Smooth Homotopy 4-Sphere

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ABSTRACT

Every smooth homotopy 4-sphere is diffeomorphic to the 4-sphere.

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1. Introduction

This paper is a paper created by adding proofs to the research announcement paper [12] without proof. So, the proofs are original in this paper. Unless otherwise stated, *manifolds*, *embeddings* and *isotopies* are considered in the *smooth category*. An n-punctured manifold of an m-manifold X is the m-manifold $X^{n(0)}$ obtained from X by removing the interiors of n mutually disjoint m-balls in the interior of X, where choices of the m-balls are independent of the diffeomorphism type of $X^{n(0)}$. The m-manifold $X^{1(0)}$ is denoted by $X^{(0)}$. By this convention, a *homotopy 4-sphere* is a 4-manifold M homotopy equivalent to the 4-sphere S^4, and a *homotopy 4-ball* is a 1-punctured manifold $M^{(0)}$ of a homotopy 4-sphere M. Ever since the positive solution of Topological 4D Poincaré Conjecture (meaning that every topological homotopy 4-sphere is homeomorphic to S^4) and existence of exotic 4-spaces, [3, 4, 16], it has been questioned whether Smooth 4D Poincaré Conjecture (meaning that every homotopy 4-sphere is diffeomorphic to S^4) holds. This paper answers this question affirmatively (see Corollary 1.3). For a positive integer n, the *stable 4-sphere of genus* n is the 4-manifold

$$\Sigma = \Sigma(n) = S^4 \# n(S^2 \times S^2) = S^4 \#_{i=1}^{n} S^2 \times S_i^2,$$

which is the union of the n-punctured manifold $(S^4)^{n(0)}$ of the 4-sphere S^4 and the 1-punctured manifolds $(S^2 \times S^2_i)^{(0)}\,(i = 1, 2, \ldots, n)$ of the 2-sphere products $S^2 \times S^2_i\,(i = 1, 2, \ldots, n)$ pasting the boundary 3-spheres of $(S^4)^{n(0)}$ to the boundary 3-spheres of $(S^2 \times S^2_i)^{(0)}\,(i = 1, 2, \ldots, n)$. An *orthogonal 2-sphere pair* or simply an *O2-sphere pair* of the stable 4-sphere Σ is a pair (S, S') of 2-spheres S and S' embedded in Σ which meet transversely at just one point with the intersection numbers $\mathrm{I}(S, S) = \mathrm{I}(S', S') = 0$ and $\mathrm{I}(S, S') = 1$. A *pseudo-O2-sphere basis* of the stable 4-sphere Σ of genus n is the system (S_*, S'_*) of n disjoint O2-sphere pairs $(S_i, S'_i)\,(i = 1, 2, \ldots, n)$ in Σ. Let $N(S_i, S'_i)$ be a regular neighborhood of the union $S_i \cup S'_i$ of the O2-sphere pair (S_i, S'_i) in Σ such that $N(S_i, S'_i)\,(i = 1, 2, \ldots, n)$ are mutually disjoint and diffeomorphic to the 1-punctured manifold $(S^2 \times S^2)^{(0)}$ of the sphere product $S^2 \times S^2$. The *region* of a pseudo-O2-sphere basis (S_*, S'_*) in Σ of genus n is a connected 4-manifold $\Omega(S_*, S'_*)$ in Σ obtained from the 4-manifolds $N(S_i, S'_i)\,(i = 1, 2, \ldots, n)$ by connecting along disjoint 1-handles $h^1_j\,(j = 1, 2, \ldots, n-1)$ in Σ. Since Σ is a simply connected 4-manifold, the region $\Omega(S_*, S'_*)$ in Σ does not depend on any choices of the 1-handles and is uniquely determined by the pseudo-O2-sphere basis (S_*, S'_*) up to isotopies of Σ (see [9]). The *residual region* of $\Omega(S_*, S'_*)$ in Σ is the 4-manifold $\Omega^c(S_*, S'_*) = \mathrm{cl}(\Sigma \setminus \Omega(S_*, S'_*))$ which is a homotopy 4-ball shown by van Kampen theorem and a homological argument. An *O2-sphere basis* of the stable 4-sphere Σ of genus n is a pseudo-O2-sphere basis (S_*, S'_*) of Σ such that the residual region $\Omega^c(S_*, S'_*)$ is diffeomorphic to the 4-ball. For example, the pseudo-O2-sphere basis

$$(S^2 \times 1_*, 1 \times S^2_*) = \{(S^2 \times 1_i, 1 \times S^2_i)|\, i = 1, 2, ..., n\}$$

of Σ is an O2-sphere basis of Σ and called the *standard* O2-sphere basis of Σ. The following result is basically the main result of this paper.

Theorem 1.1. For any two pseudo-O2-sphere bases (R_*, R'_*) and (S_*, S'_*) of the stable 4-sphere Σ of any genus $n \geq 1$, there is an orientation-preserving diffeomorphism $h : \Sigma \to \Sigma$ sending (R_i, R'_i) to (S_i, S'_i) for all $i\,(i = 1, 2, \ldots, n)$.

Since the image of an O2-sphere basis (R_*, R'_*) of Σ by an orientation-preserving diffeomorphism $f : \Sigma \to \Sigma$ is an O2-sphere basis of Σ, the following corollary is obtained from existence of the standard O2-sphere basis and Theorem 1.1.

Corollary 1.2. Every pseudo-O2-sphere basis of the stable 4-sphere Σ of any genus $n \geq 1$ is an O2-sphere basis of Σ.

The following result (4D Smooth Poincaré Conjecture) is a direct consequence of Corollary 1.2.

Corollary 1.3. Every homotopy 4-sphere M is diffeomorphic to the 4-sphere S^4.

Proof of Corollary 1.3. It is known that there is an orientation-preserving diffeomorphism

$$\kappa : M\#\Sigma \to \Sigma$$

from the connected sum $M\#\Sigma$ onto Σ for a positive integer n by Wall [19]. The connected sum $M\#\Sigma$ is taken the union $M^{(0)} \cup \Sigma^{(0)}$. By Corollary 1.2, the image $\kappa(\Sigma^{(0)})$ is the region $\Omega(S_*, S'_*)$ of an O2-sphere basis (S_*, S'_*) of Σ. and the residual region $\Omega^c(S_*, S'_*) = \kappa(M^{(0)})$ is a 4-ball and hence $M^{(0)}$ is diffeomorphic to the 4-ball D^4. The diffeomorphism $M^{(0)} \to D^4$ extends to a diffeomorphism $M \to S^4$ by $\Gamma_4 = 0$ in Cerf [2] or $\pi_0(\mathrm{Diff}^+(S^3)) = 0$ by Hatcher [6]. This completes the proof of Corollary 1.3. □

As it is seen from the proof of Corollary 1.3, Theorem 1.1 is actually equivalent to the Smooth 4D Poincaré Conjecture by $\Gamma_4 = 0$ or $\pi_0(\mathrm{Diff}^+(S^3)) = 0$. It remains unknown whether the diffeomorphism $M \to S^4$ in Corollary 1.3 made orientation-preservingly is isotopically unique, although it is concordantly unique since $\Gamma_5 = 0$ by Kervaire [15], and piecewise-linear-isotopically unique by Hudson [10], Rourke-Sanderson [17]. The Piecewise-Linear 4D Poincaré Conjecture is equivalent the Smooth 4D Poincaré Conjecture by using a compatible smoothability of a piecewise-linear 4-manifold and a basic fact that every piecewise-linear auto-homeomorphism of the 4-disk keeping the boundary identically is piecewise-linearly ∂-relatively isotopic to the identity by [10, 17]. The result of Wall [19] used for the proof of Corollary 1.3 says further that for every closed simply connected signature-zero spin 4-manifold M with the second Betti number $\beta_2(M; Z) = 2m$, there is a diffeomorphism $\kappa : M\#\Sigma(n) \to \Sigma(m + n)$ for some n. Then there is also a homeomorphism from M to $\Sigma(m)$ by [3, 4]. It should be noted that the present technique used for the proof of Theorem 1.1 cannot be directly generalized to the case of $m > 0$. In fact, it is known by Akhmedov-Park [1] that there is a closed simply connected signature-zero spin 4-manifold M with a large second Betti number $\beta_2(M; Z) = 2m$ such that M is not diffeomorphic to $\Sigma(m)$. The following corollary is what can be said in this paper.

Corollary 1.4. Let M and M' be any closed (not necessarily simply connected) 4-manifolds with the same second Betti number $\beta_2(M; Z) = \beta_2(M'; Z)$. Then an embedding $u : M^{(0)} \to M'$ induces the fundamental group isomorphism $u_\# : \pi_1(M^{(0)}, x) \to \pi_1(M', u(x))$ if and only if the embedding $u : M^{(0)} \to M'$ extends to a diffeomorphism $u^+ : M \to M'$.

Proof of Corollary 1.4. Since the proof of the "if"part is obvious, it suffices to prove the "only if"part. For this proof, confirm by van Kampen theorem and a homological argument that the closed complement $\mathrm{cl}(M' \setminus u(M^{(0)}))$ is a homotopy 4-ball, which is diffeomorphic to the 4-ball D^4 and the embedding $u : M^{(0)} \to M'$ extends to a diffeomorphism $u^+ : M \to M'$ by the proof of Corollary 1.3. This completes the proof of Corollary 1.4. □

The following corollary is obtained by combining Corollary 1.3 with the triviality condition of an S^2-link in S^4, [14].

Corollary 1.5. Every closed 4-manifold M such that the fundamental group $\pi_1(M, x)$ is a free group of rank n and $H_2(M; Z) = 0$ is diffeomorphic to theclosed 4D handlebody $Y^S = S^4 \#_{i=1}^n S^1 \times S_i^3$.

Proof of Corollary 1.5. Let $k_i\,(i = 1, 2, \ldots, n)$ be a system of mutually disjoint simple loops in M which is homotopic to a system of loops with leggs to the base point x generating the free group $\pi_1(M, x)$, and $N(k_i) = S^1 \times D_i^3\,(i = 1, 2, \ldots, n)$ a system of mutually disjoint regular neighborhoods of $k_i\,(i = 1, 2, \ldots, n)$ in M. The 4-manifold X obtained from M by replacing $S^1 \times D_i^3$ with $D^2 \times S_i^2$ for every i is a homotopy 4-sphere by van Kampen theorem and $H_2(M; Z) = 0$, and hence X is diffeomorphic to S^4 by Corollary 1.3. The S^2-link $L = \cup_{i=1}^n K_i$ in $X = S^4$ with component $K_i = S_i^2$ the core 2-sphere of $D^2 \times S_i^2$ has the free fundamental group $\pi_1(S^4 \setminus L, x)$ of rank n with a meridian basis since it is canonically isomorphic to the free fundamental group $\pi_1(M, x)$ by a general position argument. The S^2-link L is a trivial S^2-link in S^4 and hence bounds mutually disjoint 3-balls in S^4 by [14]. By returning $D^2 \times S_i^2$ to $S^1 \times D_i^3$ for every i, the 4-manifold M is seen to be diffeomorphic to the closed 4D handlebody Y^S. This completes the proof of Corollary 1.5. □

The following corollary (4D Smooth Schoenflies Conjecture) is also obtained.

Corollary 1.6. Any (smoothly) embedded 3-sphere S^3 in the 4-sphere S^4 splits S^4 into two components of 4-manifolds which are both diffeomorphic to the 4-ball.

Proof of Corollary 1.6. The splitting components are homotopy 4-balls by van Kampen theorem and a homological argument, which are diffeomorphic to the 4-ball by the proof of Corollary 1.3. This completes the proof of Corollary 1.6.

□

The paper is organized as follows: In Section 2, a trivial surface-knot in the 4-sphere S^4 is discussed to observe that the stable 4-sphere Σ of genus n is the double branched covering space $S^4(F)_2$ of S^4 branched along a trivial surface-knot F of genus n. An *O2-handle basis* of a trivial surface-knot F in S^4 is also introduced there to show that the lift $(S(D_*), S(D'_*))$ of the core system (D_*, D'_*) of any O2-handle basis $(D_* \times I, D'_* \times I)$ of F to $S^4(F)_2 = \Sigma$ is an O2-sphere basis of Σ (See Corollary 2.2). In Section 3, the proof of Theorem 1.1 is done. In Section 4, any two homotopic diffeomorphisms of the stable 4-sphere Σ are shown to be isotopic if one diffeomorphism allows a deformation by an element of $\mathrm{Diff}^+(D^4, \mathrm{rel}\partial)$ (see Theorem 4.1 and Corollaries 4.2, 4.3 for the details).

2. Double branched covering of trivial surface-knot for stable 4-sphere

A *surface-knot* of genus n in the 4-sphere S^4 is a closed surface F of genus n embedded in S^4. Two surface-knots F and F' in S^4 are *equivalent* if there is an orientation-preserving diffeomorphism $f : S^4 \to S^4$ sending F to F' orientation-preservingly. The map f is called an *equivalence*. A *trivial* surface-knot of genus n in S^4 is a surface-knot F of genus n which is the boundary of a handlebody of genus n embedded in S^4, where a handlebody of genus n means a 3-manifold which is a 3-ball for $n = 0$, a solid torus for $n = 1$ or a boundary-disk sum of n solid tori for $n \geq 2$. A *surface-link* in S^4 is a union of disjoint surface-knots in S^4, and a *trivial surface-link* is a surface-link bounding disjoint handlebodies in S^4. A trivial surface-link in S^4 is determined regardless of the embeddings and unique up to isotopies, [9].

A *symplectic basis* of a closed surface F of genus n is a system (x_*, x'_*) of element pairs $(x_j, x'_j)\,(j = 1, 2, \ldots, n)$ of $H_1(F; Z)$ with the intersection numbers

$$\mathrm{I}(x_j, x_{j'}) = \mathrm{I}(x'_j, x'_{j'}) = \mathrm{I}(x_j, x'_{j'}) = 0$$

for all j, j' except that $\mathrm{I}(x_j, x'_j) = +1$ for all j. Every pair (x_1, x'_1) with $\mathrm{I}(x_1, x'_1) = +1$ is extended to a symplectic basis (x_*, x'_*) of F by an argument on the intersection form

$$\mathrm{I} : H_1(F; Z) \times H_1(F; Z) \to Z.$$

Further, every symplectic basis $(x_*, x'_*) = \{(x_j, x'_j) |\, j = 1, 2, \ldots, n\}$ is realized by a system of oriented simple loop pairs $(e_*, e'_*) = \{(e_j, e'_j) |\, j = 1, 2, \ldots, n\}$ of F with $e_j \cap e_{j'} = e'_j \cap e'_{j'} = e_j \cap e'_{j'} = \emptyset$ for all distinct j, j' and with tranverse intersection $e_j \cap e'_j$ at just one point for all j, which is called a *loop basis* of F. For a surface-knot F in S^4, an element $x \in H_1(F; Z)$ is said to be *spin* if the Z_2-reduction $[x]_2 \in H_1(F; Z_2)$ of x has $\eta([x]_2) = 0$ for the Z_2-quadratic function

$$\eta : H_1(F; Z_2) \to Z_2$$

associated with a surface-knot F in S^4, which is defined as follows: For a simple loop e in F bounding a surface D_e in S^4 with $D_e \cap F = e$, the Z_2-self-intersection number $\mathrm{I}(D_e, D_e)$ (mod 2) with respect to the F-framing is defined to be the value $\eta([e]_2)$. For every surface-knot F in S^4, there is a spin basis of F (see [7]). Every spin pair (x_1, x_1') in F with $\mathrm{I}(x_1, x_1') = +1$ is extended to a spin symplectic basis (x_*, x_*') of F by vanishing of Arf invarinat of the Z_2-quadratic function $\eta : H_1(F; Z_2) \to Z_2$ for every surface-knot F in S^4. In particular, any spin pair (x_1, x_1') is realized by a spin loop pair (e_1, e_1') of F extendable to a spin loop basis (e_*, e_*') of F.

A 2-*handle* on a surface-knot F in S^4 is a 2-handle $D \times I$ on F embedded in S^4 such that

$$(D \times I) \cap F = (\partial D) \times I,$$

where I denotes a closed interval with 0 as the center and $D \times 0$ is called the *core* of the 2-handle $D \times I$ and identified with D. For a 2-handle $D \times I$ on F in S^4, the loop ∂D of the core disk D is a spin loop in F since $\eta([\partial D]_2) = 0$. To save notation, if an embedding $h : D \times I \cup F \to X$ is given from a 2-handle $D \times I$ on a surface F to a 4-manifold X, then the 2-handle image $h(D \times I)$ and the core image $h(D)$ on $h(F)$ are denoted by $hD \times I$ and hD, respectively. An *orthogonal 2-handle pair* or simply an *O2-handle pair* on a surface-knot F in S^4 is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I$ and $D' \times I$ on F which *meet orthogonally* on F, in other words, which meet F onlywith the attaching annuli $(\partial D) \times I$ and $(\partial D') \times I$ so that the loops ∂D and $\partial D'$ meet transversely at just one point q and the intersection $(\partial D) \times I \cap (\partial D') \times I$ is diffeomorphic to the square $Q = \{q\} \times I \times I$ [11]. For a trivial surface-knot F of genus n in S4, an O2-*handle basis* of F of genus n in S^4 is a system $(D_* \times I, D_*' \times I)$ of mutually disjoint O2-handle pairs $(D_i \times I, D_i' \times I)\,(i = 1, 2, \dots, n)$ on F in S^4 such that the loop system $(\partial D_*, \partial D_*')$ given by $\{(\partial D_i, \partial D_i') |\, i = 1, 2, \dots, n\}$ forms a spin loop basis of F. Every trivial surface-knot F in S^4 is moved into the boundary of a standard handlebody in the equatorial 3-sphere S^3 of S^4, where a *standard* O2-handle basis and a *standard* loop basis of F are taken. For any given spin loop basis of a trivial surface-knot F of genus n in S^4, there is an O2-handle basis $(D_* \times I, D_*' \times I)$ of F in S^4 with the given spin loop basis as the loop basis $(\partial D_*, \partial D_*')$. This is because there is an equivalence $f : (S^4, F) \to (S^4, F)$ sending the standard spin loop basis to the given spin loop basis of F and hence there is an O2-handle basis of F in S^4 with the given spin loop basis which is the image of the standard O2-handle basis of F by [8], [11, (2.5.1), (2.5.2)]. Further any O2-handle basis of F in S^4 with attaching part fixed is unique up to orientation-preserving diffeomorphisms of S^4 keeping F point-wise fixed, [13].

For the double branched covering projection $p : S^4(F)_2 \to S^4$ branched along F, the non-trivial covering involution of $S^4(F)_2$ is denoted by α. The preimage $p^{-1}(F)$ in Σ of F which is the fixed point set of α and diffeomorphic to F is also written by the same notation as F. The following result is a standard result.

Lemma 2.1. Let $(D_* \times I, D'_* \times I)$ be a standard O2-handle basis of a trivial surface-knot F of genus n in S^4. Then there is an orientation-preserving diffeomorphism $f : S^4(F)_2 \to \Sigma$ sending the 2-sphere pair system $(S(D_*), S(D'_*))$ with $S(D_i) = D_i \cup \alpha D_i$ and $S(D'_i) = D'_i \cup \alpha D'_i\,(i = 1, 2, \ldots, n)$ to the standard O2-sphere basis $(S^2 \times 1_*, 1 \times S^2_*)$ of the stable 4-sphere Σ of genus n. In particular, the 2-sphere pair system $(S(D_*), S(D'_*))$ is an O2-sphere basis of Σ.

Proof of Lemma 2.1. Let $A_i\,(i = 1, 2, \ldots, n)$ be mutually disjoint 4-balls which are regular neighborhoods of the 3-balls

$$D_i \times I \cup D'_i \times I \quad (i = 1, 2, \ldots, n)$$

in S^4. The closed complement $(S^4)^{n(0)} = \mathrm{cl}(S^4 \setminus \cup_{i=1}^n A_i)$ is the n-punctured manifold of S^4. Let $P = F \cap (S^4)^{n(0)}$ be a proper n-punctured 2-sphere in $(S^4)^{n(0)}$. Since the pair $((S^4)^{n(0)}, P)$ is an n-punctured pair of a trivial 2-knot space (S^4, S^2) and the double branched covering space $S^4(S^2)_2$ is diffeomorphic to S^4, the double branched covering space $(S^4)^{n(0)}(P)_2$ of $(S^4)^{n(0)}$ branched along P is diffeomorphic to $(S^4)^{n(0)}$. On the other hand, for the proper surface $P_i = F \cap A_i$ in the 4-ball A_i, the pair (A_i, P_i) is considered as a 1-punctured pair of a trivial torus-knot space (S^4, T), so that the double branched covering space $A_i(P_i)_2$ is diffeomorphic to the 1-punctured manifold of the double branched covering space $S^4(T)_2$. The trivial torus-knot space (S^4, T) is the double of the product pair $(B, o) \times I = (B \times I, o \times I)$ for a trivial loop o in the interior of a 3-ball B and an interval I, so that (S^4, T) is diffeomorphic to the boundary pair

$$\partial((B, o) \times I^2) = (\partial(B \times I^2), \partial(o \times I^2)),$$

where I^m denotes the m-fold product of I for any $m \geq 2$. Thus, the double branched covering space $S^4(T)_2$ is diffeomorphic to the boundary $\partial(B(o)_2 \times I^2)$, where $B(o)_2$ is the double branched covering space of B branched along o which is diffeomorphic to the product $S^2 \times I$. This means that the 5-manifold $B(o)_2 \times I^2$ is the product $S^2 \times I^3$, and $S^4(T)_2$ is diffeomorphic to $S^2 \times S^2$, which shows that $A_i(P_i)_2$ is diffeomorphic to $(S^2 \times S^2)^{(0)}$. This construction also shows that there is an orientation-preserving diffeomorphism

$$f_i : A_i(P_i)_2 \to (S^2 \times S^2)^{(0)}_i$$

sending the O2-sphere pair $(S(D_i), S(D'_i))$ to the standard O2-sphere pair $(S^2 \times 1_i, 1 \times S^2_i)$ of the connected summand $(S^2 \times S^2)^{(0)}_i$ of Σ for all i. A desired orientation-preserving diffeomorphism $f : S^4(F)_2 \to \Sigma$ is constructed from a diffeomorphism $f'' :$

$(S^4)^{n(0)}(P)_2 \to (S^4)^{n(0)}$ and the diffeomorphisms $f_i\,(i = 1, 2, \ldots, n)$. This completes the proof of Lemma 2.1.

□

The identification of $S^4(F)_2 = \Sigma$ is fixed by an orientation-preserving diffeomorphism $f : S^4(F)_2 \to \Sigma$ given in Lemma 2.1. The following corollary is obtained from Lemma 2.1 and [11, 13].

Corollary 2.2. For any two O2-handle bases $(D_* \times I, D'_* \times I)$ and $(E_* \times I, E'_* \times I)$ of a trivial surface-knot F of genus n in S^4, there is an orientation-preserving α-equivariant diffeomorphism f" of Σ sending the 2-sphere pair system $(S(D_*), S(D'_*))$ to the 2-sphere pair system $(S(E_*), S(E'_*))$. In particular, the 2-sphere pair system $(S(D_*), S(D'_*))$ for every O2-handle basis $(D_* \times I, D'_* \times I)$ is an O2-sphere basis of Σ.

Proof of Corollary 2.2. There is an equivalence $f : (S^4, F) \to (S^4, F)$ keeping F set-wise fixed sending the O2-handle basis $(D_* \times I, D'_* \times I)$ to the O2-handle basis $(E_* \times I, E'_* \times I)$ of F by uniqueness of an O2-handle pair, [11, 13]. By construction, the lifting diffeomorphism $f'' : S^4(F)_2 \to S^4(F)_2$ of f sends $(S(D_*), S(D'_*))$ to $(S(E_*), S(E'_*))$. From Lemma 2.1, the 2-sphere pair system $(S(D_*), S(D'_*))$ for every O2-handle basis $(D_* \times I, D'_* \times I)$ is shown to be an O2-sphere basis of Σ by taking . $(E_* \times I, E'_* \times I)$ a standard O2-handle basis of F. This completes the proof of Corollary 2.2. □

An *n-rooted disk family* is the triplet (d, d_*, b_*) where d is a disk, d_* is a system of mutually disjoint disks $d_i\,(i = 1, 2, \ldots, n)$ in the interior of d and b_* is a system of mutually disjoint bands $b_i\,(i = 1, 2, \ldots, n)$ in the n-punctured disk $\mathrm{cl}(d \setminus d_*)$ such that b_i spans an arc in the loop ∂d_i and an arc in the loop ∂d. Let $\gamma(b_*)$ denote the centerline system of the band system b_*. In the following lemma shows that there is a canonical n-rooted disk family (d, d_*, b_*) associated with an O2-handle basis $(D_* \times I, D'_* \times I)$ of a trivial surface-knot F of genus n in S^4.

Lemma 2.3. Let $(D_* \times I, D'_* \times I)$ be an O2-handle basis of a trivial surface-knot F of genus n in S^4, and (d, d_*, b_*) an n-rooted disk family. Then there is an embedding

$$\varphi : (d, d_*, b_*) \times I \to (S^4, D_* \times I, D'_* \times I)$$

such that

(1) The surface F is the boundary of the handlebody V of genus n given by

$$V = \varphi(\mathrm{cl}(d \setminus d_*) \times I),$$

(2) There is an identification

$$\varphi(d_* \times I, d_*) = (\varphi(d_*) \times I, \varphi(d_*)) = (D_* \times I, D_*)$$

as 2-handle systems on F, and

(3) There is an identification

$$\varphi(b_* \times I, \gamma(b_*) \times I) = (D'_* \times I, D'_*)$$

as 2-handle systems on F.

Lemma 2.3 says that the 2-handle system s$D_* \times I$ and $D'_* \times I$ are attached to the handlebody V bounded by F along a longitude system and a meridian system of V, respectively.

Proof of Lemma 2.3. If $(E_* \times I, E'_* \times I)$ is a standard O2-handle basis of F, then it is easy to construct such an embedding

$$\varphi' : (d, d_*, b_*) \times I \to (S^4, E_* \times I, E'_* \times I)$$

with (1)-(3) taking φ' and $(E_* \times I, E'_* \times I)$ as φ and $(D_* \times I, D'_* \times I)$, respectively. In general, there is an equivalence $f : (S^4, F) \to (S^4, F)$ keeping F set-wise fixed and sending the standard O2-handle basis $(E_* \times I, E'_* \times I)$ of F into the O2-handle basis $(D_* \times I, D'_* \times I)$ of F by uniqueness of an O2-handle pair in [11, 13]. The composite embedding

$$\varphi = f\varphi' : (d, d_*, b_*) \times I \to (S^4, D_* \times I, D'_* \times I)$$

is a desired embedding. This completes the proof of Lemma 2.3. □

In Lemma 2.3, the embedding φ, the 3-ball $B = \varphi(D \times I)$, the handlebody V and the pair (B, V) are respectively called a *bump embedding*, a *bump 3-ball*, a *bump handlebody* and a *bump pair* of F in S^4. For a bump embedding

$$\varphi : (d, d_*, b_*) \times I \to (S^4, D_* \times I, D'_* \times I),$$

there is an embedding $\varphi'' : d \times I \to S^4(F)_2$ with $p\varphi'' = \varphi$. Since $p(\varphi''(d_* \times I), \varphi''(b_* \times I)) = (D_* \times I, D'_* \times I)$ by the conditions (1)-(3) of Lemma 2.3, the images $\varphi''(d_* \times I)$ and $\varphi''(b_* \times I)$ are respectively considered as 2-handle systems on F in $S^4(F)_2$ with $p\varphi''(d_* \times I) = \tilde{D}_* \times I$ and $p\varphi''(b_* \times I) = \tilde{D}'_* \times I$ so that $(\varphi''(d_* \times I), \varphi''(b_* \times I))$ is an O2-handle basis of F in $S^4(F)_2$, which is also denoted by $(D_* \times I, D'_* \times I)$ to define an embedding

$$\varphi'' : (d, d_*, b_*) \times I \to (S^4(F)_2, D_* \times I, D'_* \times II)$$

with $p\varphi'' = \varphi$. This embedding is called a *lifting bump embedding* of the bump embedding. In this case, the *bump 3-ball* $\varphi''(d \times I)$ and the *bump handlebody* $\varphi''(\mathrm{cl}(d \setminus d_*) \times I)$ of F in $S^4(F)_2$ are also denoted by B and V by counting $p\varphi''(d \times I) = B$ and $p\varphi''(\mathrm{cl}(d \setminus d_*) \times I) = V$, respectively. For the non-trivial covering involution α of $S^4(F)_2$, the composite embedding

$$\alpha\varphi'' : (d, d_*, b_*) \times I \to (S^4(F)_2, \alpha D_* \times I, \alpha D'_* \times I)$$

is another lifting bump embedding of the bump embedding φ. For the bump 3-ball $\alpha\varphi''(d \times I) = \alpha B$ and the bump handlebody $\alpha\varphi''(\mathrm{cl}(d \setminus d_*) \times I) = \alpha V$ of F in $S^4(F)_2$, we have

$$V \cap \alpha V = B \cap \alpha B = F$$

in $S^4(F)_2$. For a lifting bump embedding, the following lemma is obtained.

Lemma 2.4. Let $\varphi'' : (d, d_*, b_*) \times I \to (\Sigma, D_* \times I, D'_* \times I)$ be a lifting bump embedding. For an embedding

$$u : \Sigma^{(0)} \to \Sigma,$$

assume that the image $\varphi''(d \times I)$ is in the interior of $\Sigma^{(0)}$ to define the composite embedding $u\varphi'' : (d, d_*, b_*) \times I \to (\Sigma, uD_* \times I, uD'_* \times I)$. Then there is a diffeomorphism $g : \Sigma \to \Sigma$ which is isotopic to the identity such that the composite embedding

$$gu\varphi'' : (d, d_*, b_*) \times I \to (\Sigma, guD_* \times I, guD'_* \times I)$$

is identical to the lifting bump embedding

$$\varphi'' : (d, d_*, b_*) \times I \to (\Sigma, D_* \times I, D'_* \times I).$$

Proof of Lemma 2.4. The 0-section $(d, d_*, b_*) \times 0$ of the line bundle $(d, d_*, b_*) \times I$ of the n-rooted disk family (d, d_*, b_*) is identified with (d, d_*, b_*). Move the disk $u\varphi''(d)$ into the disk $\varphi''(d)$ in Σ and then move the disk system $u\varphi''(d_*)$ and the band system $u\varphi''(b_*)$ into the disk system $\varphi''(d_*)$ and the band system $\varphi''(b_*)$ in the disk $\varphi''(d)$, respectively. These deformations are attained by a diffeomorphism g' of Σ which is isotopic to the identity, so that

$$g'u\varphi''(d, d_*, b_*) = \varphi''(d, d_*, b_*).$$

Further, there is a diffeomorphism g'' of Σ which is isotopic to the identity such that the composite embedding $g''g'u : \Sigma^{(0)} \to \Sigma$ preserves the normal line bundles of the disk $\varphi''(d)$ in $\Sigma^{(0)}$ and Σ, so that

$$gu\varphi''(d, d_*, b_*) \times I = \varphi''(d, d_*, b_*) \times I$$

for the diffeomorphism $g = g''g'$ of Σ isotopic to the identity. This completes the proof of Lemma 2.4. □

In Lemma 2.4, also assume that the image $\alpha\varphi''(d \times I)$ is in the interior of $\Sigma^{(0)}$. Then note that any disk interior of the disk systems $gu\alpha D_*$ and $gu\alpha D'_*$ does not meet the bump 3-ball $B = \varphi''(d \times I)$ in Σ. In fact, gu defines an embedding from

$$gu : B \cup \alpha B \to \Sigma$$

with $gu(B, F) = (B, F)$. The complement $gu\alpha B \setminus F$ of F in the 3-ball $gu\alpha B$ does not meet the bump 3-ball B since $B \cap \alpha B = F$. This means that any disk interior of the disk systems $gu\alpha D_*$ and $gu\alpha D'_*$ does not meet the bump 3-ball B. Note that this property comes from the fact that $\Sigma^{(0)}$ and Σ have the same genus n.

3. Proof of Theorem 1.1

For the proof of Theorem 1.1, the following result known by Wall [18] is used.

Lemma 3.1. For any pseudo-O2-sphere bases (R_*, R'_*) and (S_*, S'_*) of the stable 4-sphere Σ of genus n, there is an orientation-preserving diffeomorphism $f : \Sigma \to \Sigma$ which induces an isomorphism $f_* : H_2(\Sigma; Z) \to H_2(\Sigma; Z)$ such that $[fR_i] = [S_i]$ and $[fR'_i] = [S'_i]$ for all i.

Assum that (R_*, R'_*) is an O2-sphere basis of Σ with $(R_*, R'_*) = (S(D_*), S(D'_*))$ for an O2-handle basis $(D_* \times I, D'_* \times I)$ of a trivial surface-knot F of genus n in S^4. Let $u : \Sigma^{(0)} \to \Sigma$ be an embedding such that $(uS(D_*), uS(D'_*)) = (S_*, S'_*)$. By Lemma 3.1, assume that the homology classes $[uS(D_i)] = [S_i]$ and $[uS(D'_i)] = [S'_i]$ are respectively identical to the homology classes $[Ri] = [S(Di)]$ and $[R'i] = [S(D'i)]$ for all i. Let (B, V) be a bump pair of the O2-handle basis $(D_* \times I, D'_* \times I)$ of F in S^4 defined soon after Lemma 2.3. Recall that the two lifts of (B, V) to Σ under the double branched covering projection $p : S^4(F)_2 \to S^4$ are denoted by (B, V) and $(\alpha B, \alpha V)$. To complete the proof of Theorem 1.1, three lemmas are provided from here. The first lemma is stated as follows.

Lemma 3.2. There is a diffeomorphism g of Σ which is isotopic to the identity such that the composite embedding

$$gu : \Sigma^{(0)} \to \Sigma$$

preserves the bump pair (B, V) in Σ identically and has the property that every disk interior of the disk systems $gu\alpha D_*$ and $gu\alpha D'_*$ meets every disk of the disk systems αD_* and $\alpha D'_*$ transversely only with the intersection number 0.

Proof of Lemma 3.2. By Lemma 2.4, there is a diffeomorphism $g : \Sigma \to \Sigma$ which is isotopic to the identity such that the composite embedding $gu : \Sigma^{(0)} \to \Sigma$ preserves the bump pair (B, V) in $\Sigma^{(0)}$ identically and has the property that any disk interior of the disk systems $gu\alpha D_*$ and $gu\alpha D'_*$ does not meet the O2-handle basis $(D_* \times I, D'_* \times I)$ in Σ and meets transversely any disk interior of the disk systems αD_* and $\alpha D'_*$ with a finite number of points. Since

$$S(D_i) = D_i \cup \alpha D_i,\ S(D'_i) = D'_i \cup \alpha D'_i,\ guD_i = D_i\,, guD'_i = D'_i$$

and any disk interior pair of the disk systems αD_* and $\alpha D'_*$ is a disjoint pair, every disk interior of the disk systems $gu\alpha D_*$ and $gu\alpha D'_*$ meets every disk interior of the disk systems αD_* and $\alpha D'_*$ only with intersection number 0 by the homological identities

$$[guS(D_i)] = [S(D_i)], \quad [guS(D_i)] = [S(D_i)]$$

for all i and the invariance of their intersection numbers. This completes the proof of Lemma 3.2. □

By Lemma 3.2, assume that the orientation-preserving embedding $u : \Sigma^{(0)} \to \Sigma$ sends the bump pair (B, V) to itself identically and has the property that every disk interior of the disk systems $u\alpha D_*$ and $u\alpha D'_*$ meets every disk interior of the disk systems αD_* and $\alpha D'_*$ only with the intersection number 0. Then the following lemma is obtained:

Lemma 3.3. There is a diffeomorphism g of Σ which is isotopic to the identity such that the composite embedding $gu : \Sigma^{(0)} \to \Sigma$ sends the disk systems D_* and D'_* identically and the disk interiors of the disk systems $gu\alpha D_*$, $gu\alpha D'_*$ to be disjoint from the disk systems αD_* and $\alpha D'_*$ in Σ.

Proof of Lemma 3.3. Between the open disks $u\mathrm{Int}(\alpha D_i)$, $u\mathrm{Int}(\alpha D'_{i'})$ for all i, i' and the open disks $\mathrm{Int}(\alpha D_j)$, $\mathrm{Int}(\alpha D'_{j'})$ for all j, j', suppose an open disk, say $u\mathrm{Int}(\alpha D_i)$ meets an open disk, say $\mathrm{Int}(\alpha D_j)$ with a pair of points with opposite signs. A procedure to eliminate this pair of points is explained from now. Let a be a simple arc in the open disk $\mathrm{Int}(\alpha D_j)$ joining the pair of points whose interior does not meet the intersection points $\mathrm{Int}(\alpha D_j) \cap u\mathrm{Int}(\alpha D_*)$ and $\mathrm{Int}(\alpha D_j) \cap u\mathrm{Int}(\alpha D'_*)$. Let $T(a)$ be the torus obtained from the 2-sphere $uS(D_i) = D_i \cup u(\alpha D'_i)$ by a surgery along a 1-handle $h^1(a)$ on $uS(D_i)$ with core the arc a and with $h^1(a) \cap \mathrm{Int}(\alpha D_j) = a$. Slide the arc a along the open disk $\mathrm{Int}(\alpha D_j)$ without moving the endpoints so that

(*) the 1-handle $h^1(a)$ passes once time through a thickening $S'_j \times I$ of a 2-sphere S'_j parallel to the 2-sphere $S(D'_j) = D'_j \cup \alpha D'_j$ (not meeting $S(D'_j)$), and

(**) the intersection $h^1(a) \cap (S'_j \times I)$ is a 2-handle $d_j \times I$ on the torus $T(a)$ which is a strong deformation retract of the 2-handle $h^1(a)$ on $T(a)$.

After these deformations (*), (**), let $u'S(D_i)$ be the 2-sphere obtained from $T(a)$ by the surgery along the 2-handle $d_j \times I$. The resulting 2-sphere $u'S(D_i)$ is obtained from the 2-sphere $uS(D_i)$ as $u'S(D_i) = g'uS(D_i)$ for a diffeomorphism $g' : \Sigma \to \Sigma$ which is isotopic to the identity. This isotopy of g' keeps the outside of a regular neighborhood of $h^1(a)$ in the image of g' fixed. Next, regard the 2-handle $d_j \times I$ on $T(a)$ as a 1-handle of the 2-sphere $u'S(D_i)$. Let $u''S(D_i)$ be the 2-sphere obtained from $T(a)$ by the surgery along the 2-handle $\mathrm{cl}(S'_j \backslash d_j) \times I$ and regard this 2-handle as a 1-handle on the 2-sphere $u''S(D_i)$. The 2-sphere $u'S(D_i)$ is isotopically deformed into the 2-sphere $u''S(D_i)$ by a homotopy deformation of a 1-handle (see [9, Lemma 1.4]). Thus, there is a diffeomorphism $g'' : \Sigma \to \Sigma$ which is isotopic to the identity such that

$$u''S(D_i) = g'''u'S(D_i) = g''g'uS(D_i).$$

This isotopy of g'' keeps the outside of a regular neighborhood of $S'_j \times I$ fixed. By this procedure, the total geometric intersection number between the open disks $u''\mathrm{Int}(\alpha D_i)$, $u''\mathrm{Int}(\alpha D'_{i'})$ for all i, i' and the open disks $\mathrm{Int}(\alpha D_j)$, $\mathrm{Int}(\alpha D'_{j'})$ for all j, j' is reduced by 2. By continuing this process, we have a diffeomorphism $g : \Sigma \to \Sigma$ which is isotopic to the identity such that the composite embedding $gu : \Sigma^{(0)} \to \Sigma$ sends the disk systems D_* and D'_* identically and the open disks $gu\mathrm{Int}(\alpha D_i)$ and $gu\mathrm{Int}(\alpha D'_{i'})$ are disjoint from the open disks $\mathrm{Int}(\alpha D_j)$ and $\mathrm{Int}(\alpha D'_{j'})$ for all i, i', j, j'. The procedure is similarly done for the other cases that $u\mathrm{Int}(\alpha D_i)$ meets $\mathrm{Int}(\alpha D'_{j'})$ with a pair of points with opposite signs, that $u\mathrm{Int}(\alpha D'_{i'})$ meets $\mathrm{Int}(\alpha D_j)$ with a pair of points with opposite signs and that $u\mathrm{Int}(\alpha D'_{i'})$ meets $\mathrm{Int}(\alpha D'_{j'})$ with a pair of points with opposite signs. This completes the proof of Lemma 3.3. □

For the O2-sphere basis $(S(D_*), S(D'_*))$ of Σ, let

$$q_* = \{q_i = S(D_i) \cap S(D'_i) |\, i = 1, 2, \dots, n\}$$

be the transverse intersection point system between $S(D_*)$ and $S(D'_*)$. The diffeomorphism g of Σ sending the disk systems D_* and D'_* identically in Lemma 3.3 is further deformed so that, while leaving the transverse intersection point q_i, the disks guD_i and D_i are separated and then the disks guD'_i and D'_i are separated. Thus,

$$guD_i \cap D_i = guD'_i \cap D'_i = q_i$$

for all i. By this deformation, the pseudo-O2-sphere basis $(guS(D_*), guS(D'_*))$ of Σ is assumed to meet the O2-sphere basis $(S(D_*), S(D'_*))$ of Σ at just the transverse intersection point system q_*. Next, the diffeomorphism g of Σ is deformed so that a disk neighborhood system of q_* in $guS(D_*)$ and a disk neighborhood system of q_* in $S(D_*)$ are matched, and then a disk neighborhood system of q_* in $guS(D'_*)$ and a disk neighborhood system of q_* in $S(D'_*)$ are matched. Thus, there is a diffeomorphism g of Σ which is isotopic to the identity such that the meeting part of the pseudo-O2-sphere basis $(guS(D_*), guS(D'_*))$ and the O2-sphere basis $(S(D_*), S(D'_*))$ is just a disk neighborhood pair system (d_*, d'_*) of the transverse intersection point system q_*. Now, assume that for an embedding $u : \Sigma^{(0)} \to \Sigma$, the meeting part of the pseudo-O2-sphere basis $(uS(D_*), uS(D'_*))$ and the O2-sphere basis $(S(D_*), S(D'_*))$ is just a disk neighborhood pair system (d_*, d'_*) of q_*. Then the following lemma is obtained:

Lemma 3.4. There is an orientation-preserving diffeomorphism h of Σ such that the composite embedding $hu : \Sigma^{(0)} \to \Sigma$ preserves the O2-sphere basis $(S(D_*), S(D'_*))$ identically.

The proof of Lemma 3.4 is obtained by using Lemma 3.5 (*Framed Light-bulb Diffeomorphism Lemma*) which is proved easily in comparison with an isotopy version (Lemma 3.7) of this lemma using Gabai's 4D light-bulb theorem [5]. To state Lemmas 3.5,3.7, call a 4-manifold Y in S^4 which is diffeomorphic to $S^1 \times D^3$ a *4D solid torus*. A *boundary fiber circle* of the 4D solid torus Y is a fiber circle of the S^1-bundle ∂Y diffeomorphic to $S^1 \times S^2$. Let $Y^c = \mathrm{cl}(S^4 \setminus Y)$ be the exterior of Y in S^4. Let Y_* be a system of mutually disjoint 4D solid tori Y_i, $(i = 1, 2, \dots, n)$ in S^4, and Y_*^c the system of the exteriors Y_i^c of Y_i in S^4 $(i = 1, 2, \dots, n)$. Let

$$\cap Y_*^c = \cap_{i=1}^n Y_i^c.$$

Lemma 3.5 (Framed Light-bulb Diffeomorphism Lemma). Let Y_* be a system of mutually disjoint 4D solid tori Y_i $(i = 1, 2, \dots, n)$ in S^4. Let $D_* \times I$ and $E_* \times I$ be systems of mutually disjoint framed disks $D_i \times I, E_i \times I$ $(i = 1, 2, \dots, n)$ in $\cap Y_*^c$ such that

$$(D_* \times I) \cap \partial Y_i^c = (\partial D_i) \times I = (\partial E_i) \times I = (E_* \times I) \cap \partial Y_i^c$$

and $\partial D_i = \partial E_i$ is a boundary fiber circle of Y_i for all i. Then there is an orientation-preserving diffeomorphism $h : S^4 \to S^4$ sending Y_* identically such that $h(D_* \times I, D_*) = (E_* \times I, E_*)$.

Proof of Lemma 3.5. Let k_* be the system of the loops $k_i = \partial D_i = \partial E_i$ $(i = 1, 2, \dots, n)$. Let $c : k_* \times [0, 1] \to D_*$ be a boundary collar function of D_* with $c(x, 0) = x$ for all $x \in k_*$, and $c' : k_* \times [0, 1] \to E_*$ a boundary collar function of E_*

with $c'(x,0) = x$ for all $x \in k_*$. Assume that

$$c(x,t) \times I = c'(x,t) \times I$$

for all $x \in k_*$ and $t \in [0,1]$. Let

$$\nu(\partial D_*) = c(k_* \times [0,1]) \quad \text{and} \quad D_*^- = \mathrm{cl}(D_* \setminus \nu(\partial D_*)).$$

Similarly, let

$$\nu(\partial E_*) = c(k_* \times [0,1]) \quad \text{and} \quad E_*^- = \mathrm{cl}(E_* \setminus \nu(\partial E_*)).$$

Consider $D_* \times I$ and $E_* \times I$ in S^4. Let β_* be the system of arcs β_i $(i = 1, 2, \ldots, n)$ such that β_i is an arc in k_i, and β_*^c the system of the arcs $\beta_i^c = \mathrm{cl}(k_i \setminus \beta_i)$ $(i = 1, 2, \ldots, n)$. By 3-cell moves within a regular neighborhood of $c(\beta_* \times [0,1]) \times I \cup D_*^- \times I$ in S^4, there is an orientation-preserving diffeomorphism h' of S^4 such that $h'(D_* \times I) = c(\beta_*^c \times [0,1]) \times I$. Since $\nu(\partial D_*)$ is identical to $\nu(\partial E_*)$, $c(\beta_*^c \times [0,1])$ is identical to $c'(\beta_*^c \times [0,1])$. By 3-cell moves within a regular neighborhood of $c(\beta_* \times [0,1]) \times I \cup E_*^- \times I$ in S^4, there is an orientation-preserving diffeomorphism h'' of S^4 such that $h''(c'(\beta_*^c \times [0,1]) \times I) = E_* \times I$. The diffeomorphism $h''h'$ is an orientation-preserving diffeomorphism of S^4 sending $D_* \times I$ to $E_* \times I$. Let $N(k_*)$ be a regular neighborhood system of the loop system k_* in S^4 meeting $c(k_* \times [0,1])$ regularly, which is a system of n mutually disjoint 4D solid tori. Then the diffeomorphism $h''h'$ is deformed into an orientation-preserving diffeomorphism h of S^4 which sends $N(k_*)$ identically such that $h(D_* \times I, D_*) = (E_* \times I, E_*)$. Here, the 4D solid torus system $N(k_*)$ can be replaced with any given Y_* because $N(k_*)$ is isotopic to Y_* in S^4. This completes the proof of Lemma 3.5. □

The proof of Lemma 3.4 is obtained from Lemma 3.5 as follows.

Proof of Lemma 3.4. Consider the 4-manifold X diffeomorphic to the 4-sphere S^4 which is obtained from Σ by replacing a regular neighborhood system $N(S(D'_*)) = S^2 \times D^2_*$ of the 2-sphere system $S(D'_*)$ in Σ with the 4D solid torus system $Y_* = D^3 \times S^1_*$. Let $E^u_* \times I$ and $E_* \times I$ be the 2-handle systems in $X = S^4$ attached to the 4D solid torus system Y_* which are obtained from the thickening 2-sphere systems $uS(D_*) \times I$ and $S(D_*) \times I$ in Σ, respectively. Lemma 3.5 can be used for the 2-handle systems $E^u_* \times I$ and $E_* \times I$ attached to Y_*. Then, there is an orientation-preserving diffeomorphism $\rho : S^4 \to S^4$ sending Y_* identically such that

$$\rho(E^u_* \times I, E^u_*) = (E_* \times I, E_*).$$

By returning the 4D solid torus system Y_* in X to the regular neighborhood system $N(S(D'_*))$ of the 2-sphere system $S(D'_*)$ in Σ, there is an orientation-preserving diffeomorphism $\rho' : \Sigma \to \Sigma$ sending $N(S(D'_*))$ identically such that $\rho' u S(D_*) = S(D_*)$. For the pseudo-O2-sphere basis $(S(D_*), \rho' u S(D'_*))$ and the O2-sphere basis $(S(D_*), S(D'_*))$ in Σ, consider the 4-manifold X' obtained from Σ by replacing a regular neighborhood system $N(S(D_*)) = S^2 \times D^2_*$ of the 2-sphere system $S(D_*)$ in Σ with the 4D solid torus system $Y'_* = D^3 \times S^1_*$. Then the 4-manifold X' is diffeomorphic to the 4-sphere S^4. Let $E'^u_* \times I$ and $E'_* \times I$ be the 2-handle systems in $X' = S^4$ attached to the 4D solid torus system Y'_* which are obtained from the thickening 2-sphere systems $\rho' u S(D'_*) \times I$ and $S(D'_*) \times I$ in Σ, respectively. Lemma 3.5 can be used for the 2-handle systems $E'^u_* \times I$ and $E'_* \times I$ attached to Y'_*. Then, there is an orientation-preserving diffeomorphism $\rho'' : S^4 \to S^4$ sending Y'_* identically such that

$$\rho''(E'^u_* \times I, E'^u_*) = (E'_* \times I, E'_*).$$

By returning the 4D solid torus system Y'_* in X' to the regular neighborhood system $N(S(D_*))$ of the 2-sphere system $S(D_*)$ in Σ, there is an orientation-preserving diffeomorphism $\rho''' : \Sigma \to \Sigma$ sending $N(S(D_*))$ identically with $\rho''' \rho' u S(D'_*) = S(D'_*)$. For the orientation-preserving diffeomorphism $h = \rho''' \rho' : \Sigma \to \Sigma$, the composite embedding $hu : \Sigma^{(0)} \to \Sigma$ preserves the O2-sphere basis $(S(D_*), S(D'_*))$ identically. This completes the proof of Lemma 3.4. □

Completion of Proof of Theorem 1.1. Since

$$(S(D_*), S(D'_*)) = (R_*, R'_*), \quad (uS(D_*), uS(D'_*)) = (S_*, S'_*),$$

the orientation-preserving diffeomorphism h of Σ in Lemma 3.4 sends (S_*, S'_*) to (R_*, R'_*). This completes the proof of Theorem 1.1. □

Note 3.6. The diffeomorphism h in Lemma 3.4 is taken to be isotopic to the identity by using the following lemma (*Framed light-bulb isotopy lemma*) based on Gabai's 4D light-bulb theorem, [5] instead of Lemma 3.5.

Lemma 3.7 (Framed Light-bulb Isotopy Lemma). Let Y_* be a system of mutually disjoint 4D solid tori $Y_i\,(i = 1, 2, \ldots, n)$ in S^4. Let $D_* \times I$ and $E_* \times I$ be systems of mutually disjoint framed disks $D_i \times I, E_i \times I\,(i = 1, 2, \ldots, n)$ in $\cap Y^c_*$ such that

$$(D_* \times I) \cap \partial Y^c_i = (\partial D_i) \times I = (\partial E_i) \times I = (E_* \times I) \cap \partial Y^c_i$$

and $\partial D_i = \partial E_i$ is a boundary fiber circle of Y_i for all i. If the unions $D_i \cup E_i\,(i = 1, 2, \ldots, n)$ are mutually disjoint, then there is a diffeomorphism $h : S^4 \to S^4$ which is Y_*-relatively isotopic to the identity such that $h(D_* \times I, D_*) = (E_* \times I, E_*)$.

Proof of Lemma 3.7. As in the proof of Lemma 3.5, assume that a boundary collar system of D_* coincides with a boundary collar system of E_*. First, show the assertion of the special case $n = 1$. Let $Y = S^1 \times D^3$. Let D and E be proper disks in Y^c admitting trivial line bundles $D \times I, E \times I$ such that

$$(\partial D) \times I = (\partial E) \times I \subset \partial Y^c.$$

If the singular 2-sphere $D \cup (-E)$ is not null-homologous in Y^c, the disk D is ∂-relatively homologous to the 2-cycle $E + mS$ in $(Y^c, \partial Y^c)$ for a 2-sphere generator $[S]$ of $H_2(Y^c; Z)$ (which is isomorphic to Z) and some non-zero integer m. The self-intersection numbers $\mathrm{I}([D],[D])$ and $\mathrm{I}([E],[E])$ in Y^c with the given framing of $\partial D = \partial E$ in ∂Y^c and $\mathrm{I}([S],[S])$ are all 0. Thus,

$$\mathrm{I}([D],[D]) = \mathrm{I}([E],[E]) + 2m\mathrm{I}([S],[E]) = 0 \pm 2m = 0$$

and $m = 0$. This shows that the singular 2-sphere $D \cup (-E)$ is null-homologous in Y^c. By Gabai's 4D light-bulb theorem [5, Theorem 10.4], there is a diffeomorphism $\lambda : Y^c \to Y^c$ such that λ is ∂-relatively isotopic to the identity and $\lambda D = E$, which extends to a diffeomorphism $h = \lambda^+ : S^4 \to S^4$ such that h is Y-relatively isotopic to the identity and $hD = E$. Note that a diffeomorphism of S^4 preserves trivial line bundles on disks if the line bundles on the boundary circles are preserved. This is because a sole obstruction that a disk admits a trivial line bundle extending a given line bundle on the boundary circle in S^4 is that the self-intersection number of the disk with the boundary framing given by the line bundle is 0. Thus, the diffeomorphism h of S^4 has $h(D \times I, D) = (E \times I, E)$ and the assertion of the special case $n = 1$ is shown. For the proof in general case, let $K = K(k_*)$ be a connected graph in S^4 constructed from the loop system k_* of the loops $k_i = \partial D_i = \partial E_i$, $(i = 1, 2, \ldots, n)$ by adding mutually disjoint $n-1$ simple arcs a_j $(j = 1, 2, \ldots, n-1)$ not meeting any interior disk of the disk systems D_* and E_*. For every s with $1 \leq s \leq n$, let Y_s be a regular neighborhood of the disk-arc union

$$k_s \cup_{1 \leq i \leq n, i \neq s} E_i \cup_{j=1}^{n-1} a_j$$

in S^4, which is a 4D solid torus in S^4. By the proof of the special case $n = 1$, there is a diffeomorphism h_1 of S^4 such that h_1 is Y_1-relatively isotopic to the identity and $h_1 D_1 = E_1$. Next, for the disk systems $h_1(D_*)$ and E_*, there is a diffeomorphism h_2 of S^4 such that h_2 is Y_2-relatively isotopic to the identity and $h_2 h_1 D_1 = E_1$ and $h_2 h_1 D_2 = E_2$. Continuing this process, there is a diffeomorphism $h = h_n \ldots h_2 h_1$ of S^4 such that h is $N(K)$-relatively isotopic to the identity for a regular neighborhood $N(K)$ of K and $hD_i = E_i$ $(i = 1, 2, \ldots, n)$. Thus, for a regular neighborhood $N(k_*)$ of the loop system k_* in $N(K)$, this diffeomorphism h of S^4 is $N(k_*)$-relatively isotopic to the identity and $hD_i = E_i\,(i = 1, 2, \ldots, n)$. This diffeomorphism h of S^4 is $N(k_*)$-relatively isotopic to the identity for a regular neighborhood $N(k_*)$ of the loop system k_* in $N(K)$ and has $hD_i = E_i\,(i = 1, 2, \ldots, n)$, where $N(k_*)$ can be regarded as the 4D solid torus system Y_* as in the proof of Lemma 3.5. This completes the proof of Lemma 3.7. □

4. Diffeomorphisms of stable 4-sphere

Let $\mathrm{Diff}^+(D^4, \mathrm{rel}\partial)$ be the orientation-preserving diffeomorphism group of the 4-ball D^4 keeping the boundary ∂D^4 point-wise fixed. An *identity-shift* of a 4-manifold Σ is a diffeomorphism $\iota : \Sigma \to \Sigma$ obtained from the identity $1 : \Sigma \to \Sigma$ by replacing the identity on a 4-ball in Σ disjoint from F with an element of $\mathrm{Diff}^+(D^4, \mathrm{rel}\ \partial)$. The following result is a main result of this section.

Theorem 4.1. Any two homotopic diffeomorphisms of the stable 4-sphere Σ are isotopic up to a composition of one diffeomorphism and an identity-shift ι.

Because at present *it appears unknown whether* $\pi_0(\mathrm{Diff}^+(D^4, \mathrm{rel}\partial))$ *is trivial or not*, the identity-shift ι is needed in Theorem 4.1. However, it is known that any identity-shift ι is concordant to the identity since $\Gamma_5 = 0$ by Kervaire [15]). Thus, the following result is a consequence of Theorem 4.1(, whose proof is omitted).

Corollary 4.2. Any two homotopic diffeomorphisms of the stable 4-sphere Σ are concordant.

In Piecewise-Linear Category, every piecewise-linear auto-homeomorphism of the 4-disk keeping the boundary identically is piecewise-linearly ∂-relatively isotopic to the identity, [10], [17]. Thus, the following result is a consequence of Theorem 4.1(, whose proof is omitted).

Corollary 4.3. Any two homotopic piecewise-linear auto-homeomorphisms of the stable 4-sphere Σ are piecewise-linearly isotopic.

The proof of Theorem 4.1 is done as follows.

Proof of Theorem 4.1. Let $f_i\,(i = 0, 1)$ are homotopic diffeomorphisms of $\Sigma = S^4(F)_2$ for a trivial surface-knot F of genus n in S^4. Then the composite diffeomorphism $g = f_1^{-1} f_0$ of Σ is homotopic to the identity. By Lemmas 3.2, 3.3, 3.4 and Note 3.6, there is a diffeomorphism h of Σ isotopic to the identity such that the composite diffeomorphism hg of Σ sends the O2-sphere basis $(S(D_*), S(D'_*))$ identically. By the proof of Lemma 2.1, there is a proper 1-punctured trivial surface P' of F in a 4-ball A such that a region $\Omega(S(D_*), S(D'_*))$ in Σ is the double branched covering space $A(P')_2$ of the 4-ball A branched along P' and the residual region $\Omega^c(S(D_*), S(D'_*))$ is the double branched covering space $A^c(d)_2$ of the 4-ball $A^c = \mathrm{cl}(S^4 \setminus A)$ branched along the proper trivial disk-knot $d = \mathrm{cl}(F \setminus P')$ in A^c. In this situation, there is a diffeomorphism h' of Σ which is isotopic to the identity such that the composite diffeomorphism $h'hg$ of Σ preserve the region $\Omega(S(D_*), S(D'_*))$ identically, which defines a ∂-identical diffeomorphism $\delta : A^c(d)_2 \to A^c(d)_2$. Since $A^c(d)_2$ is a 4-ball and the lifting d' of the disk d to $A^c(d)_2$ is a trivial disk-knot in $A^c(d)_2$ which is ∂-parallel, there is a ∂-identical diffeomorphism δ' of $A^c(d)_2$ which is ∂-relatively isotopic to the identity such that the composite diffeomorphism $\delta'\delta$ is the identity except for a 4-ball U in $A^c(d)_2$ disjoint from d'. Let h'' be the diffeomorphism of Σ defined by δ' and the identity of $\mathrm{cl}(\Sigma \setminus A^c(d)_2)$. The composite diffeomorphism $h''h'hg$ of Σ preserves $\mathrm{cl}(\Sigma \setminus U)$ identically, which is considered as an identity-shift ι of Σ. Since $h''h'h$ is isotopic to the identity, the diffeomorphism $g = f_1^{-1}f_0$ of Σ is isotopic to ι, and thus the diffeomorphism f_0 of Σ is isotopic to the composite diffeomorphism $f_1\iota$ of Σ. This completes the proof of Theorem 4.1.

5. Conclusion

In an earlier version of this paper (see the research announcement [12]), the author tried to show Lemma 3.1 with an α-equivariant orientation-preserving diffeomorphism as f (see [12, Lemma 3.2]) by a homological argument on an O2-handle basis of a trivial surface-knot F of genus n in S^4. However, such a trial is not yet succeeded. The cause of this failure arose from a calculation error on the intersection numbers of O2-handle bases. In fact, the claim [12, Lemma 3.1] is false, which can be seen by checking the intersection numbers of the sphere-bases $(S(D_1), S(D'_1))$ and $(S(E_1), S(E'_1))$ in $\Sigma(1)$, although the Z_2-version of [12, Lemma 3.1] is true. The claim [12, Lemma 3.2] and the related claims in [12] except for [12, Lemma 3.1] are affirmatively solved by using Theorem 1.1, Corollary 2.2 and Theorem 4.1 if *every sphere-basis* $(S*, S'*)$ *of* Σ *is homotopic to the sphere-basis* $(S(E_*), S(E'_*))$ *of an O2-handle basis* (E_*, E'_*) *of* F *in* S^4. It is hoped that attempts to understand every sphere basis of Σ by using the O2-handle bases of a trivial surface-knot F in S^4 will help simplify the proof of Theorem 1.1.

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