(\mathbf{m}'-\boldsymbol{\alpha}_i)}f_i|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle$.

Lemma C.2. *The following formulas hold for* $\mathbf{m}$ *and* $i \in I$;

(1) $K^{-1}e_iK|\mathbf{m}\rangle = \mathbf{i}^{\beta_i(\mathbf{m})}e_i|\mathbf{m}\rangle$,
(2) $K^{-1}f_iK|\mathbf{m}\rangle = \mathbf{i}^{-\beta_i(\mathbf{m}-\boldsymbol{\alpha}_i)}f_i|\mathbf{m}\rangle$.

Proposition C.3. *For* u_i $(i \in I,\ u = e, f, k)$, *we have*

$$\Delta(\bar{u}_i)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle = \mathbf{i}^{\Lambda_i(\mathbf{m}+\mathbf{m}')}(K \otimes K)^{-1}\Delta^{\Gamma}(u_i)(K \otimes K)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle,$$

on $\mathcal{W}^{\otimes 2}$*. Here*

$$\Lambda_i(\mathbf{m}) = \begin{cases} m_i + m_{i+1} - (\delta_{i0} + \delta_{in})|\mathbf{m}| - n\delta_{i0} & (u = e) \\ m_i + m_{i+1} + (\delta_{i0} + \delta_{in})(|\mathbf{m}| + 1) - 2 & (u = f) \\ 2m_i - 2m_{i+1} & (u = k) \end{cases},$$

except when $i = 0$ *and for* $U_q(B^{(1)}(0,n))$*, where*

$$\Lambda_0(\mathbf{m}) = \begin{cases} 2m_1 - 2|\mathbf{m}| - 2n + 1 & (u = e) \\ 2m_1 - 2|\mathbf{m}| - 2n + 3 & (u = f) \\ 0 & (u = k) \end{cases}.$$

Here we should understand $m_0 = m_{n+1} = 0$*.*

Proof. It follows from Lemmas C.1 and C.2, and the following calculations. For instance, for $i = n$

$$\begin{aligned} \Delta(\bar{e}_n)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle &= (1 \otimes \bar{e}_n + \bar{e}_n \otimes \sigma\bar{k}_n)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle \\ &= \kappa(\mathbf{i}^{1-|\mathbf{m}'|}[m'_n]|\mathbf{m}\rangle \otimes |\mathbf{m}' - \mathbf{e}_n\rangle \\ &\quad + (-1)^{|\mathbf{m}'|}\mathbf{i}^{2-|\mathbf{m}|+2m'_n}q^{-2m'_n-1}[m_n]|\mathbf{m} - \mathbf{e}_n\rangle \otimes |\mathbf{m}'\rangle), \\ \Delta^{\Gamma}(e_n)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle &= (\Gamma^{-1}(1 \otimes e_n)\Gamma + \Gamma^{-1}(e_n \otimes 1)\Gamma \cdot (1 \otimes k_n))|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle \\ &= \kappa(\mathbf{i}^{|\mathbf{m}|-m_n+1}[m'_n]|\mathbf{m}\rangle \otimes |\mathbf{m}' - \mathbf{e}_n\rangle \\ &\quad + \mathbf{i}^{-|\mathbf{m}'|+m'_n+2}q^{-2m'_n-1}[m_n]|\mathbf{m} - \mathbf{e}_n\rangle \otimes |\mathbf{m}'\rangle), \end{aligned}$$

and for $i \neq n$

$$\begin{aligned} \Delta(\bar{e}_i)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle &= (1 \otimes \bar{e}_i + \bar{e}_i \otimes \bar{k}_i)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle \\ &= \mathbf{i}^{2m'_{i+1}}[m'_i]|\mathbf{m}\rangle \otimes |\mathbf{m}' - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle \\ &\quad + \mathbf{i}^{2m_{i+1}+2m'_i-2m'_{i+1}}q^{-2m'_i+2m'_{i+1}}[m_i]|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle \otimes |\mathbf{m}'\rangle, \\ \Delta^{\Gamma}(e_i)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle &= (\Gamma^{-1}(1 \otimes e_i)\Gamma + \Gamma^{-1}(e_i \otimes 1)\Gamma \cdot (1 \otimes k_i))|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle \\ &= \mathbf{i}^{-m_i-m_{i+1}}[m'_i]|\mathbf{m}\rangle \otimes |\mathbf{m}' - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle \\ &\quad + \mathbf{i}^{m'_i+m'_{i+1}}q^{-2m'_i+2m'_{i+1}}[m_i]|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle \otimes |\mathbf{m}'\rangle. \end{aligned}$$

□

For a quantum group such as $U = U_q(D^{(2)}_{n+1})$, $U_q(A^{(2)\dagger}_{2n})$, $U_q(C^{(2)}(n+1))$, $U_q(B^{(1)}(0,n))$ a quantum R matrix $R(z)$ is defined, if it exists, as an intertwiner satisfying

$$\check{R}(z)(\pi_x \otimes \pi_y)\Delta(u) = (\pi_y \otimes \pi_x)\Delta(u)\check{R}(z),$$

where $\check{R}(z) = PR(z)$, P is the transposition of the tensor components and $z = x/y$. We also note that the coproduct we use here is (C.1). For $U = U_q(D^{(2)}_{n+1})$ or $U_q(A^{(2)\dagger}_{2n})$, the existence of quantum R matrices are proved in [13] or Appendix B. We denote them by $\check{R}_{KO}(z)$. Let $\check{R}_{new}(z)$ be the quantum R matrices for the quantum groups $U = U_q(C^{(2)}(n+1))$ or $U_q(B^{(1)}(0,n))$. From the previous proposition, we have

Proposition C.4. *For generic $x, y \in \mathbb{Q}(q)$, $\check{R}_{new}(z)$ and $\check{R}_{KO}(z)$ have the following relation:*

$$\check{R}_{new}(z, -q) = (K \otimes K)^{-1} \Gamma^{-1} \check{R}_{KO}(z, q) \Gamma (K \otimes K).$$

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