Notes on **Abe's Bimodules**

Kaneda Masaharu
Osaka City University
Department of Mathematics
kaneda@sci.osaka-cu.ac.jp

November 2, 2020

Abstract

This is a detailed exposition of recent work by ABE noriyuki on Soergel bimodules and their action on the principal blocks of G_1T for reductive algebraic groups G in positive characteristic.

This is a sequel to my lecture note [K19], an introduction to [RW18]. In order to describe the irreducible characters of reductive algebraic groups in positive characteristic p, Lusztig [L] conjectured that they should be given in terms of the Kazhdan-Lusztig polynomials of the associated 岩堀-Hecke algebra. Although the conjecture holds for large p, Williamson [W] has recently found its failure for not so small p against the expectation for a long time.

In the monumental monograph [RW18] Riche and Williamson defined an action of the Elias-Williamson category $\mathcal{D}$ [EW16] on the pricipal block of the algebraic representations of the general linear group $\mathrm{GL}_n(\Bbbk)$ over an algebraically closed field $\Bbbk$ of characteristic $p > n$, and showed that the character formulae for the indecomposable tilting modules for $\mathrm{GL}_n(\Bbbk)$ are described by the p-Kazhdan-Lusztig basis of the associated 岩堀-Hecke algebra. Assuming the existence of an action of $\mathcal{D}$ on the principal blocks for reductive algebraic groups in general, moreover, [RW18] obtained character formulae of the indecomposable tilting modules likewise,from which the formulae for irreducibles would follow. Subsequently, using geometry, without invoking the action of $\mathcal{D}$, Achar, Makisumi, Riche, and Williamson [AMRW] obtained the characters of the indecmposable tilting modules for reductive groups in terms of the p-Kazhdan-Lusztig polynomials by for $p > h$ the Coxeter number of G, from which the irreducible characters can now be obtained thanks to Sobaje [Sob] by an elementary algorithm though not entirely in terms of the p-Kazhdan-Lusztig polynomials.

When I was finishing up an ealier version of [K19], [Ab19a] appeared and, soon after, [Ab19b]. In [K19] I gave an action of $\mathcal{D}$ on the pricipal block of the representations of G_1T, G_1 the Frobenius kernel of and T a maximal torus of $\mathrm{GL}_n(\Bbbk)$. The Elias-Williamson category $\mathcal{D}$ is a diagrammatic categorification of the 岩堀-Hecke algebra $\mathcal{H}$ for any Coxeter system $(\mathcal{W}, \mathcal{S})$, and is equivalent to the category of Soergel bimodules. In [Ab19a] Abe gives in the classical language of algebras and combinatorics his version of Soergel bimodules which categorifies $\mathcal{H}$.

Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p > h$. For G_1T-modules a guiding object is Lusztig's periodic module [L80] in place of the anti-sphereical module [RW18] for G-modules. Let $\mathcal{W}$ denote the affine Weyl group of G and $\mathrm{Rep}_0(G_1T)$ the category of G_1T-modules whose composition factors all have highest weights in the $\mathcal{W}$-orbit of 0, denoted $\hat{L}(x \bullet 0)$, $x \in \mathcal{W}$, $x \bullet 0 = px\frac{\rho}{p} - \rho$, $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$, Δ^+ a positive system of the roots of G. It contains the principal block of G_1T [J, II.9.19]. The knowledge of the characters of all $\hat{L}(x \bullet 0)$ gives the entire irreducible characters for G by Curtis' theorem, Steinberg's tensor product theorem and the translation principle. Let $\hat{\Delta}(x \bullet 0)$, $x \in \mathcal{W}$, denote the G_1T-Verma module of highest weight $x \bullet 0$, denoted $\hat{Z}_1(x \bullet 0)$ in [J, II.9.1]. Those G_1T-Verma modules uniformly have dimension $p^{|\Delta^+|}$ [J, II.9.2], and live in $\mathrm{Rep}_0(G_1T)$ [J, II.9.15]. As their characters are known [J, II.9.2] and as the matrix of the multiplicities $[\hat{\Delta}(x \bullet 0) : \hat{L}(y \bullet 0)]$ of $\hat{L}(y \bullet 0)$ in a composition series of $\hat{\Delta}(x \bullet 0)$ is unipotent, those multiplicities will yield the characters of $\hat{L}(y \bullet 0)$. Let $\mathrm{Rep}_0'(G_1T)$ denote the full subcategory of $\mathrm{Rep}_0(G_1T)$ consisting of those that admit a filtration, called a $\hat{\Delta}$-flag, whose subquotients are all of the form $\hat{\Delta}(x \bullet 0)$, $x \in \mathcal{W}$. Tilting modules for G_1T are injectives; an injective G_1T-module is also projective [J, II.9.4], admits a $\hat{\Delta}$-filtration, and also a filtration whose subquotients are all dual G_1T-Verma modules [J, II.11.4]. Let $\hat{Q}(x\bullet 0)$ be the G_1T-injective hull of $\hat{L}(x\bullet 0)$, which is also its projective cover [J, II.11.5]. Then the multiplicity $(\hat{Q}(x\bullet 0) : \hat{\Delta}(y\bullet 0))$ of $\hat{\Delta}(y\bullet 0)$, $y \in \mathcal{W}$, in a $\hat{\Delta}$-filtration of $\hat{Q}(x \bullet 0)$ is equal [J, II.11.4, 9.9] to

$$[\hat{\Delta}(y \bullet 0) : \hat{L}(x \bullet 0)] = \dim G_1T\mathrm{Mod}(\hat{\Delta}(y \bullet 0), \hat{Q}(x \bullet 0)).$$

Thus, we may focus our study on $\mathrm{Rep}_0'(G_1T)$. Let $[\mathrm{Rep}_0'(G_1T)]$ denote the Grothendieck group of $\mathrm{Rep}_0'(G_1T)$, a completion of which gives the Grothendieck group of the whole of $\mathrm{Rep}_0(G_1T)$. Letting $\mathbb{Z}[\mathcal{W}]$ denote the group algebra of $\mathcal{W}$, one has an isomorphism of abelian groups

$$\mathbb{Z}[\mathcal{W}] \to [\mathrm{Rep}_0'(G_1T)] \quad \text{via} \quad x \mapsto [\hat{\Delta}(x \bullet 0)], \quad x \in \mathcal{W}, \tag{1}$$

under which the right multiplication by $s+1$, $s \in \mathcal{S}$ the set of distinguished generators of $\mathcal{W}$, on the LHS is given by the wall-crossing functor Θ_s on the RHS [J, II.9.22]. Now, consider a quantization of $\mathbb{Z}[\mathcal{W}]$ by 岩堀-Hecke algebra $\mathcal{H}$ over the Laurent polynomial ring $\mathbb{Z}[v, v^{-1}]$. If we let $(H_x|x \in \mathcal{W})$ denote the standard basis of $\mathcal{H}$ after [S97], $\forall s \in \mathcal{S}$,

$$H_x(H_s + v) = \begin{cases} H_{xs} + vH_x & \text{if } xs > x, \\ H_{xs} + v^{-1}H_x & \text{else.} \end{cases}$$

Thus, the isomorphism (1) allows $\mathcal{H}$ to act on $[\mathrm{Rep}_0'(G_1T)]$ by specialization $v \rightsquigarrow 1$, and hence $H_s + v$ specializing to Θ_s $\forall s \in \mathcal{S}$. Let $(\underline{H}_x|x \in \mathcal{W})$ denote the Kazhdan-Lusztig basis of $\mathcal{H}$ and write $\underline{H}_x = \sum_{y \in \mathcal{W}} h_{y,x}H_y$, $h_{y,x} \in \mathbb{Z}[v, v^{-1}]$. The $h_{x,y}$ are the celebrated Kazhdan-Lusztig polynomials. Let $\mathcal{W}^{\mathrm{res}} = \{w \in \mathcal{W} | \langle w \bullet 0, \alpha^\vee \rangle \in]0, p[\ \forall \alpha \in \Delta^+ \text{ simple}\}$. Lusztig's conjecture for G_1T may be phrased to assert that, $\forall w \in \mathcal{W}^{\mathrm{res}}$, $\forall x \in \mathcal{W}$,

$$(\hat{Q}(w_0 w \bullet 0) : \hat{\Delta}(x \bullet 0)) = h_{x, w_0 w}(1),$$

where $w_0 \in \mathcal{W}$ is such that $w_0\Delta^+ = -\Delta^+$. Let now $\mathcal{A}$ be the set of alcoves, which are the connected components of $(X \otimes_{\mathbb{Z}} \mathbb{R}) \setminus \cup_{\alpha \in \Delta^+, n \in \mathbb{Z}} \{\nu \in X \otimes_{\mathbb{Z}} \mathbb{R} | \langle \nu, \alpha^\vee \rangle = n\}$. If $A^+ \in \mathcal{A}$ is the alcove containing $\frac{\rho}{p}$, there is a bijection $\mathcal{W} \to \mathcal{A}$ via $x \mapsto xA^+$, $x \in \mathcal{W}$, under which import the left and the right regular actions of $\mathcal{W}$ onto $\mathcal{A}$. Then the free $\mathbb{Z}[v, v^{-1}]$-module $\mathcal{P}$ of basis $\mathcal{A}$ is isomorphic to $\mathcal{H}$ via $H_x \mapsto xA^+$, $x \in \mathcal{W}$, and comes equipped with a structure of right

$\mathcal{H}$-module transferring the right regular action of $\mathcal{W}$, which is Lusztig's periodic module for $\mathcal{H}$. In [Ab19b] he categorifies $\mathcal{P}$ to admit an action of Soergel bimodules, inducing an action on $\mathrm{Rep}_0(G_1T)$ compatible with the wall-crossing functors.

Given a Coxeter system $(\mathcal{W}, \mathcal{S})$, Soergel's original bimodules are defined over the symmetric algebra of a reflection faithful linear representation of $\mathcal{W}$, to categorify the 岩堀-Hecke algebra of $(\mathcal{W}, \mathcal{S})$ [S92], [S07]. As the reflection faithfulness is hard to come by in positive characteristic, Elias and Williamson [EW16] start with a much less restrictive representation of $\mathcal{W}$, and Abe follows suit. In order to maintain some sort of faithfulness of the representation, Abe's version of Soegel bimodules [Ab19a] comes with a condition that they split into "weight spaces" with respect to $\mathcal{W}$ over the fractional field of the symmetric algebra started out with. They may in fact be defined over the symmetric algebra localized by the roots.

Back to G, Abe's category $\tilde{\mathcal{K}}'$ [Ab19b] of bimodules categorifying Lusztig's periodic module $\mathcal{P}$ are bimodules over the symmetric algebra S of the coweight lattice of G over $\Bbbk$ by base change. The bimodules split into the "weight spaces" with respect to the affine Weyl group $\mathcal{W}$ of G over the localization of the symmetric algebra by the coroots. As the alcoves are in bijective correspondence with $\mathcal{W}$, the decomposition may be parametrized by $\mathcal{A}$, recording the linkage principle for G [J, II.6]. The right S-module structure on bimodules in $\tilde{\mathcal{K}}'$ is designed to admit an action of his version $\mathfrak{SB}$ of Soergel bimodules associated to $(\mathcal{W}, \mathcal{S})$. As the action of $\mathcal{W}$ on the coweight lattice is not linear, however, he annihilates the translations, losing the faithfulness of the representation by $\mathcal{W}$. The eventual import of the $\mathfrak{SB}$-action onto $\mathrm{Rep}_0(G_1T)$ is performed on the projectives through the Andersen-Jantzen-Soergel combinatorial category [AJS] in the style of Fiebig [F11] such that the actions of the indecomposables in $\mathfrak{SB}$ associated to $\mathcal{S}$ are compatible with the corresponding wall-crossing translation functors. For that end, conditions (S), (LE), (ES) from Fiebig [F08a], [F08b] and Fiebig+Lanini [FL15] are imposed on $\tilde{\mathcal{K}}'$ to define a subcategory $\tilde{\mathcal{K}}_P$ of projectives, (S) standing for "sheafification", and (ES) for "exact structure" in Fiebig's theory of sheaves on moment graphs. The properties (S) and (LE) allow gluing the SL_2-theory. Finally, an ideal quotient $\mathcal{K}_P$ of $\tilde{\mathcal{K}}_P$ gives a desired equivalence with the projectives of $\mathrm{Rep}_0(G_1T)$ deformed over the completion of the symmetric algebra S. The categories $\tilde{\mathcal{K}}', \tilde{\mathcal{K}}_P, \mathcal{K}_P, \mathfrak{S}\mathfrak{G}$ are all graded, and the work is fruit of graded reprensentation theory. There is also a version for singular Soergel bimodules [Ab20].

By now there is a formula available for the irreducibles of $\mathrm{Rep}_0(G_1T)$ for reductive groups in general in terms of p-Kazhdan-Lusztig polynomials for $p > 2h-1$, due to Riche and Williamson [RW19] without invoking an action by the Soergel bimodules on the principal block. Abe's bimodules, however, certainly provide more algebraic insight to the representation theory of G_1T. The indecomposable projective of $\tilde{\mathcal{K}}$, corresponding to $\hat{Q}(w_0 \bullet 0)$, is obtained by applying the indecomposable Soergel bimodule associated to w_0 on the rank 1 standard bimodule of $\tilde{\mathcal{K}}$ corresponding to $\hat{\Delta}(w_0 \bullet 0)$. All the other projectives of $\tilde{\mathcal{K}}$ are obtained by applying $\mathfrak{SB}$ further on the seminal projective indecomposable, translations, degree shift, and taking direct summands. It is now desired that the indecomposable projective G_1T-modules $\hat{Q}(x \bullet 0)$ be described concretely by the action of $\mathfrak{SB}$ and that the properties of the characters of the indecomposables of $\mathfrak{SB}$, the p-Kazhdan-Lusztig basis of $\mathcal{H}$ in the present sense, to be clarified.

I am very much grateful to Abe for patiently explaining his work. Though Abe writes very well, it will be of my pleasure if this may be of any further help.

I. Soergel bimodules

Throughout the chapter $(\mathcal{W}, \mathcal{S})$ will denote a Coxeter system with $|\mathcal{S}| < \infty$, and $\mathbb{K}$ a noetherian domain; we will impose additional conditions on $\mathbb{K}$ as we move along. Specifically, we impose a mild condition in (3.4). From §4 on we will assume that $\mathbb{K}$ is local, so that a direct summand of a free R-module remains free, R a polynomial ring defined at the outset in (1.1). From §6 on we assume that $\mathbb{K}$ is a complete noetherian local domain, so that our categories are Krull-Schmidt. In §6 we impose the GKM condition on V, V introduced in (1.1). §7 is an exposition of [S92] and we assume that $\mathbb{K}$ is an infinite field and the characteristic of $\mathbb{K}$ is not a torsion prime so that Demazure's result [Dem] holds, and in addition that $2 \neq 0$ in $\mathbb{K}$ and also $3 \neq 0$ if type G_2 is involved as a component. In §§8 and 9 we assume that $\mathbb{K}$ is a complete DVR under the characteristic restrictions from §7.

The length function on $(\mathcal{W}, \mathcal{S})$ is denoted by ℓ, and the Chevalley-Bruhat order by $\geq$. By a graded module we will always mean a $\mathbb{Z}$-graded module. If M is one, M^i, $i \in \mathbb{Z}$, will denote the i-th homogeneous piece of M. In particular, $0 \in M^i$. For $n \in \mathbb{Z}$ we let $M(n)$ denote M with the grading shifted by n such that $M(n)^i = M^{i+n}$ $\forall i \in \mathbb{Z}$.

1. **Basic set-up**

1.1. After [EW16], let $(V, \{\alpha_s | s \in \mathcal{S}\}, \{\alpha_s^\vee | s \in \mathcal{S}\})$ be a triple of a free $\mathbb{K}$-module V of finite rank with a $\mathbb{K}$-linear action of $\mathcal{W}$, $\alpha_s \in V$, $\alpha_s^\vee \in V^\vee = \mathrm{Mod}_{\mathbb{K}}(V, \mathbb{K})$, such that $\forall s \in \mathcal{S}$,

(i) $\langle \alpha_s, \alpha_s^\vee \rangle = 2$,

(ii) $s(v) = v - \langle v, \alpha_s^\vee \rangle \alpha_s \quad \forall v \in V$,

(iii) $\alpha_s^\vee : V \twoheadrightarrow \mathbb{K}$ and $\alpha_s \neq 0$; a priori $\mathbb{K}$ may be of characteristic 2.

We let $\mathcal{W}$ act on $V^\vee$ contragrediently: $fw = f(w^{-1}?)$ $\forall f \in V^\vee, w \in \mathcal{W}$.

Let $R = \mathrm{S}_{\mathbb{K}}(V)$ the symmetric algebra of V and $Q = \mathrm{Frac}(R)$ the field of fractions of R. We endow R with a structure of graded algebra with $\deg(V) = 2$. We call $t \in \mathcal{W}$ a reflection iff $t \in \cup_{w \in \mathcal{W}} w\mathcal{S}w^{-1}$, and put $\mathcal{T} = \cup_{w \in \mathcal{W}} w\mathcal{S}w^{-1}$.

Lemma: *(i) If $s, r \in \mathcal{S}$ with $r = xsx^{-1}$ for some $x \in \mathcal{W}$, $\alpha_r \in \mathbb{K}^\times x\alpha_s$.*

(ii) If $t = wsw^{-1} \in \mathcal{T}$, $w \in \mathcal{W}$, $s \in \mathcal{S}$, $w\alpha_s$ is independent of the choices of w and s up to $\mathbb{K}^\times$.

(iii) $\forall t \in \mathcal{T}$, we choose w and s such that $t = wsw^{-1}$ and define $\alpha_t = w\alpha_s$ up to $\mathbb{K}^\times$. With $\alpha_t^\vee = w\alpha_s^\vee = \alpha_s^\vee(w^{-1}?)$, one has $\forall v \in V$,

$$tv = v - \langle v, \alpha_t^\vee \rangle \alpha_t.$$

Proof: (i) By (iii) take $\delta \in V$ with $\langle \delta, \alpha_s^\vee \rangle = 1$. Then $s\delta = \delta - \langle \delta, \alpha_s^\vee \rangle \alpha_s = \delta - \alpha_s$, and hence

$$x\delta - \langle x\delta, \alpha_r^\vee \rangle \alpha_r = rx\delta = xsx^{-1}x\delta = xs\delta = x(\delta - \alpha_s) = x\delta - x\alpha_s.$$

Thus, $x\alpha_s = \zeta\alpha_r$ with $\zeta = \langle x\delta, \alpha_r^\vee\rangle \in \mathbb{K}$. In turn, as $s = x^{-1}rx$, there is some $\zeta' \in \mathbb{K}$ such that $x^{-1}\alpha_r = \zeta'\alpha_s$. Then $\zeta'\zeta\alpha_r = \zeta' x\alpha_s = \alpha_r$. As V is free over $\mathbb{K}$, we must have $\zeta'\zeta = 1$ by (i). Thus, $\alpha_r = \zeta^{-1}x\alpha_s \in \mathbb{K}^\times x\alpha_s$.

(ii) Assume $wsw^{-1} = yry^{-1}$ for some $r \in \mathcal{S}$ and $y \in \mathcal{W}$. Then $r = y^{-1}ws(y^{-1}w)^{-1}$, and hence $\alpha_r \in \mathbb{K}^\times y^{-1}w\alpha_s$ by (i). Thus, $y\alpha_s \in \mathbb{K}^\times w\alpha_s$.

(iii) One has

$$\begin{aligned} tv &= wsw^{-1}v = w(w^{-1}v - \langle w^{-1}, \alpha_s^\vee\rangle\alpha_s) = v - \langle w^{-1}v, \alpha_s^\vee\rangle w\alpha_s \\ &= v - \langle v, w\alpha_s^\vee\rangle w\alpha_s \quad \text{by definition of the } \mathcal{W}\text{-action on } V^\vee \\ &= v - \langle v, \alpha_t^\vee\rangle\alpha_t. \end{aligned}$$

1.2. Let $\mathcal{C}'$ be the category of graded R-bimodules M with $Q \otimes_R M$ admitting a decomposition $Q \otimes_R M = \coprod_{w\in\mathcal{W}} M_w^Q$ as a (Q, R)-bimodule such that

(i) $\{w \in \mathcal{W} | M_w^Q \neq 0\}$ is finite,

(ii) $\forall a \in R$, $\forall m \in M_w^Q$, $ma = (wa)m$.

Thus, if the actions of x and y on V coincide, for distinct $x, y \in \mathcal{W}$, M_x^Q and M_y^Q are separated. A morphism $\phi \in \mathcal{C}'(M, N)$ is a homomorphism of graded R-bimodules such that $(Q \otimes_R \phi)(M_w^Q) \leq N_w^Q\ \forall w \in \mathcal{W}$. Put $\mathcal{C}'^\sharp(M, N) = \coprod_{n\in\mathbb{Z}} \mathcal{C}'(M, N(n))$. We will often abbreviate $Q \otimes_R M$ and $Q \otimes_R \phi$ as M^Q and ϕ^Q, resp.

Remarks: (i) The right action by $a \in R \setminus 0$ on each M_w^Q, $w \in \mathcal{W}$, is invertible; as $wa \neq 0$, $m = \frac{1}{wa}(ma)\ \forall m \in M_w^Q$. Thus, $?a$ is invertible on the whole of $Q \otimes_R M$, and $Q \otimes_R M$ comes equipped with a structure of Q-bimodules; $m\frac{1}{a} = \frac{1}{wa}m$ if $m \in M_w^Q$. Then the decomposition $Q \otimes_R M = \coprod_{w\in\mathcal{W}} M_w^Q$ holds as a Q-bimodules.

(ii) If the action of $\mathcal{W}$ on V is not faithful, M_x^Q and M_y^Q for distinct $x, y \in \mathcal{W}$ are distinguished by definition. Assume now that $\mathcal{W}$ acts faithfully on V. Then $\forall M \in \mathcal{C}'$, $\forall w \in \mathcal{W}$,

$$M_w^Q = \{m \in Q \otimes_R M | ma = (wa)m\ \forall a \in R\},$$

and $\mathcal{C}'$ forms a full subcategory of the category RBimodgr of graded R-bimodules. For by definition LHS $\subseteq$ RHS. Let $m \in$ RHS and write $m = \sum_{x\in\mathcal{W}} m_x$ with $m_x \in M_x^Q$. Thus, $\forall a \in V$, $\sum_x (xa)m_x = ma = (wa)m = \sum_x (wa)m_x$. If $m_x \neq 0$, $xa = wa$ as M^Q is a Q-linear space with Q a field. Then $x = w$ by the hypothesis.

Let $N \in \mathcal{C}'$ and let $\phi \in R\text{Bimod}(M, N)$. Let $m \in M_w^Q$ and write $\phi^Q(m) = \sum_x \phi^Q(m)_x$ with $\phi^Q(m)_x \in N_x^Q$. Then $\forall a \in R$,

$$\sum_x (xa)\phi^Q(m)_x = \phi^Q(m)a = \phi^Q(ma) = \phi^Q((wa)m) = (wa)\sum_x \phi^Q(m)_x.$$

If $\phi^Q(m)_x \neq 0$, $xa = wa$, and hence $x = w$. Thus, $\phi^Q(m) \in N_w^Q$.

(iii) If we equip $\mathcal{C}'(M, N)$ with a structure of R-bimodule via $(a\phi b)(m) = \phi(amb) = a\phi(m)b$, $a, b \in R, \phi \in \mathcal{C}'(M, N)$, $\mathcal{C}'$ forms an R-bilinear additive category [中岡, Def. 3.1.11, p. 124, Def.

3.2.3, p. 130]. Given $\phi \in \mathcal{C}'(M,N)$, let K be the kernel of ϕ as graded R-bimodules. By flat extension one has $K^\emptyset = \ker(\phi^\emptyset) = \coprod_{x\in\mathcal{W}} \ker(\phi_x^\emptyset)$:

$$\begin{array}{ccccc}
K & \hookrightarrow & M & \xrightarrow{\ \phi\ } & N \\
\downarrow & & \downarrow & & \downarrow \\
\coprod_{x\in\mathcal{W}} \ker(\phi_x^\emptyset) & \hookrightarrow & \coprod_{x\in\mathcal{W}} M_x^\emptyset & \xrightarrow[\coprod_{x\in\mathcal{W}} \phi_x^\emptyset]{} & \coprod_{x\in\mathcal{W}} N_x^\emptyset.
\end{array}$$

Thus, $K \hookrightarrow M$ gives the kernel of ϕ in $\mathcal{C}'$. In particular, $\mathcal{C}'(S_0)$ is Karoubian/idempotent complete [中岡, Def. 3.3.40, p. 174].

1.3. Let now $\mathcal{C}^{\mathrm{tf}}$ denote a full subcategory of $\mathcal{C}'$ consisting of the torsion-free left R-modules that are also of finite type as R-bimodules. Thus, $\forall M \in \mathcal{C}^{\mathrm{tf}}$,

(1)
$$\begin{array}{ccc}
M & & \\
\wr\!\mid & \searrow & \\
R \otimes_R M & \hookrightarrow & Q \otimes_R M,
\end{array}$$

and hence

(2) $$M \text{ is torsion-free also as a right } R\text{-module.}$$

For if $m \in M \setminus 0$ and $a \in R$ with $ma = 0$, writing $m = \sum_x m_x$ in M^Q with $m_x \in M_x^Q$,

$$0 = ma = \sum_x (xa) m_x.$$

If $m_x \neq 0$, $xa = 0$, and hence $0 = x^{-1}(xa) = a$.

One has an isomorphism of R-bimodules

(3) $$M \otimes_R Q \to M^Q \quad \text{via} \quad m \otimes \frac{a}{b} \mapsto (1 \otimes m)\frac{a}{b} = \sum_{w\in\mathcal{W}} \frac{wa}{wb} m_w,$$

where $1 \otimes m = \sum_{w\in\mathcal{W}} m_w$ with $m_w \in M_w^Q$. For the map is well-defined by Rmk. 1.2.(i). One has

$$\begin{aligned}
\sum_{\text{finite}} m_i \otimes \frac{a_i}{b_i} &= \sum m_i a_i \otimes \frac{1}{b_i} = \sum m_i a_i \otimes \frac{b}{b_i}\frac{1}{b} \quad \text{with } b = \prod b_i \\
&= \sum m_i \frac{a_i b}{b_i} \otimes \frac{1}{b},
\end{aligned}$$

and hence any element of $M \otimes_R Q$ is of the form $m \otimes \frac{1}{b}$. If $(1 \otimes m)\frac{1}{b} = 0$, $1 \otimes m = 0$, and hence

$m = 0$ as $M \hookrightarrow M^Q$. Thus, the map is injective. To see its surjectivity, one has

$$\begin{aligned}
\frac{1}{a} \otimes m &= \sum_{w\in I} \frac{1}{a} m_w \quad \text{for some finite } I \text{ by definition (1.2.i)} \\
&= \sum_{w\in I} m_w \frac{1}{w^{-1}a} = \sum_{w\in I} m_w \frac{a'}{w^{-1}a}\frac{1}{a'} \quad \text{with } a' = \prod_{w\in I} w^{-1}a \\
&= \sum_{w\in I} m_w \frac{a'c}{w^{-1}a}\frac{1}{ca'} \quad \text{for some } c \in R \text{ such that } \forall w \in I,\ m_w \frac{a'c}{w^{-1}a} \in M \cap M_w^Q \\
&= \sum_{w\in I} (1 \otimes m_w \frac{a'c}{w^{-1}a}) \frac{1}{ca'}.
\end{aligned}$$

1.4. Let $M \in \mathcal{C}^{\mathrm{tf}}$. $\forall m \in M$, let m_w denote the w-component of m under $M \hookrightarrow \coprod_{x\in\mathcal{W}} M_x^Q$. $\forall I \subseteq W$, let $M_I = M \cap \coprod_{w\in I} M_w^Q$ and $M^I = \mathrm{im}\left(\begin{array}{ccc} M & \hookrightarrow & \coprod_{w\in\mathcal{W}} M_w^Q \\ & \searrow & \downarrow \\ & & \coprod_{x\in I} M_x^Q \end{array}\right)$. Thus,

$$M_I \leq M^I \leq \prod_{w\in I} M_w^Q, \tag{1}$$

and there is a short exact sequence in $\mathcal{C}^{\mathrm{tf}}$ [中岡, Def. 3.3.29]

$$0 \to M_{\mathcal{W}\setminus I} \to M \to M^I \to 0, \tag{2}$$

i.e., $(M_{\mathcal{W}\setminus I} \to M) = \ker_{\mathcal{C}^{\mathrm{tf}}}(M \to M^I)$ and $(M \to M^I) = \mathrm{coker}_{\mathcal{C}^{\mathrm{tf}}}(M_{\mathcal{W}\setminus I} \to M)$.

Warning: In II a similar notation M_I will have different meaning.

$\forall w \in \mathcal{W}$, put for simplicity $M_w = M_{\{w\}}$ and $M^w = M^{\{w\}}$. Note that on both M_w and M^w one has $ma = (wa)m$ $\forall m \in M_w$ (resp. M^w) $\forall a \in R$, and hence their left R-module structure are completely determined by the right R-module structure and vice versa. One has

$$\begin{array}{ccc} m \quad M & \hookrightarrow & M^Q \\ \downarrow & \searrow & \uparrow \\ \sum_{w\in\mathcal{W}} m_w & & \coprod_{w\in\mathcal{W}} M^w. \end{array} \tag{3}$$

Let $\mathrm{supp}_{\mathcal{W}}(M) = \{w \in \mathcal{W} | M_w^Q \neq 0\}$, and $\forall m \in M$, $\mathrm{supp}_{\mathcal{W}}(m) = \{w \in \mathcal{W} | m_w \neq 0\}$. Thus,

$$M_I = \{m \in M | \mathrm{supp}_{\mathcal{W}}(m) \subseteq I\}. \tag{4}$$

Lemma: *(i)* $\mathrm{supp}_{\mathcal{W}}(M) = \{w \in \mathcal{W} | M_w \neq 0\} = \{w \in \mathcal{W} | M^w \neq 0\}$.

(ii) $M_I, M^I \in \mathcal{C}^{\mathrm{tf}}$ *with* $(M_I)^Q = (M^I)^Q = \coprod_{w\in I} M_w^Q$.

(iii) If $J \subseteq \mathcal{W}$, $M_I \cap M_J = M_{I\cap J} = (M_I)_J$.

(iv) If $J \subseteq I$, $M_I/M_J \in \mathcal{C}^{\mathrm{tf}}$.

(v) If $N \in \mathcal{C}^{\mathrm{tf}}$ with $\mathrm{supp}_{\mathcal{W}}(N) \subseteq I$,

$$\mathcal{C}^{\mathrm{tf}}(M, N) \simeq \mathcal{C}^{\mathrm{tf}}(M^I, N), \quad \mathcal{C}^{\mathrm{tf}}(N, M) \simeq \mathcal{C}^{\mathrm{tf}}(N, M_I).$$

Proof: (i) Let $w \in \mathrm{supp}_{\mathcal{W}}(M)$. As $M_w^Q \neq 0$, there are $m \in M$ and $q \in Q$ such that $qm \in M_w^Q \setminus 0$. If $q = \frac{b}{a}$ with $a, b \in R$, $bm \in M_w \setminus 0$. The assertion now follows from (1).

(ii) By (1) again it is enough to check that $(M_I)^Q \supseteq \coprod_{w\in I} M_w^Q$. Let $m \in \coprod_{w\in I} M_w^Q$. There is $a \in R$ with $am_w \in M \setminus 0$ $\forall w \in I$, and hence $am_w \in M_I$. Then

$$m = \frac{1}{a}(am) = \frac{1}{a}\sum_{w\in I} am_w \in (M_I)^Q.$$

(iii) One has

$$\begin{aligned} M_I \cap M_J &= (M \cap \coprod_{x\in I} M_x^Q) \cap (M \cap \coprod_{y\in J} M_y^Q) = M \cap \coprod_{x\in I} M_x^Q \cap \coprod_{y\in J} M_y^Q = M \cap \coprod_{w\in I\cap J} M_w^Q \\ &= M_{I\cap J}. \end{aligned}$$

Likewise, $(M_I)_J = M_{I\cap J}$.

(iv) One has

$$\begin{aligned} M_I/M_J &= M_I/(M_I)_J \quad \text{by (iii)} \\ &\simeq (M_I)^{\mathcal{W}\setminus J} \quad \text{by (2)} \\ &\in \mathcal{C}^{\mathrm{tf}} \quad \text{by (ii)}. \end{aligned}$$

(v) follows from (2).

1.5. **Lemma:** *Any $M \in \mathcal{C}^{\mathrm{tf}}$ is of finite type both as a left and right R-module.*

Proof: $\forall w \in \mathcal{W}$, $M \twoheadrightarrow M^w$, and hence M^w is of finite type over $R \otimes_{\mathbb{K}} R$ by definition (1.3). Moreover, $\forall m \in M^w$, $\forall a \in R$, $ma = (wa)m$, and hence M^w is of finite type as a left R-module. As $M \hookrightarrow \coprod_{w\in\mathcal{W}} M^w$ and as $\mathrm{supp}_{\mathcal{W}}(M)$ is finite, M must be of finite type as a left R-module. Likewise as a right R-module.

1.6. A prime example of an object of $\mathcal{C}^{\mathrm{tf}}$ is $R(w)$, $w \in \mathcal{W}$, which is R as the ordinary graded left R-module with a structure of R-bimodule such that

(1) $$ab = (wb)a \quad \forall a \in R(w), b \in R.$$

Thus, $R(w)^Q = R(w)_w^Q = Q$ with the (Q, R)-bimodule structure induced by (1). Put $Q(w) = Q \otimes_R R(w)$. Note that $R(w) \simeq R$ as graded right R-modules via $a \mapsto w^{-1}a$.

$\forall M \in \mathcal{C}^{\mathrm{tf}}$, $\forall n \in \mathbb{Z}$,

(2) $$\mathcal{C}^{\mathrm{tf}}(R(w), M(n)) \simeq M_w^n,$$

and hence

(3) $$\coprod_{n\in\mathbb{Z}} \mathcal{C}^{\mathrm{tf}}(R(w), M(n)) \simeq M_w.$$

Warning: If $f \in \mathcal{C}^{\mathrm{tf}}(M, N)$ is surjective, it may happen that $f_w : M_w \to N_w$ is NOT surjective for some $w \in \mathcal{W}$, cf. (2.2.15) below. Thus, $R(w)$ need not be "projective" in $\mathcal{C}^{\mathrm{tf}}$.

1.7. In order for it to be closed under taking tensor products over R, define a full subcategory $\mathcal{C}$ of $\mathcal{C}^{\mathrm{tf}}$ consisting of all M flat as a left R-module. For $I \subseteq \mathcal{W}$, M_I and M^I may not remain flat over R. If $\phi \in \mathcal{C}(M, N)$, $\ker_{\mathcal{C}^{\mathrm{tf}}}(\phi)$ may not be flat over R. If ϕ is an idempotent, however, $\ker_{\mathcal{C}^{\mathrm{tf}}}(\phi)$ from Rmk. 1.2.(iii) gives the kernel of ϕ in $\mathcal{C}$, and hence

(1) $\mathcal{C}$ is Karoubian complete [中岡, Def. 3.3.40, p. 174] .

$\forall M, N \in \mathcal{C}$, put as in (1.2)

$$\mathcal{C}^{\sharp}(M, N) = \coprod_{n\in\mathbb{Z}} \mathcal{C}(M, N(n)).$$

$\forall M, N \in \mathcal{C}$, $M \otimes_R N \hookrightarrow (M \otimes_R N)^Q \simeq M^Q \otimes_Q N^Q$, and hence $M \otimes_R N \in \mathcal{C}$ with

$$(M \otimes_R N)_w^Q = \sum_{\substack{x,y\in\mathcal{W}\\ xy=w}} M_x^Q \otimes_Q N_y^Q = \sum_{\substack{x,y\in\mathcal{W}\\ xy=w}} M_x^Q \otimes_R N_y^Q,$$

which we will denote by $M * N$. Thus, $\mathcal{C}$ comes equipped with a structure of monoidal category with the unit object $R(e)$ [中岡, Def. 3.5.2, p. 211]. In particular,

Lemma: *$\forall M, N \in \mathcal{C}$, $\mathrm{supp}_{\mathcal{W}}(M * N) = \{xy | x \in \mathrm{supp}_{\mathcal{W}}(M), y \in \mathrm{supp}_{\mathcal{W}}(N)\}$.*

1.8 **Graded rank:** Let $M = \coprod_{i\in\mathbb{Z}} M^i$ be a graded left/right R-module. If $a \in R^d$ for some $d \in \mathbb{Z}$, $M \to M(d)$ via $m \mapsto am$ is a homomorphism of graded modules, i.e., of degree 0. In particular,

(1) $$\begin{array}{ccccc} R(-d)^i = R^{i-d} \ni b & \mapsto & ab \in R^i & \quad R(-d) \xrightarrow{\ a?\ } R \\ & & & \quad R(-d) \overset{\sim}{\dashrightarrow} aR \hookrightarrow R. \end{array}$$

We say M is a graded free R-module iff $M \simeq \coprod_j R(n_j)$, $n_j \in \mathbb{Z}$, in which case its graded rank is defined to be

$$\mathrm{grk}(M) = \sum_j v^{n_j} \in \mathbb{Z}[v, v^{-1}], \quad v \text{ an indeterminate.}$$

Thus,

$$\mathrm{grk}(M(1)) = v\mathrm{grk}(M). \tag{2}$$

In particular,

$$\mathrm{grk}(M(n)(1)) = \mathrm{grk}(M(n+1)) = v^{n+1}\mathrm{grk}(M).$$

If M_1 and M_2 are both graded free,

$$\mathrm{grk}(M_1 \oplus M_2) = \mathrm{grk}(M_1) + \mathrm{grk}(M_2). \tag{3}$$

If M has a homogeneous basis $\{m_i\}_i$,

$$\mathrm{grk}(M) = \mathrm{grk}(\coprod_i Rm_i) = \sum_i \mathrm{grk}(Rm_i) = \sum_i \mathrm{grk}(R(-\deg(m_i))) = \sum_i v^{-\deg(m_i)}. \tag{4}$$

Eg. Let $M \in \mathcal{C}$ and $w \in \mathcal{W}$. If M^w is a graded free R-module, one has in $\mathcal{C}$

$$M^w \simeq \coprod_i R(w)(n_i) \quad \exists n_i \in \mathbb{Z}.$$

Likewise for M_w.

Lemma: *Assume that $\mathbb{K}$ is a field. Let M be a graded left R-module of graded rank $q \in \mathbb{Z}[v, v^{-1}]$ with a filtration of graded R-modules $0 = M_0 < M_1 < \cdots < M_r = M$. If $N_i \leq M_i/M_{i+1}$ is a graded free of graded rank q_i such that $\sum_i q_i = q$, then $N_i = M_i/M_{i+1}$ $\forall i$.*

Proof: $\forall k \in \mathbb{Z}$, $\sum_i \dim(N_i^k)$ is equal to the coefficient of v^k in $\sum_i q_i = q$, and hence

$$\sum_i \dim(N_i^k) = \dim M^k == \sum_i \dim(M_i/M_{i+1})^k.$$

Thus, $N_i = M_i/M_{i+1}$.

2. **Soergel bimodules**

2.1. $\forall s \in \mathcal{S}$, put $R^s = \{a \in R | sa = a\}$, and set $B(s) = R \otimes_{R^s} R(1)$ an ordinary R-bimodule by the multplications on the 1st and the 2nd component. To verify that $B(s)$ admits a structure of $\mathcal{C}$, let $\delta \in V$ with $\langle \delta, \alpha_s^\vee \rangle = 1$, using the standing assumption (1.1.iii). Recall first from [EW16, claim 3.11]

Lemma: $R = R^s \oplus \delta R^s = R^s \oplus (s\delta)R^s$.

Proof: We check first that $R^s \cap \delta R^s = 0$. Let $0 = x + \delta y$ with $x, y \in R^s$. Then

$$0 = s(x + \delta y) = x + (s\delta)y = x + (\delta - \alpha_s)y,$$

and hence $\alpha_s y = 0$. Thus, $y = 0$, and hence $x = 0$ also.

Now, $\forall m \in V$, $m = (m - \langle m, \alpha_s^\vee\rangle\delta) + \langle m, \alpha_s^\vee\rangle\delta$ with $m - \langle m, \alpha_s^\vee\rangle\delta \in V^s$, and hence $V = V^s \oplus \delta\mathbb{K}$. Let R^d be the d-th homogeneous piece of R and assume inductively that $R^d = (R^d)^s \oplus \delta(R^{d-1})^s$. Let $x \in (R^d)^s, y \in (R^{d-1})^s, m \in V^s, \xi \in \mathbb{K}$. Then $(x+\delta y)(m+\xi\delta) = xm + \delta(ym+\xi x) + \xi\delta^2 y$ with $\delta^2 y = \{-\delta(s\delta) + \delta(\delta+s\delta)\}y = -\delta(s\delta)y + \delta\{(\delta+s\delta)y\}$. As $\delta(s\delta) \in (R^2)^s$ and $\delta + s\delta \in V^s$,

$$(x+\delta y)(m+\xi\delta) = xm - \xi\delta(s\delta)y + \delta\{ym + \xi x + \xi(\delta+s\delta)y\} \in (R^{d+1})^s \oplus \delta(R^d)^s,$$

and hence $R = R^s \oplus \delta R^s$.

Finally, $\forall a_1, a_2 \in R^s$,

$$\begin{aligned} a_1 + \delta a_2 &= a_1 + (\delta + s\delta - s\delta)a_2 = \{a_1 + (\delta+s\delta)a_2\} + s\delta(-a_2), \\ a_1 + s\delta a_2 &= a_1 + (s\delta + \delta - \delta)a_2 = \{a_1 + (s\delta+\delta)a_2\} + \delta(-a_2). \end{aligned}$$

2.2. Keep the notation of 2.1. One has as graded left R-modules

(1)
$$\begin{aligned} B(s) &= R \otimes_{R^s} (R^s \oplus \delta R^s)(1) \simeq R(1) \oplus \delta R(1) \\ &\simeq R(1) \oplus R(-2)(1) \quad \text{by (1.8.1)} \\ &= R(1) \oplus R(-1). \end{aligned}$$

Also, as graded right R-modules

(2)
$$B(s) = (R^s \oplus \delta R^s) \otimes_{R^s} R(1) \simeq R(1) \oplus R(-1).$$

In $R \otimes_{R^s} R$ one has

$$\begin{aligned} (\delta\otimes 1 - 1\otimes s\delta)\delta &= \delta\otimes\delta - 1\otimes(s\delta)\delta = -\delta(s\delta)\otimes 1 + \delta\otimes(\delta+s\delta) - \delta\otimes s\delta \\ &= \{-\delta(s\delta) + \delta(\delta+s\delta)\}\otimes 1 - \delta\otimes s\delta = \delta^2\otimes 1 - \delta\otimes s\delta \\ &= \delta(\delta\otimes 1 - 1\otimes s\delta), \\ (\delta\otimes 1 - 1\otimes\delta)\delta &= \delta\otimes\delta - 1\otimes\delta^2 = \delta\otimes\delta - 1\otimes\{-\delta(s\delta) + \delta(\delta+s\delta)\} \\ &= \delta\otimes\delta + 1\otimes\delta(s\delta) - (\delta+s\delta)\otimes\delta = \delta(s\delta)\otimes 1 - s\delta\otimes\delta \\ &= (s\delta)(\delta\otimes 1 - 1\otimes\delta). \end{aligned}$$

As $R = R^s \oplus \delta R^s$, one obtains that $\forall a \in R$,

(3) $\quad (\delta\otimes 1 - 1\otimes s\delta)a = a(\delta\otimes 1 - 1\otimes s\delta) \quad$ and $\quad (\delta\otimes 1 - 1\otimes\delta)a = (sa)(\delta\otimes 1 - 1\otimes\delta).$

Also,

(4) $\quad 1\otimes\delta - (s\delta)\otimes 1 = 1\otimes\delta - (s\delta)\otimes 1 - 1\otimes(\delta+s\delta) + (\delta+s\delta)\otimes 1 = \delta\otimes 1 - 1\otimes s\delta.$

By (3) one obtains that $\delta\otimes 1 - 1\otimes s\delta$ and $\delta\otimes 1 - 1\otimes\delta$ are Q-linearly independent in $Q\otimes_R (R\otimes_{R^s} R)$, and hence $Q\otimes_R (R\otimes_{R^s} R) = Q(\delta\otimes 1 - 1\otimes s\delta) \oplus Q(\delta\otimes 1 - 1\otimes\delta)$ with isomorphisms of (Q, R)-bimodules

(5)
$$\begin{aligned} Q(\delta\otimes 1 - 1\otimes s\delta) &\simeq Q(e) = Q\otimes_R R(e) \quad \text{via} \quad q(\delta\otimes 1 - 1\otimes s\delta) \mapsto q\otimes\alpha_s, \\ Q(\delta\otimes 1 - 1\otimes\delta) &\simeq Q(s) = Q\otimes_R R(s) \quad \text{via} \quad q(\delta\otimes 1 - 1\otimes\delta) \mapsto q\otimes\alpha_s, \end{aligned}$$

$$\begin{array}{ccc} \delta\otimes 1 - 1\otimes s\delta & \longmapsto & \alpha_s \\ {\scriptstyle ?\delta}\big\downarrow{\scriptstyle (3)} & \circlearrowright & \big\downarrow{\scriptstyle ?\delta} \\ \delta(\delta\otimes 1 - 1\otimes s\delta) & \longmapsto & \delta\alpha_s, \end{array} \qquad \begin{array}{ccc} \delta\otimes 1 - 1\otimes \delta & \longmapsto & \alpha_s \\ {\scriptstyle ?\delta}\big\downarrow{\scriptstyle (3)} & \circlearrowright & \big\downarrow{\scriptstyle ?\delta} \\ s\delta(\delta\otimes 1 - 1\otimes \delta) & \longmapsto & (s\delta)\alpha_s. \end{array}$$

Thus, $B(s) = R\otimes_{R^s}R(1)$ comes equipped with a structure of $\mathcal{C}$ such that $B(s)^Q = B(s)^Q_e \oplus B(s)^Q_s$ with

(6) $$B(s)^Q_e = Q(\delta\otimes 1 - 1\otimes s\delta) \simeq Q(e) \quad \text{and} \quad B(s)^Q_s = Q(\delta\otimes 1 - 1\otimes \delta) \simeq Q(s).$$

Explicitly, in $B(s)^Q = Q\otimes_R (R\otimes_{R^s} R)$, $\forall a, b \in R$,

(7) $$1\otimes a\otimes b = \frac{ab}{\alpha_s}\otimes(\delta\otimes 1 - 1\otimes s\delta) + \frac{a(sb)}{\alpha_s}\otimes(\delta\otimes 1 - 1\otimes \delta).$$

For we may assume $a = 1$ and $b = \delta$; the case $b = 1$ follows from (4). Thus, it is enough to check that $\alpha_s\otimes\delta = \delta(\delta\otimes 1 - 1\otimes s\delta) + (s\delta)(\delta\otimes 1 - 1\otimes\delta)$ in $R\otimes_{R^s} R$. But

$$\begin{aligned} \text{RHS} &= (\delta\otimes 1 - 1\otimes s\delta)\delta + (\delta\otimes 1 - 1\otimes\delta)\delta \quad \text{by (3)} \\ &= (1\otimes\delta - s\delta\otimes 1)\delta + \delta\otimes\delta - 1\otimes\delta^2 \quad \text{by (4)} \\ &= -s\delta\otimes\delta + \delta\otimes\delta = \alpha_s\otimes\delta. \end{aligned}$$

Thus, together with (5) one has a CD

(8) $$\begin{array}{ccccc} & & ab & & Q(e) \\ & \nearrow & & \nearrow & \uparrow \\ a\otimes b & & R\otimes_{R^s} R & \longrightarrow & (R\otimes_{R^s} R)^Q = B(s)^Q_e\oplus B(s)^Q_s \\ & \searrow & & \searrow & \downarrow \\ & & a(sb) & & Q(s). \end{array}$$

Note that the elements $\delta\otimes 1 - 1\otimes s\delta$ and $1\otimes\delta - \delta\otimes 1$ are independent of the choice of δ; if $\delta' \in V$ with $\langle\delta', \alpha_s^\vee\rangle = 1$,

(9) $$\delta\otimes 1 - 1\otimes s\delta = \delta'\otimes 1 - 1\otimes s\delta' \quad \text{and} \quad 1\otimes\delta - \delta\otimes 1 = 1\otimes\delta' - \delta'\otimes 1.$$

For let $V^s = \{\nu \in V | s\nu = \nu\}$. $\forall\mu \in V$, $\mu = (\mu - \langle\mu, \alpha_s^\vee\rangle\delta) + \langle\mu, \alpha_s^\vee\rangle\delta$ with $\mu - \langle\mu, \alpha_s^\vee\rangle\delta \in V^s$, and hence $V = V^s \oplus \mathbb{K}\delta$. Write $\delta' = \nu + \xi\delta$ for some $\nu \in V^s$ and $\xi \in \mathbb{K}$ by (2.1). Then $1 = \langle\delta', \alpha_s^\vee\rangle = \xi$, and hence $\delta' = \nu + \delta$ and the assertion follows.

The structure of $B(s)$ is already quite intricate. For $x \in \mathcal{W}$ let $\leq x = \{w \in \mathcal{W} | w \leq x\}$,

$< x = \{w \in \mathcal{W} | w < x\}$. One has from (7)

(10) $$B(s)_e = R(\delta \otimes 1 - 1 \otimes s\delta) = R(1 \otimes \delta - s\delta \otimes 1) = B(s)_{\leq e}/B(s)_{<e} \simeq R(e)(-1)$$

as $\delta \otimes 1 - 1 \otimes s\delta = 1 \otimes \delta - s\delta \otimes 1$ has degree 1 in $B(s)$

(11) $$B(s)_s = R(\delta \otimes 1 - 1 \otimes \delta)) \simeq R(s)(-1),$$

(12) $$B(s)^e = R(\frac{1}{\alpha_s} \otimes (\delta \otimes 1 - 1 \otimes s\delta)) \simeq R(e)(1),$$

(13) $$\begin{aligned} B(s)^s &= R(\frac{1}{\alpha_s} \otimes (\delta \otimes 1 - 1 \otimes \delta)) \\ &\simeq B(s)_{\leq s}/B(s)_{<s} = B(s)/B(s)_e = R(\overline{1 \otimes 1}) \simeq R(s)(1). \end{aligned}$$

To see the last equality,

$$\begin{aligned} s\delta(1 \otimes 1) + (\delta \otimes 1 - 1 \otimes s\delta) &= s\delta(1 \otimes 1) + (1 \otimes \delta - s\delta \otimes 1) \quad \text{by (4)} \\ &= 1 \otimes \delta. \end{aligned}$$

Thus,

(14) $$\begin{array}{ccc} B(s)_s & \hookrightarrow & B(s) \\ \vdots & & \downarrow \\ \sim & & B(s)^s \\ \vdots & & \parallel \\ R(\overline{\alpha_s \otimes 1}) & \hookrightarrow & R(\overline{1 \otimes 1}) \end{array}$$

as

$$\begin{aligned} \alpha_s(1 \otimes 1) = \alpha_s \otimes 1 = (\delta - s\delta) \otimes 1 &= \delta \otimes 1 - s\delta \otimes 1 \\ &\equiv \delta \otimes 1 - s\delta \otimes 1 - (1 \otimes \delta - s\delta \otimes 1) \quad \mod B(s)_e \\ &= \delta \otimes 1 - \delta \otimes 1. \end{aligned}$$

Consider now the exact sequence $0 \to B(s)_s \to B(s) \to B(s)^e \to 0$. It induces

(15) $$\begin{array}{ccccc} (B(s)_s)^e & \longrightarrow & B(s)^e & \longrightarrow & (B(s)^e)^e \\ \parallel & & & & \parallel \\ 0 & & & & B(s)^e, \\ & & & & \\ (B(s)_s)^s & \longrightarrow & B(s)^s & \longrightarrow & (B(s)^e)^s \\ \parallel & & & & \parallel \\ B(s)_s & & & & 0, \\ & & & & \\ (B(s)_s)_e & \longrightarrow & B(s)_e & \longrightarrow & (B(s)^e)_e \\ \parallel & & \wr\! \mid & & \wr\! \mid \\ 0 & & R(e)(-1) & & R(e)(1). \end{array}$$

Note also that the decomposition of $B(s)^Q$ as in (6) holds over $R[\frac{1}{\alpha_s}]$:

(16) $$R[\frac{1}{\alpha_s}] \otimes_R B(s) = R[\frac{1}{\alpha_s}](\delta \otimes 1 - 1 \otimes s\delta) \oplus R[\frac{1}{\alpha_s}](\delta \otimes 1 - 1 \otimes \delta).$$

Consider a homomorphism of grade R-bimodules $m^s : B(s) \to R(s)(1)$ via $a \otimes b \mapsto a(sb)$. As the action of $\langle s \rangle$ on V is faithful, $m^s \in \mathcal{C}'$ by Rmk. 1.2.(ii), and hence factors through the quotient $B(s) \to B(s)^s$:

$$\begin{array}{ccccc} a \otimes b & \longmapsto & & & a(sb) \\ \Big\downarrow & & B(s) & \xrightarrow[m^s]{} & R(s)(1). \\ & & \Big\downarrow & \overset{\sim}{\nearrow} & \\ \frac{a(sb)}{\alpha_s} \otimes (\delta \otimes 1 - 1 \otimes \delta) & & B(s)^s & & \end{array} \tag{17}$$

The structure of R-bimodule on $B(s)$ endows it with a structure of graded left $R \otimes_{R_s} R$-module. Thus, if we let $(R \otimes_{R^s} R)\mathrm{Modgr}$ denote the category of graded left $R \otimes_{R^s} R$-modules, one has

$$\begin{aligned} \mathcal{C}'(B(s), B(s)) &\simeq (R \otimes_{R^s} R)\mathrm{Modgr}(B(s), B(s)) \quad \text{by Rmk. 1.2.(ii) again} \\ &\qquad\qquad\qquad \text{as the action of } \langle s \rangle \text{ on } V \text{ is faithful} \\ &\simeq (R \otimes_{R^s} R)\mathrm{Modgr}(R \otimes_{R^s} R, R \otimes_{R^s} R) \\ &\simeq (R \otimes_{R^s} R)^0 \quad \text{as } (1,1) \text{ must be sent to an element of degree } 0 \\ &= \mathbb{K}(1 \otimes 1) \simeq \mathbb{K}. \end{aligned} \tag{18}$$

In particular, $B(s)$ is indecomposable in $\mathcal{C}'$.

Now let $\mathcal{Z}' = \{(z_e, z_s) \in R(e) \oplus R(s) | z_s \equiv z_e \mod \alpha_s\}$ a graded $\mathbb{K}$-subalgebra of $R^2 = \coprod_{d \in \mathbb{N}} (R^d)^2$ equipped with a structure of R-bimodule, which is the structure algebra of a moment graph [F08a]. Under the imbedding (8) one has $B(s) \hookrightarrow \mathcal{Z}'(1)$ via $a \otimes b \mapsto (ab, a(sb))$. As $1 \otimes 1 \mapsto (1,1)$ and as $\delta \otimes 1 - 1 \otimes \delta \mapsto (\delta, \delta) - (\delta, s\delta) = (0, \alpha_s)$, one has

$$B(s) \simeq \mathcal{Z}'(1). \tag{19}$$

From (10) and (13) one has a short exact sequence in $\mathcal{C}'$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R(e)(-2) & \longrightarrow & \mathcal{Z}' & \longrightarrow & R(s) & \longrightarrow & 0 \\ & & a & \longmapsto & (a\alpha_s, 0) & & & & \\ & & & & (a, b) & \longmapsto & b. & & \end{array}$$

Let us compute the 米田-extensions of $R(s)$ by $R(e)(-2)$ in $\mathcal{C}'$. As $s\delta \neq \delta$, $\langle s \rangle$ acts on V faithfully. Then the computation of extensions in $\mathcal{C}'$ is equivalent to one in $(R \otimes_{R_s} R)\mathrm{Modgr}$ or in $\mathcal{Z}'\mathrm{Modgr}$ by Rmk. 1.2.(ii). Thus, given another exact sequence $0 \to R(e)(-2) \xrightarrow{f} M \xrightarrow{g} R(s) \to 0$ in $\mathcal{Z}'\mathrm{Modgr}$, let $m \in M^0$ with $g(m) = 1$, and let $\phi \in \mathcal{Z}'\mathrm{Modgr}(\mathcal{Z}', M)$ with $(1,1) \mapsto m$. Thus, $\forall a \in R$, $\phi(a,a) = a\phi(1,1) = am$. As $m\delta = \phi((1,1)\delta) = \phi(\delta, s\delta) = \phi(\delta, \delta - \alpha_s) = \phi(\delta - \alpha_s + \alpha_s, \delta - \alpha_s) = (\delta - \alpha_s)m + \phi(\alpha_s, 0)$, $\phi(\alpha_s, 0) = m\delta + (\alpha_s - \delta)m$. As the sequence splits as a right R-module, one has

$$\begin{array}{ccc} R(e)(-2) & \xrightarrow{f} & M \\ & \underset{\sim}{\searrow} & \Big\uparrow \\ & & M_e. \end{array}$$

As $\phi(\alpha_s, 0)$ and $f(1) \in M_e^2 \simeq R(e)(-2)^2 = \mathbb{K}$, $\phi(\alpha_s, 0) = \xi f(1)$ for some $\xi \in \mathbb{K}$. Then $\forall a, b \in R$, $\phi(a, a + b\alpha_s) = \phi(a + b\alpha_s - b\alpha_s, a + b\alpha_s) = (a + b\alpha_s)\phi(1, 1) - b\phi(\alpha_s, 0) = (a + b\alpha_s)m - b\xi f(1)$, and hence results a CD of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R(e)(-2) & \longrightarrow & \mathcal{Z}' & \longrightarrow & R(s) & \longrightarrow & 0 \\
 & & {\scriptstyle\xi}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & R(e)(-2) & \xrightarrow[f]{} & M & \xrightarrow[g]{} & R(s) & \longrightarrow & 0.
\end{array}$$

In particular, if $\xi \in \mathbb{K}^\times$, $M \simeq \mathcal{Z}'$. If $\xi = 0$, $m\delta = (\delta - \alpha_s)m = (s\delta)m$, and hence $m \in M_s$ and $M \simeq R(e)(-2) \oplus R(s)$ in $\mathcal{C}'$. In general, from [Rot, Th. 7.30]

$$\operatorname{Ext}^1_{\mathcal{Z}'\mathrm{Modgr}}(R(s), R(e)(-2)) \simeq \mathbb{K}. \tag{20}$$

For if $\xi' \in \mathbb{K}$ with $f \circ \xi' = f \circ \xi$, then $\xi f(1) = (f \circ \xi)(1) = (f \circ \xi')(1) = \xi' f(1)$. As M is torsion free over R, we must have $\xi = \xi'$.

Likewise, one has a CD of exact sequences in $\mathcal{Z}'$Modgr

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B(s)_s & \longrightarrow & B(s) & \longrightarrow & B(s)^e & \longrightarrow & 0 \\
 & & \sim\downarrow & & \downarrow\sim & & \sim\downarrow & & \\
0 & \longrightarrow & R(s)(-1) & \longrightarrow & \mathcal{Z}'(1) & \longrightarrow & R(e)(1) & \longrightarrow & 0 \\
 & & 1 & \longmapsto & (0, \alpha_s) & & & & \\
 & & & & (a, b) & \longmapsto & a, & &
\end{array}$$

and

$$\operatorname{Ext}^1_{\mathcal{Z}'\mathrm{Modgr}}(R(e), R(s)(-2)) \simeq \mathbb{K}. \tag{21}$$

On the other hand, the exact sequence

$$0 \to R[\frac{1}{\alpha_s}](e)(-2) \to \mathcal{Z}'[\frac{1}{\alpha_s}] \to R[\frac{1}{\alpha_s}](s) \to 0$$

splits in $\mathcal{Z}'[\frac{1}{\alpha_s}]$Modgr, and hence $R[\frac{1}{\alpha_s}](e)(-2)$ and $R[\frac{1}{\alpha_s}](s)$ are both projective as graded left $\mathcal{Z}'[\frac{1}{\alpha_s}]$-modules. Thus, $\forall n \in \mathbb{Z}$,

$$\operatorname{Ext}^1_{\mathcal{Z}'[\frac{1}{\alpha_s}]\mathrm{Modgr}}(R[\frac{1}{\alpha_s}](e), R[\frac{1}{\alpha_s}](s)(n)) = 0 = \operatorname{Ext}^1_{\mathcal{Z}'[\frac{1}{\alpha_s}]\mathrm{Modgr}}(R[\frac{1}{\alpha_s}](s), R[\frac{1}{\alpha_s}](e)(n)). \tag{22}$$

2.3. Let $s \in \mathcal{S}$ and $M \in \mathcal{C}$. Let us examine the structure of $B(s) * M \in \mathcal{C}$. $\forall w \in \mathcal{W}$,

$$\begin{aligned}
(B(s) * M)^Q_w &= \coprod_{\substack{x,y \in \mathcal{W} \\ xy = w}} B(s)^Q_x \otimes_Q M^Q_y = \{B(s)^Q_e \otimes_Q M^Q_w\} \oplus \{B(s)^Q_s \otimes_Q M^Q_{sw}\} \\
&\simeq Q(e) \otimes_Q M^Q_w \oplus Q(s) \otimes_Q M^Q_{sw} \\
&\simeq M^Q_w \oplus M^Q_{sw} \quad \text{via} \quad (q_1 \otimes m_1, q_2 \otimes m_2) \mapsto \\
&\qquad (q_1 m_1, (sq_2) m_2) = (m_1 w^{-1} q_1, m_2 (sw)^{-1}(sq_2)) = (m_1 w^{-1} q_1, m_2 w^{-1} q_2)
\end{aligned} \tag{1}$$

with

$$\text{(2)} \qquad \begin{aligned} B(s)^Q_e \otimes_Q M^Q_w &= Q(\delta\otimes 1 - 1\otimes s\delta)\otimes_Q M^Q_w \quad \text{by (2.2.5)}\\ &= \{(\delta\otimes 1 - 1\otimes s\delta)\otimes m | m\in M^Q_w\}\\ &\simeq \{\delta\otimes m - 1\otimes (s\delta)m | m \in M^Q_w\} \quad \text{in } R\otimes_{R^s} M^Q_w\\ &= \{1\otimes \delta m - s\delta\otimes m | m\in M^Q_w\} \quad \text{by (2.2.4)},\\ B(s)^Q_s\otimes_Q M^Q_{sw} &= Q(\delta\otimes 1 - 1\otimes\delta)\otimes_Q M^Q_{sw} \quad \text{by (2.2.5) again}\\ &\simeq \{\delta\otimes m - 1\otimes\delta m | m\in M^Q_{sw}\} \quad \text{in } R\otimes_{R^s} M^Q_{sw}. \end{aligned}$$

Likewise, $\forall w\in\mathcal{W}$,

$$\text{(3)} \qquad \begin{aligned} (M * B(s))^Q_w &= \coprod_{\substack{x,y\in\mathcal{W}\\ xy=w}} M^Q_x\otimes_Q B(s)^Q_y = \{M^Q_w\otimes_Q B(s)^Q_e\}\oplus\{M^Q_{ws}\otimes_Q B(s)^Q_s\}\\ &\simeq M^Q_w\otimes_Q Q(e)\oplus M^Q_{ws}\otimes_Q Q(s)\\ &\simeq M^Q_w\oplus M^Q_{ws} \quad \text{via} \quad (m_1\otimes q_1, m_2\otimes q_2)\mapsto\\ &\qquad\qquad (m_1q_1, m_2(sq_2)) = ((wq_1)m_1, (wq_2)m_2) \end{aligned}$$

with

$$\text{(4)} \qquad \begin{aligned} M^Q_w\otimes_Q B(s)^Q_e &= M^Q_w\otimes_Q Q(\delta\otimes 1 - 1\otimes s\delta) \quad \text{by (2.2.5)}\\ &= \{m\otimes(\delta\otimes 1 - 1\otimes s\delta) | m\in M^Q_w\}\\ &\simeq \{m\delta\otimes 1 - m\otimes s\delta | m\in M^Q_w\} \quad \text{in } M^Q_w\otimes_{R^s} R\\ &= \{m\otimes\delta - m(s\delta)\otimes 1 | m\in M^Q_w\} \quad \text{by (2.2.4)},\\ M^Q_{ws}\otimes_Q B(s)^Q_s &= M^Q_{ws}\otimes_Q Q(1\otimes\delta - \delta\otimes 1) \quad \text{by (2.2.5) again}\\ &\simeq \{m\otimes\delta - m\delta\otimes 1 | m\in M^Q_{ws}\} \quad \text{in } M^Q_{ws}\otimes_{R^s} R. \end{aligned}$$

Lemma: *Let $s\in\mathcal{S}$ and $M\in\mathcal{C}$.*

(i) The structure of $B(s)*M\in\mathcal{C}$ is such that each composite $B(s)*M\hookrightarrow (B(s)*M)^Q\to (B(s)*M)^Q_w$, $w\in\mathcal{W}$, reads

$$\begin{array}{ccc} B(s)*M & \longrightarrow & (B(s)*M)^Q_w\\ \wr\big| & & \big|\wr\\ R\otimes_{R^s} M & \dashrightarrow & M^Q_w\oplus M^Q_{sw}\\ a\otimes m & \longmapsto & (am_w, (sa)m_{sw}), \end{array}$$

and that

$$(B(s)*M)^Q_w = \{(\delta\otimes m - 1\otimes(s\delta)m | m\in M^Q_w\}\oplus\{(\delta\otimes m - 1\otimes\delta m | m\in M^Q_{sw}\}$$

in $(R\otimes_{R^s} M^Q_w)\oplus(R\otimes_{R^s} M^Q_{sw})$ with $\delta\otimes m - 1\otimes(s\delta)m = 1\otimes\delta m - (s\delta)\otimes m$, $m\in M^Q_w$.

(ii) $\forall w \in \mathcal{W}$,

$$\begin{aligned}(B(s)*M)^Q_w \oplus (B(s)*M)^Q_{sw} &= \{B(s)^Q_e \otimes_Q M^Q_w \oplus B(s)^Q_s \otimes_Q M^Q_{sw}\} \\ &\qquad \oplus \{B(s)^Q_e \otimes_Q M^Q_{sw} \oplus B(s)^Q_s \otimes_Q M^Q_w\} \\ &= \{B(s)^Q_e \oplus B(s)^Q_s\} \otimes_Q M^Q_w \oplus \{B(s)^Q_s \oplus B(s)^Q_e\} \otimes_Q M^Q_{sw} \\ &= \{B(s)^Q \otimes_Q M^Q_w\} \oplus \{B(s)^Q \otimes_Q M^Q_{sw}\} \\ &\simeq B(s) \otimes_R (M^Q_w \oplus M^Q_{sw}).\end{aligned}$$

(iii) The structure of $M * B(s) \in \mathcal{C}$ is such that each composite $M*B(s) \hookrightarrow (M*B(s))^Q \to (M*B(s))^Q_w$, $w \in \mathcal{W}$, reads

$$\begin{array}{ccc} M * B(s) & \longrightarrow & (M*B(s))^Q_w \\ \sim\big\downarrow & & \big\downarrow\sim \\ M \otimes_{R^s} R & \dashrightarrow & M^Q_w \oplus M^Q_{ws} \\ m \otimes a & \longmapsto & (m_w a, m_{ws}(sa)), \end{array}$$

and that

$$(M*B(s))^Q_w = \{m\otimes\delta - m(s\delta)\otimes 1 | m \in M^Q_w\} \oplus \{m\otimes\delta - m\delta\otimes 1 | m \in M^Q_{ws}\}$$

in $(M^Q_w \otimes_{R^s} R) \oplus (M^Q_{ws} \otimes_{R^s} R)$ *with* $m\otimes\delta - m(s\delta)\otimes 1 = m\delta\otimes 1 - m\otimes s\delta$, $m \in M^Q_w$.

(iv) $\forall w \in \mathcal{W}$,

$$\begin{aligned}(M*B(s))^Q_w \oplus (M*B(s))^Q_{ws} &= \{M^Q_w \otimes_Q B(s)^Q_e \oplus M^Q_{ws} \otimes_Q B(s)^Q_s\} \\ &\qquad \oplus \{M^Q_{ws} \otimes_Q B(s)^Q_e \oplus M^Q_w \otimes_Q B(s)^Q_s\} \\ &\simeq M^Q_w \otimes_Q \{B(s)^Q_e \oplus B(s)^Q_s\} \oplus M^Q_{ws} \otimes_Q \{B(s)^Q_s \oplus B(s)^Q_e\} \\ &\simeq \{M^Q_w \oplus M^Q_{ws}\} \otimes_R B(s).\end{aligned}$$

Proof: By (2.2.8) one has

$$\begin{array}{ccccc} B(s)*M & \longrightarrow & (B(s)*M)^Q_w = (Q(e)\otimes_Q M^Q_w) \oplus (Q(s) \otimes_Q M^Q_{sw}) & \xrightarrow{\sim} & M^Q_w \oplus M^Q_{sw} \\ a \otimes_{R^s} m \mapsto a \otimes_R 1 \otimes_{R^s} m & \longmapsto & (a \otimes m_w, a \otimes m_{sw}) & \longmapsto & (am_w, (sa)m_{sw}), \end{array}$$

$$\begin{array}{ccccc} M*B(s) & \longrightarrow & (M*B(s))^Q_w = (M^Q_w \otimes_Q Q(e)) \oplus (M^Q_{ws} \otimes_Q Q(s)) & \xrightarrow{\sim} & M^Q_w \oplus M^Q_{ws} \\ m \otimes_{R^s} a \mapsto m \otimes_R 1 \otimes_{R^s} a & \longmapsto & (m_w \otimes a, m_{ws} \otimes sa) & \longmapsto & (m_w a, m_{ws}(sa)). \end{array}$$

2.4. Let $\mathcal{BS}$ denote the full subcategory of $\mathcal{C}$ consisting of the finite direct sums of $B(s_1) * \cdots * B(s_r)(n)$, $s_1, \dots, s_r \in \mathcal{S}$, $n \in \mathbb{Z}$. As R-bimodules

$$B(s_1) * \cdots * B(s_r)(n) = (R \otimes_{R^{s_1}} R) \otimes_R \cdots \otimes_R (R \otimes_{R^{s_r}} R) \simeq R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_r}} R.$$

Let $\mathfrak{S}$Bimod denote the full subcategory of $\mathcal{C}$ consisting of the direct summands of objects of $\mathcal{BS}$. Thus, both $\mathcal{BS}$ and $\mathfrak{S}$Bimod are closed under the monoidal product. $\forall \underline{x} = (s_1 \dots, s_r) \in S^r$, set $B(\underline{x}) = B(s_1) * \cdots * B(s_r)$; we set $B(\emptyset) = R(e)$ for the empty sequence $\emptyset$. We will also write $B(s_1 \dots, s_r)$ for $B(\underline{x})$. From (1.7) one has

Lemma: $\text{supp}_{\mathcal{W}}(B(\underline{x})) = \{s_1^{e_1} \dots s_r^{e_r} | e_1, \dots, e_r \in \{0,1\}\}$. *In particular, if $\underline{x}$ is a reduced expression of x,* $\text{supp}_{\mathcal{W}}(B(\underline{x})) = \{y \in \mathcal{W} | y \leq x\}$.

2.5. $\forall M \in \mathcal{C}$, one has from (2.2)

$$\begin{aligned} B(s) * M &= \{R(1) \oplus R(-1)\} \otimes_R M \simeq M(1) \oplus M(-1) \quad \text{as graded right } R\text{-modules}, \\ M * B(s) &= M \otimes_R \{R(1) \oplus R(-1)\} \simeq M(1) \oplus M(-1) \quad \text{as graded left } R\text{-modules}. \end{aligned}$$

Lemma: $\forall \underline{x} = (s_1, \dots, s_r) \in S^r$, *$B(\underline{x})$ is gradrd free both as a left and right R-module of graded rank $(v + v^{-1})^r$.*

Proof: By definition

$$\text{grk}(B(s_i)) = \text{grk}(R(1) \oplus R(-1)\} = v + v^{-1}.$$

Thus,

$$\begin{aligned} \text{grk}(B(s_1) * B(s_2)) &= \text{grk}(B(s_2)(1) \oplus B(s_2)(-1)\} = \text{grk}(B(s_2)(1)) + \text{grk}(B(s_2)(-1)) \\ &= v(v + v^{-1}) + v^{-1}(v + v^{-1}) = (v + v^{-1})^2. \end{aligned}$$

2.6 Let RBimod denote the category of R-bimodules. For $M \in \mathcal{C}$ we regard $B(s) * M = (R \otimes_{R^s} R(1)) \otimes_R M$, as a nongraded R-bimodule, to be $R \otimes_{R^s} M$.

Lemma: *Let $M, N \in \mathcal{C}$ and $s \in \mathcal{S}$.*

(i) $\mathcal{C}(B(s) * M, N) \simeq \mathcal{C}(M, B(s) * N)$.

(ii) $\mathcal{C}(M * B(s), N) \simeq \mathcal{C}(M, N * B(s))$.

Proof: Take $\delta \in V$ with $\langle \delta, \alpha_s^\vee \rangle = 1$.

(i) Define $\Psi : R\text{Bimod}(B(s) * M, N) \to R\text{Bimod}(M, B(s) * N)$ via

$$\phi \mapsto \psi_\phi : m \mapsto 1 \otimes \phi(1 \otimes \delta m) - (s\delta) \otimes \phi(1 \otimes m).$$

To check it well-defined,

$$\begin{aligned} \psi_\phi(\delta m) &= 1 \otimes \phi(1 \otimes \delta^2 m) - (s\delta) \otimes \phi(1 \otimes \delta m) \\ &= 1 \otimes \phi(1 \otimes (-\delta(s\delta) + \delta(\delta + s\delta)))m) - (s\delta) \otimes \phi(1 \otimes \delta m) \\ &= -\delta(s\delta) \otimes \phi(1 \otimes m) + (\delta + s\delta) \otimes \phi(1 \otimes \delta m) - (s\delta) \otimes \phi(1 \otimes \delta m) \\ &= \delta\{1 \otimes \phi(1 \otimes \delta m) - (s\delta) \otimes \phi(1 \otimes m)\} = \delta\psi_\phi(m). \end{aligned}$$

Note also that ψ_ϕ is homogeneous, i.e., graded of degree 0, if ϕ is; if $m \in M^d$, $1 \otimes \delta m \in (B(s) * M)^{d+1}$ while $1 \otimes m \in (B(s) * M)^{d-1}$, and hence $1 \otimes \phi(1 \otimes \delta m) \in (B(s) * N)^{d+1-1}$ and $s\delta \otimes \phi(1 \otimes m) \in (B(s) * N)^{d-1+1}$.

Given $\psi \in R\text{Bimod}(M, B(s) * N)$ write, according to the decompostion $B(s) * N = \{R \otimes_{R_s} R(1)\} \otimes_R N \simeq \{R^s(1) \oplus \delta R^s(1)\} \otimes_{R^s} N = \{R^s(1) \oplus (-s\delta)R^s(1)\} \otimes_{R^s} N \simeq N(1) \oplus (-s\delta)N(1)$,

$$\psi(m) = 1 \otimes \psi_1(m) - s\delta \otimes \psi_2(m) \quad \exists! \psi_1(m), \psi_2(m) \in N.$$

Thus, $\psi_1, \psi_2 \in (R^s, R)\text{Bimod}(M, N)$. Define $\Phi : R\text{Bimod}(M, B(s) * N) \to R\text{Bimod}(B(s) * M, N)$ via

$$\psi \mapsto \phi_\psi : a \otimes m \mapsto a\psi_2(m), \quad a \in R, m \in M.$$

If ψ is homogeneous, $\psi_2 : M \to N(-1)$, and hence ϕ_ψ will also be homogeneous as $B(s) * M \simeq R(1) \otimes_{R^s} M$; $R(1)^i \otimes_{R^s} M^j \ni a \otimes m \mapsto a\psi_2(m) \in N^{(i+1)+(j-1)} = N^{i+j}$.

Now, $\phi_{\psi_\phi}(a \otimes m) = a(\psi_\phi)_2(m) = a\phi(1 \otimes m) = \phi(a \otimes m)$, and hence $\phi_{\psi_\phi} = \phi$. Also, $\psi_{\phi_\psi}(m) = 1 \otimes \phi_\psi(1 \otimes \delta m) - s\delta \otimes \phi_\psi(1 \otimes m) = 1 \otimes \psi_2(\delta m) - s\delta \otimes \psi_2(m)$. But

$$\begin{aligned} 1 \otimes \psi_1(\delta m) - s\delta \otimes \psi_2(\delta m) = \psi(\delta m) = \delta\psi(m) &= \delta \otimes \psi_1(m) - \delta(s\delta) \otimes \psi_2(m) \\ &= \{-s\delta + (s\delta + \delta)\} \otimes \psi_1(m) - \delta(s\delta) \otimes \psi_2(m) \\ &= -s\delta \otimes \psi_1(m) + 1 \otimes \{(s\delta + \delta)\psi_1(m) - \delta(s\delta)\psi_2(m)\}, \end{aligned}$$

and hence $\psi_1(m) = \psi_2(\delta m)$ and $\psi_1(\delta m) = (s\delta + \delta)\psi_1(m) - \delta(s\delta)\psi_2(m)$. Thus,

$$\psi_{\phi_\psi}(m) = 1 \otimes \psi_1(m) - s\delta \otimes \psi_2(m) = \psi(m),$$

and $\psi_{\phi_\psi} = \psi$. It follows that Ψ and Φ are inverse to each other.

We show next that Φ and Ψ induce bijections

$$\mathcal{C}(B(s) * M, N) \rightleftarrows \mathcal{C}(M, B(s) * N).$$

For that we have only to verify that $\forall \phi \in R\text{Bimod}(B(s) * M, N)$, $\phi \in \mathcal{C}$ iff $\psi_\phi \in \mathcal{C}$. Put $\psi = \psi_\phi$ for simplicity. We are to check that $\forall w \in \mathcal{W}$, $\psi^Q(M^Q_w) \subseteq (B(s)*N)^Q_w$ iff $\phi^Q((B(s)*M)^Q_w) \subseteq N^Q_w$. $\forall m \in M$, $\forall y \in \mathcal{W}$,

(1) $\{\psi^Q(1 \otimes m)\}_y = \{1 \otimes \phi(1 \otimes \delta m) - s\delta \otimes \phi(1 \otimes m)\}_y$ in $(B(s) * N)^Q$ with $1 \otimes m \in M^Q = Q \otimes_R M$

$$\begin{aligned} &= (\phi(1 \otimes \delta m)_y - (s\delta)\phi(1 \otimes m)_y, \phi(1 \otimes \delta m)_{sy} - \delta\phi(1 \otimes m)_{sy}) \text{ in } N^Q_y \oplus N^Q_{sy} \text{ by (2.3.i)} \\ &= (\phi(1 \otimes \delta m - s\delta \otimes m)_y, \phi(1 \otimes \delta m - \delta \otimes m)_{sy}) \\ &= (\phi(\delta \otimes m - 1 \otimes (s\delta)m)_y, \phi(1 \otimes \delta m - \delta \otimes m)_{sy}) \quad \text{by (2.3.i) again.} \end{aligned}$$

Thus, if $m \in M^Q_w$, (1) reads

(2) $$\psi^Q(m)_y = (\phi^Q(\delta \otimes m - 1 \otimes (s\delta)m)_y, \phi^Q(1 \otimes \delta m - \delta \otimes m)_{sy})$$

with $\delta \otimes m - 1 \otimes (s\delta)m \in (B(s) * M)^Q_w$ and $1 \otimes \delta m - \delta \otimes m \in (B(s) * M)^Q_{sw}$ by (2.3.i). Thus, if $\phi \in \mathcal{C}$,

$$\phi^Q(\delta \otimes m - 1 \otimes (s\delta)m) \in N^Q_w \quad \text{and} \quad \phi^Q(1 \otimes \delta m - \delta \otimes m) \in N^Q_{sw},$$

and hence $\psi^Q(m)_y = 0$ unless $y = w$. It follows that $\psi^Q(M^Q_w) \subseteq (B(s) * N)^Q_w$.

In turn, $(B(s)*M)^Q_w = \{\delta \otimes m - 1 \otimes (s\delta)m + 1 \otimes \delta m' - \delta \otimes m' | m \in M^Q_w, m' \in M^Q_{sw}\}$ by (2.3.i). If $\psi \in \mathcal{C}$, $\forall m \in M^Q_w$, $\phi^Q(\delta \otimes m - 1 \otimes (s\delta)m) \in N^Q_w$ by (2) as $\psi^Q(m) \in (B(s) * N)^Q_w = N^Q_w \oplus N^Q_{sw}$, while $\forall m' \in M^Q_{sw}$, $\phi^Q(1\otimes\delta m' - \delta\otimes m') \in N^Q_w$ by (2) again as $\psi^Q(m') \in (B(s)*N)^Q_{sw} = N^Q_{sw}\oplus N^Q_w$. Thus, $\phi^Q((B(s)*M)^Q_w) \subseteq N^Q_w$, and $\phi \in \mathcal{C}$.

(ii) Define $\Psi : R\text{Bimod}(M * B(s), N) \to R\text{Bimod}(M, N * B(s))$ via

$$\phi \mapsto \psi_\phi : m \mapsto \phi(m\delta \otimes 1) \otimes 1 - \phi(m \otimes 1) \otimes s\delta.$$

To check that ψ_ϕ is well-defined,

$$\begin{aligned}
\psi_\phi(m\delta) &= \phi(m\delta^2 \otimes 1) \otimes 1 - \phi(m\delta \otimes 1) \otimes s\delta \\
&= \phi(m(-\delta(s\delta) + \delta(\delta + s\delta)) \otimes 1) \otimes 1 - \phi(m\delta \otimes 1) \otimes s\delta \\
&= -\phi(m \otimes 1) \otimes \delta(s\delta) + \phi(m\delta \otimes 1) \otimes (\delta + s\delta) - \phi(m\delta \otimes 1) \otimes s\delta \\
&= \{\phi(m\delta \otimes 1) \otimes 1 - \phi(m \otimes 1) \otimes s\delta\}\delta = \psi_\phi(m)\delta.
\end{aligned}$$

If ϕ is graded of degree 0, so is ψ_ϕ; if $m \in M^d$, $m\delta \otimes 1 \in (M * B(s))^{d+1}$ while $m \otimes 1 \in (M * B(s))^{d-1}$, and hence $\phi(m\delta \otimes 1) \otimes 1 \in (N * B(s))^{d+1-1}$, $\phi(m \otimes 1) \otimes s\delta \in (N * B(s))^{d-1+1}$.

Given $\psi \in R\text{Bimod}(M, N * B(s))$, along the decompostion $N * B(s) = N \otimes_R \{R \otimes_{R_s} R(1)\} \simeq N \otimes_{R^s} \{R^s(1) \oplus \delta R^s(1)\} = N \otimes_{R^s} \{R^s(1) \oplus (-s\delta)R^s(1)\} \simeq N(1) \oplus (-s\delta)N(1)$, write

$$\psi(m) = \psi_1(m) \otimes 1 - \psi_2(m) \otimes s\delta \quad \exists!\psi_1(m), \psi_2(m) \in N.$$

Thus, $\psi_1, \psi_2 \in (R, R^s)\text{Bimod}(M, N)$. Define $\Phi : R\text{Bimod}(M, N * B(s)) \to R\text{Bimod}(M * B(s), N)$ via

$$\psi \mapsto \phi_\psi : m \otimes a \mapsto \psi_2(m)a, \quad a \in R, m \in M.$$

If ψ is graded of degree 0, $\psi_2 : M \to N(-1)$, so therefore is ϕ_ψ; if $m \in M^d$, $\psi_2(M) \otimes s\delta \in (N * B(s))^d = \{N \otimes_{R^s} R(1)\}^d$, and hence $\psi_2(m) \in N^{d-1} = N(-1)^d$. If $a \in R^c$, $m \otimes a \in (M * B(s))^{d+c-1}$ and $\psi_2(m)a \in N^{d-1+c}$.

Now, $\phi_{\psi_\phi}(m \otimes a) = (\psi_\phi)_2(m)a = \phi(m \otimes 1)a = \phi(m \otimes a)$, and hence $\phi_{\psi_\phi} = \phi$. Also, $\psi_{\phi_\psi}(m) = \phi_\psi(m\delta \otimes 1) \otimes 1 - \phi_\psi(m \otimes 1) \otimes s\delta = \psi_2(m\delta) \otimes 1 - \psi_2(m) \otimes s\delta$. But

$$\begin{aligned}
\psi_1(m\delta) \otimes 1 - \psi_2(m\delta) \otimes s\delta = \psi(m\delta) = \psi(m)\delta &= \psi_1(m) \otimes \delta - \psi_2(m) \otimes (s\delta)\delta \\
&= \psi_1(m) \otimes \{-s\delta + (\delta + s\delta)\} - \psi_2(m)(s\delta)\delta \otimes 1 \\
&= \{\psi_1(m)(\delta + s\delta) - \psi_2(m)(s\delta)\delta\} \otimes 1 - \psi_1(m) \otimes s\delta.
\end{aligned}$$

and hence $\psi_1(m) = \psi_2(m\delta)$. Then $\psi_{\phi_\psi}(m) = \psi_1(m) \otimes 1 - \psi_2(m) \otimes s\delta = \psi(m)$, and hence $\psi_{\phi_\psi} = \psi$. It follows that Φ and Ψ are inverse to each other.

We show next that Ψ and Φ induce bijections

$$\mathcal{C}(M * B(s), N) \rightleftarrows \mathcal{C}(M, N * B(s)).$$

For that we have only to verify that $\forall \phi \in R\text{Bimod}(M * B(s), N)$, $\phi \in \mathcal{C}$ iff $\psi_\phi \in \mathcal{C}$; if $\psi \in \mathcal{C}$, $\psi_{\phi_\psi} \in \mathcal{C}$, and hence we will have $\phi_\psi \in \mathcal{C}$. Put $\psi = \psi_\phi$ for simplicity. We must check that

$\forall w \in \mathcal{W}$, $\psi^Q(M^Q_w) \subseteq (N * B(s))^Q_w$ iff $\phi^Q((M * B(s))^Q_w) \subseteq N^Q_w$. $\forall m \in M$, $\forall y \in \mathcal{W}$,

$$
\begin{aligned}
(3) \qquad \{\psi^Q(1 \otimes m)\}_y &= \{\phi(m\delta \otimes 1) \otimes 1 - \phi(m \otimes 1) \otimes s\delta\}_y \quad \text{in } (N * B(s))^Q \\
&\qquad\qquad \text{with } 1 \otimes m \in M^Q = Q \otimes_R M \\
&= (\phi(m\delta \otimes 1)_y - \phi(m \otimes 1)_y(s\delta), \phi(m\delta \otimes 1)_{ys} - \phi(m \otimes 1)_{ys}\delta) \\
&\qquad\qquad \text{in } N^Q_y \oplus N^Q_{ys} \text{ by (2.3.iii)} \\
&= (\phi(m\delta \otimes 1 - m \otimes s\delta)_y, \phi(m\delta \otimes 1 - m \otimes \delta)_{ys}) \\
&= (\phi(m \otimes \delta - m(s\delta) \otimes 1)_y, \phi(m\delta \otimes 1 - m \otimes \delta)_{ys}) \quad \text{by (2.3.iii) again.}
\end{aligned}
$$

Thus, if $m \in M^Q_w$, (3) reads

$$
(4) \qquad \psi^Q(m)_y = (\phi^Q(m \otimes \delta - m(s\delta) \otimes 1)_y, \phi^Q(m\delta \otimes 1 - m \otimes \delta)_{ys}) \quad \text{in } N^Q_y \oplus N^Q_{ys}
$$

with $m \otimes \delta - m(s\delta) \otimes 1 \in (M * B(s))^Q_w$ and $m\delta \otimes 1 - m \otimes \delta \in (M * B(s))^Q_{ws}$ by (2.3.iii). Thus, if $\phi \in \mathcal{C}$,

$$
\phi^Q(m \otimes \delta - m(s\delta) \otimes 1) \in N^Q_w \quad \text{and} \quad \phi^Q(m\delta \otimes 1 - m \otimes \delta) \in N^Q_{ws},
$$

and hence $\psi^Q(m)_y = 0$ unless $y = w$. It follows that $\psi^Q(M^Q_w) \subseteq (N * B(s))^Q_w$.

In turn, $(M * B(s))^Q_w = \{m \otimes \delta - m(s\delta) \otimes 1 + m' \otimes \delta - m'\delta \otimes 1 | m \in M^Q_w, m' \in M^Q_{ws}\}$ by (2.3.iii). If $\psi \in \mathcal{C}$, $\forall m \in M^Q_w$, $\forall m' \in M^Q_{ws}$,

$$
\phi^Q(m \otimes \delta - m(s\delta) \otimes 1) \in N^Q_w \quad \text{and} \quad \phi^Q(m' \otimes \delta - m'\delta \otimes 1) \in N^Q_w
$$

by (4) as $\psi^Q(m) \in (N * B(s))^Q_w = N^Q_w \oplus N^Q_{ws}$ and as $\psi^Q(m') \in (N * B(s))^Q_{ws} = N^Q_{ws} \oplus N^Q_w$. Thus, $\phi^Q((M * B(s))^Q_w) \subseteq N^Q_w$, and $\phi \in \mathcal{C}$.

2.7 **Duality:** Let $M \in R\mathrm{Bimod}$. Let $\mathrm{Mod}R$ denote the category of right R-modules, and set $D(M) = \mathrm{Mod}R(M, R)$ equipped with a structure of R-bimodule such that

$$
(1) \qquad (afb)(m) = f(amb) = f(am)b \quad \forall f \in D(M), \forall m \in M, \forall a, b \in R.
$$

Assume now that $M \in \mathcal{C}$. Thus, M is of finite type either as a left or right R-module by (1.5). We equip $D(M)$ with a grading such that $D(M)^i = \{f \in D(M) | f(M^j) \subseteq R^{j+i} \ \forall j\}$, $i \in \mathbb{Z}$. If we let $\mathrm{Modgr}R$ denote the category of graded right R-modules, $D(M)^i = \mathrm{Modgr}R(M, R(i))$. As M is finite type as a right R-module, one has $D(M) = \coprod_{i \in \mathbb{Z}} D(M)^i$ [NvO, 2.4.4]. Also, M is torsion-free as a right R-module (1.3.2). Thus,

$$
\begin{aligned}
\mathrm{Mod}R(M, R) \otimes_R Q &\simeq \mathrm{Mod}R(M, Q) \quad \text{by the five lemma} \\
&\simeq \mathrm{Mod}Q(M \otimes_R Q, Q) \\
&= \mathrm{Mod}Q(\coprod_{w \in \mathcal{W}} M^Q_w, Q) \quad \text{by (1.3.3)} \\
&\simeq \coprod_{w \in \mathcal{W}} \mathrm{Mod}Q(M^Q_w, Q) \quad \text{from definition (1.2.i).}
\end{aligned}
$$

$\forall f \in \mathrm{Mod}Q(M^Q_w, Q)$, $\forall a, b \in Q$, $\forall x \in M^Q_w$,

$$
(afb)(x) = f(axb) = f(x(w^{-1}a)b) = f(x)(w^{-1}a)b = \{(wb)f(w^{-1}a)\}(x).
$$

Thus, if we let $D'(M)^Q_w = \mathrm{Mod}Q(M^Q_w, Q)$, $D(M) \otimes_R Q = \coprod_{w \in \mathcal{W}} D'(M)^Q_w$ is a decomposition as (R, Q)-bimodules. Note also that $D(M)$ is torsion free as a right R-module. For if $f \in D(M)$ and $b \in R \setminus 0$ with $fb = 0$, $\forall m \in M$, $0 = (fb)(m) = f(mb) = f(m)b$, and hence $f(m) = 0$, and $f = 0$. Then $D(M) \hookrightarrow D(M) \otimes_R Q$, and we may, as in (1.3.3), identify $D(M) \otimes_R Q$ with $Q \otimes_R D(M)$. As such

(2) $$D(M) \in \mathcal{C}^{\mathrm{tf}} \text{ with } D(M)^Q_w = D'(M)^Q_w \ \forall w \in \mathcal{W}.$$

Then $\forall f \in D(M)$,

(3) $$f_w = \mathrm{pr}_w \circ f^Q = (f \otimes_R Q)|_{M^Q_w} = f^Q|_{M^Q_w}.$$

We have obtained a contravariant functor $D : \mathcal{C} \to \mathcal{C}^{\mathrm{tf}}$.

If $N \in \mathcal{C}$, $D(M) \otimes_R N \to \mathrm{Mod}R(M, N)$ via $f \otimes n \mapsto f(?)n$ does NOT make sense!

2.8. $\forall w \in \mathcal{W}$, $\forall n \in \mathbb{Z}$, one has

(1) $$D(R(w)(n)) \simeq R(w)(-n).$$

Lemma: *$\forall I \subseteq \mathcal{W}$, $\forall M \in \mathcal{C}$, $D(M^I) \simeq D(M)_I$.*

Proof: As $M \twoheadrightarrow M^I$, $D(M^I) \leq D(M)$. $\forall f \in D(M)$,

$$\begin{aligned} f \in D(M^I) &\text{ iff } f|_{M_{\mathcal{W}\setminus I}} = 0 \quad \text{as } 0 \to M_{\mathcal{W}\setminus I} \to M \to M^I \to 0 \text{ is exact by (1.4.2)} \\ &\text{ iff } f^Q|_{\coprod_{w \in \mathcal{W}\setminus I} M^Q_w} = 0 \quad \text{by (1.4.ii)} \\ &\text{ iff } f \in D(M)_I \quad \text{by (2.7.3).} \end{aligned}$$

2.9 Let $M \in \mathcal{C}$ and $w \in \mathcal{W}$. The structure of R-bimodule on M^w may be described entirely by its left/right R-module structure.

Lemma: *Assume that M^w is graded free as a left/right R-module.*

(i) $D(M)_w$ is also left graded free over R with $\mathrm{grk}(D(M)_w) = \mathrm{grk}(M^w)(v^{-1})$.

(ii) $D(D(M^w)) \simeq M^w$ in $\mathcal{C}$.

Proof: By (1.8) we may assume that $M^w = R(w)(n)$ for some $n \in \mathbb{Z}$. Then

$$\begin{aligned} D(M)_w &\simeq D(M^w) \quad \text{by (2.8)} \\ &= D(R(w)(n)) \simeq R(w)(-n), \end{aligned}$$

and hence $\mathrm{grk}(D(M)_w) = v^{-n} = \mathrm{grk}(R(w)(n))(v^{-1}) = \mathrm{grk}(M^w)(v^{-1})$. One has also

$$\begin{aligned} M^w = R(w)(n) &\simeq D(R(w)(-n)) \\ &\simeq D(D(M)_w) \quad \text{by above} \\ &\simeq D(D(M^w)) \quad \text{by (2.8) again.} \end{aligned}$$

2.10. $\forall M \in \mathcal{C}$, recall that $D(M)$ is graded with $D(M)^n = \mathrm{Mod}R(M,R)^n = \{f \in \mathrm{Mod}R(M,R) | f(M^i) \subseteq R^{i+n} = \mathrm{Modgr}R(M, R(n))\ \forall i \in \mathbb{Z}\}\ \forall n \in \mathbb{Z}$.

Consider first the case $M = B(s)$, $s \in \mathcal{S}$. Let $\delta \in V$ with $\langle \delta, \alpha_s^\vee \rangle = 1$, and let

$$\Phi_s : D(B(s)) \to B(s) \quad \text{via} \quad f \mapsto 1 \otimes_{R_s} f(\delta \otimes_{R^s} 1) - (s\delta) \otimes_{R^s} f(1 \otimes_{R^s} 1). \tag{1}$$

We show that Φ_s is invertible in $\mathcal{C}$. $\forall a \in R$,

$$\begin{aligned} \Phi_s(fa) &= 1 \otimes (fa)(\delta \otimes 1) - (s\delta) \otimes (fa)(1 \otimes 1) = 1 \otimes f(\delta \otimes a) - (s\delta) \otimes f(1 \otimes a) \\ &= 1 \otimes f(\delta \otimes 1)a - (s\delta) \otimes f(1 \otimes 1)a = \Phi_s(f)a. \end{aligned}$$

Likewise, $\forall a \in R^s$, $\Phi_s(af) = a\Phi_s(f)$. Also,

$$\begin{aligned} \Phi_s(\delta f) &= 1 \otimes f(\delta^2 \otimes 1) - (s\delta) \otimes f(\delta \otimes 1) \\ &= 1 \otimes f((-\delta(s\delta) + \delta(\delta + s\delta)) \otimes 1) - (s\delta) \otimes f(\delta \otimes 1) \\ &= -1 \otimes f(1 \otimes 1)\delta(s\delta) + 1 \otimes f(\delta \otimes 1)(\delta + s\delta) - (s\delta) \otimes f(\delta \otimes 1) \\ &= -\delta(s\delta) \otimes f(1 \otimes 1) + (\delta + s\delta) \otimes f(\delta \otimes 1) - (s\delta) \otimes f(\delta \otimes 1) \\ &= \delta\{1 \otimes f(\delta \otimes 1) - (s\delta) \otimes f(\delta \otimes 1)\} = \delta\Phi_s(f). \end{aligned}$$

Thus, Φ_s is R-bilinear. As $\delta \otimes 1 \in B(s)^1$ and $1 \otimes 1 \in B(s)^{-1}$, if $f \in D(B(s))^k$, $k \in \mathbb{Z}$, $1 \otimes_{R_s} f(\delta \otimes 1) - (s\delta) \otimes_{R^s} f(1 \otimes 1) \in (R \otimes_{R^s} R)^{k+1} = B(s)^k$, and hence Φ_s is graded.

Now, $D(B(s))^Q = D(B(s))_e^Q \oplus D(B(s))_s^Q$ with

$$\begin{aligned} D(B(s))_e^Q &= \mathrm{Mod}Q(B(s)_e^Q, Q) \quad \text{by (2.7.2)} \\ &= \mathrm{Mod}Q(Q(\delta \otimes 1 - 1 \otimes s\delta), Q) \quad \text{by (2.2.6)}, \\ D(B(s))_s^Q &= \mathrm{Mod}Q(B(s)_s^Q, Q) = \mathrm{Mod}Q(Q(\delta \otimes 1 - 1 \otimes \delta), Q) \quad \text{likewise}. \end{aligned}$$

If $f \in D(B(s))_e^Q$, $f(\delta \otimes 1) = f(1 \otimes \delta) = f(1 \otimes 1)\delta$, and hence

$$\begin{aligned} 1 \otimes f(\delta \otimes 1) - s\delta \otimes f(1 \otimes 1) &= 1 \otimes f(1 \otimes 1)\delta - s\delta \otimes f(1 \otimes 1) \\ &= (1 \otimes \delta - s\delta \otimes 1)f(1 \otimes 1) \in B(s)_e^Q. \end{aligned}$$

If $f \in D(B(s))_s^Q$, $f(\delta \otimes 1) = f(1 \otimes s\delta) = f(1 \otimes 1)s\delta$, and hence

$$\begin{aligned} 1 \otimes f(\delta \otimes 1) - s\delta \otimes f(1 \otimes 1) &= 1 \otimes f(1 \otimes 1)s\delta - s\delta \otimes f(1 \otimes 1) = (1 \otimes s\delta - s\delta \otimes 1)f(1 \otimes 1) \\ &= (\delta \otimes 1 - 1 \otimes \delta)f(1 \otimes 1) \in B(s)_s^Q \quad \text{as} \\ 1 \otimes s\delta - s\delta \otimes 1 &= 1 \otimes ((s\delta + \delta) - \delta) - ((s\delta + \delta) - \delta) \otimes 1 \\ &= (s\delta + \delta) \otimes 1 - 1 \otimes \delta - (s\delta + \delta) \otimes 1 + \delta \otimes 1 = \delta \otimes 1 - 1 \otimes \delta. \end{aligned}$$

Thus, $\Phi_s \in \mathcal{C}'(D(B(s)), B(s))$. Finally, $\forall a, b \in R$, $1 \otimes_{R^s} a + \delta \otimes_{R^s} b = 1 \otimes_{R^s} \{a + (s\delta + \delta)b\} - s\delta \otimes_{R^s} b$, and hence

$$B(s) = \{1 \otimes_{R^s} a - s\delta \otimes_{R^s} b | a, b \in R\}. \tag{2}$$

Then,

$$\Psi_s : B(s) \to D(B(s)) \quad \text{via} \quad 1 \otimes a - s\delta \otimes b \mapsto \text{“}1 \otimes x + \delta \otimes y \mapsto bx + ay\text{”} \quad \forall a, b, x, y \in R \tag{3}$$

gives an inverse to Ψ_s:

$$f \mapsto 1 \otimes f(\delta \otimes 1) - s\delta \otimes f(1 \otimes 1) \mapsto \begin{array}{c} 1 \otimes x + \delta \otimes y \\ \downarrow \\ f(1 \otimes 1)x + f(\delta \otimes 1)y \\ \| \\ f(1 \otimes x + \delta \otimes y). \end{array}$$

Note also that

(4) $$\begin{array}{ccccc} m & B(s) & \xrightarrow[\sim]{\Psi_s} & D(B(s)) & \\ & & \searrow^{\mathrm{ev}} & \sim \big\downarrow D(\Phi_s) & \\ & \mathrm{ev}_m & & D^2(B(s)). & \end{array}$$

For let $f \in D(B(s))$. $\forall a, b \in R$,

(5) $$\begin{aligned} \{(D(\Phi_s) \circ \Psi_s)(1 \otimes a - s\delta \otimes b)\}(f) &= \{\Psi_s(1 \otimes a - s\delta \otimes b) \circ \Phi_s\}(f) \\ &= \Psi_s(1 \otimes a - s\delta \otimes b)(\Phi_s(f)) \\ &= \Psi_s(1 \otimes a - s\delta \otimes b)(1 \otimes f(\delta \otimes 1) - s\delta \otimes f(1 \otimes 1)) \\ &= \Psi_s(1 \otimes a - s\delta \otimes b)\{1 \otimes \{f(\delta \otimes 1) - (s\delta + \delta)f(1 \otimes 1)\} + \delta \otimes f(1 \otimes 1)\} \\ &= b\{f(\delta \otimes 1) - (s\delta + \delta)f(1 \otimes 1)\} + af(1 \otimes 1) \\ &= f(\delta \otimes b) - f((s\delta + \delta) \otimes b) + f(1 \otimes a) = f(1 \otimes a - s\delta \otimes b) \\ &= \mathrm{ev}_{1 \otimes a - s\delta \otimes b}(f). \end{aligned}$$

Then, $\forall \varphi \in \mathcal{C}(B(s), B(s))$,

(6) $$\begin{array}{ccccc} m & & \longmapsto & & \mathrm{ev}_m \\ B(s) & \xrightarrow[\mathrm{ev}]{\sim} & D^2(B(s)) & & \big\downarrow \\ \varphi \big\downarrow & & \big\downarrow D^2(\varphi) & & \\ B(s) & \xrightarrow[\sim]{\mathrm{ev}} & D^2(B(s)) & & \mathrm{ev}_{\varphi(m)} \end{array}$$

as $\{D^2(\varphi)(\mathrm{ev}_m)\}(f) = (\mathrm{ev}_m \circ D(\varphi))(f) = \mathrm{ev}_m(f \circ \varphi) = (f \circ \varphi)(m) = f(\varphi(m)) = \mathrm{ev}_{\varphi(m)}(f)$ $\forall f \in D(B(s))$, and hence

(7) $$D^2 \simeq \mathrm{id} \quad \text{on } B(s).$$

More generally,

Lemma: *$\forall M \in \mathcal{C}$ with $D(M) \in \mathcal{C}$, $\forall s \in \mathcal{S}$, $D(B(s) * M) \simeq B(s) * D(M)$ in $\mathcal{C}$. In particular, $\forall \underline{x} \in \mathcal{S}^r$, $D(B(\underline{x})) \simeq B(\underline{x})$ in $\mathcal{C}$, and hence $DB \simeq B$ $\forall B \in \mathfrak{S}\mathrm{Bimod}$.*

Proof: We regard $B(s)*?$ as $R(1) \otimes_{R^s} ?$. Take $\delta \in V$ with $\langle \delta, \alpha_s^\vee \rangle = 1$. $\forall f \in D(B(s) * M)^k = \mathrm{Mod}R(R(1) \otimes_{R^s} M, R)^k$, define $f_1 \in \mathrm{Mod}R(M, R)^{k-1}$ and $f_2 \in \mathrm{Mod}R(M, R)^{k+1}$ via

$$f_1(m) = f(1 \otimes m) \quad \text{and} \quad f_2(m) = f(\delta \otimes m) \quad \forall m \in M.$$

Let $\Phi : D(B(s) * M) \to B(s) * D(M) = R(1) \otimes_{R^s} \mathrm{Mod}R(M, R)$ via

(8) $$f \mapsto 1 \otimes f_2 - (s\delta) \otimes f_1.$$

$\forall a \in R$, $\forall m \in M$,

$$(fa)_1(m) = (fa)(1 \otimes m) = f(1 \otimes ma) = f(1 \otimes m)a = f_1(m)a,$$
$$(fa)_2(m) = (fa)(\delta \otimes m) = f(\delta \otimes ma) = f(\delta \otimes m)a = f_2(m)a,$$

and hence $(fa)_1 = f_1 a$, $(fa)_2 = f_2 a$. Thus, $\Phi \in \mathrm{Modgr}R$.

If $a \in R^s$, $\forall m \in M$,

$$(af)_1(m) = (af)(1 \otimes m) = f(a \otimes m) = f(1 \otimes am) = f_1(am) = (af_1)(m),$$
$$(af)_2(m) = (af)(\delta \otimes m) = f(a\delta \otimes m) = f(\delta \otimes am) = f_2(am) = (af_2)(m),$$

and hence $(af)_1 = af_1$, $(af)_2 = af_2$. Thus, $\Phi \in R^s\mathrm{Modgr}$. One has

$$\begin{aligned}(\delta f)_1(m) &= (\delta f)(1 \otimes m) = f(\delta \otimes m) = f_2(m),\\ (\delta f)_2(m) &= (\delta f)(\delta \otimes m) = f(\delta^2 \otimes m) = f((-\delta(s\delta) + \delta(\delta + s\delta)) \otimes m)\\ &= -f(1 \otimes \delta(s\delta)m) + f(\delta \otimes (\delta + s\delta)m) = -f_1(\delta(s\delta)m) + f_2(\delta + s\delta)m)\\ &= -(\delta(s\delta)f_1)(m) + ((\delta + s\delta)f_2)(m),\end{aligned}$$

and hence $(\delta f)_1 = f_2$, $(\delta f)_2 = -\delta(s\delta)f_1 + (\delta + s\delta)f_2$. Then

$$\begin{aligned}\Phi(\delta f) &= 1 \otimes (\delta f)_2 - s\delta \otimes (\delta f)_1 = 1 \otimes \{-\delta(s\delta)f_1 + (\delta + s\delta)f_2\} - s\delta \otimes f_2\\ &= -\delta(s\delta) \otimes f_1 + (\delta + s\delta) \otimes f_2 - s\delta \otimes f_2 = -\delta(s\delta) \otimes f_1 + \delta \otimes f_2\\ &= \delta\Phi(f),\end{aligned}$$

and hence $\Phi \in R\mathrm{Modgr}$ also.

We show next that $\forall w \in \mathcal{W}$,

(9) $$\Phi^Q(D(B(s) * M)^Q_w) \subseteq (B(s) * D(M))^Q_w.$$

Let $f \in D(B(s) * M)^Q_w = \mathrm{Mod}Q((B(s) * M)^Q_w, Q)$ after (2.7.2). Recall from (2.3.ii) that $\{B(s) \otimes_R M^Q_y\} \oplus \{B(s) \otimes_R M^Q_{sy}\} \simeq (B(s) * M)^Q_y \oplus (B(s) * M)^Q_{sy}$ $\forall y \in \mathcal{W}$. Then

$$f_1(M^Q_y) = f(1 \otimes M^Q_y) \subseteq f((B(s) * M)^Q_y \oplus (B(s) * M)^Q_{sy}),$$
$$f_2(M^Q_y) = f(\delta \otimes M^Q_y) \subseteq f((B(s) * M)^Q_y \oplus (B(s) * M)^Q_{sy}),$$

and hence $\forall y \notin \{w, sw\}$,
$$f_1(M^Q_y) = 0 = f_2(M^Q_y).$$

Thus, $\mathrm{supp}_{\mathcal{W}}(f_1), \mathrm{supp}_{\mathcal{W}}(f_2) \subseteq \{w, sw\}$, and $\mathrm{supp}_{\mathcal{W}}(\Phi^Q(f)) = \mathrm{supp}_{\mathcal{W}}(1 \otimes f_2 - s\delta \otimes f_1) \subseteq \{w, sw\}$ by (1.7). As $f \in D(B(s) * M)^Q_w$,

$$\begin{aligned}0 &= f((B(s) * M)^Q_{sw})\\ &= f(\{\delta \otimes m - 1 \otimes (s\delta)m + \delta \otimes m' - 1 \otimes \delta m' | m \in M^Q_{sw}, m' \in M^Q_w\}) \quad \text{by (2.3.i)}.\end{aligned}$$

In particular, $\forall m \in M^Q_{sw}$,

$$0 = f(\delta \otimes m - 1 \otimes (s\delta)m) = f_2(m) - f_1((s\delta)m) = (f_2 - (s\delta)f_1)(m),$$

and hence $(f_2)_{sw} = (s\delta)(f_1)_{sw}$ by (2.7.3). If $m \in M^Q_w$,

$$0 = f(\delta \otimes m - 1 \otimes \delta m) = f_2(m) - f_1(\delta m) = (f_2 - \delta f_1)(m),$$

and hence $(f_2)_w = \delta(f_1)_w$ also. Then

$$\begin{aligned}\Phi^Q(f)_{sw} &= (1 \otimes f_2 - (s\delta) \otimes f_1)_{sw} \\ &= ((f_2)_{sw} - (s\delta)(f_1)_{sw}, (f_2)_w - \delta(f_1)_w) \quad \text{in } D(M)^Q_{sw} \oplus D(M)^Q_w \text{ by (2.3.i)} \\ &= 0.\end{aligned}$$

Thus, $\operatorname{supp}_{\mathcal{W}}(\Phi^Q(f)) \subseteq \{w\}$, and (9) holds.

Finally, an inverse of Φ is given by

$$\Psi : g = 1 \otimes g_1 + \delta \otimes g_2 \mapsto \text{“}1 \otimes m_1 + \delta \otimes m_2 \mapsto g_1(m_2) + g_2(m_1 + (\delta + s\delta)m_2)\text{”} \quad \forall g_1, g_2 \in DM, \forall m_1, m_2 \in M,$$

i.e.,

(10) $$\begin{aligned} 1 \otimes g_1 - s\delta \otimes g_2 &= 1 \otimes g_1 + \delta \otimes g_2 - 1 \otimes (s\delta + \delta)g_2 = 1 \otimes \{g_1 - (s\delta + \delta)g_2\} + \delta \otimes g_2 \\ &\mapsto \text{“}1 \otimes m_1 + \delta \otimes m_2 \mapsto (g_1 - (s\delta + \delta)g_2)(m_2) + g_2(m_1 + (\delta + s\delta)m_2) \\ &\qquad = g_1(m_2) + g_2(m_1)\text{”}.\end{aligned}$$

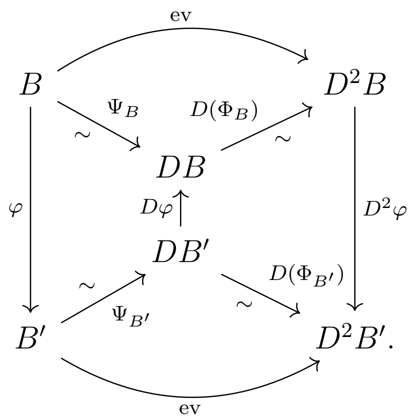
If $a \in R$,

$$\begin{aligned}\Psi(g)((1 \otimes m_1 + \delta \otimes m_2)a) &= g_1(m_2a) + g_2(m_1a) = \{g_1(m_2) + g_2(m_1)\}a \\ &= \Psi(g)(1 \otimes m_1 + \delta \otimes m_2)a,\end{aligned}$$

and hence $\Psi(g)$ is right R-linear. To see that $\Psi(g) \in D(B(s) * M)^k = \operatorname{Mod}R(R(1) \otimes_{R^s} M, R)^k$ if $g = 1 \otimes g_1 - s\delta \otimes g_2 \in (B(s) * DM)^k = \{R(1) \otimes_{R^s} \operatorname{Mod}R(M, R)\}^k$, $k \in \mathbb{Z}$, one has $g_1 \in (DM)^{k+1}, g_2 \in (DM)^{k-1}$. If $1 \otimes m_1 + \delta \otimes m_2 \in (B(s) * M)^l$, $m_1 \in M^{l+1}$ and $m_2 \in M^{l-1}$. Thus $g_1(m_2) + g_2(m_1) \in R^{l+k}$, and hence $\Psi(g) \in D(B(s) * M)^k$.

2.11. Let $B \in \mathfrak{S}\mathrm{Bimod}$, and let $\Phi_B \in \mathcal{C}(DB, B)^\times$, $\Psi_{B'} \in \mathcal{C}(B, DB)^\times$ as in (2.10).

Proposition: *$\forall \varphi \in \mathcal{C}(B, B')$, one has a commutative diagram*



In particular, $D^2 \simeq \mathrm{id}$ *on* $\mathfrak{S}\mathrm{Bimod}$ *with* $D : \mathcal{C}(B, B') \xrightarrow{\sim} \mathcal{C}(B', B)$ *via* $\varphi \mapsto D(\varphi)$.

Proof: Let $M \in \mathfrak{S}\mathrm{Bimod}$ and $\Phi_M \in \mathcal{C}(DM, M)^\times$ with an inverse $\Psi_M \in \mathcal{C}(M, DM)^\times$ such that $D(\Phi_M) \circ \Psi_M = \mathrm{ev}$ as in (4), and let $\Phi \in \mathcal{C}(D(B(s) * M), B(s) * M)^\times$ with an inverse $\Psi \in \mathcal{C}(B(s) * M, D(B(s) * M))^\times$ as in (2.11). It suffices to show that

(1)
$$\begin{array}{ccc} x & B(s) * M & \xrightarrow[\sim]{B(s)*\Psi_M} & B(s) * DM \\ & & & \downarrow{\scriptstyle \sim}\ \Psi \\ & & & D(B(s) * M) \\ & & & \downarrow D((B(s)*\Phi_M)\circ\Phi) \\ & \mathrm{ev}_x & & D^2(B(s) * M). \end{array}$$

Let $\delta \in V$ with $\langle \delta, \alpha_s^\vee \rangle = 1$. Let $a \in R$ and $m \in M$. Regarding $a \otimes m \in R(1) \otimes_{R^s} M = B(s) * M$, we are to show on $D(B(s) * M)$ that

(2) $$\{(\Psi \circ (B(s) * \Psi_M))(a \otimes m)\} \circ \{(B(s) * \Phi_M) \circ \Phi\} = \mathrm{ev}_{a\otimes m}.$$

Write $a = a_1 - (s\delta)a_2$, $a_1, a_2 \in R^s$, and let $m_1, m_2 \in M$. Then, regarding $1 \otimes m_1, \delta \otimes m_2 \in R(1) \otimes_{R^s} R = B(s) * M$, one has

$$\begin{aligned} \{(\Psi\circ(B(s) * \Psi_M))(a \otimes m)\}(1 \otimes m_1 + \delta \otimes m_2) &= \{\Psi(a \otimes \Psi_M(m))\}(1 \otimes m_1 + \delta \otimes m_2) \\ &= \{\Psi(1 \otimes a_1\Psi_M(m) - s\delta \otimes a_2\Psi_M(m))\}(1 \otimes m_1 + \delta \otimes m_2) \\ &= (a_1\Psi_M(m))(m_2) + (a_2\Psi_M(m))(m_1) \quad \text{by (2.10.10)} \end{aligned}$$

while $\forall f \in D(B(s) * M)$,

$$\begin{aligned} \{(B(s)*\Phi_M) \circ \Phi\}(f) &= (B(s) * \Phi_M)(1 \otimes f(\delta\otimes ?) - s\delta \otimes f(1\otimes ?)) \quad \text{by (2.10.8) with} \\ &\qquad f(\delta\otimes ?), f(1\otimes ?) \in DM \text{ and } 1 \otimes f(\delta\otimes ?), s\delta \otimes f(1\otimes ?) \in R(1) \otimes_{R^s} DM = B(s) * M \\ &= 1 \otimes \Phi_M(f(\delta\otimes ?)) - s\delta \otimes \Phi_M(f(1\otimes ?)) \quad \text{in } R(1) \otimes_{R^s} M = B(s) * M \\ &= 1 \otimes \Phi_M(f(\delta\otimes ?)) - (s\delta + \delta - \delta) \otimes \Phi_M(f(1\otimes ?)) \\ &= 1 \otimes \{\Phi_M(f(\delta\otimes ?)) - (s\delta + \delta) \otimes \Phi_M(f(1\otimes ?)\} + \delta \otimes \Phi_M(f(1\otimes ?)). \end{aligned}$$

Thus,

$$\begin{aligned} &[\{(\Psi \circ (B(s) * \Psi_M))(a \otimes m)\} \circ \{(B(s) * \Phi_M) \circ \Phi\}](f) \\ &\quad = (a_1\Psi_M(m))\{\Phi_M(f(1\otimes ?))\} + (a_2\Psi_M(m))\{\Phi_M(f(\delta\otimes ?)) - (s\delta + \delta)\Phi_M(f(1\otimes ?)\} \\ &\quad = (\Psi_M(a_1m))(\Phi_M(f(1\otimes ?))) + (\Psi_M(a_2m))(\Phi_M(f(\delta\otimes ?))) \\ &\qquad\quad - (\Psi_M(a_2m))(\Phi_M((s\delta + \delta)f(1\otimes ?))) \\ &\quad = (\Psi_M(a_1m))(\Phi_M(f(1\otimes ?))) + (\Psi_M(a_2m))(\Phi_M(f(\delta\otimes ?))) \\ &\qquad\quad - (\Psi_M(a_2m))(\Phi_M(f((s\delta + \delta)\otimes ?))). \end{aligned}$$

Now, $\forall m \in M$, $\forall g \in DM$,

(3) $$(\Psi_M(m))(\Phi_M(g)) = (\Psi_M(m) \circ \Phi_M)(g) = \{D(\Phi_M) \circ \Phi_M)(m)\}(g) = \mathrm{ev}_m(g) = g(m).$$

It follows that

$$
\begin{aligned}
&[\{(\Psi\circ(B(s)*\Psi_M))(a\otimes m)\}\circ\{(B(s)*\Phi_M)\circ\Phi\}](f)\\
&\quad= f(1\otimes a_1m)+f(\delta\otimes a_2m)-f((s\delta+\delta)\otimes a_2m)=f(1\otimes a_1m)-f(s\delta\otimes a_2m)\\
&\quad= f(a_1\otimes m)-f((s\delta)a_2\otimes m)\quad\text{as } a_1,a_2\in R^s\\
&\quad= \mathrm{ev}_{(a_1-a_2s\delta)\otimes m}(f)=\mathrm{ev}_{a\otimes m}(f),
\end{aligned}
$$

as desired.

2.12. Let RMod (resp. RModgr) denote the category of (resp. graded) left R-modules. $\forall M\in\mathcal{C}$, $\forall i\in\mathbb{Z}$, let $D^l(M)^i=R\mathrm{Modgr}(M,R(i))$, and set $D^l(M)=\coprod_{i\in\mathbb{Z}}D^l(M)^i\simeq R\mathrm{Mod}(M,R)$ [NvO, 2.4.4]. We equip $D^l(M)$ with a structure of R-bimodule such that $(afb)(m)=f(amb)=af(mb)$ $\forall f\in D^l(M)$, $\forall a,b\in R$, $\forall m\in M$. Then

$$
\begin{aligned}
Q\otimes_R D^l(M)&\simeq R\mathrm{Mod}(M,Q)\quad\text{by the 5 lemma}\\
&\simeq Q\mathrm{Mod}(Q\otimes_R M,Q)=Q\mathrm{Mod}(\coprod_{w\in\mathcal{W}}M_w^Q,Q)\\
&\simeq \coprod_{w\in\mathcal{W}}Q\mathrm{Mod}(M_w^Q,Q)\quad\text{by (1.2.i).}
\end{aligned}
$$

$\forall f\in Q\mathrm{Mod}(M_w^Q,Q)$, $\forall q_1,q_2\in Q$, $\forall x\in M_w^Q$, $(q_1fq_2)(x)=f(q_1xq_2)=q_1(wq_2)f(x)$, and hence $q_1fq_2=q_1(wq_2)f$. Thus, $D^l(M)$ admits a structure of $\mathcal{C}'$ with

(1) $$D^l(M)_w^Q=Q\mathrm{Mod}(M_w^Q,Q)\quad\forall w\in\mathcal{W}.$$

Also, $D^l(M)$ is torsion free as a left R-module: if $af=0$, $a\in R$, $f\in D^l(M)$, $\forall m\in M$, $0=(af)(m)=f(am)=af(m)$. As R is a domain, if $a\neq 0$, $f=0$.

In particular, $\forall w\in\mathcal{W}$, $\forall n\in\mathbb{Z}$,

(2) $$D^l(R(w)(n))\simeq R(w)(-n).$$

For let $f\in D^l(R(w))$ and $a,b\in R$. Then

$$(afb)(1)=f(a1b)=f(a(wb))=a(wb)f(1)=(a(wb)f)(1),$$

and hence $afb=a(wb)f$.

If $f\in D^l(M)$, $a\in R$, and $m\in M_w$, $f(ma)=f((wa)m)=(wa)f(m)$, which may be distinct from $af(m)$, and hence $D^l(M)$ need not be isomorphic to $D(M)$.

Lemma: *$\forall M\in\mathcal{C}$ with $D^l(M)\in\mathcal{C}$, $\forall s\in\mathcal{S}$, $D^l(M*B(s))\simeq D^l(M)*B(s)$ in $\mathcal{C}$. In particular, $\forall\underline{x}\in\mathcal{S}^r$, $D^l(B(\underline{x}))\simeq B(\underline{x})\simeq D(B(\underline{x}))$.*

Proof: We regard $?*B(s)$ as $?\otimes_{R^s}R(1)$. Take $\delta\in V$ with $\langle\delta,\alpha_s^\vee\rangle=1$. $\forall f\in D^l(M*B(s))^k=R\mathrm{Mod}(M\otimes_{R^s}R(1),R)^k$, $k\in\mathbb{Z}$, define $f_1\in D^l(M)^{k-1}=R\mathrm{Mod}(M,R)^{k-1}$ and $f_2\in D^l(M)^{k+1}=R\mathrm{Mod}(M,R)^{k+1}$ via

$$f_1(m)=f(m\otimes 1)\quad\text{and}\quad f_2(m)=f(m\otimes\delta).$$

Let $\Phi : D^l(M * B(s)) \to D^l(M) * B(s) = R\mathrm{Mod}(M, R) \otimes_{R^s} R(1)$ via $f \mapsto f_2 \otimes 1 - f_1 \otimes s\delta$. $\forall a \in R^s$, $\forall m \in M$,

$$\begin{aligned}(fa)_1(m) &= (fa)(m \otimes 1) = f(m \otimes a) = f(ma \otimes 1) = f_1(ma) = (f_1a)(m),\\(fa)_2(m) &= (fa)(m \otimes \delta) = f(m \otimes \delta a) = f(ma \otimes \delta) = f_2(ma) = (f_2a)(m),\end{aligned}$$

and hence $(fa)_1 = f_1a$, $(fa)_2 = f_2a$, $\Phi(fa) = (fa)_2 \otimes 1 - (fa)_1 \otimes s\delta = f_2a \otimes 1 - f_1a \otimes s\delta = (f_2 \otimes 1 - f_1 \otimes s\delta)a = \Phi(f)a$. Also,

$$\begin{aligned}(f\delta)_1(m) &= (f\delta)(m \otimes 1) = f(m \otimes \delta) = f_2(m),\\(f\delta)_2(m) &= (f\delta)(m \otimes \delta) = f(m \otimes \delta^2) = f(m \otimes \delta(s\delta + \delta - s\delta))\\&= f(m(s\delta + \delta) \otimes \delta) - f(m\delta s\delta \otimes 1) = (f_2(s\delta + \delta))(m) - (f_1\delta s\delta)(m),\end{aligned}$$

and hence $(f\delta)_1 = f_2, (f\delta)_2 = f_2(s\delta + \delta) - f_1\delta s\delta$. Then

$$\begin{aligned}\Phi(f\delta) &= (f\delta)_2 \otimes 1 - (f\delta)_1 \otimes s\delta = f_2(s\delta + \delta) \otimes 1 - f_1\delta s\delta \otimes 1 - f_2 \otimes s\delta\\&= f_2 \otimes (s\delta + \delta) - f_1 \otimes \delta s\delta - f_2 \otimes s\delta = f_2 \otimes \delta - f_1 \otimes \delta s\delta = (f_2 \otimes 1 - f_1 \otimes s\delta)\delta\\&= \Phi(f)\delta,\end{aligned}$$

and hence Φ is a homomorphism of graded R-bimodules.

We show next that $\forall w \in \mathcal{W}$,

$$\Phi^Q(D^l(M * B(s))^Q_w) \subseteq \{D^l(M) * B(s)\}^Q_w. \tag{3}$$

Let $f \in D^l(M * B(s))^Q_w = Q\mathrm{Mod}((M * B(s))^Q_w, Q)$. Recall from (2.3.iv) that, $\forall y \in \mathcal{W}$, $(M * B(s))^Q_y \oplus (M * B(s))^Q_{ys} \simeq \{M^Q_y \otimes_R B(s)\} \oplus \{M^Q_{ys} \otimes_R B(s)\}$. Then

$$\begin{aligned}(f_1)^Q(M^Q_y) &= f^Q(M^Q_y \otimes 1) \subseteq f^Q((M * B(s))^Q_y \oplus (M * B(s))^Q_{ys}),\\(f_2)^Q(M^Q_y) &= f^Q(M^Q_y \otimes \delta) \subseteq f^Q((M * B(s))^Q_y \oplus (M * B(s))^Q_{ys}),\end{aligned}$$

and hence $(f_1)^Q(M^Q_y) = 0 = (f_2)^Q(M^Q_y)$ unless $y \in \{w, ws\}$. Thus, $\mathrm{supp}_{\mathcal{W}}(f_1), \mathrm{supp}_{\mathcal{W}}(f_2) \subseteq \{w, ws\}$, and $\mathrm{supp}_{\mathcal{W}}(\Phi(f)) = \mathrm{supp}_{\mathcal{W}}(f_2 \otimes 1 - f_1 \otimes s\delta) \subseteq \{w, ws\}$. As $f \in D^l(M * B(s))^Q_w$,

$$\begin{aligned}0 &= f^Q((M * B(s))^Q_{ws})\\&= f^Q(\{m \otimes \delta - m(s\delta) \otimes 1 + m' \otimes \delta - m'\delta \otimes 1 | m \in M^Q_{ws}, m' \in M^Q_w\}) \quad \text{by (2.3.iii)}.\end{aligned}$$

In particular, $\forall m \in M^Q_{ws}$,

$$0 = f^Q(m \otimes \delta - m(s\delta) \otimes 1) = (f_2)^Q(m) - (f_1)^Q(m(s\delta)) = (f_2 - f_1(s\delta))^Q(m),$$

and hence $(f_2)_{ws} = (f_1)_{ws}(s\delta)$. If $m' \in M^Q_w$,

$$0 = f^Q(m' \otimes \delta - m'\delta \otimes 1) = (f_2)^Q(m') - (f_1)^Q(m'\delta) = (f_2 - f_1\delta)^Q(m'),$$

and hence $(f_2)_w = (f_1)_w\delta$. Then

$$\begin{aligned}\Phi(f)_{ws} &= (f_2 \otimes 1 - f_1 \otimes s\delta)_{ws}\\&= ((f_2)_{ws} - (f_1)_{ws}(s\delta), (f_2)_w - (f_1)_w\delta) \quad \text{in } D^l(M)^Q_{ws} \oplus D^l(M)^Q_w \text{ by (2.3.iii)}\\&= 0.\end{aligned}$$

Thus, $\mathrm{supp}_{\mathcal{W}}(\Phi(f)) \subseteq \{w\}$, and (3) holds.

Finally, an inverse of Φ is given by $\Psi : D^l(M) * B(s) \to D^l(M * B(s))$ via

$$g = g_1 \otimes 1 - g_2 \otimes s\delta \mapsto \text{“}m_1 \otimes 1 + m_2 \otimes \delta \mapsto g_1(m_2) + g_2(m_1)\text{”} \quad \forall g_1, g_2 \in D^l(M), \forall m_1, m_2 \in M.$$

$\forall a \in R$,

$$\begin{aligned}\Psi(g)(a(m_1 \otimes 1 + m_2 \otimes \delta)) &= \Psi(g)(am_1 \otimes 1 + am_2 \otimes \delta) = g_1(am_2) + g_2(am_1) \\ &= a\{g_1(m_2) + g_2(m_1)\} = a\Psi(g)(m_1 \otimes 1 + m_2 \otimes \delta),\end{aligned}$$

and hence $\Psi(g)$ is left R-linear. If $g = g_1 \otimes 1 - g_2 \otimes s\delta \in (D^l(M) * B(s))^k = \{R\mathrm{Mod}R(M, R) \otimes_{R^s} R(1)\}^k$, $k \in \mathbb{Z}$, $g_1 \in D^l(M)^{k+1}$ and $g_2 \in D^l(M)^{k-1}$. If $m_1 \otimes 1 + m_2 \otimes \delta \in (M * B(s))^r$, $m_1 \in M^{r+1}$ and $m_2 \in M^{r-1}$. Thus $g_1(m_2) + g_2(m_1) \in R^{r+k}$, and hence $\Psi(g) \in D^l(M * B(s))^k$, as desired.

3. 岩堀-Hecke algebras

3.1. Let v be an indeterminate. The 岩堀-Hecke algebra $\mathcal{H}$ of $(\mathcal{W}, \mathcal{S})$ is a $\mathbb{Z}[v, v^{-1}]$-algebra having a basis $\{H_w | w \in \mathcal{W}\}$ under the multiplication [S97] such that

(i) $(H_s + v)(H_s - v^{-1}) = 0 \quad \forall s \in \mathcal{S}$,

(ii) $H_x H_y = H_{xy} \quad \forall x, y \in \mathcal{W}$ with $\ell(xy) = \ell(x) + \ell(y)$.

$\forall s \in \mathcal{S}$, put $\underline{H}_s = H_s + v$. Thus [S97, p. 84], $\forall x \in \mathcal{W}$,

(1)
$$H_x \underline{H}_s = \begin{cases} H_{xs} + vH_x & \text{if } xs > x, \\ H_{xs} + v^{-1}H_x & \text{else,} \end{cases}$$

and likewise

(2)
$$\underline{H}_s H_x = \begin{cases} H_{sx} + vH_x & \text{if } sx > x, \\ H_{sx} + v^{-1}H_x & \text{else.} \end{cases}$$

$\forall \underline{x} = (s_1, \dots, s_r) \in \mathcal{S}^r$, put $\underline{H}_{\underline{x}} = \underline{H}_{s_1} \dots \underline{H}_{s_r}$. $\forall w \in \mathcal{W}$, define $p^w_{\underline{x}} \in \mathbb{Z}[v, v^{-1}]$ by $\underline{H}_{\underline{x}} = \sum_{w \in \mathcal{W}} p^w_{\underline{x}} H_w$. For $s \in \mathcal{S}$ we will often abbreviate $p^w_{(s)}$ as p^w_s.

Lemma: $\sum_{w \in \mathcal{W}} v^{\ell(w)} p^w_{\underline{x}}(v^{-1}) = (v + v^{-1})^r$.

Proof: One has a $\mathbb{Z}[v, v^{-1}]$-algebra homomorphism $\mathrm{sgn} : \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ via $H_w \mapsto v^{-\ell(w)}$. If $\underline{x} = (s_1, \dots, s_r)$,

$$(v + v^{-1})^r = \mathrm{sgn}(\underline{H}_{\underline{x}}) = \mathrm{sgn}(\sum_{w \in \mathcal{W}} p^w_{\underline{x}} H_w) = \sum_{w \in \mathcal{W}} p^w_{\underline{x}} v^{-\ell(w)},$$

and hence

$$(v^{-1} + v)^r = \sum_{w \in \mathcal{W}} p^w_{\underline{x}}(v^{-1}) v^{\ell(w)}.$$

3.2 **Lemma:** $\forall w \in \mathcal{W}$, $\dim_Q B(\underline{x})^Q_w = \dim_Q \{B(\underline{x})^w\}^Q = p^w_{\underline{x}}(1)$.

Proof: The first equality follows from (1.4.ii). For the 2nd equality we argue by induction on $\ell(w)$. As $\sum_{w \in \mathcal{W}} p^w_s H_w = \underline{H}_s$ $\forall s \in \mathcal{S}$,

$$p^w_s = \begin{cases} 1 & \text{if } w = s, \\ v & \text{if } w = e, \\ 0 & \text{else.} \end{cases} \tag{1}$$

Thus,

$$p^w_s(1) = \begin{cases} 1 & \text{if } w \in \{e, s\}, \\ 0 & \text{else.} \end{cases}$$

On the other hand, as $B(s)^Q = B(s)^Q_e \oplus B(s)^Q_s$ with $B(s)^Q_e \simeq Q(e)$ and $B(s)^Q_s \simeq Q(s)$ by (2.2.6),

$$\dim B(s)^Q_w = \begin{cases} 1 & \text{if } w \in \{e, s\}, \\ 0 & \text{else.} \end{cases}$$

Thus, $\dim B(s)^Q_w = p^w_s(1)$ $\forall w \in \mathcal{W}$.

Under the specialization $v \rightsquigarrow 1$ one has

$$\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H} \simeq \mathbb{Z}[\mathcal{W}] \quad \text{via} \quad 1 \otimes H_w \mapsto w \quad \forall w \in \mathcal{W}.$$

Then, $\forall \underline{x} = (s_1, \dots, s_r)$,

$$(s_1 + 1) \dots (s_r + 1) \leftarrow\!\shortmid 1 \otimes \underline{H}_{\underline{x}} = 1 \otimes \sum_{w \in \mathcal{W}} p^w_{\underline{x}} H_w \mapsto \sum_{w \in \mathcal{W}} p^w_{\underline{x}}(1) w.$$

Thus, $\forall s \in \mathcal{S}$,

$$\begin{aligned} \sum_{w \in \mathcal{W}} p^w_{(s,s_1,\dots,s_r)}(1) w &= (s+1)(s_1+1)\dots(s_r+1) = (s+1) \sum_{w \in \mathcal{W}} p^w_{(s_1,\dots,s_r)}(1) w \\ &= \sum_{x \in \mathcal{W}} \{p^x_{(s_1,\dots,s_r)}(1) sx + p^x_{(s_1,\dots,s_r)}(1) x\}. \end{aligned}$$

Then,

$$p^{sw}_{(s_1,\dots,s_r)}(1) + p^w_{(s_1,\dots,s_r)}(1) = p^w_{(s,s_1,\dots,s_r)}(1). \tag{2}$$

$\forall M \in \mathcal{C}$, $\dim(B(s) * M)^Q_w = \dim M^Q_w + \dim M^Q_{sw}$ by (2.3.i). Thus,

$$\begin{aligned} \dim B(\underline{x})^Q_w &= \dim B(s_2, \dots, s_r)^Q_w + \dim B(s_2, \dots, s_r)^Q_{s_1 w} \\ &= p^w_{(s_2,\dots,s_r)}(1) + p^{s_1 w}_{(s_2,\dots,s_r)}(1) \quad \text{by the induction hypothesis} \\ &= p^w_{(s_1,\dots,s_r)}(1) \quad \text{by (2).} \end{aligned}$$

3.3. We will eventually show, under additional conditions on $\mathbb{K}$, Soergel's categorification theorem that any B^w, $B \in \mathfrak{S}\mathrm{Bimod}$, $w \in \mathcal{W}$, is left/right graded free over R, and that there is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras $[\mathfrak{S}\text{Bimod}] \to \mathcal{H}$ via $[B] \mapsto \sum_{w \in \mathcal{W}} v^{-\ell(w)} \mathrm{grk}(B^w) H_w$, where $[\mathfrak{S}\text{Bimod}]$ is the split Grothendieck group of $\mathfrak{S}$Bimod.

Let $\underline{x} = (s_1, \dots, s_r) \in \mathcal{S}^r$ and $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_r) \in \{0,1\}^r$. Put $\underline{x}^{\mathbf{e}} = s_1^{\mathbf{e}_1} \dots s_r^{\mathbf{e}_r}$, $x_0 = e$, $x_1 = s_1^{\mathbf{e}_1}, x_2 = s_1^{\mathbf{e}_1} s_2^{\mathbf{e}_2}, \dots, x_r = \underline{x}^{\mathbf{e}}$. Assign a label U (resp. D) to $i \in [1, r]$ iff $x_{i-1} s_i > x_{i-1}$ (resp. else). The defect $d_{\underline{x}}(\mathbf{e})$ of $\mathbf{e}$ is defined by

$$d_{\underline{x}}(\mathbf{e}) = |\{i | \text{the label of } i \text{ is U and } \mathbf{e}_i = 0\}| - |\{i | \text{the label of } i \text{ is D and } \mathbf{e}_i = 0\}|.$$

One has from [EW16, Lem. 2.7]

$$p_{\underline{x}}^w = \sum_{\substack{\mathbf{e} \\ \underline{x}^{\mathbf{e}} = w}} v^{d_{\underline{x}}(\mathbf{e})}.$$

Define $u_{\underline{x}} = (1 \otimes 1) * \dots * (1 \otimes 1) \in B(s_1) * \dots * B(s_r) = B(\underline{x})$. For our purposes we will need

Assumption: *$\forall s, t \in \mathcal{S}$ distinct with $\mathrm{ord}(st) = l < \infty$, putting $\underline{x} = (s, t, \dots)$ and $\underline{y} = (t, s, \dots) \in \mathcal{S}^l$, $\exists \Phi \in \mathcal{C}(B(\underline{x}), B(\underline{y})) : \Phi(u_{\underline{x}}) = u_{\underline{y}}$; as $\mathrm{ord}(st) = l$,*

$$\underbrace{st\dots}_{l} = \underbrace{ts\dots}_{l}.$$

3.4. Let $s, t \in \mathcal{S}$ distinct with $\mathrm{ord}(st) < \infty$. Let $\mathcal{T}(s, t)$ be the set of reflections in $\langle s, t \rangle$. In the rest of §3 we will verify

Lemma: *If, $\forall t_1, t_2 \in \mathcal{T}(s, t)$ distinct, there is $v \in V$ such that $\langle v, \alpha_{t_1}^\vee \rangle = 0$ and $\langle v, \alpha_{t_2}^\vee \rangle = 1$, then Assumption (3.3) holds.*

3.5. We will be arguing sometimes over $\mathbb{K}/\mathfrak{m}$ for $\mathfrak{m} \in \mathrm{Max}(\mathbb{K})$, see (4.9) for example, in which case we will assume that (3.4) holds also for $V \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})$ in place of V.

We will argue after [S07]. We assume throughout the rest of §3 that the condition in (3.4) holds. Put $\mathcal{W}' = \langle s, t \rangle$ and $\mathcal{T}' = \mathcal{T}(s, t)$ with $\mathrm{ord}(st) = l$. Thus,

$$\mathcal{T}' = \{\underbrace{st\dots}_{n}, \underbrace{ts\dots}_{n} | n \text{ odd}\} = \{w \in \mathcal{W}' | \ell(w) \text{ odd}\}. \tag{1}$$

Recall also that a reduced expression of $w \in \mathcal{W}'$ is a sequence from $\{s, t\}$ [HRC, Th. 5.5, p. 113].

Lemma: *$\mathcal{W}'$ acts faithfully on V.*

Proof: Let $w \in \mathcal{W}'$ be trivial on V.

Just suppose $\ell(w)$ is odd. By (1) there is $u \in \{s, t\}$ with $\ell(uwu) < \ell(w)$; if $\underline{w} = (s, t, \dots, s)$ is a reduced expression of w, $sws < w$. Likewise, if $\underline{w} = (t, s, \dots, t)$.

Then uwu is also trivial on V. As $\ell(uwu) = \ell(w) - 2$, by induction on the length either s or t is trivial on V. Assume for the moment that s is trivial on V. Take $v \in V$ with $\langle v, \alpha_s^\vee \rangle = 1$ by (1.1.iii). Then $v = sv = v - \alpha_s$, and hence $\alpha_s = 0$, contradicting the standing hypothesis (1.1.iii).

Thus, $\ell(w)$ must be even. Then $w = ux$ for some $u \in \{s, t\}$ and $x \in \mathcal{T}'$ by (1) again. Just suppose $w \neq e$. Then $x \neq u$. Take $v \in V$ with $\langle v, \alpha_x^\vee \rangle = 0$ while $\langle v, \alpha_u^\vee \rangle = 1$. Then $v = wv = v - \alpha_u$, and hence $\alpha_u = 0$, absurd again.

3.6. $\forall M \in \mathcal{C}$ with $\mathrm{supp}_{\mathcal{W}}(M) \subseteq \mathcal{W}'$, the decomposition $M^Q = \coprod_{x \in \mathcal{W}'} M_x^Q$ is determined by the R-bimodule structure on M, thanks to (3.5) and Rmk. 1.2.(ii). As $\mathrm{supp}_{\mathcal{W}}(B(\underline{x}))$ and $\mathrm{supp}_{\mathcal{W}}(B(\underline{y}))$ in (3.4) are both contained in $\mathcal{W}'$, we have only to show the existence of $\Phi \in R\mathrm{Bimodgr}(B(\underline{x}), B(\underline{y}))$ with $\Phi(u_{\underline{x}}) = u_{\underline{y}}$. Note also that $\forall t_1, t_2 \in \mathcal{T}'$ distinct,

(1) $$\alpha_{t_1}^\vee \text{ and } \alpha_{t_1}^\vee \text{ are linearly independent over } \mathbb{K}.$$

For let $\xi_1 \alpha_{t_1}^\vee + \xi_2 \alpha_{t_2}^\vee = 0$ with $\xi_1, \xi_2 \in \mathbb{K}$. Take $v \in V$ with $\langle v, \alpha_{t_1}^\vee \rangle = 0$ and $\langle v, \alpha_{t_2}^\vee \rangle = 1$. Then $0 = \langle v, \xi_1 \alpha_{t_1}^\vee + \xi_2 \alpha_{t_2}^\vee \rangle = \xi_2$. As $\alpha_{t_2}^\vee \neq 0$ by the standing hypothesis, $\xi_1 = 0$ also.

Thus, $\mathrm{Frac}(\mathbb{K})\alpha_{t_1}^\vee + \mathrm{Frac}(\mathbb{K})\alpha_{t_2}^\vee$ is 2-dimensional, which is contained in $\mathrm{Frac}(\mathbb{K})\alpha_s^\vee + \mathrm{Frac}(\mathbb{K})\alpha_t^\vee$; $\alpha_{xsx^{-1}}^\vee = x\alpha_s^\vee$ $\forall x \in \mathcal{W}'$ by (1.1). It follows that

(2) $$\mathrm{Frac}(\mathbb{K})\alpha_{t_1}^\vee + \mathrm{Frac}(\mathbb{K})\alpha_{t_2}^\vee = \mathrm{Frac}(\mathbb{K})\alpha_s^\vee + \mathrm{Frac}(\mathbb{K})\alpha_t^\vee.$$

Note next that we may assume $\mathbb{K}$ is infinite by base change, e.g., to $\mathbb{K}[v]$ which is free over $\mathbb{K}$; if we let $\mathcal{C}(R[v])$ denote $\mathcal{C}$ over $R[v] = R \otimes_{\mathbb{K}} \mathbb{K}[v]$,

$$\begin{aligned}
&\mathcal{C}(R[v])(B(\underline{x}) \otimes_{\mathbb{K}} \mathbb{K}[v], B(\underline{y}) \otimes_{\mathbb{K}} \mathbb{K}[v]) \\
&\quad \simeq \mathcal{C}(R[v])(R(e) \otimes_{\mathbb{K}} \mathbb{K}[v], \cdots * B(t) * B(s) * B(\underline{y}) \otimes_{\mathbb{K}} \mathbb{K}[v]) \quad \text{by (2.6)} \\
&\quad \simeq \mathcal{C}(R(e), \cdots * B(t) * B(s) * B(\underline{y})) \otimes_{\mathbb{K}} \mathbb{K}[v] \quad \text{by (1.6.2); } \forall M \in \mathcal{C} \text{ with } \mathrm{supp}_{\mathcal{W}}(M) \subseteq \mathcal{W}', \\
&\qquad\qquad (M \otimes_{\mathbb{K}} \mathbb{K}[v])^{Q(v)} \simeq Q(v) \otimes_R M \simeq Q(v) \otimes_Q M^Q = Q(v) \otimes_Q \coprod_{x \in \mathcal{W}'} M_x^Q, \text{ and hence} \\
&\qquad\qquad\qquad\qquad (M \otimes_{\mathbb{K}} \mathbb{K}[v])_x^{Q(v)} \simeq Q(v) \otimes_Q M_x^Q \\
&\quad \simeq \mathcal{C}(B(\underline{x}), B(\underline{y})) \otimes_{\mathbb{K}} \mathbb{K}[v].
\end{aligned}$$

Thus, if $\sum_i \Phi_i \otimes v^i \in \mathcal{C}(R[v])(B(\underline{x}) \otimes_{\mathbb{K}} \mathbb{K}[v], B(\underline{y}) \otimes_{\mathbb{K}} \mathbb{K}[v])$ sends $u_{\underline{x}} \otimes 1$ to $u_{\underline{y}} \otimes 1$, $\Phi_0(u_{\underline{x}}) = u_{\underline{y}}$. We may then regard $R \otimes_{\mathbb{K}} R$ as the $\mathbb{K}$-algebra of rational functions on $V^\vee \times V^\vee$ with $V^\vee$ denoting the $\mathbb{K}$-dual of V.

$\forall x \in \mathcal{W}'$, let $\mathrm{Gr}(x) = \{(f, x^{-1}f) | f \in V^\vee\}$. $\forall A \subseteq \mathcal{W}'$, let

$$\begin{aligned}
R(A) &= (R \otimes_{\mathbb{K}} R)/I(\cup_{x \in A} \mathrm{Gr}(x)) = (R \otimes_{\mathbb{K}} R)/\cap_{x \in A} (xa \otimes 1 - 1 \otimes a | a \in R) \\
&\leq \prod_{x \in A} \{(R \otimes_{\mathbb{K}} R)/(xa \otimes 1 - 1 \otimes a | a \in R)\} \\
&\simeq \prod_A R
\end{aligned}$$

using

$$(3) \qquad (R\otimes_{\mathbb{K}} R)/(xa\otimes 1-1\otimes a|a\in R)\xrightarrow{\sim} R \quad \text{via} \quad a\otimes b\mapsto a(xb).$$

Thus,

$$(4) \qquad \begin{array}{ccc} a\otimes b & \longmapsto & (a(xb))_{x\in A} \\ R\otimes_{\mathbb{K}} R & \longrightarrow & \prod_A R, \\ \downarrow & \nearrow & \\ R(A) & & \end{array}$$

which induces by base extension

$$(5) \qquad R(A)^Q \xrightarrow{\sim} \prod_A Q.$$

To see that, note first that, as $\mathcal{W}'$ is faithful on V, $\forall x\in\mathcal{W}'\setminus\{e\}$, $\ker(x-\mathrm{id})<V$. Then $V\supset\cup_{x\in\mathcal{W}'\setminus\{e\}}\ker(x-\mathrm{id})$ as $\mathbb{K}$ is now infinite; here we could even argue over $\mathrm{Frac}(\mathbb{K})$. Take $c\in V\setminus\cup_{x\in\mathcal{W}'\setminus\{e\}}\ker(x-\mathrm{id})\subseteq R$, so $xc\neq c$ $\forall x\in\mathcal{W}'\setminus\{e\}$. $\forall x\in A$, let $c_x\in R\otimes_{\mathbb{K}} R$ such that $c_x(f,g)=\prod_{y\in A\setminus\{x\}}\{c(f)-c(yg)\}$ $\forall f,g\in V^\vee$. Thus, $c_x=0$ on $\mathrm{Gr}(y)$ $\forall y\in A\setminus\{x\}$ while $c_x\neq 0$ on $\mathrm{Gr}(x)$. Then, $\forall(q_x|x\in A)\in\prod_A Q$,

$$\sum_{x\in A}\frac{q_x}{c_x|_{\mathrm{Gr}(x)}}\otimes c_x=(q_x)_x$$

in $Q\otimes_R\{\prod_{x\in A}(R\otimes_{\mathbb{K}} R)/(xa\otimes 1-1\otimes a|a\in R)\}\simeq\prod_A Q$.

In particular, $R(A)^Q=\coprod_{x\in A}R(A)^Q_x=\coprod_{x\in\mathcal{W}'}R(A)^Q_x$ with $R(A)^Q_x=Q\otimes_R\{(R\otimes_{\mathbb{K}} R)/(xa\otimes 1-1\otimes a|a\in R)\}$, and hence

$$(6) \qquad R(A)\in\mathcal{C}^{\mathrm{tf}}.$$

One has $R(\{x\})\simeq R(x)$. We will abbreviate $R(\{x_1,\dots,x_r\})$ as $R(x_1,\dots,x_r)$.

Let $R(A)^+$ be the image of $R\otimes_{\mathbb{K}} R^s$ in $R(A)$.

Lemma: *If $As=A$ in $\mathcal{W}'$, $R(A)^+\otimes_{R^s}R\xrightarrow{\sim}R(A)$ via $\phi\otimes a\mapsto\phi(1\otimes a)$.*

Proof: We have only to show that the map is injective. Take $\delta\in V$ with $\langle\delta,\alpha_s^\vee\rangle=1$. As $R=R^s\oplus\delta R^s$ by (2.1), one has a CD

$$\begin{array}{ccc} R(A)^+\otimes_{R^s}R & \longrightarrow & R(A) \\ \| & & \uparrow \\ \{R(A)^+\otimes_{R^s}R^s\}\oplus\{R(A)^+\otimes_{R^s}\delta R^s\} & \longrightarrow & R(A)^++R(A)^+(1\otimes\delta). \end{array}$$

Thus, it is enough to show that $R(A)^+\cap R(A)^+(1\otimes\delta)=0$. Let $f=g(1\otimes\delta)$, $f,g\in R(A)^+$, which reads $f_x=g_x(x\delta)$ $\forall x\in A$ in $\prod_A R$. As $A=As$ and as both f and g belong to the image of $R \otimes_{\mathbb{K}} R^s$

$$\begin{aligned} f_x &= f_{xs} \quad \text{by (3)} \\ &= g_{xs}(xs\delta) = g_x(xs\delta) \quad \text{likewise.} \end{aligned}$$

Then $0 = g_x(x\delta - xs\delta) = g_x(x\alpha_s)$, and hence $g_x = 0\ \forall x \in A$. Thus, $g = 0$.

3.7 **Lemma:** *Let $x \in \mathcal{W}'$ with $xs > x$ and $A = \{y \in \mathcal{W}' | y \leq x\}$. Then*

$$R(A) \otimes_R B(s) \simeq \{R(A \cup As)(1)\} \oplus \{R(A \cap As)(-1)\}.$$

Proof: Assume first that $x = e$. Then $A = \{e\}$, $As = \{s\}$, $A \cup As = \{e, s\}$, and $A \cap As = \emptyset$. Thus, we are to show that $B(s) \simeq R(e, s)(1)$, and hence we have only to show that

$$\begin{array}{ccc} R \otimes_{\mathbb{K}} R & \xrightarrow{\text{qt}} & R(e,s). \\ \downarrow & \nearrow_{\sim} & \\ R \otimes_{R^s} R & & \end{array}$$

Take $\delta \in V$ with $\langle \delta, \alpha_s^\vee \rangle = 1$. As $R \otimes_{R^s} R = R \otimes_{R^s} \{R^s \oplus \delta R^s\}$ by (2.1), let $a \otimes 1 + b \otimes \delta = 0$ in $R(e, s)$, $a, b \in R$. Then, calculating in $\prod_{\{e,s\}} R$ by (3.6.3), one has

$$\begin{aligned} 0 &= (a \otimes 1 + b \otimes \delta)_e = a + b\delta, \\ 0 &= (a \otimes 1 + b \otimes \delta)_s = a + b(s\delta) = a + b(\delta - \alpha_s), \end{aligned}$$

and hence $a\alpha_s = 0$ in R. Then $a = 0$, and hence also $b = 0$. Thus, $R \otimes_{R^s} R \xrightarrow{\sim} R(e, s)$, as desired.

Thus, we may assume $x > e$. As $xs > x$, a reduced expression of x must end with t and

(1)
$$\begin{aligned} A \setminus As &= \{y \in \mathcal{W}' | y \leq x, ys \not\leq x\} \\ &= \begin{cases} \{x, tx\} & \text{if } \ell(x) \text{ is odd,} \\ \{x, sx\} & \text{else} \end{cases} \\ &= \{x, xr\} \quad \text{with } r = \begin{cases} x^{-1}tx & \text{if } \ell(x) \text{ is odd,} \\ x^{-1}sx & \text{else.} \end{cases} \end{aligned}$$

As $xs > x$ again, $r \neq s$. Take $v_0 \in V$ with $\langle v_0, \alpha_r^\vee \rangle = 0$ while $\langle v_0, \alpha_s^\vee \rangle = 1$, and put $\xi = xv_0 \otimes 1 - 1 \otimes v_0 \in V \otimes_{\mathbb{K}} R \subseteq R \otimes_{\mathbb{K}} R$. We show next that, $\forall \phi \in R(A)$,

(2)
$$\phi\xi = 0 \text{ in } R(A) \text{ iff } \phi = 0 \text{ in } R(A \cap As)$$

under the restriction $R(A) \twoheadrightarrow R(A \cap As)$.

"if" We show $\phi\xi = 0$ in $\prod_A R$, i.e., $(\phi\xi)_y = 0\ \forall y \in A$. As $\phi_y = 0\ \forall y \in A \cap As$, we have only to verify that $\xi_y = 0\ \forall y \in A \setminus As = \{x, xr\}$. But

$$\begin{aligned} \xi_y &= (xv_0 \otimes 1 - 1 \otimes v_0)_y = xv_0 - yv_0 \quad \text{by (3.6.3)} \\ &= 0 \quad \text{as } rv_0 = v_0. \end{aligned}$$

"only if" It is enough to show that $\xi_y = 0\ \forall y \in A \cap As$. Just suppose $0 \neq \xi_y = xv_0 - yv_0$ for some $y \in A \cap As$. Then $v_0 = y^{-1}xv_0 = y^{-1}xrv_0$. By (3.5.1) either $y^{-1}x$ or $y^{-1}xr \in \mathcal{T}'$, which we denote by z; $\{z\} = \{y^{-1}x, y^{-1}xr\} \cap \mathcal{T}'$. Thus, $zv_0 = v_0$, and $\langle v_0, \alpha_r^\vee \rangle = 0 = \langle v_0, \alpha_z^\vee \rangle$. But $r \neq z$; if $r = z = y^{-1}x$, $y = xr^{-1} = xr \in A \setminus As$, absurd. If $r = z = y^{-1}xr$, $y = x \notin As$, absurd again. Then by (3.6.2)

$$\mathrm{Frac}(\mathbb{K})\alpha_r^\vee + \mathrm{Frac}(\mathbb{K})\alpha_z^\vee = \mathrm{Frac}(\mathbb{K})\alpha_s^\vee + \mathrm{Frac}(\mathbb{K})\alpha_t^\vee,$$

and hence $1 = \langle v_0, \alpha_s^\vee \rangle = 0$, absurd.

If $\mathrm{Ann}(\xi) = \{\phi \in R(A) | \phi\xi = 0\}$, (2) yields in RBimod

(3)
$$\begin{array}{ccccccc}
\phi|_{A\cap As} & R(A \cap As) & \overset{\sim}{\longleftarrow} & R(A)/\mathrm{Ann}(\xi) & \overset{\sim}{\longrightarrow} & R(A)\xi & \phi\xi. \\
 & \nwarrow & & \uparrow & & \nearrow & \\
 & & & R(A) & & & \\
 & & & \phi & & &
\end{array}$$

Thus, $R(A \cap As)(-2) \simeq R(A)\xi$. Also, (3) induces

$$\begin{array}{ccc}
R(A \cap As)^+ & \overset{\sim}{\longrightarrow} & R(A)^+\xi, \\
\nwarrow & & \nearrow \\
 & R \otimes_{\mathbb{K}} R^s &
\end{array}$$

and hence

(4)
$$R(A \cap As)^+(-2) \simeq R(A)^+\xi.$$

Consider next $\mathrm{res} : R(A \cup As) \twoheadrightarrow R(A)$. Under the right multiplication of s on $A \cup As$ let $R(A \cup As)^s = \{\phi \in R(A \cup As) | \phi(f, x^{-1}f) = \phi(f, (xs)^{-1}f)\ \forall f \in V^\vee, \forall x \in A \cup As\}$. If $\phi \in R(A \cup As)^s$ with $\phi|_A = 0$, $\phi|_{As} = 0$ also, and hence

(5)
$$\begin{array}{ccc}
R(A \cup As) & \xrightarrow{\mathrm{res}} & R(A). \\
\uparrow & \nearrow & \\
R(A \cup As)^s & &
\end{array}$$

Also, $R(A \cup As)^+ \subseteq R(A \cup As)^s$; $\forall a \in R$, $\forall b \in R^s$, $\forall f \in V^\vee$, $\forall x \in A \cup As$,

(6) $(a \otimes b)(f, x^{-1}f) = a(f)b(x^{-1}f) = a(f)(sb)(x^{-1}f) = a(f)b(sx^{-1}f) = (a \otimes b)(f, (xs)^{-1}f)$.

Let M be the image of $R(A \cup As)^+$ in $R(A)$ under (5). Then we are left to show that

(7)
$$R(A) = M \oplus R(A)^+\xi,$$

in which case

$$\begin{aligned}
R(A) \otimes_R B(s) &\simeq R(A) \otimes_{R^s} R(1) \\
&\simeq \{R(A \cup As)^+ \otimes_{R^s} R(1)\} \oplus \{R(A \cap As)^+(-2) \otimes_{R^s} R(1)\} \quad \text{by (4)} \\
&\simeq R(A \cup As)(1) \oplus \{R(A \cap As)(-1) \quad \text{by (3.6)}.
\end{aligned}$$

Now,

$$\begin{aligned} M + R(A)^+\xi \twoheadleftarrow R\otimes_{\mathbb{K}} R^s + (R\otimes_{\mathbb{K}} R^s)\xi &= R\otimes_{\mathbb{K}} R^s + R\otimes_{\mathbb{K}} R^s v_0 \quad \text{as } \xi = xv_0\otimes 1 + 1\otimes v_0 \\ &= R\otimes_{\mathbb{K}} R \quad \text{as } R^s + R^s v_0 = R^s \oplus R^s v_0 = R \text{ by (2.1)}, \end{aligned}$$

and hence $M + R(A)^+\xi = R(A)$.

Let finally $\phi\xi = m$ for some $\phi \in R(A)^+$ and $m \in M$. Let $\hat{\phi}$ be a lift of ϕ in $R\otimes_{\mathbb{K}} R^s$. Consider $\hat{\phi} = (\hat{\phi}_y)$ and $m = (m_y)$ in $\prod_{A\cap As} R$. Thus, $\forall y \in A\cap As$,

$$\begin{aligned} m_y &= \hat{\phi}_y \xi_y = \hat{\phi}_y(xv_0 - yv_0) \quad \text{by (3.6.3)}, \\ m_{ys} &= \hat{\phi}_{ys}\xi_{ys} = \hat{\phi}_{ys}(xv_0 - ysv_0) \quad \text{likewise} \end{aligned}$$

with $m_{ys} = m_y$ and $\hat{\phi}_{ys} = \hat{\phi}_y$ by (6). Then $0 = \hat{\phi}_y(yv_0 - ysv_0) = \hat{\phi}_y(y\alpha_s)$. Thus, $\hat{\phi} = 0$ in $R(A\cap As)$. Then by (4) one has $\phi\xi = 0$ in $R(A)$, and (7) holds.

3.8. If $x \in \mathcal{W}'$ with $xs > x$, $(\leq x) \cup (\leq x)s = (\leq xs)$, and hence $R(\leq xs)(1)$ is a direct summand of $R(\leq x)\otimes_R B(s)$ by (3.7). Thus, for a reduced expression $\underline{w} = (\dots, s, t)$ of $w \in \mathcal{W}'$, $R(\leq w)(1)$ is a direct summand of $R(\leq wt)\otimes_R B(t)$, $R(\leq wt)(1)$ is a direct summand of $R(\leq wts)\otimes_R B(s)$,..., and hence $R(\leq w)(\ell(w))$ is a direct summand of $\cdots \otimes_R B(s)\otimes_R B(t) = B(\underline{w})$. Likewise if $(\dots, s, t)$ is a reduced expression. Thus, in either case

(1) $$R(\leq w)(\ell(w)) \text{ is a direct summand of } B(\underline{w}).$$

In particular, $R(\leq w)(\ell(w)) \in \mathcal{C}$.

In our set up (3.3), $\underline{x}$ and $\underline{y}$ are both reduced expressions of the longest element z_0 of $\mathcal{W}'$ and $l = \ell(z_0)$. Thus, $R(\leq z_0)(l)$ is a direct summand of both $B(\underline{x})$ and $B(\underline{y})$. Write

$$\begin{array}{ccc} B(\underline{x}) & \xrightarrow{\Phi} & B(\underline{y}). \\ {\scriptstyle\text{proj}}\downarrow & \nearrow & \\ R(\leq z_0)(l) & & \end{array}$$

Lemma: *One has*

$$\begin{array}{ccc} B(\underline{x}) & \xrightarrow{\Phi} & B(\underline{y}) \\ \downarrow & & \downarrow \\ B(\underline{x})^{z_0} & \xrightarrow[\sim]{} & B(\underline{y})^{z_0}. \end{array}$$

Proof: If $\underline{w}$ is a reduced expression of $w \in \mathcal{W}'$, $\dim B(\underline{w})^Q_w = 1$ by (2.3), and hence $B(\underline{w}) = R(\leq w)(\ell(w)) \oplus M$ for some M by (1) with $\mathrm{supp}_{\mathcal{W}}(M) \subseteq (< w)$ by (2.4). Thus,

$$B(\underline{x})^{z_0} = \{R(\leq z_0)(l)\}^{z_0} : \quad \begin{array}{ccc} B(\underline{x}) & \longrightarrow & R(\leq z_0)(l) \\ \downarrow & & \downarrow \\ B(\underline{x})^{z_0} & \xrightarrow[\sim]{} & \{R(\leq z_0)(l)\}^{z_0}. \end{array}$$

Likewise, $\{R(\leq z_0)(l)\}^{z_0} \simeq B(\underline{y})^{z_0}$, and hence the assertion.

3.9. We now complete the proof of (3.4). Define a homomorphism of graded R-bimodules $m^{\underline{x}} : B(\underline{x}) \to R(z_0)(l)$ via $R \otimes_{R^s} R \otimes_{R^t} R \cdots \ni a_0 \otimes a_1 \otimes \cdots \otimes a_l \mapsto a_0(sa_1)(sta_2)\dots(st\dots a_l)$. By (3.5) and Rmk. 1.2.(ii) one has $m^{\underline{x}} \in \mathcal{C}$, which induces by (1.4.v) a surjection $\overline{m^{\underline{x}}} \in \mathcal{C}(B(\underline{x})^{z_0}, R(z_0)(l))$. Then $\overline{m^{\underline{x}}}$ is invertible by consideration of rank. Likewise, $B(\underline{y})^{z_0} \simeq R(z_0)(l) \simeq B(\underline{x})^{z_0}$.

Finally, $B(\underline{x})^{-l}$ (resp. $B(\underline{y})^{-l}$) is free over $\mathbb{K}$ of basis $u_{\underline{x}}$ (resp. $u_{\underline{y}}$). Then by (3.8) we must have $\Phi(u_{\underline{x}}) = cu_{\underline{y}}$ for some $c \in \mathbb{K}^\times$, and hence $c^{-1}\Phi$ will do.

4. Light leaves

We recall from [EW16] Libedinsky's light leaves [Lib], to describe a basis of $B(\underline{x})^w$ among other things. From now on we will assume $\mathbb{K}$ is local, so that a direct summand of a graded free left R-module remains graded free [Lam, Cor. II.5.4.7, p. 79].

4.1. Let $w \in \mathcal{W}$ and $\underline{x}, \underline{y} \in \mathcal{S}^{\ell(w)}$ 2 reduced expressions of w. Thus, there is a sequence of reduced expressions $\underline{x}^0 = \underline{x}, \underline{x}^1, \dots, \underline{x}^r = \underline{y}$ such that each pair of $\underline{x}^i$ and $\underline{x}^{i+1}$ differs by a single braid relation. Under the standing hypothesis (3.3) there is $\phi_i \in \mathcal{C}(B(\underline{x}^i), B(\underline{x}^{i+1}))$ such that $u_{\underline{x}^i} \mapsto u_{\underline{x}^{i+1}}$. Their composite $B(\underline{x}) \to B(\underline{y})$ is called a rex [EW16, 16.4.2], so that

(1) $$\mathrm{rex}(u_{\underline{x}}) = u_{\underline{y}}.$$

$\forall s \in \mathcal{S}$, $\forall a \in R$, set $\partial_s(a) = \frac{a-sa}{\alpha_s}$, which is a twisted derivation: $\forall b \in R$, $\partial_s(ab) = (\partial_s a)b + (sa)\partial_s b$. Define $m^s \in R\mathrm{Bimod}(B(s), R)^1$ via

$$R \otimes_{R^s} R \ni a \otimes b \mapsto ab \in R,$$

$i_0^s \in R\mathrm{Bimod}(B(s) * B(s), B(s))^{-1}$ via

$$R \otimes_{R^s} R \otimes_{R^s} R \ni a \otimes b \otimes c \mapsto a\partial_s(b) \otimes c \in R \otimes_{R^s} R,$$

and set $i_1^s = m^s \circ i_0^s \in R\mathrm{Bimod}(B(s) * B(s), R)^0 : R \otimes_{R^s} R \otimes_{R^s} R \ni a \otimes b \otimes c \mapsto a\partial_s(b)c \in R$. As $\langle s \rangle$ acts faithfully on V, one has from Rmk. 1.2(ii) that

$$m^s \in \mathcal{C}(B(s), R(1)), \quad i_0^s \in \mathcal{C}(B(s) * B(s), B(s)(-1)), \quad i_1^s \in \mathcal{C}(B(s) * B(s), R).$$

4.2. Let $\underline{x} = (s_1, \dots, s_r) \in \mathcal{S}^r$, $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_r) \in \{0,1\}^r$, $w = \underline{x}^{\mathbf{e}}$. Fix a reduced expresson $\underline{w}$ of w. We define a light leaf $LL_{\underline{x},\mathbf{e}} \in \mathcal{C}(B(\underline{x}), B(\underline{w})(d(\mathbf{e})))$ inductively as follows; the definition will depend not only on the choice of reduced expression $\underline{w}$ but also on the choices of rex's involved, but that will not be important. Let $\underline{x}_{\leq k} = (s_1, \dots, s_k), \mathbf{e}_{\leq k} = (\mathbf{e}_1, \dots, \mathbf{e}_k)$, and $w_k = \underline{x}_{\leq k}^{\mathbf{e}_{\leq k}} = s_1^{\mathbf{e}_1} \dots s_k^{\mathbf{e}_k}$. Recall from (3.3) labels U, D, and the defect d. Fix a reduced expresson $\underline{w}_k$ of w_k and define $LL_k \in \mathcal{C}(B(\underline{x}_{\leq k}), B(\underline{w}_k)(d(\mathbf{e}_k)))$ in 4 cases as follows:

Case U0: $\mathbf{e}_k = 0$ and $w_{k-1}s_k > w_{k-1}$. Thus, $d(\mathbf{e}_{\leq k}) = d(\mathbf{e}_{\leq k-1}) + 1$, and $\underline{w}_k$ is a reduced expression also of $w_{k-1} = s_1^{\mathbf{e}_1} \dots s_{k-1}^{\mathbf{e}_{k-1}} = s_1^{\mathbf{e}_1} \dots s_k^{\mathbf{e}_k} = \underline{x}_{\le k}^{\mathbf{e}_{\le k}} = w_k$.

$$\begin{array}{ccc}
B(\underline{x}_{\le k-1}) * B(s_k) & \xrightarrow{LL_{k-1} \otimes_R B(s_k)} & B(\underline{w}_{\le k-1})(d(\mathbf{e}_{\le k-1})) * B(s_k) \\
\Big\downarrow{\scriptstyle LL_k} & & \Big\downarrow{\scriptstyle B(\underline{w}_{\le k-1}) \otimes_R m^{s_k}} \\
B(\underline{w}_k)(d(\mathbf{e}_{\le k})) & \xleftarrow[\text{rex}]{} & B(\underline{w}_{\le k-1})(d(\mathbf{e}_{\le k})),
\end{array}$$

Case U1: $\mathbf{e}_k = 1$ and $w_{k-1} s_k > w_{k-1}$. Thus, $d(\mathbf{e}_{\le k-1}) = d(\mathbf{e}_{\le k})$ and $(\underline{w}_{k-1}, s_k)$ is a reduced expression of w_k.

$$\begin{array}{ccc}
B(\underline{x}_{\le k-1}) * B(s_k) & \xrightarrow{LL_{k-1} \otimes_R B(s_k)} & B(\underline{w}_{\le k-1})(d(\mathbf{e}_{\le k-1})) * B(s_k) \\
& \searrow{\scriptstyle LL_k} & \Big\downarrow{\scriptstyle \text{rex}} \\
& & B(\underline{w}_{\le k})(d(\mathbf{e}_{\le k})),
\end{array}$$

Case D0: $\mathbf{e}_k = 0$ and $w_{k-1} s_k < w_{k-1}$. Thus, $d(\mathbf{e}_{\le k}) = d(\mathbf{e}_{\le k-1}) - 1$, $\underline{w}_k$ is a reduced expression also of $w_{k-1} = s_1^{\mathbf{e}_1} \dots s_{k-1}^{\mathbf{e}_{k-1}} = s_1^{\mathbf{e}_1} \dots s_k^{\mathbf{e}_k} = \underline{x}_{\le k}^{\mathbf{e}_{\le k}} = w_k$, and there is a reduced expression $(t_1, \dots, t_l, s_k)$ for w_{k-1}.

$$\begin{array}{ccccc}
B(\underline{x}_{\le k-1}) * B(s_k) & \xrightarrow{LL_{k-1} \otimes_R B(s_k)} & B(\underline{w}_{\le k-1})(d(\mathbf{e}_{\le k-1})) * B(s_k) & \xrightarrow{\text{rex} \otimes_R B(s_k)} & B(t_1, \dots, t_l, s_k)(d(\mathbf{e}_{\le k-1})) * B(s_k) \\
& & & & \parallel \\
& & & & B(t_1, \dots, t_l)(d(\mathbf{e}_{\le k-1})) * B(s_k) * B(s_k) \\
& \searrow{\scriptstyle LL_k} & & & \Big\downarrow{\scriptstyle B(t_1, \dots, t_l)(d(\mathbf{e}_{\le k-1})) \otimes_R i_0^{s_k}} \\
& & B(\underline{w}_k)(d(\mathbf{e}_{\le k})) & \xleftarrow[\text{rex}]{} & B(t_1, \dots, t_l, s_k)(d(\mathbf{e}_{\le k-1}) - 1).
\end{array}$$

Case D1: $\mathbf{e}_k = 1$ and $w_{k-1} s_k < w_{k-1}$. Thus, $d(\mathbf{e}_{\le k}) = d(\mathbf{e}_{\le k-1})$, there is a reduced expression $(t_1, \dots, t_l, s_k)$ of w_{k-1}, and hence $(t_1, \dots, t_{k-1})$ is a reduced expression of $w_k = w_{k-1} s_k$.

$$\begin{array}{ccccc}
B(\underline{x}_{\le k-1}) * B(s_k) & \xrightarrow{LL_{k-1} \otimes_R B(s_k)} & B(\underline{w}_{\le k-1})(d(\mathbf{e}_{\le k-1})) * B(s_k) & \xrightarrow{\text{rex} \otimes_R B(s_k)} & B((t_1, \dots, t_l, s_k))(d(\mathbf{e}_{\le k-1})) * B(s_k) \\
& \searrow{\scriptstyle LL_k} & & & \Big\downarrow{\scriptstyle B((t_1, \dots, t_l))(d(\mathbf{e}_{\le k-1})) \otimes_R i_1^{s_k}} \\
& & B(\underline{w}_k)(d(\mathbf{e}_{\le k})) & \xleftarrow[\text{rex}]{} & B((t_1, \dots, t_l))(d(\mathbf{e}_{\le k-1})).
\end{array}$$

Set now $LL_{\underline{x}, \mathbf{e}} = LL_r$. One could define $LL_{\underline{w},(1,\dots,1)} = \mathrm{id}_{B(\underline{w})}$ by taking each $\underline{w}_k$ as a subsequence of $\underline{w}$ and taking id for rex in each case U1, which, however, is not important.

4.3. Fix $\underline{x} = (s_1, \dots, s_r) \in \mathcal{S}^r$.

Lemma: *Let $\mathbf{e}, \mathbf{f} \in \{0,1\}^r$ with $\underline{x}^{\mathbf{e}} = \underline{x}^{\mathbf{f}}$. If the labels U/D of $\mathbf{e}$ and $\mathbf{f}$ coincide at each place, $\mathbf{e} = \mathbf{f}$.*

Proof: We argue by descending induction on r to show that $\mathbf{e}_i = \mathbf{f}_i \ \forall i \in [1, r]$.

If the labels of $\mathbf{e}$ and $\mathbf{f}$ at r are both U, $s_1^{\mathbf{e}_1} \dots s_{r-1}^{\mathbf{e}_{r-1}} s_r > s_1^{\mathbf{e}_1} \dots s_{r-1}^{\mathbf{e}_{r-1}}$. Assume first $\mathbf{e}_r = 0$. Just suppose $\mathbf{f}_r = 1$. Then $s_1^{\mathbf{e}_1} \dots s_{r-1}^{\mathbf{e}_{r-1}} = \underline{x}^{\mathbf{e}} = \underline{x}^{\mathbf{f}} = s_1^{\mathbf{f}_1} \dots s_{r-1}^{\mathbf{f}_{r-1}} s_r$, and hence $s_1^{\mathbf{e}_1} \dots s_{r-1}^{\mathbf{e}_{r-1}} s_r <$

$s_1^{\mathbf{e}_1}\dots s_{r-1}^{\mathbf{e}_{r-1}}$, absurd. If $\mathbf{e}_r = 1$ and $\mathbf{f}_r = 0$, $s_1^{\mathbf{f}_1}\dots s_{r-1}^{\mathbf{f}_{r-1}}s_r < s_1^{\mathbf{f}_1}\dots s_{r-1}^{\mathbf{f}_{r-1}}$ likewise, absurd again. Thus, $\mathbf{e}_r = \mathbf{f}_r$ if the labels of $\mathbf{e}$ and $\mathbf{f}$ at r are both U.

Assume next that the labels of $\mathbf{e}$ and $\mathbf{f}$ at r are both D. Then $s_1^{\mathbf{e}_1}\dots s_{r-1}^{\mathbf{e}_{r-1}}s_r < s_1^{\mathbf{e}_1}\dots s_{r-1}^{\mathbf{e}_{r-1}}$. Assume $\mathbf{e}_r = 0$ and just suppose $\mathbf{f}_r = 1$. Then

$$s_1^{\mathbf{e}_1}\dots s_{r-1}^{\mathbf{e}_{r-1}} = \underline{x}^{\mathbf{e}} = \underline{x}^{\mathbf{f}} = s_1^{\mathbf{f}_1}\dots s_{r-1}^{\mathbf{f}_{r-1}}s_r < s_1^{\mathbf{f}_1}\dots s_{r-1}^{\mathbf{f}_{r-1}},$$

and hence $s_1^{\mathbf{e}_1}\dots s_{r-1}^{\mathbf{e}_{r-1}}s_r = \underline{x}^{\mathbf{e}}s_r = \underline{x}^{\mathbf{f}}s_r = s_1^{\mathbf{f}_1}\dots s_{r-1}^{\mathbf{f}_{r-1}} > s_1^{\mathbf{e}_1}\dots s_{r-1}^{\mathbf{e}_{r-1}}$, absurd. Likewise, if $\mathbf{e}_r = 1$, we must have $\mathbf{f}_r = 1$ also. Thus, $\mathbf{e}_r = \mathbf{f}_r$ if the labels of $\mathbf{e}$ and $\mathbf{f}$ at r are both D also.

Assume now that $\mathbf{e}_j = \mathbf{f}_j\ \forall j > i$. As $s_1^{\mathbf{e}_1}\dots s_r^{\mathbf{e}_r} = \underline{x}^{\mathbf{e}} = \underline{x}^{\mathbf{f}} = s_1^{\mathbf{f}_1}\dots s_r^{\mathbf{f}_r}$, $s_1^{\mathbf{e}_1}\dots s_i^{\mathbf{e}_i} = s_1^{\mathbf{f}_1}\dots s_i^{\mathbf{f}_i}$ by the induction hypothesis. Then $\mathbf{e}_i = \mathbf{f}_i$ as in the case $i = r$.

4.4. Let $w \in \mathcal{W}$ and $\underline{x} \in \mathcal{S}^r$. By (4.3) one can introduce a total order $<_{\underline{x},w}$, abbreviated simply as $<$, on $\{\mathbf{e} \in \mathcal{S}^r | \underline{x}^{\mathbf{e}} = w\}$ in such a way that $\mathbf{f} < \mathbf{e}$ iff $\exists i \in [1, r]$:

(i) the labels of $\mathbf{e}$ and $\mathbf{f}$ are the same at $j\ \forall j < i$,

(ii) the labels of $\mathbf{e}$ at i is D,

(iii) the labels of $\mathbf{f}$ at i is U.

In particular, if (i) holds and if the label of $\mathbf{e}$ at i is U, regardless of the label of $\mathbf{f}$ at i, $\mathbf{f} \geq \mathbf{e}$. $\forall s \in \mathcal{S}$, choose $\delta_s \in V$ such that $\langle \delta_s, \alpha_s^\vee \rangle = 1$. $\forall \mathbf{e} \in \{0,1\}^r$, define $b_{\underline{x},\mathbf{e}} \in B(\underline{x})$ by $b_{\underline{x},\mathbf{e}} = b_1 \otimes_R \cdots \otimes_R b_r \in B(s_1) * \cdots * B(s_r) = B(\underline{x})$ with

$$b_i = \begin{cases} 1 \otimes 1 & \text{if the label of } \mathbf{e} \text{ at } i \text{ is U,} \\ \delta_{s_i} \otimes 1 & \text{else.} \end{cases}$$

Proposition: *Let $\mathbf{e}, \mathbf{f} \in \{0,1\}^r$ with $\underline{x}^{\mathbf{e}} = w = \underline{x}^{\mathbf{f}}$. Fix a reduced expression $\underline{w}$ of w. Under $LL_{\underline{x},\mathbf{e}} \in \mathcal{C}(B(\underline{x}), B(\underline{w})(d(\mathbf{e})))$*

$$LL_{\underline{x},\mathbf{e}}(b_{\underline{x},\mathbf{f}}) = \begin{cases} u_{\underline{w}} & \text{if } \mathbf{f} = \mathbf{e}, \\ 0 & \text{if } \mathbf{f} < \mathbf{e}. \end{cases}$$

In particular, $\{LL_{\underline{x},\mathbf{e}} | \underline{x}^{\mathbf{e}} = w\}$ is a left/right R-linearly independent set. Also, $\deg(b_{\underline{x},\mathbf{e}}) = -d(\mathbf{e}) - \ell(w)$.

Proof: We show by induction on k that

$$LL_k(b_{\underline{x}_{\leq k},\mathbf{f}_{\leq k}}) = \begin{cases} u_{\underline{w}_k} & \text{if } \mathbf{f}_{\leq k} = \mathbf{e}_{\leq k}, \\ 0 & \text{if } \mathbf{f}_{\leq k} < \mathbf{e}_{\leq k}. \end{cases}$$

To start the induction, let $k = 1$. By definition the labels of $\mathbf{e}$ and $\mathbf{f}$ at 1 are U, and hence $b_{\underline{x}_{\leq 1},\mathbf{f}_{\leq 1}} = 1 \otimes 1$. If $\mathbf{e}_1 = 0$, we are in Case U0 and

$$LL_1(b_{\underline{x}_{\leq 1},\mathbf{f}_{\leq 1}}) = \operatorname{rex} \circ m^{s_1}(1 \otimes 1) = \operatorname{rex}(1) = 1 = u_{\underline{w}_1} \quad \text{as } \underline{w}_1 = \emptyset = \underline{w}_0.$$

If $\mathbf{e}_1 = 1$, we are in Case U1 and

$$\begin{aligned} LL_1(b_{\underline{x}_{\leq 1},\mathbf{f}_{\leq 1}}) &= \operatorname{rex}(1 \otimes 1) = 1 \otimes 1 \\ &= u_{\underline{w}_1} \quad \text{as } w_1 = s_1 \text{ and } \underline{w}_1 = (s_1). \end{aligned}$$

Assume now that the labels of $\mathbf{e}_{\leq k}$ and $\mathbf{f}_{\leq k}$ are the same at all places, and hence $\mathbf{e}_{\leq k} =_{\leq \mathbf{k}}$ by (4.3). Assume first that the label of $\mathbf{e}$ at k is U, and hence $b_{\underline{x}_{\leq k},\mathbf{f}_{\leq k}} = b_{\underline{x}_{\leq k},\mathbf{e}_{\leq k}} = b_{\underline{x}_{\leq k-1},\mathbf{e}_{\leq k-1}} \otimes 1 \otimes 1$ by definition. If $\mathbf{e}_k = 0$, we are in Case U0, and, suppressing the shifts in the following, have

$$\begin{aligned} LL_k(b_{\underline{x}_{\leq k},\mathbf{f}_{\leq k}}) &= \operatorname{rex} \circ (B(\underline{w}_{k-1}) \otimes_R m^{s_k}) \circ (LL_{k-1} \otimes_R B(s_k))(b_{\underline{x}_{\leq k-1},\mathbf{f}_{\leq k-1}} \otimes 1 \otimes 1) \\ &= \operatorname{rex} \circ (B(\underline{w}_{k-1}) \otimes_R m^{s_k})(u_{\underline{w}_{k-1}} \otimes 1 \otimes 1) \quad \text{by the induction hypothesis} \\ &= \operatorname{rex}(u_{\underline{w}_{k-1}} \otimes 1) = \operatorname{rex}(u_{\underline{w}_{k-1}}) \\ &= u_{\underline{w}_k} \quad \text{by definition (4.1.1).} \end{aligned}$$

If $\mathbf{e}_k = 1$, we are in Case U1, and have

$$\begin{aligned} LL_k(b_{\underline{x}_{\leq k},\mathbf{f}_{\leq k}}) &= \operatorname{rex} \circ (LL_{k-1} \otimes_R B(s_k))(b_{\underline{x}_{\leq k-1},\mathbf{f}_{\leq k-1}} \otimes 1 \otimes 1) \\ &= \operatorname{rex}(u_{\underline{w}_{k-1}} \otimes 1 \otimes 1) \quad \text{by the induction hypothesis} \\ &= \operatorname{rex}(u_{\underline{w}_k}) = u_{\underline{w}_k}. \end{aligned}$$

Assume next that the label of $\mathbf{e}$ at k is D, and hence $b_{\underline{x}_{\leq k},\mathbf{f}_{\leq k}} = b_{\underline{x}_{\leq k},\mathbf{e}_{\leq k}} = b_{\underline{x}_{\leq k-1},\mathbf{f}_{\leq k-1}} \otimes \delta_{s_k} \otimes 1$. As $w_{k-1}s_k < w_{k-1}$, let $(t_1, \ldots, t_l, s_k)$ be a reduced expression of w_{k-1}. If $\mathbf{e}_k = 0$, we are in Case D0, and have

$$\begin{aligned} LL_k(b_{\underline{x}_{\leq k},\mathbf{f}_{\leq k}}) &= \operatorname{rex} \circ (B(t_1, \ldots, t_l) \otimes_R i_0^{s_k}) \circ (\operatorname{rex} \otimes_R B(s_k)) \\ &\qquad \circ (LL_{k-1} \otimes_R B(s_k))(b_{\underline{x}_{\leq k-1},\mathbf{f}_{\leq k-1}} \otimes \delta_{s_k} \otimes 1) \\ &= \operatorname{rex} \circ (B(t_1, \ldots, t_l) \otimes_R i_0^{s_k}) \circ (\operatorname{rex} \otimes_R B(s_k))(u_{w_{\leq k-1}} \otimes \delta_{s_k} \otimes 1) \\ &\qquad \text{by the induction hypothesis} \\ &= \operatorname{rex} \circ (B(t_1, \ldots, t_l) \otimes_R i_0^{s_k})(u_{(t_1,\ldots,t_l,s_k)} \otimes \delta_{s_k} \otimes 1) \quad \text{by (4.1.1)} \\ &= \operatorname{rex}(u_{(t_1,\ldots,t_l)} \otimes i_0^{s_k}(1 \otimes \delta_{s_k} \otimes 1)) \quad \text{as } 1 \otimes 1 \otimes \delta_{s_k} \otimes 1 \mapsto 1 \otimes \delta_{s_k} \otimes 1 \text{ under} \\ &\qquad B(s_k) * B(s_k) = R \otimes_{R^{s_k}} R \otimes_R R \otimes_{R^{s_k}} R \xrightarrow{\sim} R \otimes_{R^{s_k}} R \otimes_{R^{s_k}} R \\ &= \operatorname{rex}(u_{(t_1,\ldots,t_l)} \otimes 1 \otimes 1) = \operatorname{rex}(u_{(t_1,\ldots,t_l,s_k)}) = u_{\underline{w}_k}. \end{aligned}$$

If $\mathbf{e}_k = 1$, we are in Case D1, and have

$$\begin{aligned} LL_k(b_{\underline{x}_{\leq k},\mathbf{f}_{\leq k}}) &= \operatorname{rex} \circ (B(t_1, \ldots, t_l) \otimes_R i_1^{s_k}) \circ (\operatorname{rex} \otimes_R B(s_k)) \\ &\qquad \circ (LL_{k-1} \otimes_R B(s_k))(b_{\underline{x}_{\leq k-1},\mathbf{f}_{\leq k-1}} \otimes \delta_{s_k} \otimes 1) \\ &= \operatorname{rex} \circ (B(t_1, \ldots, t_l) \otimes_R i_1^{s_k}) \circ (\operatorname{rex} \otimes_R B(s_k))(u_{w_{\leq k-1}} \otimes \delta_{s_k} \otimes 1) \\ &\qquad \text{by the induction hypothesis} \\ &= \operatorname{rex} \circ (B(t_1, \ldots, t_l) \otimes_R i_1^{s_k})(u_{(t_1,\ldots,t_l,s_k)} \otimes \delta_{s_k} \otimes 1) \quad \text{by (4.1.1)} \\ &= \operatorname{rex} \circ (B(t_1, \ldots, t_l) \otimes_R i_1^{s_k})(u_{(t_1,\ldots,t_l)} \otimes 1 \otimes \delta_{s_k} \otimes 1) \\ &= \operatorname{rex}(u_{(t_1,\ldots,t_l)}) = u_{\underline{w}_k}. \end{aligned}$$

Thus, we are done in the case that $\mathbf{e}_{\leq k} = \mathbf{f}_{\leq k}$, and hence $LL_{\underline{x},\mathbf{e}}(b_{\underline{x},\mathbf{e}}) = u_{\underline{w}}$.

Assume finally that $\mathbf{f} < \mathbf{e}$. Take k such that the labels of $\mathbf{e}$ and $\mathbf{f}$ are the same up to $k-1$ and the labels of $\mathbf{e}$ (resp. $\mathbf{f}$) at k is D (resp. U). Then $b_{\underline{x}_{\leq k},\mathbf{f}_{\leq k}} = b_{\underline{x}_{\leq k-1},\mathbf{f}_{\leq k-1}} \otimes 1 \otimes 1$. As the label of $\mathbf{e}$ at k is D, $w_{k-1}s_k < w_{k-1}$, and hence w_{k-1} admits a reduced expression $(t_1, \dots, t_l, s_k)$. If $\mathbf{e}_k = 1$, we are in Case D1, and have

$$\begin{aligned}
LL_k(b_{\underline{x}_{\leq k},\mathbf{f}_{\leq k}}) &= \mathrm{rex} \circ (B(t_1, \dots, t_l) \otimes_R i_1^{s_k}) \circ (\mathrm{rex} \otimes_R B(s_k)) \\
&\qquad \circ (LL_{k-1} \otimes_R B(s_k))(b_{\underline{x}_{\leq k-1},\mathbf{f}_{\leq k-1}} \otimes 1 \otimes 1) \\
&= \mathrm{rex} \circ (B(t_1, \dots, t_l) \otimes_R i_1^{s_k}) \circ (\mathrm{rex} \otimes_R B(s_k))(u_{w_{\leq k-1}} \otimes 1 \otimes 1) \\
&= \mathrm{rex} \circ (B(t_1, \dots, t_l) \otimes_R i_1^{s_k})(u_{(t_1,\dots,t_l,s_k)} \otimes 1 \otimes 1) \\
&= \mathrm{rex} \circ (B(t_1, \dots, t_l) \otimes_R i_1^{s_k})(u_{(t_1,\dots,t_l)} \otimes 1 \otimes 1 \otimes 1) = 0.
\end{aligned}$$

If $\mathbf{e}_k = 0$, we are in Case D0 and

$$\begin{aligned}
LL_k(b_{\underline{x}_{\leq k},\mathbf{f}_{\leq k}}) &= \mathrm{rex} \circ (B(t_1, \dots, t_l) \otimes_R i_0^{s_k}) \circ (\mathrm{rex} \otimes_R B(s_k)) \\
&\qquad \circ (LL_{k-1} \otimes_R B(s_k))(b_{\underline{x}_{\leq k-1},\mathbf{f}_{\leq k-1}} \otimes 1 \otimes 1) \\
&= \mathrm{rex} \circ (B(t_1, \dots, t_l) \otimes_R i_0^{s_k}) \circ (\mathrm{rex} \otimes_R B(s_k))(u_{w_{\leq k-1}} \otimes 1 \otimes 1) \\
&= \mathrm{rex} \circ (B(t_1, \dots, t_l) \otimes_R i_0^{s_k})(u_{(t_1,\dots,t_l,s_k)} \otimes 1 \otimes 1) \\
&= \mathrm{rex} \circ (B(t_1, \dots, t_l) \otimes_R i_0^{s_k})(u_{(t_1,\dots,t_l)} \otimes 1 \otimes 1 \otimes 1) = 0.
\end{aligned}$$

4.5 **A basis of $B(\underline{x})^w$:** Let $w \in \mathcal{W}, \underline{x} \in \mathcal{S}^r, \mathbf{e} \in \{0,1\}^r$ with $\underline{x}^{\mathbf{e}} = w$. Let $b^w_{\underline{x},\mathbf{e}}$ be the image of $b_{\underline{x},\mathbf{e}} \in B(\underline{x})^{-d(\mathbf{e})-\ell(w)}$ in $B(\underline{x})^w$ under the projection $\pi^w_{\underline{x}} : B(\underline{x}) \to B(\underline{x})^w$. Let $\underline{w} = (t_1, \dots, t_l) \in \mathcal{S}^l$, $l = \ell(w)$, be a reduced expression of w. Recall $m^{t_i} \in \mathcal{C}(B(t_i), R(t_i)(1))$, $i \in [1, l]$, via $a \otimes b \mapsto a(t_i b)$ from (2.2.17), and set $m^{\underline{w}} = m^{t_1} \circ (B(s_1) * m^{t_2}) \circ \dots \circ (B(t_1, \dots, t_{l-2}) * m^{t_{l-1}}) \circ (B(t_1, \dots, t_{l-1}) * m^{t_l}) \in \mathcal{C}(B(\underline{w}), R(w)(\ell(w)))$. Thus, $m^{\underline{w}} : B(\underline{w}) = R \otimes_{R^{t_1}} R \cdots \otimes_{R^{t_l}} R \ni a_0 \otimes a_1 \otimes \dots \otimes a_l \mapsto a_0(t_1 a_1) \dots (t_1 \dots t_l a_l)$.

Theorem: *(i) $B(\underline{x})^w$ is left/right graded free over R having a basis $\{b^w_{\underline{x},\mathbf{e}} | \underline{x}^{\mathbf{e}} = w\}$, so*

$$\mathrm{grk}(B(\underline{x})^w) = \sum_{\substack{\mathbf{e} \in \{0,1\}^r \\ \underline{x}^{\mathbf{e}} = w}} v^{d(\mathbf{e})+\ell(w)} = v^{\ell(w)} p^w_{\underline{x}}.$$

In particular, $B(\underline{x})^x \simeq R(x)(\ell(x))$ of basis $b^x_{\underline{x},(1,\dots,1)} = \pi^x_{\underline{x}}(u_{\underline{x}})$.

(ii) $\{m^{\underline{w}} \circ LL_{\underline{x},\mathbf{e}} | \underline{x}^{\mathbf{e}} = w\}$ forms a left/right R-linear basis of $\mathcal{C}^\sharp(B(\underline{x}), R(w))$.

Proof: By (4.4)

$$m^{\underline{w}}(LL_{\underline{x},\mathbf{e}}(b_{\underline{x},\mathbf{f}})) = \begin{cases} m^{\underline{w}}(u_{\underline{w}}) = 1 & \text{if } \mathbf{f} = \mathbf{e}, \\ 0 & \text{if } \mathbf{f} < \mathbf{e}. \end{cases}$$

As $m^{\underline{w}} \circ LL_{\underline{x},\mathbf{e}} \in \mathcal{C}(B(\underline{x}), R(w)(l+d(\mathbf{e}))$, one obtains from (1.4.v)

$$\begin{array}{ccc} B(\underline{x}) & \xrightarrow{m^{\underline{w}} \circ LL_{\underline{x},\mathbf{e}}} & R(w)(l+d(\mathbf{e})) \\ \downarrow & \overset{\psi_{\mathbf{e}}}{\nearrow} & \\ B(\underline{x})^w & & \end{array}$$

such that

$$\psi_{\mathbf{e}}(b^w_{\underline{x},\mathbf{f}}) = \begin{cases} 1 & \text{if } \mathbf{f} = \mathbf{e}, \\ 0 & \text{if } \mathbf{f} < \mathbf{e}. \end{cases}$$

Thus, $\{b^w_{\underline{x},\mathbf{e}} | \underline{x}^{\mathbf{e}} = w\}$ is a left/right R-linearly independent set. Moreover, by descending induction on $\mathbf{e}$ there is $\psi'_{\mathbf{e}} \in \psi_{\mathbf{e}} + \sum_{\mathbf{e}' > \mathbf{e}} R\psi_{\mathbf{e}'}$ such that $\forall \mathbf{f}$ with $\underline{x}^{\mathbf{f}} = w$, $\psi'_{\mathbf{e}}(b^w_{\underline{x},\mathbf{f}}) = \delta_{\mathbf{f},\mathbf{e}}$. Then $\coprod_{\underline{x}^{\mathbf{e}}=w} Rb^w_{\underline{x},\mathbf{e}} \hookrightarrow B(\underline{x})^w$ splits via $\sum_{\underline{x}^{\mathbf{e}}=w} \psi'_{\mathbf{e}}(m) b^w_{\underline{x},\mathbf{e}} \leftarrow\!\shortmid m$, and hence one can write $B(\underline{x})^w = N \oplus \coprod_{\underline{x}^{\mathbf{e}}=w} Rb^w_{\underline{x},\mathbf{e}}$ with some left/right R-module N. Then in $B(\underline{x})^Q_w$

$$(B(\underline{x})^w)^Q = N^Q \oplus \coprod_{\underline{x}^{\mathbf{e}}=w} Qb^w_{\underline{x},\mathbf{e}}.$$

But

$$\begin{aligned} \dim_Q (B(\underline{x})^w)^Q &= p^w_{\underline{x}}(1) \quad \text{by (3.2)} \\ &= \dim_Q (\coprod_{\underline{x}^{\mathbf{e}}=w} Qb^w_{\underline{x},\mathbf{e}}) \quad \text{by (3.3)}, \end{aligned}$$

and hence $N^Q = 0$. As $N \leq B(\underline{x})^w$ is torsion-free over R, we must have $N = 0$, and hence $B(\underline{x})^w = \coprod_{\underline{x}^{\mathbf{e}}=w} Rb^w_{\underline{x},\mathbf{e}}$. Then

$$\begin{aligned} \mathrm{grk}(B(\underline{x})^w) &= \sum_{\underline{x}^{\mathbf{e}}=w} v^{d(\mathbf{e})+l} = v^l \sum_{\underline{x}^{\mathbf{e}}=w} v^{d(\mathbf{e})} \\ &= v^l p^w_{\underline{x}} \quad \text{by (3.3)}. \end{aligned}$$

(ii) As $(\psi'_{\mathbf{e}} | \underline{x}^{\mathbf{e}} = w)$ forms a dual basis of $(b^w_{\underline{x},\mathbf{e}} | \underline{x}^{\mathbf{e}} = w)$,

$$R\mathrm{Mod}(B(\underline{x})^w, R) = \coprod_{\underline{x}^{\mathbf{e}}=w} R\psi'_{\mathbf{e}},$$

where the left R-linear structure on the LHS is such that $(a\phi)(m) = \phi(am) = a\phi(m)$, $m \in B(\underline{x})^w$. As $\psi'_{\mathbf{e}} \in \psi_{\mathbf{e}} + \sum_{\mathbf{e}' > \mathbf{e}} R\psi'_{\mathbf{e}}$, $(\psi_{\mathbf{e}} | \underline{x}^{\mathbf{e}} = w)$ also forms a left R-linear basis of $R\mathrm{Mod}(B(\underline{x})^w, R)$. As $\mathcal{C}^\sharp(B(\underline{x}), R(w)) \simeq \mathcal{C}^\sharp(B(\underline{x})^w, R(w)) \simeq R\mathrm{Mod}(B(\underline{x})^w, R)$ and as $\psi_{\mathbf{e}}(b_{\underline{x},\mathbf{f}}) = (m^{\underline{w}} \circ LL_{\underline{x},\mathbf{e}})(b^w_{\underline{x},\mathbf{f}})$ $\forall \mathbf{f} \leq \mathbf{e}$, $(m^{\underline{w}} \circ LL_{\underline{x},\mathbf{e}} | \underline{x}^{\mathbf{e}} = w)$ forms a left R-linear basis of $\mathcal{C}^\sharp(B(\underline{x}), R(w))$. Likewise as a right R-module.

4.6 $\forall B \in \mathfrak{S}\mathrm{Bimod}$, $\forall w \in \mathcal{W}$, B^w is graded free over R by (4.5), and hence

$$B^w \simeq \coprod_{i \in \mathbb{Z}} \{R(w)(i)\}^{\oplus m_i} \quad \exists m_i \in \mathbb{N}.$$

Corollary: *B_w is left/right graded free over R. In particular, $\forall \underline{x} \in \mathcal{S}^r$,*

$$\operatorname{grk}(B(\underline{x})_w) = v^{-\ell(w)} p^w_{\underline{x}}(v^{-1}).$$

Proof: We may assume $B = B(\underline{x})$ for some $\underline{x} \in \mathcal{S}^r$. Then

$$\begin{aligned} B(\underline{x})_w &\simeq D(B(\underline{x}))_w \quad \text{by (2.10)} \\ &\simeq D(B(\underline{x})^w) \quad \text{by (2.8).} \end{aligned}$$

As $B(\underline{x})^w$ is R-graded free of graded rank $v^{\ell(w)} p^w_{\underline{x}}$ by (4.5), so therefore is $B(\underline{x})_w$ by (2.9) with

$$\operatorname{grk}(B(\underline{x})_w) = \operatorname{grk}(B(\underline{x})^w)(v^{-1}) = v^{-\ell(w)} p^w_{\underline{x}}(v^{-1}).$$

4.7 **Corollary:** *Let $\underline{w}$ be a reduced expression of $w \in \mathcal{W}$. $\forall B \in \mathfrak{S}\mathrm{Bimod}$, one has, as graded left/right R-modules,*

$$\begin{array}{ccc} \mathcal{C}^\sharp(B, B(\underline{w})) & \dashrightarrow & R\mathrm{Mod}(B^w, B(\underline{w})^w) \\ & \searrow\!\!\!\!\searrow & \wr\,| \\ & & \mathcal{C}^\sharp(B, B(\underline{w})^w). \end{array}$$

Proof: Note first that

$$\begin{aligned} \mathcal{C}^\sharp(B, B(\underline{w})^w) &\simeq \mathcal{C}^\sharp(B^w, B(\underline{w})^w) \quad \text{by (1.4.v)} \\ &= R\mathrm{Mod}(B^w, B(\underline{w})^w). \end{aligned}$$

We may assume $B = B(\underline{x})$ for some $\underline{x} \in \mathcal{S}^r$. By (4.5.i) one has a CD

$$\begin{array}{ccc} \mathcal{C}^\sharp(B(\underline{x}), B(\underline{w})) & \longrightarrow & \mathcal{C}^\sharp(B(\underline{x}), B(\underline{w})^w) \\ & \searrow_{\mathcal{C}^\sharp(B(\underline{x}), m^{\underline{w}})} & \wr\,| \\ & & \mathcal{C}^\sharp(B(\underline{x}), R(w)(\ell(w))) \end{array}$$

with $\mathcal{C}^\sharp(B(\underline{x}), m^{\underline{w}})(LL_{\underline{x},\mathbf{e}} | \underline{x}^{\mathbf{e}} = w)$ forming a basis of $\mathcal{C}^\sharp(B(\underline{x}), R(w)(\ell(w)))$ by (4.5).

4.8. We say $I \subseteq \mathcal{W}$ is $\mathcal{W}$-open iff $\forall w \in I$, $\forall w' \in \mathcal{W}$ with $w' \leq w$, $w' \in I$; such a subset is called "closed" in [Ab19a]. The present terminology appears in better accordance, however, with the one in [Ab19b]. See (8.1) for more details.

Lemma: *Let I be a finite $\mathcal{W}$-open subset of $\mathcal{W}$ and w a maximal element of I. There exists enumeration $w_1, w_2, \dots$ of elements of $\mathcal{W}$ such that $\forall i \in \mathbb{N}^+$, $\{w_1, w_2, \dots, w_i\}$ is $\mathcal{W}$-open, $w = w_{|I|}$, and $I = \{w_1, \dots, w_{|I|}\}$.*

Proof: Put $k = |I|$. Let $w_1, \dots, w_{k-1}$ be enumeration of elements of $I \setminus \{w\}$ such that $\forall i, j \in [1, k[$, $w_i \leq w_j \Rightarrow i \leq j$. Let $w_{k+1}, \dots$ be enumeration of elements of $\mathcal{W} \setminus I$. Put $w = w_k$. Then $\{w_1, \dots, w_i\}$ is $\mathcal{W}$-open $\forall i \leq k$ as I is $\mathcal{W}$-open. Let $i > k$ and let $w_j \leq w_i$, $j \in \mathbb{N}$. If $j \leq k$, $j < i$. If $j > k$, $k+1 \leq j \leq i$ by construction. Thus, $\{w_1, \dots, w_i\}$ is $\mathcal{W}$-open $\forall i \in \mathbb{N}^+$.

4.9. $\forall LL_{\underline{x},\mathbf{e}} \in \mathcal{C}(B(\underline{x}), B(\underline{w})(d(\mathbf{e})))$, let $LL^\vee_{\underline{x},\mathbf{e}} = D(LL_{\underline{x},\mathbf{e}})$. Thus, one has from (2.10) a CD

$$\begin{array}{ccc}
B(\underline{w})(-d(\mathbf{e})) & \xrightarrow{LL^\vee_{\underline{x},\mathbf{e}}} & B(\underline{x}) \\
\wr| & & |\wr \\
D(B(\underline{w})(d(\mathbf{e}))) & \xrightarrow{D(LL_{\underline{x},\mathbf{e}})} & D(B(\underline{x})) \\
\| & & \| \\
\mathrm{Mod}R(B(\underline{w})(d(\mathbf{e})), R) & \xrightarrow[\mathrm{Mod}R(LL_{\underline{x},\mathbf{e}},R)]{} & \mathrm{Mod}R(B(\underline{x}), R).
\end{array}$$

Let $\pi^w_{\underline{x}} : B(\underline{x}) \to B(\underline{x})^w$ be the projection. Let I be a $\mathcal{W}$-open and $w \in I$. Then $B(\underline{x})_{I\setminus\{w\}} = B(\underline{x}) \cap \coprod_{y\in I\setminus\{w\}} B(\underline{x})^Q_y = \ker(B(\underline{x})_I \to B(\underline{x})^w)$, and hence

$$\begin{array}{ccc}
B(\underline{x}) & \xrightarrow{\pi^w_{\underline{x}}} & B(\underline{x})^w \\
\uparrow & & \uparrow \\
B(\underline{x})_I & \dashrightarrow & B(\underline{x})_I/B(\underline{x})_{I\setminus\{w\}},
\end{array}$$

which we will still denote by $\pi^w_{\underline{x}}$.

Theorem: *Assume that w is a maximal element of I. Let $\underline{w}$ be a reduced expression of w and $\underline{x} \in \mathcal{S}^r$. Then $(\pi^w_{\underline{x}}(LL^\vee_{\underline{x},\mathbf{e}}(u_{\underline{w}}))|\underline{x}^\mathbf{e} = w)$ forms a left/right R-linear basis of $B(\underline{x})_I/B(\underline{x})_{I\setminus\{w\}}$.*

Proof: Put $I' = I \setminus \{w\}$. By (2.4) one has $\mathrm{supp}_\mathcal{W}(B(\underline{w})) = \{y \in \mathcal{W}|y \leq w\} \subseteq I$. By (1.4.v)

(1)
$$\begin{array}{ccccc}
B(\underline{w})(-d(\mathbf{e})) & \xrightarrow{LL^\vee_{\underline{x},\mathbf{e}}} & B(\underline{x}) & \xrightarrow{\pi^w_{\underline{x}}} & B(\underline{x})^w \\
 & \searrow & \uparrow & & \uparrow \\
 & & B(\underline{x})_I & \longrightarrow & B(\underline{x})_I/B(\underline{x})_{I'},
\end{array}$$

and hence $\pi^w_{\underline{x}}(LL^\vee_{\underline{x},\mathbf{e}}(u_{\underline{w}})) \in B(\underline{x})_I/B(\underline{x})_{I'}$. One has

$$B(\underline{x})_I = B(\underline{x}) \cap \coprod_{y\in I} B(\underline{x})^Q_y = B(\underline{x}) \cap \coprod_{y\in I\cap\mathrm{supp}_\mathcal{W}(B(\underline{x}))} B(\underline{x})^Q_y = B(\underline{x})_{I\cap\mathrm{supp}_\mathcal{W}(B(\underline{x}))},$$

and $B(\underline{x})_{I'} = B(\underline{x})_{I'\cap\mathrm{supp}_\mathcal{W}(B(\underline{x}))}$. Thus, there is nothing to show if $I \cap \mathrm{supp}_\mathcal{W}(B(\underline{x})) = I' \cap \mathrm{supp}_\mathcal{W}(B(\underline{x}))$, and hence we may assume $w \in \mathrm{supp}_\mathcal{W}(B(\underline{x}))$. Also, $I \cap \mathrm{supp}_\mathcal{W}(B(\underline{x})) = \{I \cap (\cup_{y\in\mathrm{supp}_\mathcal{W}(B(\underline{x}))}(\leq y))\} \cap \mathrm{supp}_\mathcal{W}(B(\underline{x}))$. Then

$$\begin{aligned}
B(\underline{x})_I &= B(\underline{x})_{I\cap\mathrm{supp}_\mathcal{W}(B(\underline{x}))} = B(\underline{x})_{I\cap\{\cup_{y\in\mathrm{supp}_\mathcal{W}(B(\underline{x}))}(\leq y)\}\cap\mathrm{supp}_\mathcal{W}(B(\underline{x}))}, \\
B(\underline{x})_{I'} &= B(\underline{x})_{I'\cap\mathrm{supp}_\mathcal{W}(B(\underline{x}))} = B(\underline{x})_{I'\cap\{\cup_{y\in\mathrm{supp}_\mathcal{W}(B(\underline{x}))}(\leq y)\}\cap\mathrm{supp}_\mathcal{W}(B(\underline{x}))} \\
&= B(\underline{x})_{\{I\cap(\cup_{y\in\mathrm{supp}_\mathcal{W}(B(\underline{x}))}(\leq y))\cap\mathrm{supp}_\mathcal{W}(B(\underline{x}))\}\setminus\{w\}}.
\end{aligned}$$

Thus, replacing I by $I \cap \{\cup_{y\in\mathrm{supp}_\mathcal{W}(B(\underline{x}))}(\leq y)\}$, we may assume that I is finite.

We first show that the $\pi^w_{\underline{x}}(LL^\vee_{\underline{x},\mathbf{e}}(u_{\underline{w}}))$, $\underline{x}^\mathbf{e} = w$, are left/right R-linearly independent in $B(\underline{x})^w$. As $B(\underline{x})^w$ is graded free over R by (4.5), it is enough to show that they are linearly independent over Q. From (4.5.i) one has a CD

(2)
$$\begin{array}{ccccc}
 & & B(\underline{w})(-d(\mathbf{e})) & \xrightarrow{LL^{\vee}_{\underline{x},\mathbf{e}}} & B(\underline{x}) \\
 & \swarrow^{m^{\underline{w}}(-d(\mathbf{e}))} & \downarrow{\scriptstyle \pi^{w}_{\underline{w}}(-d(\mathbf{e}))} & & \downarrow{\scriptstyle \pi^{w}_{\underline{x}}} \\
R(w)(\ell(w)-d(\mathbf{e})) & \simeq & B(\underline{w})^{w}(-d(\mathbf{e})) & \underset{(1.4.\mathrm{v})}{\dashrightarrow} & B(\underline{x})^{w} \\
 & & {\scriptstyle (1.4.1)}\uparrow & & \uparrow{\scriptstyle (1.4.1)} \\
 & & B(\underline{w})_{w}(-d(\mathbf{e})) & \underset{(1.4.\mathrm{v})}{\dashrightarrow} & B(\underline{x})_{w}.
\end{array}$$

As $m^{\underline{w}}(u_{\underline{w}}) = 1$, $\pi^{w}_{\underline{w}}(-d(\mathbf{e}))(u_{\underline{w}}) \neq 0$. On the other hand, from (2.8) and (2.10) one has a CD

(3)
$$\begin{array}{ccc}
D(B(\underline{w})) & \xleftarrow{D(m^{uw})} & D(R(w)(\ell(w))) \\
\wr & & \wr \\
B(\underline{w}) & & D(B(\underline{w})^{w}) \\
{\scriptstyle \pi^{w}_{\underline{w}}}\downarrow & \nwarrow & \wr \\
B(\underline{w})^{w} & \underset{(1.4.1)}{\dashleftarrow} & B(\underline{w})_{w}.
\end{array}$$

As $(B(\underline{w})^{w})^{Q} \simeq Q \simeq B(\underline{w})^{Q}_{w}$ by (1.4.ii), letting $u^{*}_{\underline{w}} \in B(\underline{w})^{Q}_{w}$ denote the element corresponding to $(\pi^{w}_{\underline{w}})^{Q}(1 \otimes u_{\underline{w}})$, one has from (2) and (3)

$$(\pi^{w}_{\underline{x}} \circ LL^{\vee}_{\underline{x},\mathbf{e}})^{Q}(1 \otimes u_{\underline{w}}) = (LL^{\vee}_{\underline{x},\mathbf{e}} \circ D(m^{\underline{w}})(-d(\mathbf{e})))^{Q}(u^{*}_{\underline{w}}).$$

Thus, we have only to show that the $(LL^{\vee}_{\underline{x},\mathbf{e}} \circ D(m^{\underline{w}})(-d(\mathbf{e})))^{Q}(u^{*}_{\underline{w}})$, $\underline{x}^{\mathbf{e}} = w$, are linearly independent over Q.

Now, by (4.5) the $m^{\underline{w}} \circ LL_{\underline{x},\mathbf{e}}$, $\underline{x}^{\mathbf{e}} = w$, are linearly independent over R in $\mathcal{C}^{\sharp}(B(\underline{x}), R(w))$. Recall from (4.5) (resp. (2.7)) that the R-bimodule structure on $\mathcal{C}^{\sharp}(X, Y)$ (resp. $D(X) = \mathrm{Mod}R(X, R)$) is given by $(a\phi b)(x) = \phi(axb) = a\phi(x)b$ (resp. $(afb)(x) = f(axb) = f(ax)b$) $\forall a, b \in R, \forall x \in X$. Then

$$\begin{aligned}
\{(a(D\phi))(f)\}(x) &= \{a(D\phi)(f)\}(x) = \{a(f \circ \phi)\}(x) = (f \circ \phi)(ax) = f(\phi(ax)) = f(a\phi(x)) \\
&= f((a\phi)(x)) = (f \circ (a\phi))(x) = \{D(a\phi)(f)\}(x),
\end{aligned}$$

and hence $a(D\phi) = D(a\phi)$, likewise $(D\phi)a = D(\phi a)$. As $\mathcal{C}^{\sharp}(B(\underline{x}), R(w))$ is graded free over R by (4.5), $LL_{\underline{x},\mathbf{e}} \circ D(m^{\underline{w}}) = (D(m^{\underline{w}} \circ LL_{\underline{x},\mathbf{e}})$, $\underline{x}^{\mathbf{e}} = w)$, are linearly independent over R in

$$\begin{aligned}
\mathcal{C}^{\sharp}(D(R(w)), D(B(\underline{x}))) &\leq R\mathrm{Bimod}(D(B(\underline{w})^{w}), D(B(\underline{x}))) \quad \text{by (3)} \\
&\simeq R\mathrm{Bimod}(B(\underline{w})_{w}, B(\underline{x})) \quad \text{by (2.8) and (2.10)}.
\end{aligned}$$

Then, the $(LL_{\underline{x},\mathbf{e}} \circ D(m^{\underline{w}})(-d(\mathbf{e})))^{Q}$ are linearly independent over Q in $Q\mathrm{Mod}(B(\underline{w})^{Q}_{w}, B(\underline{x})^{Q})$. As $B(\underline{w})^{Q}_{w} \simeq Q$, we must have the $(LL_{\underline{x},\mathbf{e}} \circ D(m^{\underline{w}})(-d(\mathbf{e})))^{Q}(u^{*}_{\underline{w}})$, $\underline{x}^{\mathbf{e}} = w$, linearly independent over Q in $B(\underline{x})^{Q}$, as desired.

Let next $w_1, w_2, \dots$ be an enumeration of elements of $\mathcal{W}$ as in (4.8). Fix a reduced expression $\underline{w_k}$ of w_k for each $k \in \mathbb{N}^{+}$. Put $I(k) = \{w_1, \dots, w_k\}$ and consider a filtration $B(\underline{x})_{w_1} =$

$B(\underline{x})_{I(1)} \leq B(\underline{x})_{I(2)} \leq \dots$ of $B(\underline{x})$ with

$$\coprod_{\underline{x}^{\mathbf{e}}=w_k} R\pi_{\underline{x}}^{w_k}(LL_{\underline{x},\mathbf{e}}^{\vee}(u_{\underline{w_k}})) \subseteq B(\underline{x})_{I(k)}/B(\underline{x})_{I(k-1)}. \tag{4}$$

We must show that the containment is an equality. Assume first that $\mathbb{K}$ is a field. As $\deg(u_{\underline{w_k}}) = -\ell(w_k)$ and as $LL_{\underline{x},\mathbf{e}}^{\vee} \in \mathcal{C}(B(\underline{w})(-\deg(\mathbf{e})), B(\underline{x}))$, $R\pi_{\underline{x}}^{w_k}(LL_{\underline{x},\mathbf{e}}^{\vee}(u_{\underline{w_k}})) \simeq R(w_k)(\ell(w_k) - \deg(\mathbf{e}))$. Then

$$\begin{aligned} \operatorname{grk}(\coprod_{\underline{x}^{\mathbf{e}}=w_k} R\pi_{\underline{x}}^{w_k}(LL_{\underline{x},\mathbf{e}}^{\vee}(u_{\underline{w_k}}))) &= \sum_{\underline{x}^{\mathbf{e}}=w_k} v^{\ell(w_k)-\deg(\mathbf{e})} \\ &= v^{\ell(w_k)} p_{\underline{x}}^{w_k}(v^{-1}) \quad \text{by (3.3).} \end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned} \sum_k v^{\ell(w_k)} p_{\underline{x}}^{w_k}(v^{-1}) &= \sum_{w\in\mathcal{W}} v^{\ell(w)} p_{\underline{x}}^{w}(v^{-1}) \\ &= (v+v^{-1})^r \quad \text{by (3.1)} \\ &= \operatorname{grk}(B(\underline{x})) \quad \text{by (2.5).} \end{aligned}$$

Thus, if $\mathbb{K}$ is a field, one obtains from (1.8) that (4) is an equality.

In general, let $\mathfrak{m}$ be the maximal ideal of $\mathbb{K}$. By above one has a CD

$$\begin{array}{ccc} & & \{B(\underline{x})_{I(k)}/B(\underline{x})_{I(k-1)}\} \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \\ & \nearrow & \downarrow \\ \coprod_{\underline{x}^{\mathbf{e}}=w_k} (R/\mathfrak{m}R)\pi_{\underline{x}}^{w_k}(LL_{\underline{x},\mathbf{e}}^{\vee}(u_{\underline{w_k}})) & = & \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{I(k)}/\{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{I(k-1)}, \end{array} \tag{6}$$

and hence

$$\{B(\underline{x})_{I(k)}/B(\underline{x})_{I(k-1)}\} \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \twoheadrightarrow \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{I(k)}/\{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{I(k-1)} \quad \forall k. \tag{7}$$

Then, also, $B(\underline{x})_{I(k)} \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \twoheadrightarrow \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{I(k)}$.

We show by descending induction on k that (7) is invertible, and hence (4) will turn an isomorphism upon base change to $\mathbb{K}/\mathfrak{m}$. To begin the induction, take $k \gg 0$ so $B(\underline{x}) = B(\underline{x})_{I(k)}$. Assume now inductively that $B(\underline{x})_{I(k')}/B(\underline{x})_{I(k'-1)}$ is graded free for $k' > k$, so therefore is $B(\underline{x})/B(\underline{x})_{I(k)}$. Then $B(\underline{x})_{I(k)}$ is a direct summand of $B(\underline{x})$, and hence $B(\underline{x})_{I(k)} \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \twoheadrightarrow \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{I(k)}$ is injective as well, and hence invertible. Thus, one has a CD of exact rows

$$\begin{array}{ccccccc} B(\underline{x})_{I(k-1)} \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) & \longrightarrow & B(\underline{x})_{I(k)} \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) & \longrightarrow & \{B(\underline{x})_{I(k)}/B(\underline{x})_{I(k-1)}\} \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) & \longrightarrow & 0 \\ \downarrow & & \downarrow \wr & & \downarrow & & \\ \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{I(k-1)} & \to & \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{I(k)} & \to & \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{I(k)}/\{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{I(k-1)} & \to & 0. \end{array}$$

Then by the 5-lemma [中岡, Lem. 4.2.23, p. 248] (7) must be injective as well.

Now,

$$R \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) = S_{\mathbb{K}}(V) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \simeq S_{\mathbb{K}/\mathfrak{m}}(V \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})).$$

As $B(\underline{x})_{I(k)}/B(\underline{x})_{I(k-1)}$ is a subquotient of $B(\underline{x})$, it is of finite type over R, and hence each homogeneous piece $\{B(\underline{x})_{I(k)}/B(\underline{x})_{I(k-1)}\}^i$, $i \in \mathbb{Z}$, is of finite type over $\mathbb{K}$. One thus obtains by graded NAK [BH, Ex. 1.5.24(b)]

$$\coprod_{\underline{x}^{\mathbf{e}}=w_k} R\pi_{\underline{x}}^{w_k}(LL^\vee_{\underline{x},\mathbf{e}}(u_{\underline{w_k}})) \xrightarrow{\sim} \{B(\underline{x})_{I(k)}/B(\underline{x})_{I(k-1)}\}.$$

4.10. **Remarks:** (i) As (4.9.7) is invertible, one obtains also $B(\underline{x})_{I(k)} \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \simeq \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{I(k)}$. Any finite $\mathcal{W}$-open I can be realized as $I(k)$. Thus, $\forall B \in \mathfrak{S}\mathrm{Bimod}$, $\forall$ $\mathcal{W}$-open I, one has

$$B_I \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \simeq \{B \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_I.$$

(ii) Let $R^\emptyset = R[\frac{1}{\alpha_t}|t \in \mathcal{T}]$. We know from (2.2.16) that any $B \in \mathfrak{S}\mathrm{Bimod}$ splits already over $R^\emptyset$: $R^\emptyset \otimes_R B = \coprod_{w\in\mathcal{W}} B_w^\emptyset$ with $B_w^\emptyset = (R^\emptyset \otimes_R B) \cap B_w^Q$. Then, $\forall J \subseteq \mathcal{W}$, one has exact sequences

$$0 \to B_J \to B \to \coprod_{w\in\mathcal{W}\setminus J} B_w^\emptyset$$

and

$$0 \to \{B \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_J \to B \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \to \coprod_{w\in\mathcal{W}\setminus J} (B \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))_w^\emptyset,$$

where $(B \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))_w^\emptyset$ is the w-piece of $(B \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))^\emptyset = (R \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))[\frac{1}{\alpha_s}|s \in \mathcal{S}] \otimes_{R\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})} (B \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))$. As $B(\underline{x})_w^\emptyset \otimes (\mathbb{K}/\mathfrak{m}) \simeq (B(\underline{x}) \otimes (\mathbb{K}/\mathfrak{m}))_w^\emptyset$ for all $\underline{x} \in \mathcal{S}^n$ and $w \in \mathcal{W}$, one has $B_w^\emptyset \otimes (\mathbb{K}/\mathfrak{m}) \simeq (B \otimes (\mathbb{K}/\mathfrak{m}))_w^\emptyset$ also. Then

$$B_J \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \twoheadrightarrow \{B \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_J, \tag{1}$$

which induces (4.9.7).

(iii) Assume in (ii) that $\mathbb{K}$ is a DVR. We show that (1) is invertible. Write $\mathfrak{m} = \xi\mathbb{K}$. Let $b \in B_J$ vanishing in $\{B \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_J$. Then $b \in \mathfrak{m}B$, and hence $b = \xi b'$ for some $b' \in B$. If we write $b' = \sum_{w\in\mathcal{W}} b'_w$ in $(B \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))^\emptyset$ with $b'_w \in (B \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))_w^\emptyset$, $\xi b'_w = 0$ $\forall w \in \mathcal{W} \setminus J$. As $(B \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))_w^\emptyset$ is torsion-free, we must have $b'_w = 0$ $\forall w \in \mathcal{W} \setminus J$. Then $b' \in B_J$, and hence $b \in \mathfrak{m}B_J$. Thus, $B_J \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \simeq \{B \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_J$.

4.11. Let $B \in \mathfrak{S}\mathrm{Bimod}$.

Corollary: *(i) $\forall w \in \mathcal{W}$, B^w is left graded free over R.*

(ii) $\forall I$ $\mathcal{W}$-open, $\forall w \in \mathcal{W}$ maximal in I, $B_{\leq w}/B_{<w} \xrightarrow{\sim} B_I/B_{I\setminus\{w\}}$.

(iii) If $\underline{w}$ is a reduced expression of $w \in \mathcal{W}$,

$$
\begin{array}{ccccc}
f & & \mathcal{C}^\sharp(B(\underline{w}), B) & \xleftarrow{\sim} & \mathcal{C}^\sharp(B(\underline{w}), B_{\leq w}) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
f(u_{\underline{w}}) & & B_{\leq w} & & \mathcal{C}^\sharp(B(\underline{w}), B_{\leq w}/B_{<w}) \\
 & & \downarrow & & \wr\big| \\
 & & B_{\leq w}/B_{<w} & & \mathcal{C}^\sharp(B(\underline{w}), (B_{\leq w})^w) \\
 & & \wr\big| & & \wr\big| \\
 & & (B_{\leq w})^w & \xleftarrow[\sim]{} & \mathcal{C}^\sharp(B(\underline{w})^w, (B_{\leq w})^w).
\end{array}
$$

Proof: We may assume $B = B(\underline{x})$ for some $\underline{x} \in \mathcal{S}^r$. (i) follows from (4.5.i). One has

$$
\begin{array}{ccc}
B(\underline{x})_{\leq w} & \hookrightarrow & B(\underline{x})_I \\
\big\downarrow & & \big\downarrow \\
B(\underline{x})_{\leq w}/B(\underline{x})_{<w} & \rightarrowtail\cdots\rightarrow & B(\underline{x})_I/B(\underline{x})_{I\setminus\{w\}}.
\end{array}
$$

As the $\pi^w_{\underline{x}}(LL^\vee_{\underline{x},\mathbf{e}}(u_{\underline{w}}))$, $\underline{x}^{\mathbf{e}} = w$, give a basis of both $B(\underline{x})_{\leq w}/B(\underline{x})_{<w}$ and $B(\underline{x})_I/B(\underline{x})_{I\setminus\{w\}}$ by (4.9), (ii) follows.

(iii) As $\mathrm{supp}_{\mathcal{W}}(B(\underline{w})) = (\leq w) = \{y \in \mathcal{W} | y \leq w\}$ by (2.4), one has

$$
\begin{array}{ccc}
\mathcal{C}^\sharp(B(\underline{w}), B(\underline{x})) & \xleftarrow[\text{(1.4.v)}]{\sim} & \mathcal{C}^\sharp(B(\underline{w}), B(\underline{x})_{\leq w}) \\
\vdots & & \big\downarrow \\
\vdots & & \mathcal{C}^\sharp(B(\underline{w}), (B(\underline{x})_{\leq w})^w) \\
\vdots & & \wr\big| \\
\vdots & & \mathcal{C}^\sharp(B(\underline{w}), B(\underline{x})_{\leq w}/B(\underline{x})_{<w}) \\
\vdots & & \wr\big|\,\text{(1.4.v)} \\
\vdots & & \mathcal{C}^\sharp(B(\underline{w})^w, B(\underline{x})_{\leq w}/B(\underline{x})_{<w}) \\
\vdots & & \wr\big|\,\text{(4.5)} \\
\downarrow & & \\
B(\underline{x})_{\leq w}/B(\underline{x})_{<w} & \xleftarrow[\text{(1.6)}]{\sim} & \mathcal{C}^\sharp(R(w)(\ell(w)), B_{\leq w}/B_{<w}),
\end{array}
$$

under which $LL^\vee_{\underline{x},\mathbf{e}} \mapsto \pi^w_{\underline{x}}(LL^\vee_{\underline{x},\mathbf{e}}(u_{\underline{w}}))$ by (4.9.1). As the $\pi^w_{\underline{x}}(LL^\vee_{\underline{x},\mathbf{e}}(u_{\underline{w}}))$, $\underline{x}^{\mathbf{e}} = w$, form a basis of $B(\underline{x})_{\leq w}/B(\underline{x})_{<w}$ by (4.9), the assertion follows.

4.12. Recall the set $\mathcal{T} = \cup_{w\in\mathcal{W}} w\mathcal{S}w^{-1}$ of reflections. Let $w \in \mathcal{W}$ and put $f = \prod_{\substack{t\in T \\ tw<w}} \alpha_t \in R$, which is well-defined up to $\mathbb{K}^\times$ (1.1). As $\ell(w) = |\{t \in T | tw < w\}|$ [BB, Cor. 1.4.5], $\deg(f) = 2\ell(w)$.

Proposition: $\forall B \in \mathfrak{S}\mathrm{Bimod}$ *and* $w \in \mathcal{W}$*, there is an isomorphism of left/right graded free*

R-modules $B_w \simeq f(B_{\leq w}/B_{<w}) \simeq (B_{\leq w}/B_{<w})(-2\ell(w))$ *such that*

$$\begin{array}{ccc} B_w & \hookrightarrow & B_{\leq w} \\ \wr\downarrow\vdots & & \downarrow \\ f(B_{\leq w}/B_{<w}) & \hookrightarrow & B_{\leq w}/B_{<w}. \end{array}$$

Proof: We may assume that $B = B(\underline{x})$ for some $\underline{x} \in \mathcal{S}^r$. From (4.6) one has $B(\underline{x})_w$ is left/right graded free over R of graded rank $v^{-\ell(w)}p_{\underline{x}}^w(v^{-1})$ while $(\pi_{\underline{x}}^w(LL_{\underline{x},\mathbf{e}}^\vee(u_{\underline{w}}))|\underline{x}^{\mathbf{e}} = w)$ gives a left R-linear basis of $B(\underline{x})_{\leq w}/B(\underline{x})_{<w}$ by (4.9). Thus, $B(\underline{x})_{\leq w}/B(\underline{x})_{<w}$ is graded free over R of graded rank $v^{\ell(w)}p_{\underline{x}}^w(v^{-1})$ by (4.9.4), and

(1) $$\mathrm{grk}(f(B(\underline{x})_{\leq w}/B(\underline{x})_{<w})) = v^{-\ell(w)}p_{\underline{x}}^w(v^{-1}) = \mathrm{grk}(B(\underline{x})_w),$$

and hence $B(\underline{x})_w$ and $f(B(\underline{x})_{\leq w}/B(\underline{x})_{<w})$ are isomorphic as graded R-modules.

We know from (4.10.i) that

(2) $$(B(\underline{x})_{\leq w}/B(\underline{x})_{<w}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \simeq \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{\leq w}/\{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{<w}.$$

We show also that

(3) $$B(\underline{x})_w \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \simeq \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_w.$$

$B(\underline{x}) \twoheadrightarrow B(\underline{x})^w \hookrightarrow B(\underline{x})_w^\emptyset$ From (4.10.ii) one has a sequence $B(\underline{x}) \twoheadrightarrow B(\underline{x})^w \hookrightarrow B(\underline{x})_w^\emptyset$, which induces a CD

$$\begin{array}{ccccc} B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) & \twoheadrightarrow & B(\underline{x})^w \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) & \longrightarrow & B(\underline{x})_w^\emptyset \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \\ & \searrow\!\!\!\rightarrow & \vdots\twoheaddownarrow & & \wr\downarrow \\ & & \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}^w & \hookrightarrow & \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_w^\emptyset. \end{array}$$

As $B(\underline{x})^w$ is graded free by (4.5), $B(\underline{x})^w \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \simeq \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}^w$ by rank. Then, letting $D_{\mathbb{K}/\mathfrak{m}} = \mathrm{Mod}(R/\mathfrak{m}R)(?, R/\mathfrak{m}R)$, one has

$$\begin{aligned} B(\underline{x})_w \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) &\simeq D(B(\underline{x})^w) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \quad \text{by (2.8) and (2.10)} \\ &\simeq D_{\mathbb{K}/\mathfrak{m}}(B(\underline{x})^w \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})) \quad \text{as } B(\underline{x})^w \text{ is graded free of finite rank over } R \text{ again} \\ &\simeq D_{\mathbb{K}/\mathfrak{m}}((B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))^w) \\ &\simeq \{B(\underline{x}) \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_w \quad \text{by (2.8) and (2.10) again.} \end{aligned}$$

Assume next that $\underline{x}$ is a reduced expression of w. Let us thus write $\underline{w}$ for $\underline{x}$. One has $B(\underline{w})_{\leq w}/B(\underline{w})_{<w} \simeq B(\underline{w})^w$. We show that

$$\begin{array}{ccc} B(\underline{w})_w & \xhookrightarrow{(1.4.1)} & B(\underline{w})^w \\ & \searrow\!\vdots & \uparrow \\ & & fB(\underline{w})^w. \end{array}$$

It will then follow from (1)-(3) and by graded NAK that $B(\underline{w})_w \xrightarrow{\sim} fB(\underline{w})^w$. We argue by induction on $\ell(w)$. The assertion holds if $\ell(w) = 0$ with $f = 1$. If $\ell(w) = 1$, see (2.2.14). Write $\underline{w} = (s_1, \ldots, s_r)$. Put $s = s_1$ and $\underline{sw} = (s_2, \ldots, s_r)$ a reduced expression of $sw < w$. Let $\delta \in V$ with $\langle \delta, \alpha_s^\vee \rangle = 1$. As $R = R^s \oplus \delta R^s$ by (2.1), any element of $B(\underline{w}) = B(s) * B(\underline{sw}) = R \otimes_{R^s} B(\underline{sw})(1)$ is of the form $1 \otimes m + \delta \otimes m'$ for some $m, m' \in B(\underline{sw})$. Let $1 \otimes m + \delta \otimes m' \in B(\underline{w})_w$. Then $\mathrm{supp}_{\mathcal{W}}(m), \mathrm{supp}_{\mathcal{W}}(m') \subseteq \{w, sw\}$ by (2.3.i). As $B(\underline{sw})^w = 0$ by (2.3),

$$(4) \qquad m'_w = 0 = m_w.$$

Then $m' \in B(\underline{sw})_{sw}$, and hence by the induction hypothesis

$$m' \in (\prod_{\substack{t \in T \\ tsw < sw}} \alpha_t) B(\underline{sw})^{sw}.$$

As $1 \otimes m + \delta \otimes m' \in B(\underline{w})_w$, $(1 \otimes m + \delta \otimes m')_{sw} = 0$. Then

$$\begin{aligned} 0 &= (m_{sw} + \delta m'_{sw}, m_w + (s\delta)m'_w) \quad \text{in } B(\underline{w})^Q_{sw} \oplus B(\underline{w})^Q_w \text{ by (2.3.i)} \\ &= (m_{sw} + \delta m'_{sw}, 0) \quad \text{by (2)}, \end{aligned}$$

and hence $m_{sw} = -\delta m'_{sw}$. Thus,

$$m_{sw} + (s\delta)m'_{sw} = -\alpha_s m'_{sw} = -\alpha_s m' \in \alpha_s (\prod_{\substack{t \in T \\ tsw < sw}} \alpha_t) B(\underline{sw})^{sw}.$$

One has

$$|\{t \in T | tw < w\}| = \ell(w) = 1 + \ell(sw) = |\{s\} \sqcup \{sts^{-1} | t \in T, tsw < sw\}|.$$

If $tsw < sw$, $stsw < w$ as $sw < w$, and hence $\{t \in T | tw < w\} = \{s\} \sqcup \{sts^{-1} | t \in T, tsw < sw\}$. Then, up to $\mathbb{K}^\times$,

$$f = \alpha_s s(\prod_{\substack{t \in T \\ tsw < sw}} \alpha_t) = s(-\alpha_s \prod_{\substack{t \in T \\ tsw < sw}} \alpha_t),$$

and hence $m_{sw} + (s\delta)m'_{sw} \in (sf)B(\underline{sw})^{sw}$. Take $n \in B(\underline{sw})$ with $m_{sw} + (s\delta)m'_{sw} = (sf)n_{sw}$. Then

$$\begin{aligned} 1 \otimes m + \delta \otimes m' &= (1 \otimes m + \delta \otimes m')_w \\ &= (m_w + \delta m'_w, m_{sw} + (s\delta)m'_{sw}) \quad \text{in } B(\underline{w})^Q_w \oplus B(\underline{w})^Q_{sw} \text{ by (2.3.i)} \\ &= (0, m_{sw} + (s\delta)m'_{sw}) \quad \text{by (4) again} \\ &= (0, (sf)n_{sw}) \\ &= f(0, n_{sw}) \quad \text{by (2.3.i)} \\ &= f(n_w, n_{sw}) \quad \text{as } n_w = 0 \text{ by (4)} \\ &= f(1 \otimes n)_w \quad \text{by (2.3.i)} \\ &\in f(B(\underline{w}))^w \quad \text{with } B(\underline{w}) = R \otimes_{R^s} B(\underline{sw}), \end{aligned}$$

as desired.

Consider finally $\underline{x} \in \mathcal{S}^r$ in general. By (1)-(3) and by graded NAK again we have only to verify that $f(B(\underline{x})_{\leq w}/B(\underline{x})_{<w}) \leq B(\underline{x})_w$ in $B(\underline{x})_{\leq w}/B(\underline{x})_{<w}$. Let $m \in B(\underline{x})_{\leq w}$. By (4.9) we may assume

$$\begin{aligned} m &= \pi_{\underline{x}}^w(LL_{\underline{x},\mathbf{e}}^\vee(u_{\underline{w}})) \quad \exists\mathbf{e} \text{ with } \underline{x}^{\mathbf{e}} = w \\ &= LL_{\underline{x},\mathbf{e}}^\vee(\pi_{\underline{x}}^w(u_{\underline{w}})) \quad \text{as} \end{aligned}$$

$$\begin{array}{ccc} B(\underline{w}) & \xrightarrow{LL_{\underline{x},\mathbf{e}}^\vee} & B(\underline{x})(d(\mathbf{e})) \\ \downarrow{\scriptstyle \pi_{\underline{w}}^w} & & \downarrow{\scriptstyle \pi_{\underline{x}}^w} \\ B(\underline{w})^w & \dashrightarrow & B(\underline{x})^w(d(\mathbf{e})). \end{array}$$

Then

$$\begin{aligned} fm &= LL_{\underline{x},\mathbf{e}}^\vee(f\pi_{\underline{w}}^w(u_{\underline{w}})) \quad \text{with } f\pi_{\underline{w}}^w(u_{\underline{w}}) \in B(\underline{w})_w \text{ by the case above} \\ &\in LL_{\underline{x},\mathbf{e}}^\vee(B(\underline{w})_w) \subseteq B(\underline{x})_w. \end{aligned}$$

5. Categorification

In this section we assume that $\mathbb{K}$ is a complete noetherian local domain. Thus, $\mathcal{C}$ is Krull-Schmidt [CR, pf of (6.10), p. 126]; [AJS, E.6] does not apply.

5.1 **Indecomposable Soergel bimodules:** Recall from (2.2.18) that each $B(s)$, $s \in \mathcal{S}$, is indecomposable in $\mathcal{C}'$.

Theorem: *(i)* $\forall w \in \mathcal{W}$, $\exists!$ *up to isomorphism indecomposable* $B(w) \in \mathfrak{S}\mathrm{Bimod}$*:* $\mathrm{supp}_{\mathcal{W}}(B(w)) \subseteq \{x \in \mathcal{W} | x \leq w\}$ *and* $B(w)^w \simeq R(w)(\ell(w))$ *in* $\mathcal{C}$.

(ii) $\forall$ *indecomposable* $B \in \mathfrak{S}\mathrm{Bimod}$, $\exists!(w,n) \in \mathcal{W} \times \mathbb{Z}$*:* $B \simeq B(w)(n)$ *in* $\mathcal{C}$.

(iii) $D(B(w)) \simeq B(w)$.

(iv) $\forall$ *reduced expression* $\underline{w}$ *of* w, $\exists m_{n,y} \in \mathbb{N}$*:* $B(\underline{w}) \simeq B(w) \oplus \coprod_{y<w,n\in\mathbb{Z}} \{B(y)(n)\}^{\oplus m_{n,y}}$.

Proof: Fix a reduced expression $\underline{w}$ of w. Recall from (4.5.i) that $B(\underline{w})^w \simeq R(w)(\ell(w))$. As $\mathrm{supp}_{\mathcal{W}}(B(\underline{w})) = \{y \in \mathcal{W} | y \leq w\}$ by (2.4), there is a unique indecomposable direct summand $B(w)$ of $B(\underline{w})$ such that $B(w)^w \simeq R(w)(\ell(w))$. Then

$$\begin{aligned} D(B(w))_w &\simeq D(B(w)^w) \quad \text{by (2.8)} \\ &\simeq D(R(w)(\ell(w))) \simeq R(w)(-\ell(w)). \end{aligned}$$

If M is an indecomposable direct summand of $B(\underline{w})$ not isomorphic to $B(w)$, $M_w \leq M^w = 0$. As $D(B(\underline{w})) \simeq B(\underline{w})$ by (2.10), $D(B(w))$ is a direct summand of $B(\underline{w})$, and hence we must have $D(B(w)) \simeq B(w)$. By (4.5.i) again there remains only to show that an indecomposable of $\mathfrak{S}\mathrm{Bimod}$ is of the form $B(w)(n)$ for some $w \in \mathcal{W}$ and $n \in \mathbb{Z}$. Let $B \in \mathfrak{S}\mathrm{Bimod}$ indecomposable. Let $w \in \mathcal{W}$ with $\ell(w)$ maximal such that $B^w \neq 0$. Put $I = \{y \in \mathcal{W} | \ell(y) \leq \ell(w)\}$. Thus, I

is $\mathcal{W}$-open and $B_I = B$. Then $B_I/B_{I\setminus\{w\}} \simeq B^w$. By (4.5) and graded Quillen-Suslin B^w is left graded free over R. As $B(\underline{w})^w \simeq R(w)(\ell(w))$, $\{B(\underline{w})(n)\}^w$ is a direct summand of B^w for some $n \in \mathbb{Z}$. Let $\{B(\underline{w})(n)\}^w \underset{\pi}{\overset{i}{\rightleftarrows}} B^w$ be the associated imbedding and the projection. By (4.7) let $\hat{\pi} \in \mathcal{C}(B, B(\underline{w})(n))$ be a lift of π and by (4.11) let $\hat{i} \in \mathcal{C}(B(\underline{w})(n), B)$ be a lift of i; $B^w \simeq B_I/B_{I\setminus\{w\}} \simeq B_{\leq w}/B_{<w}$. Write

$$\begin{array}{ccc} B(w)(n) & \hookrightarrow & B(\underline{w})(n) \\ & \overset{\tilde{i}}{\searrow} & \downarrow \hat{i} \\ & & B \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \xrightarrow{\hat{\pi}} & B(\underline{w})(n) \\ & \overset{\tilde{\pi}}{\searrow} & \downarrow \\ & & B(w)(n). \end{array}$$

Then

$$\begin{array}{ccc} B(w)(n) & \xrightarrow{1-\tilde{\pi}\circ\tilde{i}} & B(w)(n) \\ \downarrow & & \downarrow \\ B(w)(n)^w & \underset{0}{\longrightarrow} & B(w)(n)^w, \end{array}$$

and hence $1 - \tilde{\pi} \circ \tilde{i} \notin \mathcal{C}(B(w)(n), B(w)(n))^\times$. Then $\tilde{\pi} \circ \tilde{i} \in \mathcal{C}(B(w)(n), B(w)(n))^\times$ [AF, 15.15]. Thus, $B(w)(n)$ is a direct summand of B, and hence $B(w)(n) \simeq B$.

5.2. Let $[\mathfrak{S}\text{Bimod}]$ denote the split Grothendieck group of $\mathfrak{S}$Bimod. Thus, $[\mathfrak{S}\text{Bimod}]$ admits a structure of $\mathbb{Z}[v, v^{-1}]$-algebra such that $v[B] = [B(1)]$ and $[B][B'] = [B * B']$ $\forall B, B' \in \mathfrak{S}\text{Bimod}$. By (5.1.i) (resp. (5.1.iv)) $([B(w)]|w \in \mathcal{W})$ (resp. $([B(\underline{w})]|w \in \mathcal{W})$ with $\underline{w}$ a chosen reduced expression of each w) forms a $\mathbb{Z}[v, v^{-1}]$-linear basis of $[\mathfrak{S}\text{Bimod}]$. Thus,

$$[\mathfrak{S}\text{Bimod}] = \sum_{\substack{r\in\mathbb{N} \\ \underline{x}\in\mathcal{S}^r}} \mathbb{Z}[v, v^{-1}][B(\underline{x})] = \coprod_{\substack{\underline{w} \\ \text{reduced}}} \mathbb{Z}[v, v^{-1}][B(\underline{w})] = \coprod_{w\in\mathcal{W}} \mathbb{Z}[v, v^{-1}][B(w)].$$

By (4.5) and by graded Quillen-Suslin each B^w, $B \in \mathfrak{S}\text{Bimod}$, $w \in \mathcal{W}$, is left graded free over R. Define $\text{ch} : [\mathfrak{S}\text{Bimod}] \to \mathcal{H}$ via

$$[B] \mapsto \sum_{w\in\mathcal{W}} v^{-\ell(w)} \text{grk}(B^w) H_w.$$

We will abbreviate $\text{ch}([B])$ as $\text{ch}(B)$. In particular, $\forall s \in \mathcal{S}$, $\text{ch}(B(s)) = \underline{H}_s$ by (2.2.12, 13).

Proposition: $\forall \underline{x} \in \mathcal{S}^r$, $\text{ch}(B(\underline{x})) = \underline{H}_{\underline{x}}$.

Proof: One has

$$\begin{aligned} \text{LHS} &= \sum_{w\in\mathcal{W}} v^{-\ell(w)} \text{grk}(B(\underline{x})^w) H_w \\ &= \sum_{w\in\mathcal{W}} v^{-\ell(w)} v^{\ell(w)} p_{\underline{x}}^w H_w \quad \text{by (4.5.i)} \\ &= \sum_{w\in\mathcal{W}} p_{\underline{x}}^w H_w \\ &= \underline{H}_{\underline{x}} \quad \text{by definition (3.1).} \end{aligned}$$

5.3. $\forall \underline{x} \in \mathcal{S}^r$, $\forall \underline{y} \in \mathcal{S}^k$, one has from (5.2)

$$\mathrm{ch}(B(\underline{x}))\mathrm{ch}(B(\underline{y})) = \underline{H}_{\underline{x}}\underline{H}_{\underline{y}} = \underline{H}_{\underline{xy}} = \mathrm{ch}(B(\underline{xy})).$$

As $[\mathfrak{S}\mathrm{Bimod}] = \sum_{\underline{x}} \mathbb{Z}[v, v^{-1}][B(\underline{x})]$, $\mathrm{ch} : [\mathfrak{S}\mathrm{Bimod}] \to \mathcal{H}$ is a $\mathbb{Z}[v, v^{-1}]$-algebra homomorphism. If $\underline{w}$ is a reduced expression of $w \in \mathcal{W}$, $\underline{H}_{\underline{w}} \in H_w + \sum_{y<w} \mathbb{Z}[v, v^{-1}]H_y$ from definition. Thus, $(\underline{H}_{\underline{w}}|w \in \mathcal{W})$ with $\underline{w}$ a chosen reduced expression of w is a $\mathbb{Z}[v, v^{-1}]$-linear basis of $\mathcal{H}$, and we have obtained a categorification of $\mathcal{H}$:

Theorem: $\mathrm{ch} : [\mathfrak{S}\mathrm{Bimod}] \to \mathcal{H}$ *is an isomorphism of* $\mathbb{Z}[v, v^{-1}]$*-algebras.*

5.4. Recall from [S97, p. 84] a ring involution $\bar{?}$ on $\mathcal{H}$ such that

$$\sum_{w\in\mathcal{W}} a_w H_w \mapsto \sum_{w\in\mathcal{W}} a_w(v^{-1}) H_{w^{-1}}^{-1}, \quad a_w \in \mathbb{Z}[v, v^{-1}],$$

define a ring anti-involution ω such that

$$\sum_{w\in\mathcal{W}} a_w H_w \mapsto \sum_{w\in\mathcal{W}} a_w(v^{-1}) H_w^{-1},$$

and a $\mathbb{Z}[v, v^{-1}]$-linear map ε such that

$$\sum_{w\in\mathcal{W}} a_w H_w \mapsto a_e.$$

Let also $\bar{\varepsilon} = \bar{?} \circ \varepsilon \circ \bar{?}$. Recall from (3.1.i) that

$$H_s^2 = v^{-1}H_s - vH_s + 1 \quad \forall s \in \mathcal{S}. \tag{1}$$

Lemma: ε *is a trace, i.e.,* $\varepsilon(hh') = \varepsilon(h'h)$ $\forall h, h' \in \mathcal{H}$*, and so is* $\bar{\varepsilon}$*.*

Proof: It is enough to check that $\varepsilon(H_xH_s) = \varepsilon(H_sH_x)$ $\forall x \in \mathcal{W}$, $\forall s \in \mathcal{S}$; if $y \in \mathcal{W}$ with $sy > y$,

$$\begin{aligned}\varepsilon(H_x(H_sH_y)) &= \varepsilon((H_xH_s)H_y)\\ &= \varepsilon(H_y(H_xH_s)) \quad \text{by induction on } \ell(y)\\ &= \varepsilon((H_yH_x)H_s) = \varepsilon(H_s(H_yH_x)) = \varepsilon((H_sH_y)H_x).\end{aligned}$$

Assume first $xs > x$. Then $H_xH_s = H_{xs}$, and hence $\varepsilon(H_xH_s) = 0$. If $sx > x$, $\varepsilon(H_sH_x) = 0$ likewise. If $sx < x$, write $x = sy$ with $y < x$. As $xs > x$, $y > e$. Then

$$\varepsilon(H_sH_x) = \varepsilon(H_sH_sH_y) = \varepsilon((v^{-1}H_s - vH_s + 1)H_y) = 0.$$

Assume next $xs < s$, and write $x = zs$ with $z < x$. If $z = e$, $x = s$ and the assertion holds. Thus, we may assume $z > e$. Then

$$\varepsilon(H_xH_s) = \varepsilon(H_zH_s^2) = \varepsilon(H_z(v^{-1}H_s - vH_s + 1)) = 0.$$

If $sx > x$, $\varepsilon(H_sH_x) = 0$ as well. If $sx < x$, write $x = sy$ with $y < x$. As $y \neq e$, $\varepsilon(H_sH_x) = \varepsilon(H_s^2H_y) = 0$.

5.5. One has from (5.4.1)

(1) $$H_s^{-1} = H_s + v - v^{-1} \quad \forall s \in \mathcal{S}.$$

Lemma: *Let* $s_1, \dots, s_r \in \mathcal{S}$.

(i) $\omega(\underline{H}_{(s_1,\dots,s_r)}) = \underline{H}_{(s_r,\dots,s_1)}$, $\quad \overline{\underline{H}_{(s_1,\dots,s_r)})} = \underline{H}_{(s_1,\dots,s_r)}$.

(ii) $\forall w \in \mathcal{W}$, $p^w_{(s_1,\dots,s_r)} = p^{w^{-1}}_{(s_r,\dots,s_1)}$.

Proof: (i) We know $\overline{\underline{H}_s} = \underline{H}_s$ $\forall s \in \mathcal{S}$. Also,

$$\omega(\underline{H}_s) = \omega(H_s + v) = H_s^{-1} + v^{-1} = H_s + v - v^{-1} + v^{-1} = H_s + v = \underline{H}_s.$$

(ii) One has

$$\begin{aligned}\sum_{w\in\mathcal{W}} p^{w^{-1}}_{(s_1,\dots,s_r)} H_w = \sum_{w\in\mathcal{W}} p^{w}_{(s_1,\dots,s_r)} H_{w^{-1}} &= \overline{\omega(\sum_{w\in\mathcal{W}} p^{w}_{(s_1,\dots,s_r)} H_w)} \\ &= \overline{\omega(\underline{H}_{(s_1,\dots,s_r)})} \quad \text{by definition} \\ &= \underline{H}_{(s_r,\dots,s_1)} \quad \text{by (i)} \\ &= \sum_{w\in\mathcal{W}} p^{w}_{(s_r,\dots,s_1)} H_w \quad \text{by definition again.}\end{aligned}$$

5.6. $\forall B \in \mathfrak{S}$Bimod, $\forall s_1, \dots, s_r \in \mathcal{S}$, one has

(1) $$\begin{aligned}\mathcal{C}^\sharp(B(s_1,\dots,s_r), B) &\simeq \mathcal{C}^\sharp(R(e), B * B(s_r,\dots,s_1)) \quad \text{by (2.6)} \\ &\simeq \{B * B(s_r,\dots,s_1)\}_e \quad \text{by (1.6.3),}\end{aligned}$$

which is left/right graded free over R by (4.6).

Theorem: $\forall B, B' \in \mathfrak{S}$Bimod, *$\mathcal{C}^\sharp(B, B')$ is left/right graded free over R with*

$$\mathrm{grk}(\mathcal{C}^\sharp(B, B')) = \bar{\varepsilon}\{\omega(\mathrm{ch}(B))\mathrm{ch}(B')\}.$$

Proof: $\forall s \in \mathcal{S}$,

$$\begin{aligned}\bar{\varepsilon}(\omega(\mathrm{ch}(B * B(s)))\mathrm{ch}B') &= \bar{\varepsilon}(\omega(\mathrm{ch}(B)\underline{H}_s)\mathrm{ch}B') \quad \text{by (5.3)} \\ &= \bar{\varepsilon}(\underline{H}_s\omega(\mathrm{ch}B)\mathrm{ch}B') \quad \text{by (5.5.i)} \\ &= \bar{\varepsilon}(\omega(\mathrm{ch}B)\mathrm{ch}(B')\underline{H}_s) \quad \text{as } \varepsilon \text{ is also an anti-involution} \\ &= \bar{\varepsilon}(\omega(\mathrm{ch}B)\mathrm{ch}(B' * B(s))) \quad \text{by (5.3) again.}\end{aligned}$$

We may assume $B = B(s_1, \dots, s_r)$ and $B' = B(t_1, \dots, t_l)$ for some $s_1, \dots, s_r, t_1, \dots, t_l \in \mathcal{S}$. It is now enough by (1) to show that

$$\mathrm{grk}(B(t_1, \dots, t_l, s_r \dots, s_1)_e) = \bar{\varepsilon}(\omega(\mathrm{ch}R(e))\mathrm{ch}(B(t_1, \dots, t_l, s_r \dots, s_1))).$$

One has

$$\begin{aligned}
\mathrm{RHS} &= \bar{\varepsilon}(\mathrm{ch}(B(t_1, \dots, t_l, s_r \dots, s_1))) \\
&= \bar{\varepsilon}(\underline{H}_{(t_1,\dots,t_l,s_r\dots,s_1)}) \quad \text{by (5.2)} \\
&= \overline{\varepsilon(\underline{H}_{(t_1,\dots,t_l,s_r\dots,s_1)})} \quad \text{by (5.5.i)} \\
&= \overline{\varepsilon(\sum_{w\in\mathcal{W}} p^w_{(t_1,\dots,t_l,s_r\dots,s_1)} H_w)} = \overline{p^e_{(t_1,\dots,t_l,s_r\dots,s_1)}} = p^e_{(t_1,\dots,t_l,s_r\dots,s_1)}(v^{-1}) \\
&= \mathrm{LHS} \quad \text{by (4.6)},
\end{aligned}$$

as desired.

5.7 **Formula for the morphism space:** Recall from [Lib, 4.3] that

(1) $$\varepsilon(H_x H_y) = \delta_{xy,e} \quad \forall x, y \in \mathcal{W}.$$

Corollary: $\forall \underline{x} \in \mathcal{S}^r$, $\forall \underline{y} \in \mathcal{S}^l$,

$$\mathrm{grk}(\mathcal{C}^\sharp(B(\underline{x}), B(\underline{y}))) = \sum_{w\in\mathcal{W}} (p^w_{\underline{x}} p^w_{\underline{y}})(v^{-1}).$$

Proof: Write $\underline{x} = (s_1, \dots, s_r)$ and $\underline{x}' = (s_r, \dots, s_1)$. Then

$$\begin{aligned}
\mathrm{grk}(\mathcal{C}^\sharp(B(\underline{x}), B(\underline{y}))) &= \bar{\varepsilon}(\omega(\mathrm{ch}B(\underline{x}))\mathrm{ch}B(\underline{y})) \quad \text{by (5.6)} \\
&= \bar{\varepsilon}(\omega(\underline{H}_{\underline{x}})\underline{H}_{\underline{y}}) \quad \text{by (5.2)} \\
&= \bar{\varepsilon}(\underline{H}_{\underline{x}'}\underline{H}_{\underline{y}}) \quad \text{by (5.5.i)} \\
&= \overline{\varepsilon(\overline{\underline{H}_{\underline{x}'}\underline{H}_{\underline{y}}})} \\
&= \overline{\varepsilon(\underline{H}_{\underline{x}'}\underline{H}_{\underline{y}})} \quad \text{by (5.5.i) again} \\
&= \overline{\varepsilon(\sum_{w\in\mathcal{W}} p^w_{\underline{x}'} H_w \sum_{z\in\mathcal{W}} p^z_{\underline{y}} H_z)} = \overline{\sum_{w,z\in\mathcal{W}} p^w_{\underline{x}'} p^z_{\underline{y}} \varepsilon(H_w H_z)} \\
&= \overline{\sum_{w\in\mathcal{W}} p^{w^{-1}}_{\underline{x}'} p^w_{\underline{y}}} \quad \text{by (1)} \\
&= \overline{\sum_{w\in\mathcal{W}} p^w_{\underline{x}} p^w_{\underline{y}}} \quad \text{by (5.5.ii)} \\
&= \sum_{w\in\mathcal{W}} p^w_{\underline{x}}(v^{-1}) p^w_{\underline{y}}(v^{-1}).
\end{aligned}$$

5.8 **Double leaves:** Let $\underline{x} \in \mathcal{S}^r, \underline{y} \in \mathcal{S}^l, \mathbf{e} \in \{0,1\}^r, \mathbf{f} \in \{0,1\}^l$ with $\underline{x}^{\mathbf{e}} = \underline{y}^{\mathbf{f}}$. Fix a reduced expression $\underline{w}$ of $w = \underline{x}^{\mathbf{e}} = \underline{y}^{\mathbf{f}}$. Thus, one has $LL_{\underline{x},\mathbf{e}} \in \mathcal{C}(B(\underline{x}), B(\underline{w})(d(\mathbf{e})))$ and $LL^\vee_{\underline{y},\mathbf{f}} \in$

$\mathcal{C}(B(\underline{w}), B(\underline{y})(d(\mathbf{f})))$. Put $LL_{\mathbf{e},\mathbf{f}} = LL^{\vee}_{\underline{y},\mathbf{f}}(d(\mathbf{e})) \circ LL_{\underline{x},\mathbf{e}} \in \mathcal{C}(B(\underline{x}), B(\underline{y})(d(\mathbf{e}) + d(\mathbf{f})))$, which we call a double leaf from $B(\underline{x})$ to $B(\underline{y})$:

$$\begin{array}{ccc} B(\underline{x}) & \xrightarrow{\quad LL_{\mathbf{e},\mathbf{f}} \quad} & B(\underline{y})(d(\mathbf{e}) + d(\mathbf{f})). \\ & {\scriptstyle LL_{\underline{x},\mathbf{e}}} \searrow \qquad \nearrow {\scriptstyle LL^{\vee}_{\underline{y},\mathbf{f}}(d(\mathbf{e}))} & \\ & B(\underline{w})(d(\mathbf{e})) & \end{array}$$

Theorem: *$(LL_{\mathbf{e},\mathbf{f}} | \underline{x}^{\mathbf{e}} = \underline{y}^{\mathbf{f}}, \mathbf{e} \in \mathcal{S}^r, \mathbf{f} \in \mathcal{S}^l)$ forms a left/right graded R-linear basis of $\mathcal{C}^\sharp(B(\underline{x}), B(\underline{y}))$.*

Proof: One has

$$\begin{aligned} \operatorname{grk}\Big(\coprod_{\substack{\mathbf{e}\in\mathcal{S}^r, \mathbf{f}\in\mathcal{S}^l \\ \underline{x}^{\mathbf{e}} = \underline{y}^{\mathbf{f}}}} R(LL_{\mathbf{e},\mathbf{f}})\Big) &= \sum_{\substack{\mathbf{e},\mathbf{f} \\ \underline{x}^{\mathbf{e}} = \underline{y}^{\mathbf{f}}}} v^{-d(\mathbf{e}) - d(\mathbf{f})} = \sum_{w\in\mathcal{W}} \sum_{\substack{\mathbf{e} \\ \underline{x}^{\mathbf{e}} = w}} v^{-d(\mathbf{e})} \sum_{\substack{\mathbf{f} \\ \underline{y}^{\mathbf{f}} = w}} v^{-d(\mathbf{f})} \\ &= \sum_{w\in\mathcal{W}} p^w_{\underline{x}}(v^{-1}) p^w_{\underline{y}}(v^{-1}) \\ &= \mathcal{C}^\sharp(B(\underline{x}), B(\underline{y})) \quad \text{by (5.7).} \end{aligned}$$

Then, arguing as in (4.9) using (1.8) and graded NAK [BH, Ex. 1.5.24(b)], one has only to show that the $LL_{\mathbf{e},\mathbf{f}}$ are linearly independent over R.

Let $\sum_{\underline{x}^{\mathbf{e}} = \underline{y}^{\mathbf{f}}} c_{\mathbf{e},\mathbf{f}} LL_{\mathbf{e},\mathbf{f}} = 0$, $c_{\mathbf{e},\mathbf{f}} \in R$, $\mathbf{e} \in \mathcal{S}^r, \mathbf{f} \in \mathcal{S}^l$. Put $I = \{\underline{x}^{\mathbf{e}} = \underline{y}^{\mathbf{f}} \in \mathcal{W} | c_{\mathbf{e},\mathbf{f}} \neq 0, \mathbf{e} \in \mathcal{S}^r, \mathbf{f} \in \mathcal{S}^l\}$ and just suppose $I \neq \emptyset$. Let $\hat{I} = \cup_{z\in I}(\leq z)$. Thus, $\hat{I}$ is $\mathcal{W}$-open. If $\underline{w}$ is a reduced expression of $w \in I$, $\operatorname{supp}_{\mathcal{W}}(B(\underline{w})) = (\leq w) \subseteq \hat{I}$ by (2.4). Then by (1.4.v)

$$\begin{array}{ccc} B(\underline{x}) & \xrightarrow{c_{\mathbf{e},\mathbf{f}} LL_{\mathbf{e},\mathbf{f}}} & B(\underline{y})(d(\mathbf{e}) + d(\mathbf{f})) \\ & \searrow & \uparrow \\ & & \{B(\underline{y})(d(\mathbf{e}) + d(\mathbf{f}))\}_{\hat{I}}. \end{array}$$

Let w be a maximal element of I. Then w remains maximal in $\hat{I}$. Put $J = \hat{I} \setminus \{w\}$. Consider the projection $\pi^w_{\underline{y}} : B(\underline{y}) \to (B(\underline{y}))^w$. As $\pi^w_{\underline{y}} \circ LL_{\mathbf{e},\mathbf{f}} = 0$ unless $w = \underline{x}^{\mathbf{e}} = \underline{y}^{\mathbf{f}}$ by the maximality of w, one has

$$0 = \pi^w_{\underline{y}} \circ \sum_{\underline{x}^{\mathbf{e}} = \underline{y}^{\mathbf{f}}} c_{\mathbf{e},\mathbf{f}} LL_{\mathbf{e},\mathbf{f}} = \sum_{\substack{\mathbf{e},\mathbf{f} \\ \underline{x}^{\mathbf{e}} = \underline{y}^{\mathbf{f}} = w}} c_{\mathbf{e},\mathbf{f}} \pi^w_{\underline{y}} \circ LL_{\mathbf{e},\mathbf{f}}.$$

Put $E = \{\mathbf{e} \in \mathcal{S}^r | c_{\mathbf{e},\mathbf{f}} \neq 0, \underline{x}^{\mathbf{e}} = w\}$. Recall from (4.4) the total order on E. Let $\mathbf{e}'$ be the minimum element of E. $\forall \mathbf{e} \in E$, $LL_{\underline{x},\mathbf{e}}(b_{\underline{x},\mathbf{e}'}) = 0$ unless $\mathbf{e}' = \mathbf{e}$ by (4.4), and hence

$$\begin{aligned} 0 &= \sum_{\substack{\mathbf{e},\mathbf{f} \\ \underline{x}^{\mathbf{e}}=\underline{y}^{\mathbf{f}}=w}} c_{\mathbf{e},\mathbf{f}} \pi^w_{\underline{y}} \circ LL_{\mathbf{e},\mathbf{f}}(b_{\underline{x},\mathbf{e}'}) = \sum_{\substack{\mathbf{f} \\ \underline{y}^{\mathbf{f}}=w}} c_{\mathbf{e}',\mathbf{f}} \pi^w_{\underline{y}} \circ LL_{\mathbf{e}',\mathbf{f}}(b_{\underline{x},\mathbf{e}'}) \\ &= \sum_{\underline{y}^{\mathbf{f}}=w} c_{\mathbf{e}',\mathbf{f}} \pi^w_{\underline{y}} (LL^\vee_{\underline{y},\mathbf{f}}(u_{\underline{w}}) \quad \text{by (4.4) again.} \end{aligned}$$

Then $c_{\mathbf{e}',\mathbf{f}} = 0$ $\forall \mathbf{f}$ with $\underline{y}^{\mathbf{f}} = w$ by (4.9), absurd.

5.9. $\forall B \in \mathfrak{S}\mathrm{Bimod}$, $\forall w \in \mathcal{W}$, put $B^{\mathrm{ou}}_w = B_{\leq w}/B_{<w}$, which is graded free over R by (4.12). Let $(B^{\mathrm{ou}}_w : R(w)(i))$, $i \in \mathbb{Z}$, denote the multiplicity of $R(w)(i)$ appearing in B^{ou}_w.

Let $x \in \mathcal{W}$. One has

$$\begin{aligned} B(x)^w &\simeq D(B(x)_w) \quad \text{by (2.9) and (5.1)} \\ &\simeq D(B(x)^{\mathrm{ou}}_w(-2\ell(w))) \quad \text{by (4.12)} \\ &\simeq D(\coprod_{i\in\mathbb{Z}} R(w)(i-2\ell(w))^{\oplus(B(x)^{\mathrm{ou}}_w : R(w)(i))}) \simeq \coprod_{i\in\mathbb{Z}} R(w)(2\ell(w)-i)^{\oplus(B(x)^{\mathrm{ou}}_w : R(w)(i))}. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{ch}(B(x)) &= \sum_{w\in\mathcal{W}} v^{-\ell(w)} \mathrm{grk}(B(x)^w) H_w = \sum_{w\in\mathcal{W}} v^{-\ell(w)} \sum_{i\in\mathbb{Z}} (B(x)^{\mathrm{ou}}_w : R(w)(i)) v^{2\ell(w)-i} H_w \\ &= \sum_{w\in\mathcal{W}} \sum_{j\in\mathbb{Z}} v^{-j} (B(x)^{\mathrm{ou}}_w : R(w)(\ell(w)+j)) H_w. \end{aligned}$$

As the $[B(x)]$, $x \in \mathcal{W}$, form a basis of $[\mathfrak{S}\mathrm{Bimod}]$, we have obtained an analogue of [S07, Prop. 5.9]

Proposition: *$\forall B \in \mathfrak{S}\mathrm{Bimod}$,*

$$\mathrm{ch}(B) = \sum_{w\in\mathcal{W}} \sum_{j\in\mathbb{Z}} v^{-j} (B^{\mathrm{ou}}_w : R(w)(\ell(w)+j)) H_w.$$

5.10. Back to complete noetherian local domain $\mathbb{K}$, put $\underline{H}^{\mathbb{K}}_x = \mathrm{ch}(B(x))$ $\forall x \in \mathcal{W}$. Recall from (5.1) that $\mathrm{supp}_{\mathcal{W}}(B(x)) \subseteq (\leq x)$ and that $B(x)^x \simeq R(x)(\ell(x))$. Thus,

$$\underline{H}^{\mathbb{K}}_x = \mathrm{ch}(B(x)) = \sum_{y\in\mathcal{W}} v^{-\ell(y)} \mathrm{grk}(B(x)^y) H_y = H_x + \sum_{y<x} v^{-\ell(y)} \mathrm{grk}(B(x)^y) H_y. \tag{1}$$

Put $h^{\mathbb{K}}_{y,x} = v^{-\ell(y)} \mathrm{grk}(B(x)^y) \in \mathbb{N}[v, v^{-1}]$. In particular, $\underline{H}^{\mathbb{K}}_s = \underline{H}_s$ $\forall s \in \mathcal{S}$. By (5.3) the $\underline{H}^{\mathbb{K}}_x$, $x \in \mathcal{W}$, form a $\mathbb{Z}[v, v^{-1}]$-linear basis of $\mathcal{H}$. In case $\mathbb{K}$ is a field of characteristic p, we call $(\underline{H}^{\mathbb{K}}_x | x \in \mathcal{W})$ (resp. $h^{\mathbb{K}}_{y,x}$) the p-KL basis (resp. a p-KL polynomial) of $\mathcal{H}$.

Lemma: *(i) $\forall B \in \mathfrak{S}\mathrm{Bimod}$, $\overline{\mathrm{ch}(B)} = \mathrm{ch}(DB)$.*

(ii) $\forall x \in \mathcal{W}$, $\forall s \in \mathcal{S}$ with $sx < x$, $\underline{H}_s \underline{H}^{\mathbb{K}}_x = (v + v^{-1}) \underline{H}^{\mathbb{K}}_x$, and hence $B(s) * B(x) \simeq B(x)(1) \oplus B(x)(-1)$.

Proof: (i) It is enough to show that $\overline{\underline{H}^{\mathbb{K}}_x} = \underline{H}^{\mathbb{K}}_x$ $\forall x \in \mathcal{W}$. Induction on $\ell(x)$. We may assume $\ell(x) > 1$. Take $s \in \mathcal{S}$ with $sx < x$. Recall from [BB, Prop. 2.2.7] that $\forall y < sx$, $sy < x$. Then

$$\sum_{w\in\mathcal{W}} v^{-\ell(w)}\mathrm{grk}((B(s) * B(sx))^w)H_w = \mathrm{ch}(B(s) * B(sx)) \tag{2}$$

$$\begin{aligned}
&= \underline{H}_s\underline{H}^{\mathbb{K}}_{sx} \quad \text{by (5.3)}\\
&= \underline{H}_s(H_{sx} + \sum_{y<sx} h^{\mathbb{K}}_{y,sx}H_y)\\
&= H_x + \sum_{y<x} m_y H_y \qquad \exists m_y \in \mathbb{N}[v, v^{-1}] \text{ by (3.1.2)} .
\end{aligned}$$

In particular, $v^{-\ell(x)}\mathrm{grk}((B(s) * B(sx))^x) = 1$, and hence $(B(s) * B(sx))^x \simeq R(x)(\ell(x))$. As $\mathrm{supp}_{\mathcal{W}}(B(s) * B(sx)) \subseteq (\leq x)$ by (1.7), one can write

$$B(s) * B(sx) \simeq B(x) \oplus \coprod_{\substack{y<x\\ n\in\mathbb{Z}}} B(y)(n)^{\oplus m_{y,n}} \quad \exists m_{y,n} \in \mathbb{N}.$$

Put $m'_y = \sum_{n\in\mathbb{Z}} m_{y,n}v^n \in \mathbb{N}[v, v^{-1}]$. As $D(B(s) * B(sx)) \simeq B(s) * B(sx)$ by (2.10), one has $m_{y,n} = m_{y,-n}$ $\forall y \in \mathcal{W}$, $\forall n \in \mathbb{Z}$, and hence

$$\overline{m'_y} = \overline{\sum_{n\in\mathbb{Z}} m_{y,n}v^n} = \sum_{n\in\mathbb{Z}} m_{y,n}v^{-n} = \sum_{n\in\mathbb{Z}} m_{y,-n}v^{-n} = m'_y.$$

Then

$$\begin{aligned}
\overline{\underline{H}^{\mathbb{K}}_x} + \sum_{y<x} m'_y\overline{\underline{H}^{\mathbb{K}}_y} &= \overline{\underline{H}^{\mathbb{K}}_x + \sum_{y<x} m'_y\underline{H}^{\mathbb{K}}_y} = \overline{\underline{H}_s\underline{H}^{\mathbb{K}}_{sx}}\\
&= \underline{H}_s\underline{H}^{\mathbb{K}}_{sx} \quad \text{by the induction hypothesis}\\
&= \underline{H}^{\mathbb{K}}_x + \sum_{y<x} m'_y\underline{H}^{\mathbb{K}}_y\\
&= \underline{H}^{\mathbb{K}}_x + \sum_{y<x} m'_y\overline{\underline{H}^{\mathbb{K}}_y} \quad \text{by the induction hypothesis again,}
\end{aligned}$$

and hence $\overline{\underline{H}^{\mathbb{K}}_x} = \underline{H}^{\mathbb{K}}_x$.

(ii) As in (2) one has

$$\underline{H}_s\underline{H}^{\mathbb{K}}_x = (v + v^{-1})H_x + \sum_{y<x} a_yH_y \quad \exists a_y \in \mathbb{N}[v, v^{-1}],$$

and hence one can write

$$B(s) * B(x) \simeq B(x)(1) \oplus B(x)(-1) \oplus \coprod_{\substack{y<x\\ n\in\mathbb{Z}}} B(y)(n)^{\oplus b_{y,n}} \quad \exists b_{y,n} \in \mathbb{N}.$$

As the graded left R-rank coincides with the graded right R-rank on $\mathfrak{S}$Bimod, $\mathrm{grk}(B(s) * B(x)) = (v + v^{-1})\mathrm{grk}(B(x))$ by (2.5). We must then have $B(s) * B(x) \simeq B(x)(1) \oplus B(x)(-1)$.

5.11. Recall from (2.2.16) that $B(s)$, $s \in \mathcal{S}$, splits over $R^\emptyset = R[\frac{1}{\alpha_t}|t \in \mathcal{S}]$ in the sense that $R^\emptyset \otimes_R B(s) = B(s)^\emptyset_e \oplus B(s)^\emptyset_s$ with $B(s)^\emptyset_e \simeq R^\emptyset(e)$ and $B(s)^\emptyset_s \simeq R^\emptyset(s)$. Let $\mathcal{C}_{\mathbb{K}}$ be $\mathcal{C}$ and $\mathcal{C}_{\mathbb{K}/\mathfrak{m}}$ denote $\mathcal{C}$ over $R_{\mathbb{K}/\mathfrak{m}} = \mathrm{S}_{\mathbb{K}/\mathfrak{m}}(V \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m})$ and denote an object in $\mathcal{C}_{\mathbb{K}}$ (resp. $\mathcal{C}_{\mathbb{K}/\mathfrak{m}}$) with subscript $\mathbb{K}$ (resp. $\mathbb{K}/\mathfrak{m}$). Thus, $B_{\mathbb{K}}(s) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m} \simeq B_{\mathbb{K}/\mathfrak{m}}(s)$ in $\mathcal{C}_{\mathbb{K}/\mathfrak{m}}$ graded free over $R_{\mathbb{K}/\mathfrak{m}}$ of rank $1+v$. Then by (2.3) inductively

$$(1) \qquad B_{\mathbb{K}}(\underline{w}) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m} \simeq B_{\mathbb{K}/\mathfrak{m}}(\underline{w}) \quad \forall \underline{w} \in \mathcal{S}^r,$$

and by (4.5) and (4.6)

$$(2) \qquad B_{\mathbb{K}}(\underline{w})_e \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m} \simeq B_{\mathbb{K}/\mathfrak{m}}(\underline{w})_e.$$

It follows for any $\underline{x} = (x_1, \dots, x_r) \in \mathcal{S}^r$ and $\underline{y} = (y_1, \dots, y_k) \in \mathcal{S}^k$ that

$$\begin{aligned}(3) \qquad \mathcal{C}^\sharp_{\mathbb{K}}(B_{\mathbb{K}}(\underline{x}), B_{\mathbb{K}}(\underline{y})) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m} &\simeq B_{\mathbb{K}}(y_1, \dots, y_k, x_r, \dots, x_1)_e \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m} \quad \text{by as in (5.6.1)} \\ &\simeq B_{\mathbb{K}/\mathfrak{m}}(y_1, \dots, y_k, x_r, \dots, x_1)_e \\ &\simeq \mathcal{C}^\sharp_{\mathbb{K}/\mathfrak{m}}(B_{\mathbb{K}/\mathfrak{m}}(\underline{x}), B_{\mathbb{K}/\mathfrak{m}}(\underline{y})) \\ &\simeq \mathcal{C}^\sharp_{\mathbb{K}/\mathfrak{m}}(B_{\mathbb{K}}(\underline{x}) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m}, B_{\mathbb{K}}(\underline{y}) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m}).\end{aligned}$$

If $\underline{x}$ is a reduced expression of $x \in \mathcal{W}$, taking direct summands yields

$$(4) \qquad \mathcal{C}_{\mathbb{K}}(B_{\mathbb{K}}(x), B_{\mathbb{K}}(x)) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m} \simeq \mathcal{C}_{\mathbb{K}/\mathfrak{m}}(B_{\mathbb{K}}(x) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m}, B_{\mathbb{K}}(x) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m}).$$

As $\mathcal{C}_{\mathbb{K}}(B_{\mathbb{K}}(x), B_{\mathbb{K}}(x))$ is local, so is $\mathcal{C}_{\mathbb{K}/\mathfrak{m}}(B_{\mathbb{K}}(x) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m}, B_{\mathbb{K}}(x) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m})$. As $B_{\mathbb{K}}(x) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m}$ is a direct summand of $B_{\mathbb{K}/\mathfrak{m}}(\underline{x})$ with $(B_{\mathbb{K}}(x) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m})^x \simeq R_{\mathbb{K}/\mathfrak{m}}(\ell(x))$ by (4.5), we must have by (5.1)

$$(5) \qquad B_{\mathbb{K}}(x) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m} \simeq B_{\mathbb{K}/\mathfrak{m}}(x).$$

Likewise,

$$(6) \qquad \mathcal{C}_{\mathbb{K}}(B_{\mathbb{K}}(x), B_{\mathbb{K}}(x)) \otimes_{\mathbb{K}} \mathrm{Frac}(\mathbb{K}) \simeq \mathcal{C}_{\mathrm{Frac}(\mathbb{K})}(B_{\mathbb{K}}(x) \otimes_{\mathbb{K}} \mathrm{Frac}(\mathbb{K}), B_{\mathbb{K}}(x) \otimes_{\mathbb{K}} \mathrm{Frac}(\mathbb{K})).$$

Does LHS remain local for $p \gg 0$ if $\mathbb{K} = \mathbb{Z}_p$?

$\forall x \in \mathcal{W}$, put $\underline{H}_x^{\mathbb{K}/\mathfrak{m}} = \mathrm{ch}(B_{\mathbb{K}/\mathfrak{m}}(x))$ and $\underline{H}_x^{\mathrm{Frac}(\mathbb{K})} = \mathrm{ch}(B_{\mathrm{Frac}(\mathbb{K})}(x))$. As $\mathrm{ch}(B_{\mathbb{K}/\mathfrak{m}}(x)) = \mathrm{ch}(B_{\mathbb{K}}(x) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m}) = \mathrm{ch}(B_{\mathbb{K}}(x)) = \mathrm{ch}(B_{\mathbb{K}}(x) \otimes_{\mathbb{K}} \mathrm{Frac}(\mathbb{K}))$ by (5), one has

$$B_{\mathbb{K}}(x) \otimes_{\mathbb{K}} \mathrm{Frac}(\mathbb{K}) = B_{\mathrm{Frac}(\mathbb{K})}(x) \oplus \coprod_{\substack{y<x \\ n \in \mathbb{Z}}} B_{\mathrm{Frac}(\mathbb{K})}(y)(n)^{\oplus m_{y,n}} \quad \exists m_{y,n} \in \mathbb{N}.$$

If we put $m_{y,x} = \sum_{n \in \mathbb{Z}} m_{y,n} v^n \in \mathbb{N}[v, v^{-1}]$,

$$\begin{aligned}(7) \qquad \underline{H}_x^{\mathbb{K}/\mathfrak{m}} &= \underline{H}_x^{\mathrm{Frac}(\mathbb{K})} + \sum_{y<x} \sum_{n \in \mathbb{Z}} m_{y,n} v^n \underline{H}_y^{\mathrm{Frac}\mathbb{K}} \\ &= \underline{H}_x^{\mathrm{Frac}(\mathbb{K})} + \sum_{y<x} m_{y,x} \underline{H}_y^{\mathrm{Frac}(\mathbb{K})} \quad \text{with } \overline{m_{y,x}} = m_{y,x} \text{ by (5.10.i).}\end{aligned}$$

5.12. We now compare $\underline{H}_x^{\mathbb{K}} = \mathrm{ch}B(x)$, $x \in \mathcal{W}$, over various $\mathbb{K}$, arguing after [JW17].

If $\underline{x}$ is a reduced expression of $x \in \mathcal{W}$, recall from (5.1) and (5.2) that

$$B(\underline{x}) = B(x) \oplus \coprod_{\substack{y<x \\ n\in\mathbb{Z}}} B(y)(n)^{\oplus m(y,n)} \quad \exists m(y,n) \in \mathbb{N},$$

and hence

$$\underline{H}_{\underline{x}} = \mathrm{ch}B(\underline{x}) = \underline{H}_x^{\mathbb{K}} + \sum_{\substack{y<x \\ n\in\mathbb{Z}}} m(y,n) v^n \underline{H}_y^{\mathbb{K}}.$$

Accordingly, to determine $\underline{H}_x^{\mathbb{K}}$, we may compute the multiplicities $(B(\underline{w}) : B(y)(n))$ of $B(y)(n)$, $y \in \mathcal{W}$, $n \in \mathbb{Z}$, in a decomposition of $B(\underline{w})$ into indecomposables for a reduced expression $\underline{w}$ of each $w \leq x$.

Fix a reduced expression $\underline{w} \in \mathcal{S}^r$ and a reduced expression $\underline{x}$ of x. Let $\mathcal{C}^{\not< x}$ denote the ideal quotient [中岡, Def. 3.2.43] of $\mathfrak{S}$Bimod by the set of morphisms factoring through $B(\underline{y})(n)$ for all reduced expressions $\underline{y}$ of $y < x$ and $n \in \mathbb{Z}$. Then

$$\begin{aligned}
(1) \qquad \mathcal{C}^{\not< x}(B(\underline{w}), B(\underline{x})) &= \mathcal{C}^{\not< x}(B(\underline{w}), B(x) \oplus \coprod_{\substack{y<x \\ n\in\mathbb{Z}}} B(y)(n)^{\oplus m(y,n)}) \\
&\simeq \mathcal{C}^{\not< x}(B(\underline{w}), B(x)) \oplus \coprod_{\substack{y<x \\ n\in\mathbb{Z}}} \mathcal{C}^{\not< x}(B(\underline{w}), B(y)(n))^{\oplus m(y,n)} = \mathcal{C}^{\not< x}(B(\underline{w}), B(x))
\end{aligned}$$

as $\mathcal{C}^{\not< x}(B(\underline{w}), B(y)(n)) \leq \mathcal{C}^{\not< x}(B(\underline{w}), B(\underline{y})(n)) = 0 \ \forall y < x, \forall n \in \mathbb{Z}$. In particular,

$$\begin{aligned}
(2) \qquad \mathcal{C}^{\not< x}(B(\underline{x}), B(\underline{x})) &\simeq \mathcal{C}^{\not< x}(B(x), B(x)) \oplus \prod_{\substack{y<x \\ n\in\mathbb{Z}}} \mathcal{C}^{\not< x}(B(y)(n), B(x))^{\oplus m(y,n)} \\
&= \mathcal{C}^{\not< x}(B(x), B(x))
\end{aligned}$$

as $\mathcal{C}^{\not< x}(B(y)(n), B(x)) \leq \mathcal{C}^{\not< x}(B(\underline{y})(n), B(x)) = 0 \ \forall y < x, \forall n \in \mathbb{Z}$. Also, one has from (2.11)

$$(3) \qquad \mathcal{C}^{\not< x}(B(x)(n), B(\underline{w})) \simeq \mathcal{C}^{\not< x}(B(\underline{w}), B(x)(-n)) \quad \text{via} \quad f \mapsto Df.$$

Assume from now on that $\mathbb{K}$ is a field, unless otherwise specified. Recall from (5.8) that $(LL_{\mathbf{e},\mathbf{f}} = LL^{\vee}_{\underline{x},\mathbf{f}}(d(\mathbf{e})) \circ LL_{\underline{w},\mathbf{e}} | \underline{w}^{\mathbf{e}} = \underline{x}^{\mathbf{f}}, \mathbf{e} \in \mathcal{S}^r, \mathbf{f} \in \mathcal{S}^{\ell(x)})$ forms an R-linear basis of $\mathcal{C}^{\sharp}(B(\underline{w}), B(\underline{x}))$. Thus,

$$\mathcal{C}^{\not< x,\sharp}(B(\underline{w}), B(\underline{x})) = \coprod_{n\in\mathbb{Z}} \mathcal{C}^{\not< x}(B(\underline{w}), B(\underline{x})(n)) = \sum_{\substack{\mathbf{e}\in\mathcal{S}^r \\ \underline{w}^{\mathbf{e}}=x}} R LL_{\underline{w},\mathbf{e}}$$

with $LL_{\underline{w},\mathbf{e}} \in \mathcal{C}^{\not< x}(B(\underline{w}), B(\underline{x})(d(\mathbf{e})))$. In particular,

$$\mathcal{C}^{\not< x,\sharp}(B(x), B(x)) \simeq \mathcal{C}^{\not< x,\sharp}(B(\underline{x}), B(\underline{x})) = R LL_{\underline{x},(1,\dots,1)}$$

with $LL_{\underline{x},(1,\dots,1)} \in \mathcal{C}^{\not< x}(B(x), B(x))$ as $d(1,\dots,1) = 0$ by definition (3.3).

Lemma: *(i)* $\mathcal{C}^{\not<x,\sharp}(B(\underline{w}),B(\underline{x}))$ *remains graded free over* R *with basis* $LL_{\underline{w},\mathbf{e}}$, $\underline{w}^{\mathbf{e}}=x$. *In particular,* $\mathcal{C}^{\not<x,\sharp}(B(x),B(x))$ *is graded* R*-free of basis* $LL_{\underline{x},(1,\dots,1)}$.

(ii) $\forall n\neq 0$, $B(x)(n)\not\simeq B(x)$ *in* $\mathcal{C}^{\not<x}$.

Proof: As $\mathrm{supp}_{\mathcal{W}}(B(\underline{y})(n))\not\ni x$ for any reduced expression $\underline{y}$ of $y<x$ and $n\in\mathbb{Z}$, under $m^{\underline{x}}:B(\underline{x})\to R(x)(\ell(x))$ from (4.5) one has

$$\begin{array}{ccc}\mathcal{C}^{\sharp}(B(\underline{w}),B(\underline{x})) & \longrightarrow & \mathcal{C}^{\sharp}(B(\underline{w}),R(x)(\ell(x))).\\ \downarrow & \nearrow & \\ \mathcal{C}^{\not<x,\sharp}(B(\underline{w}),B(\underline{x})) & & \end{array}$$

As the images of $LL_{\underline{w},\mathbf{e}}$, $\underline{w}^{\mathbf{e}}=x$, remain R-linearly independent, $(LL_{\underline{w},\mathbf{e}}|\underline{w}^{\mathbf{e}}=x)$ forms a basis of $\mathcal{C}^{\not<x,\sharp}(B(\underline{w}),B(\underline{x}))\simeq\mathcal{C}^{\not<x,\sharp}(B(\underline{w}),B(x))$. In particular, $\mathcal{C}^{\not<x,\sharp}(B(x),B(x))=RLL_{\underline{x},(1,\dots,1)}\simeq R$, and hence

$$\mathcal{C}^{\not<x}(B(x),B(x)(n))=R^nLL_{\underline{x},(1,\dots,1)}\quad\forall n\in\mathbb{Z}.$$

5.13. Keep the notation of (5.12). We have seen above that each $\mathcal{C}^{\not<x}(B(\underline{w}),B(x)(n))$, $n\in\mathbb{Z}$, is finite dimensional over $\mathbb{K}$. Being a quotient of $\mathcal{C}(B(x),B(x))$, $\mathcal{C}^{\not<x}(B(x),B(x))$ remains local, and hence $B(x)(n)$ remains indecomposable in $\mathcal{C}^{\not<x}$ $\forall n\in\mathbb{Z}$.

Consider the local intersection form, cf. [JW17],

$$\begin{array}{ccc}\mathcal{C}^{\not<x}(B(\underline{w}),B(x)(n))\times\mathcal{C}^{\not<x}(B(x)(n),B(\underline{w})) & \xrightarrow{I_{\underline{w},x,n}} & \mathbb{K}\\ & & \wr|\\ (f,g) & \searrow & \mathcal{C}^{\not<x}(B(x),B(x))\\ & & \wr|\\ \mapsto\ f\circ g & & \mathcal{C}^{\not<x}(B(x)(n),B(x)(n)).\end{array}$$

Let $f_1,\dots,f_a$ (resp. $g_1,\dots,g_b$) be a $\mathbb{K}$-linear basis of $\mathcal{C}^{\not<x}(B(\underline{w}),B(x)(n))=\sum\limits_{\underline{w}^{\mathbf{e}}=x}R^{n-d(\mathbf{e})}LL_{\underline{w},x,\mathbf{e}}$ (resp. $\mathcal{C}^{\not<x}(B(\underline{w}),B(x)(-n))=\sum\limits_{\underline{w}^{\mathbf{e}}=x}R^{-n-d(\mathbf{e})}LL_{\underline{w},x,\mathbf{e}}$, and put $\mathrm{rk}\,(I_{\underline{w},x,n})=\mathrm{rk}\,[\![f_i\circ Dg_j]\!]_{i\in[1,a],\mathrm{J}\in[1,b]}$.

Lemma: $\mathrm{rk}\,(I_{\underline{w},x,n})=(B(\underline{w}):B(x)(n))$ *the multiplicity of* $B(x)(n)$ *in* $B(\underline{w})$ *in* $\mathcal{C}$.

Proof: Put $I=I_{\underline{w},x,n}$, and write $B(\underline{w})=B(x)(n)^{\oplus m}\oplus B$ for some $m\in\mathbb{N}$ with $(B(\underline{w}):B(x)(n))=m$ in $\mathcal{C}$. By (5.12) the same holds in $\mathcal{C}^{\not<x}$. Then

$$\mathcal{C}^{\not<x}(B(\underline{w}),B(x)(n))\simeq\mathcal{C}^{\not<x}(B(x)(n)^{\oplus m},B(x)(n))\oplus\mathcal{C}^{\not<x}(B,B(x)(n)),$$
$$\mathcal{C}^{\not<x}(B(x)(n),B(\underline{w}))\simeq\mathcal{C}^{\not<x}(B(x)(n),B(x)(n)^{\oplus m})\oplus\mathcal{C}^{\not<x}(B(x)(n),B)$$

with $I(\mathcal{C}^{\not<x}(B(x)(n)^{\oplus m},B(x)(n)),\mathcal{C}^{\not<x}(B(x)(n),B))=0=I(\mathcal{C}^{\not<x}(B,B(x)(n)),\mathcal{C}^{\not<x}(B(x)(n),B(x)(n)^{\oplus m}))$. If there are $f\in\mathcal{C}^{\not<x}(B,B(x)(n))$ and $g\in\mathcal{C}^{\not<x}(B(x)(n),B)$ with $0\neq I(f,g)=$

$f \circ g$, we may assume $f \circ g = \mathrm{id}_{B(x)(n)}$ as $f \circ g \in \mathbb{K}^\times$, and hence $B(x)(n)$ would be a direct summand of B, absurd. Thus, I induces a perfect pairing $\bar{I}$:

$$\begin{array}{ccc} \mathcal{C}^{\not<x}(B(\underline{w}), B(x)(n)) \times \mathcal{C}^{\not<x}(B(x)(n), B(\underline{w})) & \xrightarrow{I} & \mathbb{K} \\ \downarrow & \overset{\bar{I}}{\nearrow} & \\ \mathcal{C}^{\not<x}(B(x)(n)^{\oplus m}, B(x)(n)) \times \mathcal{C}^{\not<x}(B(x)(n), B(x)(n)^{\oplus m}) & & \end{array}$$

with $\mathrm{rk}\,(I) = \mathrm{rk}\,(\bar{I}) = m$.

5.14. Keep the notation from (5.12). Recall that the $LL_{\underline{w},x,\mathbf{e}}$, $\underline{w}^{\mathbf{e}} = x$, are all defined over a complete noetherian local domain, a fortiori, over the prime fields $\mathbb{F}_p$ or $\mathbb{Q}$. Thus, $\mathrm{rk}\,(I_{\underline{w},x,n})$ depends only on $\mathrm{ch}(\mathbb{K})$. Also, if $p \gg 0$, $\mathrm{rk}\,(I_{\underline{w},x,n})$ over $\mathbb{F}_p$ coincides with the one over $\mathbb{Q}$ by (5.11). Let us $B_{\mathbb{K}}(x)$ denote the indecomposable $B(x)$ over $R = \mathrm{S}_{\mathbb{K}}(V)$ to emphasize the reference to $\mathbb{K}$. We have obtained

Proposition: *Let $x \in \mathcal{W}$.*

(i) If $\mathbb{K}$ is a field, $\mathrm{ch}(B_{\mathbb{K}}(x))$ depends only on $\mathrm{ch}(\mathbb{K})$.

(ii) If $p \gg 0$ depending on x, $\mathrm{ch}(B_{\mathbb{F}_p}(x)) = \mathrm{ch}(B_{\mathbb{Q}}(x))$.

6. Sheaves on moment graphs

Assume that $\mathbb{K}$ is a complete noetherian local domain, "local" to ensure the Quillen-Souslin.

6.1 Recall from [F08a], [F08b] an R-algebra, called the structure algebra of the moment graph associated to $(\mathcal{W}, \mathcal{S})$,

$$\mathcal{Z} = \{(z_w) \in \coprod_{d \in \mathbb{N}} \prod_{\mathcal{W}} R^d | z_{tw} \equiv z_w \mod \alpha_t \ \forall w \in \mathcal{W}\ \forall t \in T\}$$

with $a(z_w) = (az_w)\ \forall a \in R\ \forall (z_w) \in \mathcal{Z}$. Thus, $\mathcal{Z}$ is a graded R-algebra with $\mathcal{Z}^d \subseteq \prod_{\mathcal{W}} R^d$ [NvO, 1.2.3].

Fiebig [F08a] proved that the category of Soergel bimodules as in [S07] is equivalent to a certain full subcategory of $\mathcal{Z}$-modules if V is reflection faithful. We will give a version corresponding to our $\mathfrak{S}$Bimod.

Let M be a graded left $\mathcal{Z}$-module. Then M is equipped with a structure of left R by module via $R \hookrightarrow \mathcal{Z}$ via $a \mapsto a1_{\mathcal{Z}} = (a, \dots, a)$. One has also $(wa)_w \in \mathcal{Z}$. We define a right action of R by letting a act by $(wa)_w \in \mathcal{Z}$, which makes M into an R-bimodule:

$$ma = (wa)_w m \quad \forall m \in M. \tag{1}$$

To equip M with a structure of $\mathcal{C}'$, we need further some finiteness condition on M. $\forall I \subseteq \mathcal{W}$, put

$$\mathcal{Z}^I = \{(z_w) \in R^I | z_{tw} \equiv z_w \mod \alpha_t \ \forall w \in I\ \forall t \in T \text{ with } tw \in I\}.$$

If I is finite, one has from [F08b, 3.2]/[JGr, 2.7.1, 2.10]

$$(\mathcal{Z}^I)^Q = Q \otimes_R \mathcal{Z}^I \simeq Q^I = \prod_I Q. \tag{2}$$

For let $\mathcal{E} = \{(x,t) \in I \times \mathcal{T} | tx \in I, x < tx\}$. $\forall E = (x,t) \in \mathcal{E}$, put $\alpha_E = \alpha_t$ and let $\pi_{x,E}, \pi_{tx,E} : R \to R/(\alpha_E)$ be the quotients. Then

$$\mathcal{Z}^I \simeq \ker\{(\prod_{x\in I} R) \times (\prod_{E\in\mathcal{E}} R/(\alpha_E)) \to \prod_{x\in I, E\in\mathcal{E}} R/(\alpha_E)\} \quad \text{via} \quad ((a_x),(b_E)) \mapsto (\pi_{x,E}(a_x) - b_E),$$

and hence

$$\begin{aligned}(\mathcal{Z}^I)^Q &\simeq \ker\{Q \otimes_R \{(\prod_{x\in I} R) \times (\prod_{E\in\mathcal{E}} R/(\alpha_E))\} \to Q \otimes_R (\prod_{x\in I, E\in\mathcal{E}} R/(\alpha_E))\} \\ &\qquad\qquad \text{as } Q \text{ is flat over } R \\ &\simeq \ker\{\prod_{x\in I}(Q \otimes_R R) \times \prod_{E\in\mathcal{E}}(Q \otimes_R (R/(\alpha_E))\} \to \prod_{x\in I, E\in\mathcal{E}} (Q \otimes_R (R/(\alpha_E))\} \\ &\qquad\qquad \text{as } I \text{ is finite} \\ &\simeq \ker(Q^I \to 0) = Q^I.\end{aligned}$$

Let now $\mathcal{Z}\mathrm{Mod}^{\mathrm{f}}$ denote the full subcategory of graded left $\mathcal{Z}$-modules such that the action of $\mathcal{Z}$ factors through the projection $\mathcal{Z} \to \mathcal{Z}^I$ for some I finite $\subseteq \mathcal{W}$. In [F08a] the image of $\mathcal{Z}$ in $\mathcal{Z}^I$ is denoted $\mathcal{Z}^I$, and the natural map $\mathcal{Z} \to \mathcal{Z}^I$ may not be surjective. The present definition of $\mathcal{Z}\mathrm{Mod}^{\mathrm{f}}$ itself, however, remains the same as his. Let $M \in \mathcal{Z}\mathrm{Mod}^{\mathrm{f}}$ with the action of $\mathcal{Z}$ factoring through $\mathcal{Z}^I$, I finite. Then M^Q is a $(\mathcal{Z}^I)^Q$-module. As $(\mathcal{Z}^I)^Q \simeq Q^I$ by (2), $M^Q = \coprod_{x\in I} e_x M^Q$ with $e_x = (0,\dots,0,1,0,\dots,0)$, 1 at the x-th place. Put $M_x^Q = e_x M^Q$. Then

$$\begin{aligned} M_x^Q &= \{m \in M^Q | m = e_x m\} \\ &= \{m \in M^Q | zm = z_x m \ \forall z \in \mathcal{Z}^I\} \quad \text{with } z_x m = (z_x,\dots,z_x)m \text{ by definition} \\ &= \{m \in M^Q | zm = z_x m \ \forall z \in \mathcal{Z}\}. \end{aligned} \tag{3}$$

For let $m \in M^Q$ with $zm = z_x m \ \forall z \in \mathcal{Z}^I$. Write $e_x = \sum_i q_i \otimes z_i$, $z_i \in \mathcal{Z}^I, q_i \in Q$. Then

$$e_x m = (\sum_i q_i \otimes z_i)m = \sum_i q_i (z_i)_x m = m.$$

Thus, $\forall m \in M_x^Q$, $\forall a \in R$, $ma = (ya)_{y\in\mathcal{W}} m = (xa)m$, and hence M comes equipped with a structure of $\mathcal{C}'$, which is independent of the choice of finite I by the 3rd equality of (3). If $f \in \mathcal{Z}\mathrm{Mod}^{\mathrm{f}}(M,N)$, there is finite I the actions of $\mathcal{Z}$ on M and N both factor through $\mathcal{Z}^I$. $\forall m \in M_x^Q$, $x \in I$, one has $e_x f^Q(m) = f^Q(e_x m) = f^Q(m)$, and hence $f^Q(m) \in N_x^Q$. One thus obtains a faithful functor $F : \mathcal{Z}\mathrm{Mod}^{\mathrm{f}} \to \mathcal{C}'$.

Proposition: $\forall M, N \in \mathcal{Z}\mathrm{Mod}^{\mathrm{f}}$ *with N torsion-free as a left R-module,*

$$\mathcal{Z}\mathrm{Mod}^{\mathrm{f}}(M,N) \simeq \mathcal{C}'(F(M),F(N)).$$

Proof: Let $\phi \in \mathcal{C}'(F(M), F(N))$. Thus, $\forall w \in \mathcal{W}$,

$$\begin{array}{ccc} M^Q & \xrightarrow{\phi^Q} & N^Q \\ \uparrow & & \uparrow \\ M^Q_w & \underset{\phi_w}{\dashrightarrow} & N^Q_w. \end{array}$$

$\forall m \in M$, $\forall z = (z_w) \in \mathcal{Z}$, one has in N^Q_w

$$\begin{aligned} \phi(zm)_w &= \phi_w((zm)_w) = \phi_w((ze_w)m) = \phi_w(z_w e_w m) = z_w \phi_w(e_w m) = z_w \phi_w(m_w) = z_w \phi(m)_w \\ &= z\phi(m)_w \quad \text{by (3)} \\ &= ze_w\phi(m) = e_w z\phi(m) = (z\phi(m))_w. \end{aligned}$$

As $N \hookrightarrow N^Q = \coprod_{w \in \mathcal{W}} N^Q_w$ by the hypothesis, one obtains that $\phi(zm) = z\phi(m)$, and hence ϕ is $\mathcal{Z}$-linear.

6.2. $\forall s \in \mathcal{S}$, let $\mathcal{Z}^s = \{(z_w) \in \mathcal{Z} | z_{ws} = z_w \ \forall w \in \mathcal{W}\}$, which forms a subalgebra of $\mathcal{Z}$. We say that the GKM condition holds on V iff $\forall t, t' \in \mathcal{T}$ distinct, α_t and $\alpha_{t'}$ are linearly independent over $\mathbb{K}$.

Lemma: *Assume the GKM condition on V. Let $s \in \mathcal{S}$ and choose $\delta \in V$ with $\langle \delta, \alpha_s^\vee \rangle = 1$. Then $\mathcal{Z} = \mathcal{Z}^s \oplus (w\delta)_{w \in \mathcal{W}} \mathcal{Z}^s$.*

Proof: Let $z = (z_w) \in \mathcal{Z}$. $\forall w \in \mathcal{W}$, define $y_w \in R$ such that $z_w - z_{ws} = z_w - z_{wsw^{-1}w} = (w\alpha_s)y_w$. Let $t \in T \setminus \{wsw^{-1}\}$. Then

$$\begin{aligned} z_{tw} - z_{tws} &= (tw\alpha_s)y_{tw} \quad \text{by definition} \\ &\equiv (w\alpha_s)y_{tw} \quad \text{mod } \alpha_t \quad \text{by (1.1.iii)}, \end{aligned}$$

and hence modulo α_t

$$(w\alpha_s)(y_w - y_{tw}) \equiv (z_w - z_{ws}) - (z_{tw} - z_{tws}) = (z_w - z_{tw}) - (z_{tws} - z_{ws}) \equiv 0.$$

As $t \neq wsw^{-1}$, $w\alpha_s$ and α_t are linearly independent by (1.1.ii) and the GKM condition, and hence $y_w = y_{tw}$. On the other hand, if $t = wsw^{-1}$, $tw = ws$, and hence

$$(ws\alpha_s)y_w = -(w\alpha_s)y_w = z_{ws} - z_w = z_{tw} - z_{tws} = (tw\alpha_s)y_{tw} = (ws\alpha_s)y_{tw},$$

and hence $y_{ws} = y_{tw} = y_w$ again. Thus, $(y_w)_w \in \mathcal{Z}^s$.

Now put $y = (y_w)_w$ and $x = z - (w\delta)_{w \in \mathcal{W}} y \in \mathcal{Z}$. Then, $\forall s \in \mathcal{S}$,

$$\begin{aligned} x_{ws} &= z_{ws} - (ws\delta)y_{ws} = z_{ws} - (ws\delta)y_w \quad \text{as } y \in \mathcal{Z}^s \\ &= z_w - (w\alpha_s)y_w - w(\delta - \alpha_s)y_w = x_w, \end{aligned}$$

and hence $x \in \mathcal{Z}^s$ and $z = x + (w\delta)_{w \in \mathcal{W}} y$. Finally, assume $x + (w\delta)_w y = 0$ for $x, y \in \mathcal{Z}^s$. Then, $\forall w \in \mathcal{W}$, $\forall s \in \mathcal{S}$,

$$\begin{aligned} x_w + (w\delta)y_w = 0 &= x_{ws} + (ws\delta)y_{ws} \\ &= x_w + w(\delta - \alpha_s)y_w \quad \text{as } x, y \in \mathcal{Z}^s, \end{aligned}$$

and hence $(w\alpha_s)y_w = 0$. Then $y_w = 0$, and $y = 0$, hence also $x = 0$.

6.3. $\forall M \in \mathcal{Z}\mathrm{Mod}^{\mathrm{f}}$, $\forall s \in \mathcal{S}$, $\mathcal{Z} \otimes_{\mathcal{Z}^s} M$ remains in $\mathcal{Z}\mathrm{Mod}^{\mathrm{f}}$. For let $I \subseteq \mathcal{W}$ be a finite set and $\pi : \mathcal{Z} \to \mathcal{Z}^I$ be the natural map factoring through which $\mathcal{Z}$ acts on M. If $z \in \mathcal{Z}$, write $z = z_1 + (w\delta)_{w\in\mathcal{W}} z_2$ with $z_1, z_2 \in \mathcal{Z}^s$ by (6.2). Then z acts on $\mathcal{Z} \otimes_{\mathcal{Z}^s} M$ via

$$\pi(z_1) + (w\delta)_{w\in\mathcal{W}}\pi(z_2) = \pi(z_1) + (w\delta)_{w\in I}\pi(z_2) = \pi(z_1 + (w\delta)_{w\in\mathcal{W}} z_2) = \pi(z),$$

and hence $\mathcal{Z}$ acts on $\mathcal{Z} \otimes_{\mathcal{Z}^s} M$ through π.

Proposition: *Assume that the GKM condition holds on V. $\forall M \in \mathcal{Z}\mathrm{Mod}^{\mathrm{f}}$, $\forall s \in \mathcal{S}$,*

$$F(\mathcal{Z} \otimes_{\mathcal{Z}^s} M) \simeq F(M) * B(s)(-1).$$

Proof: Take $\delta \in V$ with $\langle \delta, \alpha_s^\vee \rangle = 1$. Define a map $\phi : F(M) * B(s)(-1) \to \mathcal{Z} \otimes_{\mathcal{Z}^s} M$ via $F(M) \otimes_R B(s)(-1) = M \otimes_{R^s} R \ni m \otimes a \mapsto (wa)_{w\in\mathcal{W}} \otimes m$; if $b \in R^s$, $wb = wsb$ $\forall w \in \mathcal{W}$, and hence $(wb)_{w\in\mathcal{W}} \in \mathcal{Z}^s$. Then $(w(ab))_w \otimes m = (wa)_w \otimes (wb)_w m = (wa)_w \otimes mb$ by definition (6.1.1), and hence ϕ is well-defined. Also, $\forall b \in R$,

$$\begin{aligned}(w(ab))_{w\in\mathcal{W}} \otimes m = (wb)_{w\in\mathcal{W}}(wa)_{w\in\mathcal{W}} \otimes m &= (wb)_{w\in\mathcal{W}}\{(wa)_{w\in\mathcal{W}} \otimes m\} \\ &= \{(wa)_{w\in\mathcal{W}} \otimes m\}b \quad \text{by definition (6.1.1) again.}\end{aligned}$$

Thus, ϕ is a homomorphism of graded R-bimodules. Moreover, $F(M)*B(s)(-1) = \{F(M)\otimes_{R^s} R^s\} \oplus \{F(M) \otimes_{R^s} \delta R^s\}$ while

$$\begin{aligned}\mathcal{Z} \otimes_{\mathcal{Z}^s} M &= \{\mathcal{Z}^s \oplus (w\delta)_w \mathcal{Z}^s\} \otimes_{\mathcal{Z}^s} M \quad \text{by (6.2)} \\ &\simeq \{\mathcal{Z}^s \otimes_{\mathcal{Z}^s} M\} \oplus \{(w\delta)_w \mathcal{Z}^s \otimes_{\mathcal{Z}^s} M\}.\end{aligned}$$

and hence ϕ is bijective.

Finally, we show that $\phi^Q((F(M) * B(s)(-1))^Q_w) \subseteq (\mathcal{Z} \otimes_{\mathcal{Z}^s} M)^Q_w$ $\forall w \in I$. By (2.3.iii) any element of $(F(M)*B(s)(-1))^Q_w$ is of the form $m\otimes\delta - m(s\delta)\otimes 1 + m'\otimes\delta - m'\delta\otimes 1$, $m \in M^Q_w, m' \in M^Q_{ws}$. Thus, we are to check that $(w\delta)_w \otimes m - 1\otimes m(s\delta) + (w\delta)_w \otimes m' - 1 \otimes m'\delta \in (\mathcal{Z}\otimes_{\mathcal{Z}^s} M)^Q_w$. For that by (6.1.3) it is enough to show that, $\forall z \in \mathcal{Z}$,

$$z\{(w\delta)_w\otimes m - 1\otimes m(s\delta) + (w\delta)_w\otimes m' - 1\otimes m\delta\} = z_w\{(w\delta)_w\otimes m - 1\otimes m(s\delta) + (w\delta)_w\otimes m' - 1\otimes m'\delta\}.$$

If $z \in \mathcal{Z}^s$,

$$\begin{aligned}\mathrm{LHS} &= (w\delta)_w \otimes zm - 1 \otimes zm(s\delta) + (w\delta)_w \otimes zm' - 1 \otimes zm'\delta \\ &= (w\delta)_w \otimes z_w m - 1 \otimes z_w m(s\delta) + (w\delta)_w \otimes z_{ws} m' - 1 \otimes z_{ws} m'\delta \quad \text{by (6.1.3) on } M \\ &= (w\delta)_w \otimes z_w m - 1 \otimes z_w m(s\delta) + (w\delta)_w \otimes z_w m' - 1 \otimes z_w m'\delta \quad \text{as } z \in \mathcal{Z}^s \\ &= z_w\{(w\delta)_w \otimes m - 1 \otimes m(s\delta) + (w\delta)_w \otimes m' - 1 \otimes m'\delta\} \quad \text{as } z_w \text{ reads } (z_w, \dots, z_w) \in \mathcal{Z}^s.\end{aligned}$$

Also,

$$\begin{aligned}
&(x\delta)_x\{(w\delta)_w\otimes m-1\otimes m(s\delta)+(w\delta)_w\otimes m'-1\otimes m\delta\}\\
&\quad=\{(w\delta)_w\otimes m-1\otimes m(s\delta)+(w\delta)_w\otimes m'-1\otimes m\delta\}\delta \quad \text{by definition (6.1.3)}\\
&\quad=\phi(m\otimes\delta-m(s\delta)\otimes 1+m'\otimes\delta-m'\delta\otimes 1)\delta\\
&\quad=\phi((m\otimes\delta-m(s\delta)\otimes 1+m'\otimes\delta-m'\delta\otimes 1)\delta) \quad \text{as } \phi \text{ is right } R\text{-linear}\\
&\quad=\phi(m\otimes\delta^2-m(s\delta)\otimes\delta+m'\otimes\delta^2-m'\delta\otimes\delta)\\
&\quad=\phi(m\otimes(\delta(\delta+s\delta)-\delta s\delta)-m(s\delta)\otimes\delta+m'\otimes(\delta(\delta+s\delta)-\delta s\delta)-m'\delta\otimes\delta)\\
&\quad=\phi(m(\delta+s\delta)\otimes\delta-m\delta s\delta\otimes 1-m(s\delta)\otimes\delta+m'(\delta+s\delta)\otimes\delta-m'\delta s\delta\otimes 1-m'\delta\otimes\delta)\\
&\quad=\phi(m\delta\otimes\delta-w(\delta s\delta)m\otimes 1+m's\delta\otimes\delta-(ws)(\delta s\delta)m'\otimes 1)\\
&\quad=\phi((w\delta)m\otimes\delta-(w\delta)(ws\delta)m\otimes 1+(ws)(s\delta)m'\otimes\delta-(ws)(\delta)(w\delta)m'\otimes 1)\\
&\quad=\phi((w\delta)(m\otimes\delta-(ws\delta)m\otimes 1+m'\otimes\delta-(ws\delta)m'\otimes 1))\\
&\quad=(w\delta)\phi(m\otimes\delta-(ws\delta)m\otimes 1+m'\otimes\delta-(ws\delta)m'\otimes 1) \quad \text{as } \phi \text{ is left } R\text{-linear}\\
&\quad=(w\delta)\phi(m\otimes\delta-ms\delta\otimes 1+m'\otimes\delta-m'\delta\otimes 1)\\
&\quad=(w\delta)\{(w\delta)_w\otimes m-1\otimes m(s\delta)+(w\delta)_w\otimes m'-1\otimes m'\delta\},
\end{aligned}$$

as desired.

6.4. Define a structure of graded $\mathcal{Z}$-module on R via $za=z_e a\ \forall a\in R\ \forall z\in\mathcal{Z}$ with z_e denoting the e-th component of z, which we will denote by $R_{\mathcal{Z}}$. Thus, $R_{\mathcal{Z}}\in\mathcal{Z}\mathrm{Mod}^{\mathrm{f}}$ with $F(R_{\mathcal{Z}})\simeq R(e)$. $\forall s\in\mathcal{S}$, (6.3) yields

$$F(\mathcal{Z}\otimes_{\mathcal{Z}^s}R_{\mathcal{Z}})\simeq F(R_{\mathcal{Z}})*B(s)(-1)\simeq R(e)*B(s)(-1)\simeq B(s)(-1),$$

and hence

$$F(\mathcal{Z}\otimes_{\mathcal{Z}^s}R_{\mathcal{Z}}(1))\simeq B(s). \tag{1}$$

Let $\mathcal{Z}\mathrm{Mod}^{\mathfrak{S}}$ denote the full subcategory of $\mathcal{Z}\mathrm{Mod}^{\mathrm{f}}$ consisting of the direct summands of direct sums of $\mathcal{Z}\otimes_{\mathcal{Z}^{s_1}}\cdots\otimes_{\mathcal{Z}^{s_{r-1}}}\mathcal{Z}\otimes_{\mathcal{Z}^{s_r}}R_{\mathcal{Z}}(n)$, $n\in\mathbb{Z}$, $s_1,\dots,s_r\in\mathcal{S}$. As an element of $\mathcal{Z}\mathrm{Mod}^{\mathfrak{S}}$ is torsion free over R by (6.2), from (1) one obtains

Theorem: *If the GKM condition holds on V, F induces an equivalence $\mathcal{Z}\mathrm{Mod}^{\mathfrak{S}}\to\mathfrak{S}\mathrm{Bimod}$.*

7. Deformation of Schubert calculus [S92]

Soergel bimodules were originally thought of as the algebras of regular functions of some subvarieties of $V^*\times V^*$ over $\mathbb{C}$ with V^* denoting the complexification of the geometric representation of $\mathcal{W}$ [S92]. Thus, V is the $\mathbb{C}$-linear dual of the V^*; in [S92] the present V is denoted V^*. In this section we will verify that Soergel's results carry over to our set-up in case $\mathcal{W}$ is the Weyl group of a root system Δ and V denoting a weight lattice of Δ under the base change to $\mathbb{K}$. We will assume, unless otherwise specified, that $\mathbb{K}$ is an infinite field and that, in order for Demazure's result [Dem] holds, the characteristic of $\mathbb{K}$ is not a torsion prime of Δ and the weight lattice, cf. [JMW, 2.6]. In the simply-connected simple cases the torsion primes are

A_n, C_n	B_n ($n \geq 3$), D_n, G_2	E_6, E_7, F_4	E_8
none	2	2, 3	2,3,5

In addition, we assume that $2 \neq 0$ in $\mathbb{K}$ and also that $3 \neq 0$ if G_2 is involved as a component. Thus, the GKM condition holds on V^*.

7.1. Under the standing assumptions on $\mathbb{K}$ two distinct coroots remain distinct in V^*, and hence

Lemma: *The representation V^* of $\mathcal{W}$ is faithful.*

Proof: Let $w \in \mathcal{W}$ be trivial on V^*. Then w fixes every coroot in $\Delta^\vee$, and hence $w = e$ [HLA, 10.3].

7.2. Throughout the rest of §7 we will consider the $\mathcal{W}$-action on $V^* \times V^*$, acting only on the 2nd component. We will regard $R \otimes_{\mathbb{K}} R$ as the set of rational functions on $V^* \times V^*$ with induced $\mathcal{W}$-action: $\forall f \in R \otimes_{\mathbb{K}} R$, $\forall w \in \mathcal{W}$, $\forall (\nu, \mu) \in V^* \times V^*$, $(wf)(\nu, \mu) = f(\nu, w^{-1}\mu)$.

$\forall s \in \mathcal{S}$, define a twisted derivation $\partial_s : R \to R$ via $f \mapsto \frac{f - sf}{2\alpha_s}$, $f \in R$, which is unfortunately distinct from ∂_s introduced in (4.1) by a factor of $\frac{1}{2}$. Thus, $\forall g \in R$,

$$\partial_s(fg) = (\partial_s f)g + (sf)\partial_s g. \tag{1}$$

For $\mathfrak{X} \subseteq V^* \times V^*$ let $I(\mathfrak{X}) = \mathrm{Ann}(\mathfrak{X}) = \{f \in R \otimes_{\mathbb{K}} R | f|_{\mathfrak{X}} = 0\}$ and put $R(\mathfrak{X}) = (R \otimes_{\mathbb{K}} R)/I(\mathfrak{X})$. If $\mathfrak{X}$ is a closed s-stable subset of $V^* \times V^*$, $\forall f \in I(\mathfrak{X})$,$\forall x \in \mathfrak{X}$, $(sf)(x) = f(sx) = 0$, and hence s acts on $R(\mathfrak{X})$. If, moreover, no irreducible component of $\mathfrak{X}$ lies in $(V^* \times V^*)^s = \{(\nu, \mu) \in V^* \times V^* | s\mu = \mu\}$, ∂_s acts on $R(\mathfrak{X})$. For let $f \in I(\mathfrak{X})$. It is enough to show that $\partial_s f \in I(\mathfrak{X}')$ for each irreducible component $\mathfrak{X}'$ of $\mathfrak{X}$. One has

$$(\partial_s f)|_{\mathfrak{X}'}(2\alpha_s)|_{\mathfrak{X}'} = (f - sf)|_{\mathfrak{X}'} = -(sf)|_{\mathfrak{X}'} = 0.$$

Just suppose $(2\alpha_s)|_{\mathfrak{X}'} = 0$. Thus, $\forall (\nu, \mu) \in \mathfrak{X}'$, $0 = (2\alpha_s)(\nu, \mu) = (2\alpha_s)(\mu)$. As $\mathfrak{X}' \not\subseteq (V^* \times V^*)^s$, however, there is $(\nu, \mu) \in \mathfrak{X}'$ with $\mu \neq s\mu = \mu - \alpha_s(\mu)\alpha_s^\vee$, and hence $\alpha_s(\mu) \neq 0$, absurd. Then $2\alpha_s \neq 0$ in $R(\mathfrak{X}')$, and hence $(\partial_s f)|_{\mathfrak{X}'} \neq 0$ as $R(\mathfrak{X}')$ is a domain.

Note also that ∂_s on $R(\mathfrak{X})$ is left R-linear as

$$\partial_s(a \otimes b) = a \otimes \partial_s b \quad \forall a, b \in R. \tag{2}$$

$\forall w \in \mathcal{W}$, put $\mathfrak{X}_w = \{(\nu, w^{-1}\nu) \in V^* \times V^* | \nu \in V^*\}$. Thus, $\forall y \in \mathcal{W}$, $y^{-1}\mathfrak{X}_w = \mathfrak{X}_{wy}$. One has

$$R(\mathfrak{X}_w) = (R \otimes_{\mathbb{K}} R)/(a \otimes 1 - 1 \otimes w^{-1}a | a \in R) \simeq R \quad \text{via} \quad a \otimes b \mapsto a(wb) \quad \text{with inverse } a \otimes 1 \leftarrow\!\shortmid a, \tag{3}$$

under which $R(\mathfrak{X}_w)$ comes equipped with a structure of $\mathcal{C}$ such that

$$R(\mathfrak{X}_w) \simeq R(w). \tag{4}$$

This is the reason why we defined $\mathfrak{X}_w$ in the present form rather than the one in [S92].

If $A \subseteq \mathcal{W}$ is right s-stable, i.e., $As = A$, $\mathfrak{X}_A = \cup_{w \in A} \mathfrak{X}_w$ is closed and s-stable in $V^* \times V^*$. As $\mathcal{W}$ is faithful on V^*, $\mathfrak{X}_w \not\subseteq (V^* \times V^*)^s$ $\forall w \in \mathcal{W}$. Thus, for any right s-stable $A \subseteq \mathcal{W}$ one may consider the action of s and ∂_s on $R(\mathfrak{X}_A)$.

7.3. Let $R^{\mathcal{W}}$ be the set of $\mathcal{W}$-invariants of R. $\forall a \in R^{\mathcal{W}}$, $\forall f \in R(\mathfrak{X}_{\mathcal{W}})$, $w \in \mathcal{W}$, $\forall \nu \in V^*$,

$$(fa)(\nu, w^{-1}\nu) = a(w^{-1}\nu) f(\nu, w^{-1}\nu) = a(\nu) f(\nu, w^{-1}\nu) = (af)(\nu, w^{-1}\nu),$$

and hence the right and the left actions of $R^{\mathcal{W}}$ on $R(\mathfrak{X}_{\mathcal{W}})$ coincide. Thus,

$$R \otimes_{R^{\mathcal{W}}} R \twoheadleftarrow R \otimes_{\mathbb{K}} R$$

$$R \otimes_{R^{\mathcal{W}}} R \dashrightarrow R(\mathfrak{X}_{\mathcal{W}}) \twoheadleftarrow R \otimes_{\mathbb{K}} R$$

$$R(\mathfrak{X}_{\mathcal{W}}) = (R \otimes_{\mathbb{K}} R) / \cap_{w \in \mathcal{W}} (a \otimes 1 - 1 \otimes w^{-1}a | a \in R).$$

Lemma: *(i) There is an isomorphism of graded* $\mathbb{K}$*-algebras* $R \otimes_{R^{\mathcal{W}}} R \to R(\mathfrak{X}_{\mathcal{W}})$.

(ii) $R(\mathfrak{X}_{\mathcal{W}}) \in \mathcal{C}$ *with* $R(\mathfrak{X}_{\mathcal{W}})^Q = \prod_{\mathcal{W}} Q$ *such that* $R(\mathfrak{X}_{\mathcal{W}}) \ni f \mapsto (f_w)_{w \in \mathcal{W}} \in \prod_{\mathcal{W}} Q$ *with* $f_w = f|_{\mathfrak{X}_w} \in R(\mathfrak{X}_w) \simeq R(w)$.

Proof: Let $K = \ker(R \otimes_{R^{\mathcal{W}}} R \to R(\mathfrak{X}_{\mathcal{W}}))$. There is an exact sequence

$$0 \to Q \otimes_R K \to Q \otimes_R R \otimes_{R^{\mathcal{W}}} R \to Q \otimes_R R(\mathfrak{X}_{\mathcal{W}}) \to 0$$

with

$$\begin{aligned} Q \otimes_R R \otimes_{R^{\mathcal{W}}} R &\simeq Q \otimes_R R \otimes_{R^{\mathcal{W}}} (R^{\mathcal{W}})^{\oplus |\mathcal{W}|} \quad \text{by [Dem]} \\ &\simeq Q^{|\mathcal{W}|} \end{aligned}$$

while

$$Q \otimes_R R(\mathfrak{X}_{\mathcal{W}}) \leq Q \otimes_R \prod_{w \in \mathcal{W}} (R \otimes_{\mathbb{K}} R)/(a \otimes 1 - 1 \otimes w^{-1}a | a \in R) \simeq \prod_{\mathcal{W}} Q.$$

As $\mathcal{W}$ is faithful on V, $\ker(w - \mathrm{id}_V) < V$ $\forall w \in \mathcal{W} \setminus \{e\}$, and hence $\cup_{w \in \mathcal{W} \setminus \{e\}} \ker(w - \mathrm{id}_V) \subset V$. Take $\gamma \in V \setminus \cup_{w \in \mathcal{W} \setminus \{e\}} \ker(w - \mathrm{id}_V)$, and hence $w\gamma \neq \gamma$ $\forall w \neq e$. $\forall x \in \mathcal{W}$, define $f^x \in R \otimes_{\mathbb{K}} R$ via $f^x(\nu, \mu) = \prod_{y \in \mathcal{W} \setminus \{x\}} \{\gamma(\nu) - \gamma(y\mu)\}$ $\forall (\nu, \mu) \in V^* \times V^*$. Then $f^x = 0$ on $\mathfrak{X}_y$ $\forall y \in \mathcal{W} \setminus \{x\}$ while $f^x|_{\mathfrak{X}_x} \neq 0$, and hence $f^x \neq 0$ in $R(\mathfrak{X}_x) \simeq R$. Then $\forall (q_x)_x \in Q^{|\mathcal{W}|}$,

$$\sum_{w \in \mathcal{W}} \frac{q_w}{f^w|_{\mathfrak{X}_w}} \otimes f^w = (q_x)_{x \in \mathcal{W}} \quad \text{in } Q \otimes_R \prod_{x \in \mathcal{W}} R(\mathfrak{X}_x) \simeq \prod_{\mathcal{W}} Q,$$

and hence $Q \otimes_R R(\mathfrak{X}_{\mathcal{W}}) \simeq \prod_{\mathcal{W}} Q$. It follows that $Q \otimes_R K = 0$. As $R \otimes_{R^{\mathcal{W}}} R \simeq R^{\oplus |\mathcal{W}|}$ is torsion-free over R, so is K, and hence $K = 0$.

7.4. $\forall w \in \mathcal{W}$, let $(s_1, \ldots, s_r) \in \mathcal{S}^r$ be a reduced expression of w, and set $\partial_w = \partial_{s_1} \ldots \partial_{s_r} : R \to R$, which is independent of the choice of the reduced expression [Dem, Th. 1, p. 291]. Put $\partial_e = \mathrm{id}_R$. By (7.2) the ∂_w's act on $R(\mathfrak{X}_{\mathcal{W}})$, which are left R-linear as they act only on the 2nd component. Let w_0 be the longest element of $\mathcal{W}$.

Lemma: *$\forall f \in R(\mathfrak{X}_{\mathcal{W}})$, $\forall y \in \mathcal{W}$, $\partial_{w_0} f = (\partial_{w_0} f)_y \otimes 1$ in $R(\mathfrak{X}_{\mathcal{W}})$ regarding $(\partial_{w_0} f)_y \in R(y)$ such that $(\partial_{w_0} f)_y(\nu) = (\partial_{w_0} f)(\nu, y^{-1}\nu)$ $\forall \nu \in V^*$.*

Proof: Let $f \in R \otimes_{\mathbb{K}} R$. Writing $f = \sum_i a_i \otimes b_i$, $\forall s \in \mathcal{S}$,

$$\begin{aligned} s\partial_s f &= \sum_i f_i \otimes s\partial_s b_i \\ &= \sum_i f_i \otimes \partial_s b_i \quad \text{as } s\partial_s b_i = \frac{sb_i - b_i}{-2\alpha_s} = \partial_s b_i \\ &= \partial_s f, \end{aligned}$$

and hence, by the independence of the choice of the reduced expression of w_0, $\partial_{w_0} f \in (R \otimes_{\mathbb{K}} R)^{\mathcal{W}}$. Then, writing $\partial_{w_0} f = \sum_i a'_i \otimes b'_i$ with the a'_i $\mathbb{K}$-linearly independent, we see that $b'_i \in R^{\mathcal{W}}$ $\forall i$. Thus, $\partial_{w_0} f = \sum_i a'_i b'_i \otimes 1$ in $R(\mathfrak{X}_{\mathcal{W}})$ under

$$\begin{array}{ccc} R \otimes_{\mathbb{K}} R & \twoheadrightarrow & R(\mathfrak{X}_{\mathcal{W}}). \\ \downarrow & \nearrow & \\ R \otimes_{R^{\mathcal{W}}} R & & \end{array}$$

$\forall \nu \in V^*$, $\forall y, w \in \mathcal{W}$,

$$(\partial_{w_0} f)(\nu, w^{-1}\nu) = \sum_i (a'_i b'_i)(\nu) = (\partial_{w_0} f)(\nu, y^{-1}\nu) = (\partial_{w_0} f)_y(\nu) = ((\partial_{w_0} f)_y \otimes 1)(\nu, w^{-1}\nu).$$

7.5 **Lemma:** *$\forall I \trianglelefteq R(\mathfrak{X}_{\mathcal{W}})$, $\forall s \in \mathcal{S}$, $I + \partial_s I \trianglelefteq R(\mathfrak{X}_{\mathcal{W}})$.*

Proof: As ∂_s is left R-linear (7.2.2), it is enough to check that $\forall a \in R$, $\forall f \in I$, $(\partial_s f)a = (\partial_s f)(1 \otimes a) \in I + \partial_s I$. One has

$$\partial_s I \ni \partial_s(fa) = \partial_s(f(1 \otimes a)) = (\partial_s f)(1 \otimes a) + f\partial_s(1 \otimes a) \quad \text{by (7.2.1).}$$

As $f\partial_s(1 \otimes a) \in I$, $(\partial_s f)(1 \otimes a) \in I + \partial_s I$.

7.6. Choose $\hat{f} \in R(\mathfrak{X}_{\mathcal{W}})^d \setminus 0$ for some $d \in \mathbb{N}$ with $\hat{f}|_{\mathfrak{X}_w} = 0$ $\forall w \in \mathcal{W} \setminus \{w_0\}$; $\mathfrak{X}_{w_0} \not\subseteq \cup_{w \in \mathcal{W} \setminus \{w_0\}} \mathfrak{X}_w$ by the irreducibility of $\mathfrak{X}_{w_0} \simeq V^*$, and hence such $\hat{f}$ is available. Then $\hat{f} \in R(\mathfrak{X}_{\mathcal{W}})^d_{w_0} \setminus 0$ by (7.3.ii).

Lemma: *(i) $\forall y \in \mathcal{W}$, $(\partial_y \hat{f})|_{\mathfrak{X}_{w_0 y^{-1}}} \neq 0$. In particular, $d \geq 2\ell(w_0)$.*

(ii) $\forall w \in \mathcal{W}$ with $(\partial_y \hat{f})|_{\mathfrak{X}_{w_0 w^{-1}}} \neq 0$, $w \leq y$.

Proof: Put for simplicity $\mathfrak{X}'_w = \mathfrak{X}_{w^{-1}}$. One first check that, $\forall g \in R(\mathfrak{X}_{\mathcal{W}})$,

(1) if $g|_{\mathfrak{X}'_w} = 0$ and $g|_{\mathfrak{X}'_{sw}} = 0$, $(\partial_s g)|_{\mathfrak{X}'_w} = 0$ and $(\partial_s g)|_{\mathfrak{X}'_{sw}} = 0$,

(2) if $g|_{\mathfrak{X}'_w} = 0$ but $g|_{\mathfrak{X}'_{sw}} \neq 0$, $(\partial_s g)|_{\mathfrak{X}'_w} \neq 0$ and $(\partial_s g)|_{\mathfrak{X}'_{sw}} \neq 0$.

To see (1), just suppose that $(\partial_s g)|_{\mathfrak{X}'_w} \neq 0$. There is $\nu \in V^*$ with $(\partial_s g)(\nu, w\nu) \neq 0$ but $2\alpha_s(w\nu) = 0$. Then

$$(\partial_s g)(\nu, w\nu) = \frac{g(\nu, w\nu) - (sg)(\nu, w\nu)}{2\alpha_s(w\nu)} = \frac{0 - g(\nu, sw\nu)}{2\alpha_s(w\nu)} = 0,$$

absurd. Likewise, $(\partial_s g)|_{\mathfrak{X}'_{sw}} = 0$. For (2), if $g(\nu, sw\nu) \neq 0$,

$$(2\alpha_s)(w\nu)(\partial_s g)(\nu, w\nu) = g(\nu, w\nu) - (sg)(\nu, w\nu) = -g(\nu, sw\nu) \neq 0,$$
$$(2\alpha_s)(sw\nu)(\partial_s g)(\nu, sw\nu) = g(\nu, sw\nu) - (sg)(\nu, sw\nu) = g(\nu, sw\nu) \neq 0.$$

We now argue by induction on $y \in \mathcal{W}$. If $y = e$, the assertions hold as $\hat{f}|_{\mathfrak{X}'_{w_0}} \neq 0$ by the choice of $\hat{f}$. If $y > e$, write $y = sx > x$ for some $s \in \mathcal{S}$. By the induction hypothesis $(\partial_x \hat{f})|_{\mathfrak{X}'_{xw_0}} \neq 0$ while $(\partial_x \hat{f})|_{\mathfrak{X}'_{sxw_0}} = 0$. Then $(\partial_y \hat{f})|_{\mathfrak{X}'_{yw_0}} = \partial_s(\partial_x \hat{f})|_{\mathfrak{X}'_{sxw_0}} \neq 0$ by (2), and hence (i). Assume next that $(\partial_y \hat{f})|_{\mathfrak{X}'_{ww_0}} \neq 0$. Then $\partial_s(\partial_x \hat{f})|_{\mathfrak{X}'_{ww_0}} \neq 0$, and hence by (1) either $(\partial_x \hat{f})|_{\mathfrak{X}'_{ww_0}} \neq 0$ or $(\partial_x \hat{f})|_{\mathfrak{X}'_{sww_0}} \neq 0$. If the former, $w \leq x < y$ by the induction hypothesis. If the latter, $sw \leq x$ by the induction hypothesis, and hence $w \leq y$, as desired.

7.7. Keep the notation of (7.6). As $\hat{f} \in R(\mathfrak{X}_{\mathcal{W}})_{w_0}$, one has

(1) $$R\hat{f} = \hat{f}R \lhd R(\mathfrak{X}_{\mathcal{W}}).$$

Then, $\forall s \in \mathcal{S}$, $R\hat{f} + R\partial_s \hat{f} = R\hat{f} + \partial_s(R\hat{f}) \unlhd R(\mathfrak{X}_{\mathcal{W}})$ by (7.5). Assume now that $\sum_{x<w} \partial_x(R\hat{f}) \unlhd R(\mathfrak{X}_{\mathcal{W}})$ and write $w = sy > y$. $\forall a \in R$,

$$\begin{aligned}(\partial_w \hat{f})a &= (\partial_s \partial_y \hat{f})a \\ &\in \sum_{x \leq y} \partial_x(R\hat{f}) + \partial_s \sum_{x \leq y} \partial_x(R\hat{f}) \quad \text{by the hypothesis and by (7.5) again} \\ &= \sum_{x \leq y} \partial_x(R\hat{f}) + \sum_{x \leq y} \partial_s \partial_x(R\hat{f}) = \sum_{x \leq w} \partial_x(R\hat{f}),\end{aligned}$$

and hence by (7.2.2) and by induction one obtains that

(2) $$\sum_{w \in \mathcal{W}} R(\partial_w \hat{f}) = \sum_{w \in \mathcal{W}} \partial_w(R\hat{f}) \unlhd R(\mathfrak{X}_{\mathcal{W}}).$$

By (7.6.i) one has $(\partial_{w_0} \hat{f})_e \neq 0$. As $\partial_{w_0} \hat{f} = (\partial_{w_0} \hat{f})_e \otimes 1$ by (7.4) and as $\sum_{w \in \mathcal{W}} R\partial_w \hat{f} \unlhd R(\mathfrak{X}_{\mathcal{W}})$ by (2),

$$(\partial_{w_0} \hat{f})_e R(\mathfrak{X}_{\mathcal{W}}) = (\partial_{w_0} \hat{f})_e (1 \otimes 1) R(\mathfrak{X}_{\mathcal{W}}) = ((\partial_{w_0} \hat{f})_e \otimes 1) R(\mathfrak{X}_{\mathcal{W}}) = (\partial_{w_0} \hat{f}) R(\mathfrak{X}_{\mathcal{W}}) \subseteq \sum_{w \in \mathcal{W}} R\partial_w \hat{f}.$$

Lemma: $\sum_{w \in \mathcal{W}} R\partial_w \hat{f} = (\partial_{w_0} \hat{f})_e R(\mathfrak{X}_{\mathcal{W}})$ *with the* $\partial_w \hat{f}$, $w \in \mathcal{W}$, *left* R*-linearly independent.*

Proof: Let first $\sum_{w \in \mathcal{W}} a_w \partial_w \hat{f} = 0$, $a_w \in R$. Then on $\mathfrak{X}'_e = \mathfrak{X}'_{w_0 w_0}$

$$0 = \sum_{w \in \mathcal{W}} (a_w \partial_w \hat{f})|_{\mathfrak{X}'_e} = (a_{w_0} \partial_{w_0} \hat{f})|_{\mathfrak{X}'_{w_0 w_0}} \quad \text{with } (\partial_{w_0} \hat{f})|_{\mathfrak{X}'_{w_0 w_0}} \neq 0 \text{ by (7.6)},$$

and hence $a_{w_0} \neq 0$ as $R(\mathfrak{X}'_e) \simeq R$ is a domain. If $s \in \mathcal{S}$,

$$0 = \sum_{w < w_0} (a_w \partial_w \hat{f})|_{\mathfrak{X}'_s} = (a_{sw_0} \partial_{w_0} \hat{f})|_{\mathfrak{X}'_{sw_0w_0}} \quad \text{with } (\partial_{sw_0} \hat{f})|_{\mathfrak{X}'_{sw_0w_0}} \neq 0,$$

and hence $a_{sw_0} = 0$. Likewise, by descending induction on w, get all $a_w = 0$. Thus, the $\partial_w \hat{f}$ are left R-linearly independent, and hence $(\partial_{w_0} \hat{f})_e R(\mathfrak{X}_{\mathcal{W}}) \subseteq \coprod_{w \in \mathcal{W}} R \partial_w \hat{f}$.

Recall from [Dem] that, letting $R^{\mathcal{W}}$ denote the $\mathcal{W}$-invariants of R, R has an $R^{\mathcal{W}}$-linear basis $(u_w | w \in \mathcal{W})$ with $\deg(u_w) = 2\ell(w)$ $\forall w \in \mathcal{W}$. Thus, the $1 \otimes u_w$, $w \in \mathcal{W}$, form a left R-linear basis of $R(\mathfrak{X}_{\mathcal{W}}) \simeq R \otimes_{R^{\mathcal{W}}} R$ by (7.2). Then by counting the dimension of both sides in each degree one obtains that $(\partial_{w_0} \hat{f})_e R(\mathfrak{X}_{\mathcal{W}}) = \coprod_{w \in \mathcal{W}} R \partial_w \hat{f}$.

7.8. Let $\hat{f} \in R(\mathfrak{X}_{\mathcal{W}})^d_{w_0}$ as before. By (7.7) there is $\phi \in R(\mathfrak{X}_{\mathcal{W}})$ such that $\hat{f} = (\partial_{w_0} \hat{f})_e \phi$. Then $\deg(\phi) = 2\ell(w_0)$ and $\phi \in R(\mathfrak{X}_{\mathcal{W}})_{w_0}$. Thus,

Proposition: *(i)* $\phi \in R(\mathfrak{X}_{\mathcal{W}})^{2\ell(w_0)}_{w_0} \setminus 0$.

(ii) $(\partial_w \phi | w \in \mathcal{W})$ *forms a left* R*-linear basis of* $R(\mathfrak{X}_{\mathcal{W}})$.

Proof: (ii) As $\partial_{w_0} \phi \neq 0$ by (7.6) of degree 0, $\partial_{w_0} \phi \in \mathbb{K}^\times$. Then by (7.7)

$$(\partial_{w_0} \hat{f})_e R(\mathfrak{X}_{\mathcal{W}}) = \sum_{w \in \mathcal{W}} R \partial_w ((\partial_{w_0} \hat{f})_e \phi) = (\partial_{w_0} \hat{f})_e \sum_{w \in \mathcal{W}} R \partial_w \phi.$$

As $R(\mathfrak{X}_{\mathcal{W}})$ is left R-free by [Dem], we must have

$$R(\mathfrak{X}_{\mathcal{W}}) = \sum_{w \in \mathcal{W}} R \partial_w \phi = \coprod_{w \in \mathcal{W}} R \partial_w \phi.$$

7.9. K the notation of (7.8).

Corollary: *(i)* $\forall w \in \mathcal{W}$, $\partial_{w^{-1}w_0} \phi \in R(\mathfrak{X}_{\mathcal{W}})^{2\ell(w)}_{\geq w}$ *with* $(\partial_{w^{-1}w_0} \phi)_w \neq 0$, *i.e.,* $\partial_{w^{-1}w_0} \phi \in R(\mathfrak{X}_{\mathcal{W}})^{2\ell(w)}$, $(\partial_{w^{-1}w_0} \phi)|_{\mathfrak{X}_w} \neq 0$, *and* $\forall y \in \mathcal{W}$ *with* $(\partial_{w^{-1}w_0} \phi)|_{\mathfrak{X}_y} \neq 0$, $y \geq w$.

(ii) $\forall w \in \mathcal{W}$,

$$R(\mathfrak{X}_{\mathcal{W}})_{\geq w} = \coprod_{\substack{y \in \mathcal{W} \\ y \geq w}} R \partial_{y^{-1}w_0} \phi \quad \textit{and} \quad R(\mathfrak{X}_{\mathcal{W}})_{\not< w} = \coprod_{\substack{y \in \mathcal{W} \\ y \not< w}} R \partial_{y^{-1}w_0} \phi.$$

In particular, $R(\mathfrak{X}_{\mathcal{W}})_{\geq w} / R(\mathfrak{X}_{\mathcal{W}})_{> w} \simeq R(w)(-2\ell(w)) \simeq R(\mathfrak{X}_{\mathcal{W}})_{\not< w} / R(\mathfrak{X}_{\mathcal{W}})_{\not\leq w}$ *and* $R(\mathfrak{X}_{\mathcal{W}})_{w_0} = R\phi \simeq R(w_0)(-2\ell(w_0))$.

Proof: $\forall x, y \in \mathcal{W}$, if $(\partial_x \phi)|_{\mathfrak{X}_y} \neq 0$, $(\partial_x \hat{f})|_{\mathfrak{X}_y} \neq 0$ as $\{\nu \in V^* | (\partial_x \phi)(\nu, y^{-1}\nu) \neq 0\} \not\subseteq \{\nu \in V^* | (\partial_{w_0} \hat{f})_e(\nu) = (\partial_{w_0} \hat{f})(\nu, \nu) = 0\}$. Thus, (7.6) holds with $\hat{f}$ replaced by ϕ.

(i) One has $(\partial_{w^{-1}w_0}\phi)_w = (\partial_{w^{-1}w_0}\phi)|_{\mathfrak{X}_{w_0(w^{-1}w_0)^{-1}}} \neq 0$ by (7.6.i). If $0 \neq (\partial_{w^{-1}w_0}\phi)_y = (\partial_{w^{-1}w_0}\phi)|_{\mathfrak{X}_{w_0(y^{-1}w_0)^{-1}}}$, $y^{-1}w_0 \leq w^{-1}w_0$ by (7.6.ii), and hence $y \geq w$. Also, $\deg(\partial_{w^{-1}w_0}\phi) = 2\ell(w_0) - 2\ell(w^{-1}w_0) = 2\ell(w)$.

(ii) Let $\sum_{y\in\mathcal{W}} a_y\partial_{y^{-1}w_0}\phi \in R(\mathfrak{X}_{\mathcal{W}})_{\geq w}$, $a_y \in R$. If $a_y \neq 0$, $a_y(\partial_{y^{-1}w_0}\phi)_y \neq 0$ as $(\partial_{y^{-1}w_0}\phi)_y \neq 0$ by (i) and as $\{\nu \in V^*|(\partial_{y^{-1}w_0}\phi)(\nu, y^{-1}\nu) \neq 0\} \not\subseteq \{\nu \in V^*|a_y(\nu) = 0\}$, and hence the assertions.

7.10. Let now $\psi = w_0\phi$ with ϕ as in (7.8). Then $\psi \in R(\mathfrak{X}_{\mathcal{W}})_e^{2\ell(w_0)} \setminus 0$. In particular, $\psi|_{\mathfrak{X}_e} \neq 0$ while $\psi|_{\mathfrak{X}_w} = 0\ \forall w > e$. Then, using (7.6.1,2), one checks that

(1) $$\forall y \in \mathcal{W}, \quad (\partial_y\psi)|_{\mathfrak{X}_{y^{-1}}} \neq 0,$$

(2) $$\forall w \in \mathcal{W} \text{ with } (\partial_y\psi)|_{\mathfrak{X}_{w^{-1}}} \neq 0, \quad w \leq y.$$

Arguing as in (7.7.2), one also obtains that

(3) $$\sum_{w\in\mathcal{W}} R\partial_w\psi \trianglelefteq R(\mathfrak{X}_{\mathcal{W}}).$$

with the $\partial_w\psi$, $w \in \mathcal{W}$, left R-linearly independent as in (7.7). As $(\partial_{w_0}\psi)_{w_0} \in \mathbb{K}^\times$ by (1) and as $\partial_{w_0}\psi = (\partial_{w_0}\psi)_{w_0} \otimes 1$ by (7.4), $R(\mathfrak{X}_{\mathcal{W}}) = (\partial_{w_0}\psi)_{w_0}R(\mathfrak{X}_{\mathcal{W}}) \subseteq \sum_{w\in\mathcal{W}} R\partial_w\psi$, and hence by [Dem] again

(4) $$R(\mathfrak{X}_{\mathcal{W}}) = \sum_{w\in\mathcal{W}} R\partial_w\psi = \coprod_{w\in\mathcal{W}} R\partial_w\psi.$$

Corollary: *(i)* $\forall w \in \mathcal{W}$, $\partial_{w^{-1}}\psi \in R(\mathfrak{X}_{\mathcal{W}})_{\leq w}^{2\ell(w_0w)}$ *with* $(\partial_{w^{-1}}\psi)_w \neq 0$, *i.e.,* $\partial_{w^{-1}}\psi \in R(\mathfrak{X}_{\mathcal{W}})^{2\ell(w_0w)}$, $(\partial_{w^{-1}}\psi)|_{\mathfrak{X}_w} \neq 0$ *and* $\forall y \in \mathcal{W}$ *with* $(\partial_{w^{-1}}\psi)|_{\mathfrak{X}_y} \neq 0$, $y \leq w$.

(ii) $\forall w \in \mathcal{W}$,

$$R(\mathfrak{X}_{\mathcal{W}})_{\leq w} = \coprod_{\substack{y\in\mathcal{W}\\ y\leq w}} R\partial_{y^{-1}}\psi \quad \textit{and} \quad R(\mathfrak{X}_{\mathcal{W}})_{\not> w} = \coprod_{\substack{y\in\mathcal{W}\\ y\not> w}} R\partial_{y^{-1}}\psi.$$

In particular, $R(\mathfrak{X}_{\mathcal{W}})_{\leq w}/R(\mathfrak{X}_{\mathcal{W}})_{<w} \simeq R(w)(-2\ell(w_0w)) \simeq R(\mathfrak{X}_{\mathcal{W}})_{\not> w}/R(\mathfrak{X}_{\mathcal{W}})_{\not\geq w}$ *and* $R(\mathfrak{X}_{\mathcal{W}})_e = R\psi \simeq R(e)(-2\ell(w_0))$.

Proof: (i) One has $(\partial_{w^{-1}}\psi)_w = (\partial_{w^{-1}}\psi)|_{\mathfrak{X}_w} \neq 0$ by (1). If $0 \neq (\partial_{w^{-1}}\psi)_y = (\partial_{w^{-1}}\psi)|_{\mathfrak{X}_y}$, $y^{-1} \leq w^{-1}$ by (2), and hence $y \leq w$. Also, $\deg(\partial_{w^{-1}}\psi) = 2\ell(w_0) - 2\ell(w^{-1}) = 2\ell(w_0w)$.

(ii) Argue as in (7.9.ii).

8. Properties (S) and (LE)

Assume in this section that $\mathcal{W}$ is a finite Weyl group and V the $\mathbb{K}$-linear space by base change of a weight lattice of the root system associated to $\mathcal{W}$. We will preview the properties (S) and (LE) of [Ab19b], which in turn are modelled after the ones in [FL15], applied to $\mathcal{C}$ to extend the character homomorphism ch : $[\mathfrak{S}\mathrm{Bimod}] \to \mathcal{H}$ to the Grothendieck groups of the objects of $\mathcal{C}$ admitting a Δ-flag (resp. ∇-flag) and express it in terms of the multiplicities of the Δ- (resp. ∇-) subquoteints as in (5.9). These are analogues of Soergel's formulae [S07] in case V is reflection faithful. Precisely, all the above hold if $\mathbb{K}$ is a field satisfying the characteristic condition of §7. Over a complete DVR $\mathbb{K}$, however, we will have also to work over the residue field of $\mathbb{K}$ as in (4.9), for which the objects in $\mathcal{C}$ have to split already over $R^\emptyset = R[\frac{1}{\alpha_t}|t \in \mathcal{T}]$ rather than over Q. Thus, let $\mathcal{C}^\emptyset$ denote the full subcategory of $\mathcal{C}^{\mathrm{tf}}$ consisting of those M splitting over $R^\emptyset$: $R^\emptyset \otimes_R M = \coprod_{w\in\mathcal{W}} M_w^\emptyset$ with $M_w^\emptyset = (R^\emptyset \otimes_R M) \cap M_w^Q$. Note that $\mathfrak{S}\mathrm{Bimod}$ is a subcategory of $\mathcal{C}^\emptyset$.

We assume throughout the section that $\mathbb{K}$ is a complete DVR, unless otherwise specified, with the hypotheses in §7 on the characteristic of $\mathbb{K}$ and of $\mathbb{K}/\mathfrak{m}$ for the maximal ideal $\mathfrak{m}$ of $\mathbb{K}$.

8.1. $\forall x \in \mathcal{W}$, put $(\leq x) = \{w \in \mathcal{W} | w \leq x\}$ and $(\geq x) = \{w \in \mathcal{W} | w \geq x\}$. Define $(> w)$ and $(< w)$ likewise. We say that $I \subseteq \mathcal{W}$ is $\mathcal{W}$-open iff $I = \cup_{x\in I}(\leq x)$. The $\mathcal{W}$-opens define a topology on the set $\mathcal{W}$. Thus, $J \subseteq \mathcal{W}$ is closed iff $J = \cup_{x\in J}(\geq x)$, in which case we will say J is $\mathcal{W}$-closed. $\forall t \in \mathcal{T}$, let $R^{\alpha_t} = R[\frac{1}{\alpha_u}|u \in \mathcal{T} \setminus \{t\}]$. Under the standing hypothesis one has

$$\cap_{t\in\mathcal{T}} R^{\alpha_t} = R. \tag{1}$$

$\forall M \in \mathcal{C}^\emptyset$ put $M^{\alpha_t} = R^{\alpha_t} \otimes_R M$. $\forall J \subseteq \mathcal{W}$, one has

$$\begin{aligned}(2)\quad (M_J)^{\alpha_t} &= R^{\alpha_t} \otimes_R (M \cap \coprod_{w\in J} M_w^Q) \\ &= (R^{\alpha_t} \otimes_R M) \cap (R^{\alpha_t} \otimes_R \coprod_{w\in J} M_w^Q) \quad \text{as } R^{\alpha_t} \text{ is flat over } R \text{ [BCA, Lem. I.2.6.7]} \\ &= M^{\alpha_t} \cap \coprod_{w\in J} M_w^Q = (M^{\alpha_t})_J.\end{aligned}$$

We say that M belongs to $\mathcal{C}^{\mathrm{ou}}$ iff the following two properties (S^{ou}) and (LE) hold on M:

$$\forall \mathcal{W}\text{-open } I_1 \text{ and } I_2, \quad M_{I_1\cup I_2} = M_{I_1} + M_{I_2}, \tag{S^{ou}}$$

$$\forall t \in \mathcal{T}, \quad M^{\alpha_t} = \coprod_{\Omega\in\langle t\rangle\backslash\mathcal{W}} (M^{\alpha_t} \cap \coprod_{x\in\Omega} M_x^Q) = \coprod_{\Omega\in\langle t\rangle\backslash\mathcal{W}} (M^{\alpha_t} \cap \coprod_{x\in\Omega} M_x^\emptyset). \tag{LE}$$

$\forall M \in \mathcal{C}^\emptyset$, $\forall J \subseteq \mathcal{W}$, arguing as in (4.10.iii) yields that

$$M_J/\mathfrak{m}(M_J) \simeq M_J \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \simeq \{M \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_J \simeq (M/\mathfrak{m}M)_J. \tag{3}$$

Then, $\forall M \in \mathcal{C}^{\mathrm{ou}}$,

$$\text{properties } (\mathrm{S}^{\mathrm{ou}}) \text{ and (LE) carry over to } M \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}). \tag{4}$$

For

$$
\begin{aligned}
(M/\mathfrak{m}M)_{I_1\cup I_2} &\simeq (M_{I_1\cup I_2})/\mathfrak{m}(M_{I_1\cup I_2}) \quad \text{by (3)}\\
&= (M_{I_1}+M_{I_2})/\mathfrak{m}(M_{I_1}+M_{I_2}) \simeq (M_{I_1}+M_{I_2})\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\\
&\simeq M_{I_1}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) + M_{I_2}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \quad \text{as } (M_{I_1}+M_{I_2})\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}),\\
&\qquad M_{I_1}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \text{ and } M_{I_2}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) \text{ all lie in } M\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\\
&\simeq (M\mathfrak{m}M)_{I_1} + (M\mathfrak{m}M)_{I_2} \quad \text{by (3) again.}
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
(M/\mathfrak{m}M)^{\alpha_t} &\simeq \{M\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\}^{\alpha_t} \simeq (R/\mathfrak{m}R)^{\alpha_t}\otimes_R M \simeq M^{\alpha_t}/\mathfrak{m}(M^{\alpha_t}) \simeq M^{\alpha_t}\otimes_{\Bbbk}(\mathbb{K}/\mathfrak{m})\\
&= \coprod_{\Omega\in\langle t\rangle\backslash\mathcal{W}} (M^{\alpha_t}\cap\coprod_{x\in\Omega} M_x^{\emptyset})\}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m}) = \coprod_{\Omega\in\langle t\rangle\backslash\mathcal{W}} \{(M^{\alpha_t})_\Omega\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\}\\
&= \coprod_{\Omega\in\langle t\rangle\backslash\mathcal{W}} (M^{\alpha_t}/\mathfrak{m}M^{\alpha_t})_\Omega \quad \text{as in (3)}\\
&= \coprod_{\Omega\in\langle t\rangle\backslash\mathcal{W}} \{(M/\mathfrak{m}M)^{\alpha_t}\cap\coprod_{x\in\Omega}(M/\mathfrak{m}M)_x^{\emptyset}\}.
\end{aligned}
$$

8.2. Let $M\in\mathcal{C}^{\emptyset}$, $t\in\mathcal{T}$, $w\in\mathcal{W}$, and $n\in\mathbb{Z}$.

Lemma: *(i) If* $\mathrm{supp}_{\mathcal{W}}(M)\subseteq\{w,tw\}$, $(\mathrm{S}^{\mathrm{ou}})$ *holds on* M.

(ii) If (LE) holds on M*, so does* $(\mathrm{S}^{\mathrm{ou}})$ *on* M^{α_t}.

(iii) $R(w)(n)\in\mathcal{C}^{\mathrm{ou}}$.

Proof: Let I_1 and I_2 be 2 $\mathcal{W}$-opens. Recall that either $w<tw$ or $tw<w$ [HRC, 5.9]

(i) We may assume that $I_1\cap\{w,tw\}\supseteq I_2\cap\{w,tw\}$. Let I'_j be the smallest $\mathcal{W}$-open containing $I_j\cap\{w,tw\}$, $j\in[1,2]$. Then $I'_1\supseteq I'_2$, $I'_1\cap\{w,tw\}=I_1\cap\{w,tw\}$, $I'_2\cap\{w,tw\}=I_2\cap\{w,tw\}$, and hence

$$
\begin{aligned}
M_{I'_1} &= M\cap(\coprod_{x\in I'_1} M_x^Q) \quad \text{by definition}\\
&= M\cap(\coprod_{x\in I'_1\cap\{w,tw\}} M_x^Q) = M\cap(\coprod_{x\in I_1\cap\{w,tw\}} M_x^Q) = M\cap(\coprod_{x\in I_1} M_x^Q)\\
&\qquad\qquad \text{as } \mathrm{supp}_{\mathcal{W}}(M)\subseteq\{w,tw\}\\
&= M_{I_1}.
\end{aligned}
$$

Likewise, $M_{I'_2}=M_{I_2}$, $M_{I'_1\cup I'_2}=M_{I_1\cup I_2}$. Then $M_{I_1\cup I_2}=M_{I'_1\cup I'_2}=M_{I'_1}=M_{I_1}=M_{I_1}+M_{I_2}$ as $M_{I_2}=M_{I'_2}\subseteq M_{I'_1}=M_{I_1}$.

(ii) Put $\beta=\alpha_t$. Assume now that (LE) holds on M. $\forall\Omega\in\langle t\rangle\backslash\mathcal{W}$, put $M_\Omega^\beta=M^\beta\cap$

$(\coprod_{x\in\Omega} M_x^Q)$. Thus, $M^\beta = \coprod_\Omega M_\Omega^\beta$ by (LE). If I is $\mathcal{W}$-open, one has

$$\begin{aligned}(M^\beta)_I &= M^\beta \cap \{\coprod_{x\in I}(M^\beta)_x^Q\} = (\coprod_\Omega M_\Omega^\beta) \cap \{\coprod_{x\in I}(\coprod_\Omega M_\Omega^\beta)_x^Q\} = \coprod_\Omega \{M_\Omega^\beta \cap (\coprod_{x\in I}(M_\Omega^\beta)_x^Q)\} \\ &= \coprod_\Omega (M_\Omega^\beta)_I \quad \text{by definition,}\end{aligned}$$

and hence

$$\begin{aligned}(M^\beta)_{I_1\cup I_2} &= \coprod_\Omega (M_\Omega^\beta)_{I_1\cup I_2} \\ &= \coprod_\Omega \{(M_\Omega^\beta)_{I_1} + (M_\Omega^\beta)_{I_2}\} \quad \text{as } (\mathrm{S}^{\mathrm{ou}}) \text{ holds on } M_\Omega^\beta \text{ by (i)} \\ &= \{\coprod_\Omega (M_\Omega^\beta)_{I_1}\} + \{\coprod_\Omega (M_\Omega^\beta)_{I_2}\} = (M^\beta)_{I_1} + (M^\beta)_{I_2}.\end{aligned}$$

8.3. Let K be a $\mathcal{W}$-locally closed subset and write $K = I \cap J$ with I $\mathcal{W}$-open and J $\mathcal{W}$-closed. $\forall M \in \mathcal{C}^{\mathrm{ou}}$, set $M_K^{\mathrm{ou}} = M_I/M_{I\setminus J}$. If $K = I' \cap J'$ with I' $\mathcal{W}$-open and J' $\mathcal{W}$-closed,

$$(I \cup I') \cap (J \cap J') = (I \cap J \cap J') \cup (I' \cap J \cap J') = (K \cap J') \cup (K \cap J) = K \cup K = K.$$

Also,

$$I \cup I' = I \cup \{(I \cup I') \setminus (J \cap J')\}, \tag{1}$$

$$I \cap \{(I \cup I') \setminus (J \cap J')\} = I \setminus J. \tag{2}$$

For let $x \in I' \setminus I$. As $(I' \setminus I) \cap (J \cap J') \subseteq (I' \cap J') \setminus I = (I \cap J) \setminus I = \emptyset$, $x \notin J \cap J'$. Then $x \in (I \cup I') \setminus (J \cap J')$, and (1) holds. Let next $y \in I \cap \{(I \cup I') \setminus (J \cap J')\} = I \setminus (J \cap J') \supseteq I \setminus J$. Just suppose $y \in J$. Then $y \in I \cap J = I' \cap J' \subseteq J'$, and hence $y \in J \cap J'$, absurd, and hence also (2). Then

$$\begin{aligned}M_{I\cup I'}/M_{(I\cup I')\setminus(J\cap J')} &= M_{I\cup\{(I\cup I')\setminus(J\cap J')\}}/M_{(I\cup I')\setminus(J\cap J')} \quad \text{by (1)} \\ &= \{M_I + M_{(I\cup I')\setminus(J\cap J')}\}/M_{(I\cup I')\setminus(J\cap J')} \quad \text{by } (\mathrm{S}^{\mathrm{ou}}) \\ &\simeq M_I/\{M_I \cap M_{(I\cup I')\setminus(J\cap J')}\} \\ &= M_I/M_{I\cap\{(I\cup I')\setminus(J\cap J')\}} \quad \text{by (1.4.iii)} \\ &= M_I/M_{I\setminus J} \quad \text{by (2).}\end{aligned} \tag{3}$$

Lemma: *(i) $M_K^{\mathrm{ou}} \in \mathcal{C}^{\mathrm{ou}}$ with $M_K^{\mathrm{ou}} \leq M^\emptyset$ and is, in $M^\emptyset$, independent of the choice of I and J to express K .*

(ii) $\mathrm{supp}_{\mathcal{W}}(M_K^{\mathrm{ou}}) = \mathrm{supp}_{\mathcal{W}}(M) \cap K$.

(iii) If $\mathrm{supp}_{\mathcal{W}}(M) \subseteq K$, $M_K^{\mathrm{ou}} = M$.

(iv) $M_K \otimes (\mathbb{K}/\mathfrak{m}) \simeq \{M \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_K$.

Proof: (iii) One has

$$\begin{aligned} M_K^{\mathrm{ou}} &= M_I/M_{I\setminus J} \\ &= M_I/0 \quad \text{as } (I\setminus J)\cap \mathrm{supp}_{\mathcal{W}}(M) \subseteq (I\setminus J)\cap(I\cap J)=\emptyset \\ &= M \quad \text{as } \mathrm{supp}_{\mathcal{W}}(M)\subseteq I. \end{aligned}$$

(i), (ii) By (1.4.2) one has $M_K^{\mathrm{ou}} = M_I/(M_I)_{I\setminus J} \simeq (M_I)^{I\cap J}$ torsion-free over R. In particular,

$$\begin{aligned} M_K^{\mathrm{ou}} &\leq (M_K^{\mathrm{ou}})^Q = (M_I/M_{I\setminus J})^Q \simeq (M_I)^Q/(M_{I\setminus J})^Q \\ &= (\coprod_{x\in I} M_x^Q)/(\coprod_{x\in I\setminus J} M_x^Q) \quad \text{by (1.4.ii)} \\ &\simeq \coprod_{x\in I\cap J} M_x^Q = \coprod_{x\in K} M_x^Q, \end{aligned}$$

and hence $M_K^{\mathrm{ou}} \in \mathcal{C}^{\emptyset}$ with

$$\begin{aligned} \mathrm{supp}_{\mathcal{W}}(M_K^{\mathrm{ou}}) &= \{x\in\mathcal{W} | (M_K^{\mathrm{ou}})_x^Q \neq 0\} = \{x\in\mathcal{W} | (\coprod_{y\in K} M_y^Q)_x \neq 0\} = \{x\in K | M_x^Q\neq 0\} \\ &= \mathrm{supp}_{\mathcal{W}}(M)\cap K. \end{aligned}$$

To see that (S^{ou}) and (LE) hold on M_K^{ou}, we first show that $\forall I'$ $\mathcal{W}$-open,

(4) $$(M_K^{\mathrm{ou}})_{I'} = M_{K\cap I'}^{\mathrm{ou}}.$$

If K is $\mathcal{W}$-open, the assertion follows from (1.4.iii). If K is $\mathcal{W}$-closed, put $I_1 = \mathcal{W}\setminus K$. Then

$$\begin{aligned} (M_K^{\mathrm{ou}})_{I'} &= M_K^{\mathrm{ou}} \cap \coprod_{x\in I'} (M_K^{\mathrm{ou}})_x^Q \\ &= (M/M_{I_1}) \cap \coprod_{x\in I'\cap K} M_x^Q \quad \text{by (ii)} \end{aligned}$$

while

$$\begin{aligned} M_{K\cap I'}^{\mathrm{ou}} &= M_{I'}/M_{I'\setminus K} = M_{I'}/M_{I'\cap I_1} \\ &= M_{I'}/(M_{I'}\cap M_{I_1}) \quad \text{by (1.4.iii) again} \\ &\simeq (M_{I'}+M_{I_1})/M_{I_1}. \end{aligned}$$

As $M_{K\cap I'}^{\mathrm{ou}} \leq (M_{K\cap I'}^{\mathrm{ou}})^Q = \coprod_{x\in I'\cap K} M_x^Q$, $M_{K\cap I'}^{\mathrm{ou}} \leq (M_K^{\mathrm{ou}})_{I'}$. Let $m\in M$ with $m+M_{I_1} \in \coprod_{x\in I'\cap K} M_x^Q$. Then $m_x = 0$ unless $x\in I'\cup I_1$, and hence

$$\begin{aligned} m &\in M_{I'\cup I_1} \\ &= M_{I'} + M_{I_1} \quad \text{as } (\mathrm{S}^{\mathrm{ou}}) \text{ holds on } M. \end{aligned}$$

Thus, $M_{K\cap I'}^{\mathrm{ou}} \simeq (M_K^{\mathrm{ou}})_{I'}$. In general, write $K = I\cap J$ with I $\mathcal{W}$-open and J $\mathcal{W}$-closed. One has $M_K^{\mathrm{ou}} = M_{J\cap I}^{\mathrm{ou}} \simeq (M_J^{\mathrm{ou}})_I$ by what we have just verified, and hence

$$\begin{aligned} (M_K^{\mathrm{ou}})_{I'} &\simeq ((M_J^{\mathrm{ou}})_I)_{I'} = (M_J^{\mathrm{ou}})_{I\cap I'} \quad \text{by (1.4.iii)} \\ &\simeq M_{J\cap I\cap I'}^{\mathrm{ou}} \quad \text{by above} \\ &= M_{K\cap I'}^{\mathrm{ou}}, \quad \text{as desired.} \end{aligned}$$

We now show that (S^{ou}) holds on M_K^{ou}. Given $\mathcal{W}$-open I_1 and I_2, one has

$$\begin{aligned}
(M_K^{\mathrm{ou}})_{I_1\cup I_2} &= M^{\mathrm{ou}}_{K\cap(I_1\cup I_2)} \quad \text{by (3)}\\
&= M^{\mathrm{ou}}_{(K\cap I_1)\cup(K\cap I_2)} = M^{\mathrm{ou}}_{(I\cap I_1\cap J)\cup(I\cap I_2\cap J)} = M^{\mathrm{ou}}_{\{(I\cap I_1)\cup(I\cap I_2)\}\cap J}\\
&= M_{I\cap(I_1\cup I_2)}/M_{\{I\cap(I_1\cup I_2)\}\setminus J} = M_{(I\cap I_1)\cup(I\cap I_2)}/M_{\{I\cap(I_1\cup I_2)\}\setminus J}\\
&= \{M_{I\cap I_1}+M_{I\cap I_2}\}/M_{\{I\cap(I_1\cup I_2)\}\setminus J} \quad \text{as } (\mathrm{S}^{\mathrm{ou}}) \text{ holds on } M\\
&\simeq M_{I\cap I_1}/M_{(I\cap I_1)\setminus J}+M_{I\cap I_2}/M_{(I\cap I_2)\setminus J} = M^{\mathrm{ou}}_{K\cap I_1}+M^{\mathrm{ou}}_{K\cap I_2}\\
&= (M_K^{\mathrm{ou}})_{I_1}+(M_K^{\mathrm{ou}})_{I_2} \quad \text{by (3) again,}
\end{aligned}$$

and hence $(M_K^{\mathrm{ou}})_{I_1\cup I_2} = (M_K^{\mathrm{ou}})_{I_1}+(M_K^{\mathrm{ou}})_{I_2}$.

We show finally that (LE) holds on M_K^{ou}. Let $t\in\mathcal{T}$ and put $\beta=\alpha_t$. As $(M_K^{\mathrm{ou}})^\beta = (M_I/M_{I\setminus J})^\beta \simeq (M_I)^\beta/(M_{I\setminus J})^\beta$, we have only to verify (LE) holding on M_I. Let $m\in(M_I)^\beta\leq M^\beta$. As (LE) holds on M, one can write $m=\sum_{\Omega\in\langle t\rangle\setminus\mathcal{W}} m_\Omega$ with $m_\Omega\in M^\beta\cap\coprod_{x\in\Omega}M_x^Q$. As $m\in(M_I)^\beta\leq(M_I)^Q=\coprod_{y\in I}M_y^Q$, however, $m_x=0$ unless $x\in I$. Thus, $m_\Omega\in(M_I)^\beta\cap\coprod_{x\in\Omega}(M_I)_x^Q$, as desired.

(iv) follows from (8.1.3).

8.4. Let $M\in\mathcal{C}^{\mathrm{ou}}$ and K_1 $\mathcal{W}$-locally closed. By (8.3.i) one has $M_{K_1}^{\mathrm{ou}}\in\mathcal{C}^{\mathrm{ou}}$.

Lemma: *If K_2 is another $\mathcal{W}$-locally closed, $(M_{K_1}^{\mathrm{ou}})_{K_2}^{\mathrm{ou}}\simeq M^{\mathrm{ou}}_{K_1\cap K_2}$.*

Proof: Write $K_i=I_i\cap J_i$ with I_i $\mathcal{W}$-open and J_i $\mathcal{W}$-closed, $i\in\{1,2\}$. Then

$$\begin{aligned}
(M_{K_1}^{\mathrm{ou}})_{K_2} &= (M_{K_1}^{\mathrm{ou}})_{I_2}/(M_{K_1}^{\mathrm{ou}})_{I_2\setminus J_2}\\
&= M^{\mathrm{ou}}_{K_1\cap I_2}/M^{\mathrm{ou}}_{K_1\cap(I_2\setminus J_2)} \quad \text{by (8.3.4)}
\end{aligned}$$

with

$$\begin{aligned}
M^{\mathrm{ou}}_{K_1\cap I_2} &= M^{\mathrm{ou}}_{I_1\cap I_2\cap J_1} = M_{I_1\cap I_2}/M_{(I_1\cap I_2)\setminus J_1},\\
M^{\mathrm{ou}}_{K_1\cap(I_2\setminus J_2)} &= M^{\mathrm{ou}}_{I_1\cap(I_2\setminus J_2)\cap J_1} = M_{I_1\cap(I_2\setminus J_2)}/M_{\{I_1\cap(I_2\setminus J_2)\}\setminus J_1} = M_{(I_1\cap I_2)\setminus J_2}/M_{(I_1\cap I_2)\setminus(J_1\cup J_2)},
\end{aligned}$$

and hence

$$\begin{aligned}
(M_{K_1}^{\mathrm{ou}})^{\mathrm{ou}}_{K_2} &= M_{I_1\cap I_2}/\{M_{(I_1\cap I_2)\setminus J_1}+M_{(I_1\cap I_2)\setminus J_2}\}\\
&= M_{I_1\cap I_2}/M_{(I_1\cap I_2)\setminus J_1\}\cup\{(I_1\cap I_2)\setminus J_2\}} \quad \text{as } (\mathrm{S}^{\mathrm{ou}}) \text{ holds on } M\\
&= M_{I_1\cap I_2}/M_{(I_1\cap I_2)\setminus(J_1\cap J_2)} = M^{\mathrm{ou}}_{I_1\cap I_2\cap J_1\cap J_2} = M^{\mathrm{ou}}_{K_1\cap K_2}.
\end{aligned}$$

8.5. Let $M\in\mathcal{C}^{\mathrm{ou}}$. $\forall w\in\mathcal{W}$, $\{w\}=(\leq w)\cap(>w)$ is $\mathcal{W}$-locally closed. Put $M_w^{\mathrm{ou}}=M^{\mathrm{ou}}_{\{w\}}=M^{\mathrm{ou}}_{\leq w}/M^{\mathrm{ou}}_{(\leq w)\setminus(>w)}$ for simplicity.

The filtration $M^{\mathrm{ou}}_{\leq i}$, $i\in\mathbb{N}$, of M^{ou} by length with $(\leq i)=\{w\in\mathcal{W}|\ell(w)\leq i\}$ admits a refinement by $\mathcal{W}$-opens such that each subquotient is of the form M_w^{ou}, $w\in\mathcal{W}$. We say that M admits a ∇-flag iff each M_w^{ou}, $w\in\mathcal{W}$, is graded free over R, i.e., $M_w^{\mathrm{ou}}\simeq\coprod_{i\in\mathbb{Z}}R(w)(i)^{\oplus n_i}$ for some $n_i\in\mathbb{N}$. Let $\mathcal{C}_\nabla$ denote the full subcategory of $\mathcal{C}^\emptyset$ consisting of the objects with ∇-flags.

Lemma: *$\forall M \in \mathcal{C}_\nabla$, $\forall K$ $\mathcal{W}$-locally closed, $M_K^{\mathrm{ou}} \in \mathcal{C}_\nabla$ and is left/right graded free over R. In particular, $M_K^{\mathrm{ou}} \in \mathcal{C}$.*

Proof: By (8.3.i) we know that $M_K^{\mathrm{ou}} \in \mathcal{C}^{\mathrm{ou}}$. $\forall w \in \mathcal{W}$, one has

$$(1) \qquad (M_K^{\mathrm{ou}})_w^{\mathrm{ou}} = M_{K\cap\{w\}}^{\mathrm{ou}} \quad \text{by (8.4)}$$
$$= \begin{cases} M_w^{\mathrm{ou}} & \text{if } w \in K, \\ 0 & \text{else.} \end{cases}$$

Let $I_0 = \emptyset \subset I_1 \subset \cdots \subset I_{|\mathcal{W}|} = \mathcal{W}$ be a filtration of $\mathcal{W}$ by $\mathcal{W}$-opens. Then $M_K^{\mathrm{ou}} = (M_K^{\mathrm{ou}})_{I_{|\mathcal{W}|}}$ with all $(M_K^{\mathrm{ou}})_{I_j}/(M_K^{\mathrm{ou}})_{I_{j-1}} \simeq (M_K^{\mathrm{ou}})_{I_j\setminus I_{j-1}}^{\mathrm{ou}}$, $j \in [1, |\mathcal{W}|]$, graded free by (1), and so therefore is M_K^{ou}.

8.6. Let $M \in \mathcal{C}$, $s \in \mathcal{S}$, $t \in \mathcal{T}$ and put $\alpha = \alpha_s$, $\beta = \alpha_t$. $\forall \Omega \in \langle t\rangle\backslash\mathcal{W}$, put $M^\Omega = M^\beta \cap (\coprod_{w\in\Omega} M_w^Q) = M^\beta \cap (M_w^Q \oplus M_{tw}^Q)$. Let $\delta \in V$ with $\langle \delta, \alpha^\vee\rangle = 1$.

Lemma: *(i) If $\Omega = \Omega s$, $(M * B(s))^\Omega \simeq M^\Omega \otimes_R B(s)$.*

(ii) If $\Omega \neq \Omega s$, the right actions of α on both M^Ω and $(M * B(s))^\Omega$ are invertible and

$$(M * B(s))^\Omega \simeq \{M^\Omega \otimes_R R(\delta\otimes 1 - 1\otimes s\delta)\} \oplus \{M^{\Omega s} \otimes_R R(\delta \otimes 1 - 1 \otimes \delta)\}.$$

(iii) If (LE) holds on M, so does it on $M * B$ $\forall B \in \mathfrak{S}\mathrm{Bimod}$.

Proof: (i) One has

$$\begin{aligned}
(M * B(s))^\Omega &= (M * B(s))^\beta \cap \coprod_{w\in\Omega} (M * B(s))_w^Q \\
&= (M * B(s))^\beta \cap \coprod_{w\in\Omega} \{(M_w^Q \otimes_R B(s)_e^Q) \oplus (M_{ws}^Q \otimes_R B(s)_s^Q)\} \quad \text{by (2.3.3)} \\
&= (M * B(s))^\beta \cap \coprod_{w\in\Omega} \{(M_w^Q \otimes_R B(s)_e^Q) \oplus (M_w^Q \otimes_R B(s)_s^Q)\}\} \quad \text{as } \Omega s = \Omega \\
&= (M \otimes_R B(s))^\beta \cap \coprod_{w\in\Omega} (M_w^Q \otimes_R B(s)^Q) \\
&\simeq (M^\beta \otimes_R B(s)) \cap \coprod_{w\in\Omega} (M_w^Q \otimes_R B(s)) \simeq (M^\beta \otimes_R B(s)) \cap \{(\coprod_{w\in\Omega} M_w^Q) \otimes_R B(s)\} \\
&= (M^\beta \cap \coprod_{w\in\Omega} M_w^Q) \otimes_R B(s) \quad \text{as } B(s) \text{ is free over } R \\
&= M^\Omega \otimes_R B(s).
\end{aligned}$$

(ii) Let $w \in \Omega$ and put $\gamma = w\alpha$. Thus, $\Omega = \{w, tw\}$ and $\Omega s = \{ws, tws\}$. As $\Omega s \neq \Omega$, $\Omega s \cap \Omega = \emptyset$, and hence $\gamma \neq \pm\beta$, $t\gamma \neq \pm\beta$. Then $\gamma, t\gamma \in (R^\beta)^\times$. Let $m \in M^\Omega = M^\beta \cap (M_w^Q \oplus M_{tw}^Q)$ and write $m = m_1 + m_2$ with $m_1 \in M_w^Q$ and $m_2 \in M_{tw}^Q$. Take $\delta_\beta \in V$ with $\langle \delta_\beta, \beta^\vee\rangle = 1$.

$\forall a \in R$, $m_1 a = (wa)m_1 = \gamma m_1$ and $m_2 a = (twa)m_2 = (t\gamma)m_2$. Thus, $m\alpha = \gamma m_1 + (t\gamma)m_2$, $mw^{-1}\delta_\beta = \delta_\beta m_1 + (t\delta_\beta)m_2 = \delta_\beta m_1 + (\delta_\beta - \beta)m_2$. Then in M^β

$$\begin{aligned}
&\{\frac{1}{\gamma}m + \frac{\langle\gamma,\beta^\vee\rangle}{\gamma(t\gamma)}(\delta_\beta m - mw^{-1}\delta_\beta)\}\alpha \\
&\quad = (\frac{1}{\gamma} + \frac{\langle\gamma,\beta^\vee\rangle\delta_\beta}{\gamma(t\gamma)})(\gamma m_1 + (t\gamma)m_2) - \frac{\langle\gamma,\beta^\vee\rangle}{\gamma(t\gamma)}\{\delta_\beta\gamma m_1 + (\delta_\beta - \beta)(t\gamma)m_2\} \\
&\quad = (1 + \frac{\langle\gamma,\beta^\vee\rangle\delta_\beta}{t\gamma} - \frac{\langle\gamma,\beta^\vee\rangle\delta_\beta}{t\gamma})m_1 + (\frac{t\gamma}{\gamma} + \frac{\langle\gamma,\beta^\vee\rangle\delta_\beta}{\gamma} - \frac{\langle\gamma,\beta^\vee\rangle(\delta_\beta - \beta)}{\gamma})m_2 \\
&\quad = m_1 + m_2 = m.
\end{aligned}$$

Thus, $M^\Omega\alpha = M^\Omega$. As M is right torsion-free over R by (1.3.2), the right multiplication by α on M^Ω is invertible, and on $(M * B(s))^\Omega$ as $M * B(s) \in \mathcal{C}$ by (2.3). Thus,

$$(M * B(s))^\Omega = (M * B(s))^\Omega \otimes_R R[\frac{1}{\alpha}].$$

Put $B(s)[\frac{1}{\alpha}] = B(s) \otimes_R R[\frac{1}{\alpha}]$. As $(\delta\otimes 1 - 1\otimes s\delta)\alpha = \alpha(\delta\otimes 1 - 1\otimes s\delta)$ and as $(\delta\otimes 1 - 1\otimes\delta)\alpha = (s\alpha)(\delta\otimes 1 - 1\otimes\delta) = -\alpha(\delta\otimes 1 - 1\otimes\delta)$, one has from (2.2.16)

$$B(s)[\frac{1}{\alpha}] = R[\frac{1}{\alpha}](\delta\otimes 1 - 1\otimes s\delta) \oplus R[\frac{1}{\alpha}](\delta\otimes 1 - 1\otimes\delta) \tag{1}$$

with $R[\frac{1}{\alpha}](\delta\otimes 1 - 1\otimes s\delta) \subseteq B(s)^Q_e$ and $R[\frac{1}{\alpha}](\delta\otimes 1 - 1\otimes\delta) \subseteq B(s)^Q_s$. Thus,

$$\begin{aligned}
&(M * B(s))^\Omega \otimes_R R[\frac{1}{\alpha}] = (M * B(s)[\frac{1}{\alpha}])^\beta \cap \coprod_{w\in\Omega}(M * B(s))^Q_w \quad \text{[BCA, Lem. I.2.6.7]} \\
&\quad = (M^\beta \otimes_R B(s)[\frac{1}{\alpha}]) \cap \{(M * B(s))^Q_w \oplus (M * B(s))^Q_{tw}\} \\
&\quad = (M^\beta \otimes_R B(s)[\frac{1}{\alpha}]) \cap \\
&\qquad\quad \{(M^Q_w \otimes_R B(s)^Q_e) \oplus (M^Q_{ws} \otimes_R B(s)^Q_s) \oplus (M^Q_{tw} \otimes_R B(s)^Q_e) \oplus (M^Q_{tws} \otimes_R B(s)^Q_s)\} \\
&\quad = M^\beta \otimes_R \{R[\frac{1}{\alpha}](\delta\otimes 1 - 1\otimes s\delta) \oplus R[\frac{1}{\alpha}](\delta\otimes 1 - 1\otimes\delta)\} \\
&\qquad\quad \cap \{M^Q_w \otimes_R R(\delta\otimes 1 - 1\otimes s\delta) \oplus M^Q_{tw} \otimes_R R(\delta\otimes 1 - 1\otimes s\delta) \\
&\qquad\qquad \oplus M^Q_{ws} \otimes_R R(\delta\otimes 1 - 1\otimes\delta) \oplus M^Q_{tws} \otimes_R R(\delta\otimes 1 - 1\otimes\delta)\} \\
&\quad = M^\beta \otimes_R \{R(\delta\otimes 1 - 1\otimes s\delta) \oplus R(\delta\otimes 1 - 1\otimes\delta)\} \\
&\qquad\quad \cap \{(M^Q_w \oplus M^Q_{tw}) \otimes_R R(\delta\otimes 1 - 1\otimes s\delta) \oplus (M^Q_{ws} \oplus M^Q_{tws}) \otimes_R R(\delta\otimes 1 - 1\otimes\delta))\} \\
&\qquad\quad \text{as } \alpha \in (R^\beta)^\times \\
&\quad = \{M^\beta \cap (M^Q_w \oplus M^Q_{tw})\} \otimes_R R(\delta\otimes 1 - 1\otimes s\delta) \\
&\qquad\quad \oplus \{M^\beta \cap (M^Q_{ws} \oplus M^Q_{tws})\} \otimes_R R(\delta\otimes 1 - 1\otimes\delta) \\
&\quad = \{M^\Omega \otimes_R R(\delta\otimes 1 - 1\otimes s\delta)\} \oplus \{M^{\Omega s} \otimes_R R(\delta\otimes 1 - 1\otimes\delta)\}.
\end{aligned}$$

(iii) We may assume that $B = B(s)$. We are to show that $(M * B(s))^\beta = \coprod_{\Omega\in\langle t\rangle\backslash\mathcal{W}}(M * B(s))^\Omega$. Write $\{\Omega \in \langle t\rangle\backslash\mathcal{W} | \Omega s \neq \Omega\}/\langle s\rangle = \{\Omega_1, \dots, \Omega_r\}$. Thus, $\{\Omega \in \langle t\rangle\backslash\mathcal{W} | \Omega s \neq \Omega\} =$

$\{\Omega_i, \Omega_i s | i \in [1, r]\}$. Then

$$\begin{aligned}
\coprod_{\Omega \in \langle t \rangle \backslash \mathcal{W}} (M * B(s))^\Omega &= \{\coprod_{\Omega s = \Omega} (M * B(s))^\Omega\} \oplus \coprod_{i=1}^r \{(M * B(s))^{\Omega_i} \oplus (M * B(s))^{\Omega_i s}\} \\
&= \{\coprod_{\Omega s = \Omega} (M^\Omega \otimes_R B(s))\} \oplus \coprod_{i=1}^r \{M^{\Omega_i} \otimes_R R(\delta \otimes 1 - 1 \otimes s\delta) \\
&\quad \oplus M^{\Omega_i s} \otimes_R R(\delta \otimes 1 - 1 \otimes \delta) \oplus M^{\Omega_i s} \otimes_R R(\delta \otimes 1 - 1 \otimes s\delta) \\
&\quad \oplus M^{\Omega_i} \otimes_R R(\delta \otimes 1 - 1 \otimes \delta)\} \quad \text{by (i) and (ii)} \\
&= \{\coprod_{\Omega s = \Omega} (M^\Omega \otimes_R B(s))\} \\
&\quad \oplus \coprod_{i=1}^r \{M^{\Omega_i} \otimes_R \{R[\frac{1}{\alpha}](\delta \otimes 1 - 1 \otimes s\delta) \oplus R[\frac{1}{\alpha}](\delta \otimes 1 - 1 \otimes \delta)\}\} \\
&\quad \oplus \{M^{\Omega_i s} \otimes_R \{R[\frac{1}{\alpha}](\delta \otimes 1 - 1 \otimes \delta) \oplus R[\frac{1}{\alpha}](\delta \otimes 1 - 1 \otimes s\delta)\}\} \\
&\quad \text{as the right multiplications by } \alpha \text{ on } M^{\Omega_i} \text{ and on } M^{\Omega_i s} \text{ are both invertible} \\
&= \{\coprod_{\Omega s = \Omega} (M^\Omega \otimes_R B(s))\} \oplus \coprod_{i=1}^r \{(M^{\Omega_i} \otimes_R B(s)[\frac{1}{\alpha}])\} \oplus (M^{\Omega_i s} \otimes_R B(s)[\frac{1}{\alpha}])\} \\
&\quad \text{by (1)} \\
&= \{\coprod_{\Omega s = \Omega} (M^\Omega \otimes_R B(s))\} \oplus \coprod_{i=1}^r \{(M^{\Omega_i} \otimes_R B(s)) \oplus (M^{\Omega_i s} \otimes_R B(s))\} \\
&\quad \text{as the right multiplications by } \alpha \text{ on } M^{\Omega_i} \text{ and on } M^{\Omega_i s} \text{ are invertible again} \\
&= \coprod_{\Omega \in \langle t \rangle \backslash \mathcal{W}} (M^\Omega \otimes_R B(s)) = (\coprod_{\Omega \in \langle t \rangle \backslash \mathcal{W}} M^\Omega) \otimes_R B(s) \\
&= M^\beta \otimes_R B(s) \quad \text{as (LE) holds on } M \\
&= (M * B(s))^\beta.
\end{aligned}$$

8.7. Let $s \in \mathcal{S}$ and $I \subseteq \mathcal{W}$ with $Is = I$.

Lemma: $\forall M \in \mathcal{C}$, $(M * B(s))_I \simeq M_I \otimes_R B(s)$.

Proof: One has

$$\begin{aligned}
\{(M * B(s))_I\}^Q &= \coprod_{w \in I} (M * B(s))^Q_w \quad \text{by (1.4.ii)} \\
&= \coprod_{w \in I} \{(M^Q_w \otimes_R B(s)^Q_e) \oplus (M^Q_{ws} \otimes_R B(s)^Q_s)\} \quad \text{by (2.3)} \\
&= \coprod_{w \in I} \{((M_I)^Q_w \otimes_R B(s)^Q_e) \oplus ((M_I)^Q_{ws} \otimes_R B(s)^Q_s)\} \quad \text{as } Is = I \\
&= (M_I \otimes_R B(s))^Q,
\end{aligned}$$

and hence

$$\begin{aligned}(M*B(s))_I &= (M*B(s))\cap(M_I\otimes_R B(s))^Q = (M\otimes_{R^s}R(1))\cap\{(M_I)^Q\otimes_{R^s}R(1)\}\\ &\simeq (M\cap(M_I)^Q)\otimes_{R^s}R(1) \quad \text{as } R \text{ is free over } R^s\\ &= M_I\otimes_{R^s}R(1)\simeq M_I\otimes_R B(s).\end{aligned}$$

8.8. Let $M\in\mathcal{C}^{\mathrm{ou}}$, $s\in\mathcal{S}$, $w\in\mathcal{W}$ with $w<ws$. Let I (resp. J) be a $\mathcal{W}$-open (resp. $\mathcal{W}$-closed) with $I\cap J=\{w,ws\}$. Thus, $I\setminus\{w,ws\}=I\setminus J$ and $I\setminus\{ws\}=(I\setminus J)\cup(\leq w)$ are both $\mathcal{W}$-open. As $B(s)$ is free over R, $M\otimes_R B(s)\in\mathcal{C}^{\emptyset}$, which may, however, not belong to $\mathcal{C}^{\mathrm{ou}}$.

Lemma: *If $I=Is$, there are isomorphisms of left graded R-modules*

$$\begin{aligned}(M\otimes_R B(s))_{I\setminus\{ws\}}/(M\otimes_R B(s))_{I\setminus\{w,ws\}} &\simeq M^{\mathrm{ou}}_{\{w,ws\}}(-1),\\ (M\otimes_R B(s))_I/(M\otimes_R B(s))_{I\setminus\{ws\}} &\simeq M^{\mathrm{ou}}_{\{w,ws\}}(1).\end{aligned}$$

Proof: Put $N=M\otimes_R B(s)$. By (1.4) one has all $N_I, N_{I\setminus\{ws\}}, N_{I\setminus\{w,ws\}}\in\mathcal{C}^{\emptyset}$. Put $L_1=N_{I\setminus\{ws\}}/N_{I\setminus\{w,ws\}}$, $L=N_I/N_{I\setminus\{w,ws\}}$, $L_2=N_I/N_{I\setminus\{ws\}}$, and consider an exact sequence

(1) $$0\to L_1\to L\to L_2\to 0.$$

Thus, one has a CD of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1^Q & \longrightarrow & L^Q & \longrightarrow & L_2^Q & \longrightarrow & 0\\ & & \wr\big| & & \wr\big| & & \wr\big| & & \\ 0 & \longrightarrow & N_w^Q & \longrightarrow & N_w^Q\oplus N_{ws}^Q & \longrightarrow & N_{ws}^Q & \longrightarrow & 0.\end{array}$$

By (1.4) again all $L_1, L, L_2\in\mathcal{C}^{\emptyset}$. In particular,

(2) $$L_1=L_1\cap(L_1)^Q\simeq L_1\cap N_w^Q=L_1\cap L_w^Q\simeq L\cap L_w^Q.$$

To see the last isomorphism, if $x\in L\cap L_w^Q$, $x=0$ in $L_2\leq L_2^Q$, and hence $x\in L_1$ by (1).

Now,

$$\begin{aligned}L &= (M_I\otimes_R B(s))/(M_{I\setminus\{w,ws\}}\otimes_R B(s)) \quad \text{by (8.7)}\\ &\simeq (M_I/M_{I\setminus\{w,ws\}})\otimes_{R^s}R(1)\\ &\simeq M^{\mathrm{ou}}_{\{w,ws\}}\otimes_{R^s}R(1) \quad \text{by (8.3) as } M\in\mathcal{C}^{\mathrm{ou}}\\ &= M^{\mathrm{ou}}_{\{w,ws\}}\otimes_R B(s).\end{aligned}$$

Then $L_1\simeq L\cap L_w^Q\simeq(M^{\mathrm{ou}}_{\{w,ws\}}\otimes_R B(s))\cap L_w^Q$.

By (2.3.iii) one has

$$
\begin{array}{ccc}
L(-1) \simeq (M^{\mathrm{ou}}_{\{w,ws\}} \otimes_R B(s))(-1) & \longrightarrow & L^Q_w \simeq (M^{\mathrm{ou}}_{\{w,ws\}} \otimes_R B(s))^Q_w \simeq (M^{\mathrm{ou}}_{\{w,ws\}})^Q_w \oplus (M^{\mathrm{ou}}_{\{w,ws\}})^Q_{ws} \\
\wr\| & & \\
M^{\mathrm{ou}}_{\{w,ws\}} \otimes_{R^s} R^s \oplus M^{\mathrm{ou}}_{\{w,ws\}} \otimes_{R^s} R^s\delta & (m_1 \otimes 1, m_2 \otimes \delta) \longmapsto & (m_{1,w} + m_{2,w}\delta, m_{1,ws} + m_{2,ws}s\delta) \\
\downarrow & \big\downarrow & \\
L^Q_{ws} \simeq (M^{\mathrm{ou}}_{\{w,ws\}} \otimes_R B(s))^Q_{ws} & & \\
\wr| & & \\
(M^{\mathrm{ou}}_{\{w,ws\}})^Q_{ws} \oplus (M^{\mathrm{ou}}_{\{w,ws\}})^Q_w & (m_{1,ws} + m_{2,ws}\delta, m_{1,w} + m_{2,w}s\delta). &
\end{array}
$$

As $\mathrm{supp}_{\mathcal{W}}(L) = \{w, ws\}$, one has that

$$
\begin{aligned}
(3) \qquad (m_1 \otimes 1, m_2 \otimes \delta) \in L^Q_w \quad &\text{iff} \quad (m_{1,ws} + m_{2,ws}\delta, m_{1,w} + m_{2,w}s\delta) = 0 \\
&\text{iff} \quad \begin{cases} m_{1,w} = -m_{2,w}s\delta = -(ws\delta)m_{2,w}, \\ m_{1,ws} = -m_{2,ws}\delta = -(ws\delta)m_{2,ws}, \end{cases} \\
&\text{iff} \quad m_1 = -(ws\delta)m_2 \quad \text{as } \mathrm{supp}_{\mathcal{W}}(m_1), \mathrm{supp}_{\mathcal{W}}(m_2) \subseteq \{w, ws\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
L_1 \simeq L \cap L^Q_w &\simeq \{(-(ws\delta)m \otimes 1, m \otimes \delta) | m \in M^{\mathrm{ou}}_{\{w,ws\}}\}(1) \\
&\simeq M^{\mathrm{ou}}_{\{w,ws\}}(-1) \quad \text{as } \deg(\delta) = 2 = \deg(ws\delta).
\end{aligned}
$$

Consider next an epi of graded left R-modules

$$
\phi : L \simeq M^{\mathrm{ou}}_{\{w,ws\}} \otimes_{R^s} R(1) \to M^{\mathrm{ou}}_{\{w,ws\}}(1) \quad \text{via} \quad m \otimes a \mapsto (wsa)m,
$$

under which $(m_1 \otimes 1, m_2 \otimes \delta) \mapsto m_1 + (ws\delta)m_2$. Then $\ker \phi = L \cap L^Q_w$ by (3), and hence

$$
\begin{aligned}
M^{\mathrm{ou}}_{\{w,ws\}}(1) &\simeq L/(L \cap L^Q_w) = L/L_1 \quad \text{by (2)} \\
&\simeq L_2.
\end{aligned}
$$

8.9. Let $s \in \mathcal{S}$, $w \in \mathcal{W}$ with $w < ws$. Recall from [BB, Prop. 2.2.7] that

(1) $\qquad \forall x \in \mathcal{W}$ with $x < ws$ and $x < xs$, $x \leq w$ and $xs \leq ws$.

Likewise,

(2) $\qquad$ if $x > w$ and $x > xs$, then $xs \geq w$.

For x has a reduced expression $(s_1, \dots, s_r)$ with $s_r = s$. As $ws > w$, a subexpression of $(s_1, \dots, s_{r-1})$ gives w, and hence $xs = s_1 \dots s_{r-1} \geq w$.

Note also that

(3) $\qquad \forall I$ $\mathcal{W}$-open, $I \cup Is$ remains $\mathcal{W}$-open.

For if $x \in Is$ and $y < x$, $xs \in I$, and hence we may assume $x > xs$. If $y < ys$, $y \leq xs$ by (1), and hence $y \in I$. If $y > ys$, $ys \leq xs$ by (1) again. Then $ys \in I$, and hence $y \in Is$.

Lemma: *Let $M \in \mathcal{C}_\nabla$ and I $\mathcal{W}$-open, J $\mathcal{W}$-closed. There are isomorphisms of graded left R-modules*

$$(M * B(s))_I/(M * B(s))_{I\setminus J} \simeq \begin{cases} M^{\mathrm{ou}}_{\{w,ws\}}(1) & \text{if } I\cap J = \{ws\}, \\ M^{\mathrm{ou}}_{\{w,ws\}}(-1) & \text{if } I \cap J = \{w\}. \end{cases}$$

Proof: Put $N = M * B(s) \in \mathcal{C}$; $M \in \mathcal{C}$ by (8.5).

Assume first that $I \cap J = \{ws\}$. Put $I_1 = (\leq ws)$, which is right s-invariat by (1). As I is $\mathcal{W}$-open with $ws \in I$, $I_1 \subseteq I$. Thus

$$N_{I_1}/N_{I_1\setminus\{ws\}} \hookrightarrow N_I/N_{I\setminus\{ws\}} = N_I/N_{I\setminus J}.$$

As $I_1 \cap (\geq w) = \{w, ws\}$, $N_{I_1}/N_{I_1\setminus\{ws\}} \simeq M^{\mathrm{ou}}_{\{w,ws\}}(1)$ by (8.8), and hence $M^{\mathrm{ou}}_{\{w,ws\}}(1) \leq N_I/N_{I\setminus J}$.

Let $t \in \mathcal{T}$ and put $\beta = \alpha_t$. As $R^\beta = R[\frac{1}{\alpha_u}|u \in \mathcal{T}\setminus\{t\}]$ is flat over R, $M^\beta = R^\beta \otimes_R M \in \mathcal{C}^{\mathrm{ou}}(R^\beta)$ the category $\mathcal{C}^{\mathrm{ou}}$ over R^β [BCA, Lem. I.2.6.7]. Then (LE) holds on $N^\beta \simeq M^\beta * B(s)$ by (8.6.iii), and hence (S^{ou}) holds on $N^\beta = (N^\beta)^\beta$ by (8.2). Thus, $N^\beta \in \mathcal{C}^{\mathrm{ou}}(R^\beta)$. In particular, $(N^\beta)^{\mathrm{ou}}_{I\cap J}$ does not depend on the choice of I and J by (8.3), and hence

$$\begin{aligned}(N^\beta)_I/(N^\beta)_{I\setminus J} \simeq (N^\beta)^{\mathrm{ou}}_{\{ws\}} &\simeq (N^\beta)_{I_1}/(N^\beta)_{I_1\setminus\{ws\}} \\ &\simeq (M^\beta)^{\mathrm{ou}}_{\{w,ws\}}(1) \quad \text{by (8.8) again.}\end{aligned}$$

As M admits a ∇-flag, $M^{\mathrm{ou}}_{\{w,ws\}}$ is graded free over R by (8.5). Then

$$\begin{aligned} M^{\mathrm{ou}}_{\{w,ws\}}(1) &= \cap_{t\in\mathcal{T}}\{M^{\mathrm{ou}}_{\{w,ws\}}(1)\}^{\alpha_t} \quad \text{by (8.1.1)} \\ &= \cap_{t\in\mathcal{T}}(M^{\alpha_t})^{\mathrm{ou}}_{\{w,ws\}}(1) = \cap_{t\in\mathcal{T}}\{(N^{\alpha_t})_I/(N^{\alpha_t})_{I\setminus J}\} \\ &\geq N_I/N_{I\setminus J} \quad \text{as } N_I/N_{I\setminus J} \in \mathcal{C}^\emptyset \text{ by (1.4).}\end{aligned}$$

Thus, $N_I/N_{I\setminus J} \simeq M^{\mathrm{ou}}_{\{w,ws\}}(1)$.

Assume next that $I \cap J = \{w\}$. Let us first observe that

(4) $$N_I/N_{I\setminus J} \hookrightarrow M^{\mathrm{ou}}_{\{w,ws\}}(-1).$$

As $I \setminus J = I \setminus (\geq w)$, we may assume $J = (\geq w)$. Then $J = Js$ by (2). Put $I_2' = I \cup Is$, which is $\mathcal{W}$-open by (3). Then $I_2' \cap J = (I\cap J) \cup (Is \cap J) = (I \cap J) \cup (Is \cap Js) = (I\cap J) \cup (I \cap J)s = \{w, ws\}$, and hence $I_2' \setminus \{w, ws\} = I_2' \setminus J$ and $I_2' \setminus \{ws\} = I_2' \setminus (\geq ws)$ are both $\mathcal{W}$-open. Also, $I_2' \setminus \{ws\} \supseteq I$; if $I \ni ws$, $I \supseteq \{w, ws\}$ implying $I \cap J \supseteq \{w, ws\}$, absurd. As $I \not\ni ws$ again, $I \setminus \{w, ws\} = I \setminus \{w\} = I \setminus J$, and hence $N_I/N_{I\setminus J} \hookrightarrow N_{I_2'\setminus\{ws\}}/N_{I_2'\setminus\{w,ws\}} \simeq M^{\mathrm{ou}}_{\{w,ws\}}(-1)$ by (8.8) again, and (4) holds.

Take now a sequence of $\mathcal{W}$-opens $\emptyset = I_0 \subset \cdots \subset I_{|\mathcal{W}|} = \mathcal{W}$ with $|I_{i+1}| = |I_i| + 1$ $\forall i$ such that $I_k = I$ and $I_{k-1} = I \setminus \{w\}$ for some $k \in [1, |\mathcal{W}|]$. Put $l = |\mathcal{W}|$ and write $I_i = I_{i-1} \sqcup \{w_i\}$.

Assume for the moment that $\mathbb{K}$ is a field. Then $\dim_\mathbb{K} N^d = \sum_{j=1}^l \dim_\mathbb{K}(N_{I_j}/N_{I_{j-1}})^d$. By the case $I \cap J = \{ws\}$ and by (4) one has

$$\dim_\mathbb{K}(N_{I_j}/N_{I_{j-1}})^d \leq \dim_\mathbb{K}(M^{\mathrm{ou}}_{\{w_j,w_js\}})^{d+\varepsilon(w_j)} \quad \text{with} \quad \varepsilon(w_j) = \begin{cases} -1 & \text{if } w_j < w_js, \\ 1 & \text{else.}\end{cases}$$

Then

$$\begin{aligned}\sum_{j=1}^{l}\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{\{w_j,w_js\}})^{d+\varepsilon(w_j)} &= \sum_{j=1}^{l}\{\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{w_j})^{d+\varepsilon(w_j)}+\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{w_js})^{d+\varepsilon(w_j)}\}\\
&= \sum_{w_js>w_j}\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{\{w_j\}})^{d-1}+\sum_{w_js>w_j}\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{\{w_js\}})^{d-1}\\
&\quad+\sum_{w_js<w_j}\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{\{w_j\}})^{d+1}+\sum_{w_js<w_j}\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{\{w_js\}})^{d+1}\\
&= \sum_{w_js>w_j}\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{\{w_j\}})^{d-1}+\sum_{w_js<w_j}\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{\{w_j\}})^{d+1}\\
&\quad+\sum_{w_js<w_j}\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{\{w_j\}})^{d+1}+\sum_{w_js<w_j}\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{\{w_j\}})^{d-1}\\
&= \sum_{j}\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{\{w_j\}})^{d-1}+\sum_{j}\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{\{w_j\}})^{d+1}=\dim_{\mathbb{K}}M^{d-1}+\dim_{\mathbb{K}}M^{d+1}.\end{aligned}$$

On the other hand, taking $\delta\in V$ with $\langle\delta,\alpha_s^\vee\rangle=1$, one has $N=M\otimes_{R^s}R(1)=M(1)\otimes_{R^s}R^s\oplus M(1)\otimes_{R^s}R^s\delta$, and hence

$$\begin{aligned}\dim_{\mathbb{K}}N^d &= \dim_{\mathbb{K}}M(1)^d+\dim_{\mathbb{K}}M(1)^{d-2}\quad\text{as }\deg\delta=2\\
&= \dim_{\mathbb{K}}M^{d+1}+\dim_{\mathbb{K}}M^{d-1}=\sum_j\dim_{\mathbb{K}}(M^{\mathrm{ou}}_{\{w_j,w_js\}})^{d+\varepsilon(w_j)}\\
&\geq \sum_j\dim_{\mathbb{K}}(N_{I_j}/N_{I_{j-1}})^d=\dim_{\mathbb{K}}N^d.\end{aligned}$$

We must then have in (4) an isomorphism

$$N_I/N_{I\setminus J}\xrightarrow{\sim}M^{\mathrm{ou}}_{\{w,ws\}}(-1). \tag{5}$$

Back to general complete DVR $\mathbb{K}$ with maximal ideal $\mathfrak{m}$, write $\mathfrak{m}=(a)$. By (8.1.3) one has $N_{I_j}\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\simeq\{N\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\}_{I_j}$, and hence we may regard $(N_{I_j}\otimes_{\mathbb{K}}\mathbb{K}/\mathfrak{m})_j$ giving a filtration of $N\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$ with $(N_I/N_{I\setminus J})\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})\simeq M^{\mathrm{ou}}_{\{w,ws\}}(-1)\otimes_{\mathbb{K}}(\mathbb{K}/\mathfrak{m})$ by (5). It then follows from (4) and by graded NAK that $N_I/N_{I\setminus J}\xrightarrow{\sim}M^{\mathrm{ou}}_{\{w,ws\}}(-1)$.

8.10. Let $M\in\mathcal{C}_\nabla$ and $s\in\mathcal{S}$.

Lemma: *$\forall I_1,I_2$ $\mathcal{W}$-open with $I_1\supseteq I_2$, $(M*B(s))_{I_1}/(M*B(s))_{I_2}$ is left graded free over R.*

Proof: Put $N=M*B(s)$. Take a sequence $I_2=I'_0\subset I'_1\subset\cdots\subset I'_r=I_1$ of $\mathcal{W}$-opens with $|I'_j|=|I'_{j-1}|+1$ $\forall i\in[1,r]$, and write $I'_j=I'_{j-1}\sqcup\{w_j\}$. As $\{w_j\}=I_j\setminus I_{j-1}=I_j\cap(\mathcal{W}\setminus I_{j-1})$, one has from (8.9)

$$N_{I'_j}/N_{I'_{j-1}}\simeq M^{\mathrm{ou}}_{\{w_j,w_js\}}(\varepsilon(w_j))\quad\exists\varepsilon(w_j)\in\{\pm1\},$$

which is graded free over R by (8.5). Thus, $N_{I_1}/N_{I_2}=N_{I'_r}/N_{I'_0}$ is graded free over R.

8.11. Though we do not know if $\mathcal{C}^{\mathrm{ou}}*\mathfrak{S}\mathrm{Bimod}=\mathcal{C}^{\mathrm{ou}}$,

Proposition: *$\mathcal{C}_\nabla * \mathfrak{S}\mathrm{Bimod} = \mathcal{C}_\nabla$. In particular, $\mathfrak{S}\mathrm{Bimod} \le \mathcal{C}_\nabla$.*

Proof: Let $M \in \mathcal{C}_\nabla$. We have by (8.10) only to show that $M * B(s) \in \mathcal{C}^{\mathrm{ou}}$, $s \in \mathcal{S}$. Put $N = M * B(s)$.

We know from (8.6) that (LE) holds on N. To see that (S^{ou}) holds on N, let I_1 and I_2 be 2 $\mathcal{W}$-opens. Consider $N_{I_1}/N_{I_1\cap I_2} \hookrightarrow N_{I_1\cup I_2}/N_{I_2}$, both terms of which are graded free over R by (8.10). Let $t \in \mathcal{T}$ and put $\beta = \alpha_t$. Then (S^{ou}) holds on N^{α_t} by (8.2), and hence the imbedding turns invertible upon base extension to R^β by (8.3). Thus,

$$\begin{aligned} N_{I_1\cup I_2}/N_{I_2} &= \cap_{t\in\mathcal{T}} (N_{I_1\cup I_2}/N_{I_2})^{\alpha_t} \quad \text{by (8.1.1)} \\ &\simeq \cap_{t\in\mathcal{T}} (N^{\alpha_t}_{I_1\cup I_2}/N^{\alpha_t}_{I_2}) \quad \text{by [BCA, Lem. I.2.6.7]} \\ &\simeq \cap_{t\in\mathcal{T}} (N^{\alpha_t}_{I_1}/N^{\alpha_t}_{I_1\cap I_2}) = N_{I_1}/N_{I_1\cap I_2}, \end{aligned}$$

and hence $N_{I_1\cup I_2} = N_{I_1} + N_{I_2}$.

8.12. Let $[\mathcal{C}_\nabla]$ denote the split Grothendieck group of $\mathcal{C}_\nabla$ and define $\mathrm{ch}_\nabla : [\mathcal{C}_\nabla] \to \mathcal{H}$ by

$$[M] \mapsto \sum_{w\in\mathcal{W}} v^{\ell(w)} \overline{\mathrm{grk}(M^{\mathrm{ou}}_w)} H_w = \sum_{w\in\mathcal{W}} \sum_{j\in\mathbb{Z}} v^{-j} (M^{\mathrm{ou}}_w : R(w)(\ell(w)+j)) H_w \quad \forall M \in \mathcal{C}_\nabla,$$

extending (5.9) to $[\mathcal{C}_\nabla]$. We will abbreviate $\mathrm{ch}_\nabla([M])$ as $\mathrm{ch}_\nabla(M)$. In particular, $\forall s \in \mathcal{C}$,

$$\begin{aligned} \mathrm{ch}_\nabla(B(s)) &= \overline{\mathrm{grk}(B(s)^{\mathrm{ou}}_e)} + \overline{v\mathrm{grk}(B(s)^{\mathrm{ou}}_s)} \\ &= \overline{\mathrm{grk}(B(s)_e)} + \overline{v\mathrm{grk}(B(s)/B(s)_e)} H_s = \overline{\mathrm{grk}(B(s)_e)} + \overline{v\mathrm{grk}(B(s)^s)} H_s \\ &= v + \overline{v\mathrm{grk}(R(e)(1))} H_s \quad \text{by (2.2.10, 13)} \\ &= v + H_s = \underline{H}_s \\ &= \mathrm{ch}(B(s)) \quad \text{from (5.2)}. \end{aligned}$$

Then, one has as in [S07, Prop. 5.9],

Corollary: *ch_∇ is $\mathcal{H}$-linear in the sense that $\forall M \in \mathcal{C}_\nabla$, $\forall B \in \mathfrak{S}\mathrm{Bimod}$,*

$$\mathrm{ch}_\nabla(M * B) = \mathrm{ch}_\nabla(M)\mathrm{ch}(B).$$

Proof: We may assume $B = B(s)$ for some $s \in \mathcal{S}$. One has from (8.9)

$$\begin{aligned} \mathrm{grk}((M * B(s))^{\mathrm{ou}}_w) &= \begin{cases} v\mathrm{grk}(M^{\mathrm{ou}}_{\{w,ws\}}) & \text{if } ws < w \\ v^{-1}\mathrm{grk}(M^{\mathrm{ou}}_{\{w,ws\}}) & \text{else} \end{cases} \\ &= \begin{cases} v\{\mathrm{grk}(M^{\mathrm{ou}}_w) + \mathrm{grk}(M^{\mathrm{ou}}_{\{ws\}})\} & \text{if } ws < w \\ v^{-1}\{\mathrm{grk}(M^{\mathrm{ou}}_w) + \mathrm{grk}(M^{\mathrm{ou}}_{\{ws\}})\} & \text{else.} \end{cases} \end{aligned}$$

Then

$$\begin{aligned} \mathrm{ch}_\nabla(M * B(s)) &= \sum_{w\in\mathcal{W}} v^{\ell(w)} \overline{\mathrm{grk}((M * B(s))^{\mathrm{ou}}_w)} H_w \\ &= \sum_{\substack{w\in\mathcal{W} \\ ws<w}} v^{\ell(w)-1} \overline{\mathrm{grk}(M^{\mathrm{ou}}_w) + \mathrm{grk}(M^{\mathrm{ou}}_{ws})} H_w + \sum_{\substack{w\in\mathcal{W} \\ ws>w}} v^{\ell(w)+1} \overline{\mathrm{grk}(M^{\mathrm{ou}}_w) + \mathrm{grk}(M^{\mathrm{ou}}_{ws})} H_w \end{aligned}$$

while

$$
\begin{aligned}
\mathrm{ch}_\nabla(M)\underline{H}_s &= \{\sum_{w\in\mathcal{W}} v^{\ell(w)}\overline{\mathrm{grk}(M_w^{\mathrm{ou}})}H_w\}(H_s+v)\\
&= \sum_{w\in\mathcal{W}} v^{\ell(w)}\overline{\mathrm{grk}(M_w^{\mathrm{ou}})}\begin{cases} H_{ws}+vH_w & \text{if } ws>w\\ H_{ws}+v^{-1}H_w & \text{else}\end{cases}\\
&= \sum_{\substack{w\in\mathcal{W}\\ ws>w}} v^{\ell(w)}\overline{\mathrm{grk}(M_w^{\mathrm{ou}})}(H_{ws}+vH_w)+\sum_{\substack{w\in\mathcal{W}\\ ws<w}} v^{\ell(w)}\overline{\mathrm{grk}(M_w^{\mathrm{ou}})}(H_{ws}+v^{-1}H_w),
\end{aligned}
$$

and hence $\mathrm{ch}_\nabla(M*B(s))=\mathrm{ch}_\nabla(M)\underline{H}_s=\mathrm{ch}_\nabla(M)\mathrm{ch}(B(s))$, as desired.

8.13. As the category $\mathcal{C}_\nabla$ is additive, but not necessarily abelian, we define an exact structure after [F08a, 2.5], [F08b, 4.1].

Definition: *We say that condition (ES) holds on a complex* $M_1\to M_2\to M_3$ *in* $\mathcal{C}_\nabla$ *iff the sequence* $0\to(M_1)_w^{\mathrm{ou}}\to(M_2)_w^{\mathrm{ou}}\to(M_3)_w^{\mathrm{ou}}\to 0$ *is exact* $\forall w\in\mathcal{W}$ *as graded R-modules. We define a category* $\mathcal{C}_P^{\mathrm{ou}}$ *to be the full category of* $\mathcal{C}_\nabla$ *consisting of M such that* $\forall$ *complex* $M_1\to M_2\to M_3$ *in* $\mathcal{C}_\nabla$ *with (ES), the induced sequence* $0\to\mathcal{C}(M,M_1(n))\to\mathcal{C}(M,M_2(n))\to\mathcal{C}(M,M_3(n))\to 0$ *is exact* $\forall n\in\mathbb{Z}$.

Thus, $\mathcal{C}_P^{\mathrm{ou}}$ consists of the "projectives" in $\mathcal{C}_\nabla$. As $R(e)\in\mathcal{C}_\nabla$ and as $M_e=M_e^{\mathrm{ou}}$ $\forall M\in\mathcal{C}_\nabla$, one has by (1.4.3)

$$R(e)\in\mathcal{C}_P^{\mathrm{ou}}. \tag{1}$$

We will show that $\mathfrak{S}\mathrm{Bimod}=\mathcal{C}_P^{\mathrm{ou}}$.

8.14. Let $M_1\to M_2\to M_3$ be a complex in $\mathcal{C}_\nabla$ with (ES) holding. Consider a refinement I_n by $\mathcal{W}$-opens of the length filtration $(M_i)_{\leq l}$, $l\in\mathbb{N}$, of each M_i, $i\in[1,3]$, such that $I_0=\emptyset$, $I_n=I_{n-1}\sqcup\{x_n\}$ for some $x_n\in\mathcal{W}$, $n\in[1,|\mathcal{W}|]$. Thus, $I_{|\mathcal{W}|}=\mathcal{W}$. Consider a CD

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (M_1)_{I_1} & \longrightarrow & (M_2)_{I_1} & \longrightarrow & (M_3)_{I_1} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (M_1)_{I_2} & \longrightarrow & (M_2)_{I_2} & \longrightarrow & (M_3)_{I_2} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (M_1)_{x_2}^{\mathrm{ou}} & \longrightarrow & (M_2)_{x_2}^{\mathrm{ou}} & \longrightarrow & (M_3)_{x_2}^{\mathrm{ou}} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

As $I_1=\{x_1\}$ and as (ES) ensures an exact sequence $0\to(M_1)_{x_n}^{\mathrm{ou}}\to(M_2)_{x_n}^{\mathrm{ou}}\to(M_3)_{x_n}^{\mathrm{ou}}\to 0$ $\forall n$, the top and the bottom rows are both exact. As the columns are all split exact at least as left R-modules, the middle row must be exact. Repeating the argument, one obtains that $0 \to M_1 \to M_2 \to M_3 \to 0$ must be exact. Moreover,

Lemma: *$\forall K$ $\mathcal{W}$-locally closed, $0 \to (M_1)^{\mathrm{ou}}_K \to (M_2)^{\mathrm{ou}}_K \to (M_3)^{\mathrm{ou}}_K \to 0$ is exact.*

Proof: By (8.4) on the complex $(M_1)^{\mathrm{ou}}_K \to (M_2)^{\mathrm{ou}}_K \to (M_3)^{\mathrm{ou}}_K$ the property (ES) holds, and hence the assertion by above.

8.15. We now show

Theorem: *$\mathcal{C}^{\mathrm{ou}}_P * \mathfrak{S}\mathrm{Bimod} = \mathcal{C}^{\mathrm{ou}}_P = \mathfrak{S}\mathrm{Bimod}$.*

Proof: For the first equality we have only to show that $M * B(s) \in \mathcal{C}^{\mathrm{ou}}_P$ $\forall M \in \mathcal{C}^{\mathrm{ou}}_P$ $\forall s \in \mathcal{S}$. We know from (8.11) that $M * B(s) \in \mathcal{C}_\nabla$. Assume that (ES) holds on a complex $M_1 \to M_2 \to M_3$ in $\mathcal{C}_\nabla$. By adjunction (2.6) one has a CD

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C}(M * B(s), M_1) & \longrightarrow & \mathcal{C}(M * B(s), M_2) & \longrightarrow & \mathcal{C}(M * B(s), M_3) & \longrightarrow & 0 \\ & & \wr\vert & & \wr\vert & & \wr\vert & & \\ 0 & \longrightarrow & \mathcal{C}(M, M_1 * B(s)) & \longrightarrow & \mathcal{C}(M, M_2 * B(s)) & \longrightarrow & \mathcal{C}(M, M_3 * B(s)) & \longrightarrow & 0. \end{array}$$

Thus, the exactness of the top row will follow if (ES) holds on the complex $M_1 * B(s) \to M_2 * B(s) \to M_3 * B(s)$; we know that the complex lies in $\mathcal{C}_\nabla$ by (8.11). $\forall w \in \mathcal{W}$, one has by (8.9) a CD

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (M_1 * B(s))^{\mathrm{ou}}_w & \longrightarrow & (M_2 * B(s))^{\mathrm{ou}}_w & \longrightarrow & (M_3 * B(s))^{\mathrm{ou}}_w & \longrightarrow & 0 \\ & & \wr\vert & & \wr\vert & & \wr\vert & & \\ 0 & \longrightarrow & (M_1)^{\mathrm{ou}}_{\{w,ws\}}(\pm 1) & \longrightarrow & (M_2)^{\mathrm{ou}}_{\{w,ws\}}(\pm 1) & \longrightarrow & (M_3)^{\mathrm{ou}}_{\{w,ws\}}(\pm 1) & \longrightarrow & 0 \end{array}$$

with ± 1 varying simultaneously, the bottom row of which is exact by (8.14). The first equality holds.

As $R(e)(n) \in \mathcal{C}^{\mathrm{ou}}_P$ by (8.13.1), one obtains by above that $\mathfrak{S}\mathrm{Bimod} \subseteq \mathcal{C}^{\mathrm{ou}}_P$. Assume now that $M \in \mathcal{C}^{\mathrm{ou}}_P$ is indecomposable. Refining the length filtration of M, take $\mathcal{W}$-opens I and I' with $I' = I \sqcup \{w\}$ for some w such that $\mathrm{supp}_{\mathcal{W}}(M) \setminus I = \{w\}$. Thus, $\mathrm{supp}_{\mathcal{W}}(M) \subseteq I'$ and $\mathrm{supp}_{\mathcal{W}}(M/M_I) = \{w\}$. Then

$$\begin{aligned} (M/M_I)^{\mathrm{ou}}_w &= M/M_I \quad \text{by (8.3.iii)} \\ &= M_{I'}/M_I \quad \text{as } I' \supseteq \mathrm{supp}_{\mathcal{W}}(M) \\ &\simeq M^{\mathrm{ou}}_w, \end{aligned}$$

and hence (ES) holds on the complex $M_I \to M \xrightarrow{q} M/M_I$ in $\mathcal{C}_\nabla$. Let $R(w)(n) \underset{\pi}{\overset{i}{\rightleftarrows}} M/M_I$ such that $\pi \circ i = \mathrm{id}_{R(w)(n)}$ for some $n \in \mathbb{Z}$. As $B(w)^w \simeq R(w)(\ell(w))$ by (5.1) and as $B(w) \in \mathcal{C}^{\mathrm{ou}}_P$, one obtains from (1.4.v)

$$\begin{array}{ccc} \mathcal{C}(B(w)(n-\ell(w)), M) & \twoheadrightarrow & \mathcal{C}(B(w)(n-\ell(w)), M/M_I) \\ & \searrow & \wr\vert \\ & & \mathcal{C}(R(w)(n), M/M_I). \end{array}$$

Let $\hat{i} \in \mathcal{C}(B(w)(n-\ell(w)), M)$ be a lift of i. Likewise, let $\widehat{\pi \circ q} \in \mathcal{C}(M, B(w)(n-\ell(w)))$ be a lift of $\pi \circ q$ along

$$\begin{array}{ccc} \mathcal{C}(M, B(w)(n-\ell(w))) & \twoheadrightarrow & \mathcal{C}(M, R(w)(n)) \\ & \searrow & \wr\| \\ & & \mathcal{C}(M/M_I, R(w)(n)). \end{array}$$

Then $\mathrm{id} - \widehat{\pi \circ q} \circ \hat{i} \notin \mathcal{C}(B(w)(n-\ell(w)), B(w)(n-\ell(w)))^\times$. As $B(w)(n-\ell(w))$ is indecomposable, we must have $\widehat{\pi \circ q} \circ \hat{i} \notin \mathcal{C}(B(w)(n-\ell(w)), B(w)(n-\ell(w)))^\times$. Thus, $\hat{i}$ splits, and hence is invertible.

8.16. Turning to $\mathcal{W}$-closed, we say that $M \in \mathcal{C}^\emptyset$ belongs to $\mathcal{C}^{\mathrm{fe}}$ iff the following two properties (S^{fe}) and (LE) hold on M:

$$(\mathrm{S}^{\mathrm{fe}}) \qquad \forall \mathcal{W}\text{-closed } I_1 \text{ and } I_2, M_{I_1 \cup I_2} = M_{I_1} + M_{I_2},$$

$$(\mathrm{LE}) \qquad \forall t \in \mathcal{T}, M^{\alpha_t} = \coprod_{\Omega \in \langle t \rangle \backslash \mathcal{W}} (M^{\alpha_t} \cap \coprod_{x \in \Omega} M_x^Q).$$

Writing $K = I \cap J$ with I $\mathcal{W}$-closed and J $\mathcal{W}$-open for a $\mathcal{W}$-locally closed subset K of $\mathcal{W}$, $\forall M \in \mathcal{C}^{\mathrm{fe}}$, put $M_K^{\mathrm{fe}} = M_I / M_{I \setminus J}$. Then the analogues of (8.3) and (8.4) hold for M_K^{fe}.

8.17. Let $M \in \mathcal{C}^{\mathrm{fe}}$. $\forall w \in \mathcal{W}$, let $M_w^{\mathrm{fe}} = M_{\{w\}}^{\mathrm{fe}}$, and say that M admits a Δ-flag iff each M_w^{fe}, $w \in \mathcal{W}$, is graded free over R. Let $\mathcal{C}_\Delta$ denote the full subcategory of $\mathcal{C}^\emptyset$ consisting of those with Δ-flags.

Let $M \in \mathcal{C}_\Delta$ and I $\mathcal{W}$-closed, J $\mathcal{W}$-open, $s \in \mathcal{S}$, $w \in \mathcal{W}$ with $ws < w$; note that the order is reveresed here. As in (8.9) there are isomorphisms of left graded R-modules

$$(1) \qquad (M * B(s))_I / (M * B(s))_{I \setminus J} \simeq \begin{cases} M_{\{w,ws\}}^{\mathrm{ou}}(1) & \text{if } I \cap J = \{ws\}, \\ M_{\{w,ws\}}^{\mathrm{ou}}(-1) & \text{if } I \cap J = \{w\}. \end{cases}$$

Then, as in (8.11), one obtains that

$$(2) \qquad \mathcal{C}_\Delta * \mathfrak{S}\mathrm{Bimod} = \mathcal{C}_\Delta.$$

As $R(e)(n) \in \mathcal{C}_\Delta$ $\forall n \in \mathbb{Z}$, together with (8.11) one has

$$(3) \qquad \mathfrak{S}\mathrm{Bimod} \leq \mathcal{C}_\Delta \cap \mathcal{C}_\nabla.$$

Let now $[\mathcal{C}_\Delta]$ denote the split Grothendieck group of $\mathcal{C}_\Delta$, and define $\mathrm{ch}_\Delta : [\mathcal{C}_\Delta] \to \mathcal{H}$ via

$$[M] \mapsto \sum_{w \in \mathcal{W}} v^{\ell(w)} \mathrm{grk}(M_w^{\mathrm{fe}}) H_w = \sum_{w \in \mathcal{W}} \sum_{i \in \mathbb{Z}} v^i (M_w^{\mathrm{fe}} : R(w)(-\ell(w)+i)) H_w \quad \forall M \in \mathcal{C}_\Delta.$$

In particular, $\forall s \in \mathcal{S}$,

$$\begin{aligned} (4) \qquad \mathrm{ch}_\Delta(B(s)) &= \mathrm{gr}(B(s)_e^{\mathrm{fe}}) + v\mathrm{gr}(B(s)_s^{\mathrm{fe}}) H_s = \mathrm{gr}(B(s)^e) + v\mathrm{gr}(B(s)_s) H_s \quad \text{by (1.4.2)} \\ &= v + vv^{-1} H_s \quad \text{by (2.2.12, 11)} \\ &= \underline{H}_s. \end{aligned}$$

One shows as in (8.12) that ch_Δ is $\mathcal{H}$-linear: $\forall M \in \mathcal{C}_\Delta$, $\forall B \in \mathfrak{S}\mathrm{Bimod}$,

$$\mathrm{ch}_\Delta(M * B) = \mathrm{ch}_\Delta(M)\mathrm{ch}(B), \tag{5}$$

obtaining an analogue of [S07, Prop. 5.7]. Thus, together with (8.12),

$$\mathrm{ch}_\Delta = \mathrm{ch} = \mathrm{ch}_\nabla \quad \text{on } [\mathfrak{S}\mathrm{Bimod}]. \tag{6}$$

Then, $\forall B \in \mathfrak{S}\mathrm{Bimod}$, $\forall w \in \mathcal{W}$, $v^{\ell(w)}\mathrm{grk}(B_w^{\mathrm{fe}}) = v^{-\ell(w)}\mathrm{grk}(B^w) = v^{\ell(w)}\overline{\mathrm{grk}(B_w^{\mathrm{ou}})}$, and hence

$$B_w^{\mathrm{fe}}(2\ell(w)) \simeq B^w \simeq D(B_w^{\mathrm{ou}})(2\ell(w)). \tag{7}$$

8.18. If I is $\mathcal{W}$-open (resp. $\mathcal{W}$-closed), one has from (1.4.2)

$$M^I \simeq M/M_{\mathcal{W}\setminus I} \simeq M_I^{\mathrm{fe}} \text{ (resp. } M_I^{\mathrm{ou}}).$$

It follows from (8.5) (resp. (8.16)) and (8.17.3), in accordance with [F08b, Def. 2.8],

Proposition: *$\forall M \in \mathcal{C}_\nabla$ (resp. $\mathcal{C}_\Delta$), $\forall I$ $\mathcal{W}$-closed (resp. $\mathcal{W}$-open), M^I is graded free over R. In particular, $\forall B \in \mathfrak{S}\mathrm{Bimod}$, $\forall I$ $\mathcal{W}$-closed/$\mathcal{W}$-open, M^I is graded free over R.*

8.19. Recall from [L85, 1.4]/[S07, pf of Th. 5.15] an $\mathbb{Z}[v, v^{-1}]$-bilinear form $\langle\ ,\ \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ such that $\langle H_x, H_y \rangle = \delta_{x,y}$ $\forall x, y \in \mathcal{W}$. $\forall s \in \mathcal{S}$,

$$\begin{aligned}\langle H_x \underline{H}_s, H_y \rangle &= \begin{cases} \langle H_{xs} + vH_x, H_y \rangle & \text{if } xs > x, \\ \langle H_{xs} + v^{-1}H_x, H_y \rangle & \text{else} \end{cases} \\ &= \begin{cases} \delta_{xs,y} + v\delta_{x,y} & \text{if } xs > x, \\ \delta_{xs,y} + v^{-1}\delta_{x,y} & \text{else} \end{cases}\end{aligned}$$

while

$$\begin{aligned}\langle H_x, H_y \underline{H}_s \rangle &= \begin{cases} \langle H_x, H_{ys} + vH_y \rangle & \text{if } ys > y, \\ \langle H_x, H_{ys} + v^{-1}H_y \rangle & \text{else} \end{cases} \\ &= \begin{cases} \delta_{x,ys} + v\delta_{x,y} & \text{if } ys > y, \\ \delta_{x,ys} + v^{-1}\delta_{x,y} & \text{else} \end{cases}\end{aligned}$$

If $xs > x$,

$$\begin{aligned}\delta_{xs,y} + v\delta_{x,y} = \delta_{x,ys} + v\delta_{x,y} &= \begin{cases} v & \text{if } x = y \text{ only if } ys > y, \\ 1 & \text{if } x = ys \text{ only if } y > ys, \\ 0 & \text{else} \end{cases} \\ &= \langle H_x, H_y \underline{H}_s \rangle.\end{aligned}$$

If $xs < x$,

$$\begin{aligned}\delta_{xs,y} + v^{-1}\delta_{x,y} &= \begin{cases} 1 & \text{if } xs = y \text{ only if } y < ys, \\ v^{-1} & \text{if } x = y \text{ only if } ys < y, \\ 0 & \text{else} \end{cases} \\ &= \langle H_x, H_y \underline{H}_s \rangle.\end{aligned}$$

Thus, in either case $\langle H_x \underline{H}_s, H_y\rangle = \langle H_x, H_y \underline{H}_s\rangle$, and hence $\forall H, H', H'' \in \mathcal{H}$,

(1) $$\langle HH'', H'\rangle = \langle H, H'H''\rangle.$$

We now obtain an analogue of [S07, Th. 5.15]

Theorem: $\forall B \in \mathfrak{S}\mathrm{Bimod}$, $\forall M \in \mathcal{C}_\nabla$,

$$\mathrm{grk}(\mathcal{C}^\sharp(B, M)) = \sum_{w\in\mathcal{W}} \sum_{i,j\in\mathbb{Z}} (B_w^{\mathrm{fe}} : R(w)(-\ell(w)+i))(M_w^{\mathrm{ou}} : R(w)(\ell(w)+j))v^{j-i}.$$

Proof: We have only to show by (8.12) and (8.18) that

$$\overline{\mathrm{grk}(\mathcal{C}^\sharp(B, M))} = \langle \mathrm{ch}_\Delta(B), \mathrm{ch}_\nabla(M)\rangle.$$

By (1), (8.12) and (8.17.5) we are further reduced to showing that

$$\overline{\mathrm{grk}(\mathcal{C}^\sharp(R(e), M))} = \langle \mathrm{ch}_\Delta(R(e)), \mathrm{ch}_\nabla(M)\rangle.$$

One has

$$\begin{aligned} \mathrm{LHS} &= \overline{\mathrm{grk}(M_e)} \quad \text{by (1.6.3)} \\ &= \overline{\mathrm{grk}(M_e^{\mathrm{ou}})} \end{aligned}$$

while

$$\mathrm{RHS} = \langle 1, \sum_{w\in\mathcal{W}} v^{\ell(w)} \overline{\mathrm{grk}(M_w^{\mathrm{ou}})} H_w\rangle = \overline{\mathrm{grk}(M_e^{\mathrm{ou}})},$$

as desired.

8.20. As in (8.13) we define an "exact structure" on $\mathcal{C}_\Delta$ as follows.

Definition: *We say that condition (ES) holds on a complex $M_1 \to M_2 \to M_3$ in $\mathcal{C}_\Delta$ iff the sequence $0 \to (M_1)_w^{\mathrm{fe}} \to (M_2)_w^{\mathrm{fe}} \to (M_3)_w^{\mathrm{fe}} \to 0$ is exact $\forall w \in \mathcal{W}$ as graded R-modules. We define a category $\mathcal{C}_P^{\mathrm{fe}}$ to be the full category of $\mathcal{C}_\Delta$ consisting of M such that $\forall$ complex $M_1 \to M_2 \to M_3$ in $\mathcal{C}_\Delta$ with (ES) holding, the induced sequence $0 \to \mathcal{C}(M, M_1(n)) \to \mathcal{C}(M, M_2(n)) \to \mathcal{C}(M, M_3(n)) \to 0$ is exact $\forall n \in \mathbb{Z}$.*

Thus, $\mathcal{C}_P^{\mathrm{fe}}$ consists of the "projectives" in $\mathcal{C}_\Delta$. Note that $R(e) \in \mathcal{C}_\nabla \cap \mathcal{C}_\Delta$. As observed in (2.2.15), however,

(1) $$R(e) \in \mathcal{C}_P^{\mathrm{ou}} \setminus \mathcal{C}_P^{\mathrm{fe}},$$

and hence $\mathfrak{S}\mathrm{Bimod} \not\subseteq \mathcal{C}_P^{\mathrm{fe}}$.

8.21. Nonetheless, as in (8.15) one has

Proposition: $\mathcal{C}_P^{\mathrm{fe}} * \mathfrak{S}\mathrm{Bimod} = \mathcal{C}_P^{\mathrm{fe}}$.

9. $\mathcal{Z}$ for finite Weyl groups

Assume that $\mathbb{K}$ is a complete DVR under the characteristic restrictions from §8. Recall $\mathcal{Z}$ over the present $\mathbb{K}$ from (6.1). As $\mathcal{Z}$ is torsion-free over R, $F(\mathcal{Z}) \in \mathcal{C}^{\mathrm{tf}}$. The argument of (6.1.2) actually shows that $F(\mathcal{Z}) \in \mathcal{C}^{\emptyset}$: $F(\mathcal{Z})^{\emptyset} = \coprod_{w\in\mathcal{W}} R^{\emptyset}(w)$, andalso that $\mathrm{supp}_{\mathcal{W}}(F(\mathcal{Z})) = \mathcal{W}$. We will give an isomorphism $R\otimes_{R^{\mathcal{W}}} R \to F(Z)$ of graded $\mathbb{K}$-algebras compatible with the structures of R-bimodules, and show that $F(\mathcal{Z})(\ell(w_0)) \simeq B(w_0)$ in $\mathcal{C}$. We will suppress F.

9.1. We start with

Lemma: $\mathcal{Z} \in \mathcal{C}^{\mathrm{ou}} \cap \mathcal{C}^{\mathrm{fe}}$.

Proof: We check first that (LE) holds on $\mathcal{Z}$. Let $t \in \mathcal{T}$ and put $\beta = \alpha_t$. Then $R^{\beta} = R[\frac{1}{\alpha_u}|u \in \mathcal{T} \setminus \{t\}]$ and hence

$$\begin{aligned}\mathcal{Z}^{\beta} &= R^{\beta} \otimes_R \mathcal{Z} = \{(z_w) \in (R^{\beta})^{\mathcal{W}} | z_w \equiv z_{tw} \mod \beta\ \forall w \in \mathcal{W}\}\\ &= \coprod_{\substack{w\in\mathcal{W}\\ w<tw}} \{(0,\dots,0,a,0,\dots,0,a+b\beta,0\dots,0)|a,b \in R^{\beta}\}\\ &\qquad\qquad \text{with } a \text{ at the } w\text{-th and } a+b\beta \text{ at the } tw\text{-th}\\ &= \coprod_{\Omega\in\langle t\rangle\backslash\mathcal{W}} (\mathcal{Z}^{\beta} \cap \coprod_{w\in\Omega} \mathcal{Z}^{Q}_{w}).\end{aligned}$$

The same argument shows also that (LE) holds on each $\mathcal{Z}^{\beta}$.

To check (S^{ou}) on $\mathcal{Z}$, let I_1 and I_2 be $\mathcal{W}$-open. Then $\mathcal{Z}_{I_1} + \mathcal{Z}_{I_2} \subseteq \mathcal{Z}_{I_1\cup I_2}$. Also,

$$\begin{aligned}(\mathcal{Z}_{I_1} + \mathcal{Z}_{I_2})^{\beta} &= (\mathcal{Z}_{I_1})^{\beta} + (\mathcal{Z}_{I_2})^{\beta}\\ &= (\mathcal{Z}^{\beta})_{I_1} + (\mathcal{Z}^{\beta})_{I_2} \quad \text{by (8.1.2)}\\ &= (\mathcal{Z}^{\beta})_{I_1\cup I_2} \quad \text{by (8.2.ii) as (LE) holds on } \mathcal{Z}^{\beta}\\ &= (\mathcal{Z}_{I_1\cup I_2})^{\beta} \quad \text{by (8.1.2) again.}\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{Z}_{I_1} + \mathcal{Z}_{I_2} &= \cap_{t\in\mathcal{T}} (\mathcal{Z}_{I_1} + \mathcal{Z}_{I_2})^{\alpha_t} \quad \text{as } \cap_{t\in\mathcal{T}} R^{\alpha_t} = R \text{ in each component}\\ &= \cap_{t\in\mathcal{T}} (\mathcal{Z}_{I_1\cup I_2})^{\alpha_t} = \mathcal{Z}_{I_1\cup I_2},\end{aligned}$$

and hence $\mathcal{Z} \in \mathcal{C}^{\mathrm{ou}}$. Likewise $\mathcal{Z} \in \mathcal{C}^{\mathrm{fe}}$.

9.2. We show next that $\mathcal{Z} \in \mathcal{C}_{\nabla} \cap \mathcal{C}_{\Delta}$. More precisely,

Lemma: *$\forall w \in \mathcal{W}$, $\mathcal{Z}^{\mathrm{ou}}_{w} \simeq R(w)(-2\ell(w_0w))$ and $\mathcal{Z}^{\mathrm{fe}}_{w} \simeq R(w)(-2\ell(w))$.*

Proof: By definition $\mathcal{Z}_{\leq w} = \{(z_x) \in \mathcal{Z} | z_x = 0\ \forall x \not\leq w\}$ and $\mathcal{Z}_{<w} = \{(z_x) \in \mathcal{Z} | z_x = 0\ \forall x \not< w\}$. Put $f = \prod_{\substack{t\in\mathcal{T}\\ tw>w}} \alpha_t$. Then $\forall (a_x) \in \mathcal{Z}_{\leq w}$, $f|a_w$, and hence the projectioin $\pi_w : \mathcal{Z}_{\leq w} \to R$ onto the

w-th component induces an imbedding $\mathcal{Z}^{\mathrm{ou}}_w = \mathcal{Z}_{\le w}/\mathcal{Z}_{<w} \hookrightarrow fR$:

$$\begin{array}{ccc} \mathcal{Z}_{\le w} & \xrightarrow{\pi_w} & R \\ \downarrow & & \uparrow \\ \mathcal{Z}^{\mathrm{ou}}_w & \hookrightarrow & fR. \end{array}$$

To see the first assertion, we have only to show that π_w induces a surjection $\pi'_w : \mathcal{Z}_{\le w} \twoheadrightarrow fR$.

As $\mathcal{Z}_{\le w} \otimes_{\mathbb{K}} \mathbb{K}[v] \simeq (\mathcal{Z} \otimes_{\mathbb{K}} \mathbb{K}[v])_{\le w}$ and as $R \otimes_{\mathbb{K}} \mathbb{K}[v] \simeq \mathrm{S}_{\mathbb{K}[v]}(V \otimes_{\mathbb{K}} \mathbb{K}[v])$, we may assume that $\mathbb{K}$ is infinite; the assumption that $\mathbb{K}$ is a complete noetherian local domain is irrelevant for the surjectivity of π'_w. Now, the surjectivity of π'_w follows from that of $\pi'_w \otimes_{\mathbb{K}} \mathbb{K}_{\mathfrak{m}}$ $\forall \mathfrak{m} \in \mathrm{Max}(\mathbb{K})$ [AM, Prop. 3.9], which in turn will follow from the surjectivity of $\pi'_w \otimes_{\mathbb{K}_{\mathfrak{m}}} \mathbb{K}_{\mathfrak{m}}/\mathfrak{m}\mathbb{K}_{\mathfrak{m}} \simeq \pi'_w \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m}$ by graded NAK. Thus, we may further assume that $\mathbb{K}$ is a field, and hence an infinite field by base extension.

One has a homomorphism of graded $\mathbb{K}$-algebras $\eta : R \otimes_{R^{\mathcal{W}}} R \to \mathcal{Z}$ via $a \otimes b \mapsto (a(wb))_{w \in \mathcal{W}}$ compatible with the structures of R-bimodules. For $g \in R \otimes_{R^{\mathcal{W}}} R$ write $g = \sum_i a_i \otimes b_i$. Then $\forall y \in \mathcal{W}$, $\forall \nu \in V$, one has a CD

$$\begin{array}{ccccc} g & & \longmapsto & & \sum_i a_i(wb_i) \\ R \otimes_{R^{\mathcal{W}}} R & \xrightarrow{\pi_y \circ \eta} & R & & \downarrow \\ & \searrow^{\mathrm{ev}_{(\nu, y^{-1}\nu)}} & \downarrow \mathrm{ev}_\nu & & \\ & & \mathbb{K} & & \sum_i a_i(\nu)(yb_i)(\nu) = \sum_i a_i(\nu) b_i(y^{-1}\nu). \end{array}$$

Thus, $\eta(\partial_{w^{-1}}\psi) \in \mathcal{Z}^{2\ell(w_0 w)}_{\le w} \setminus 0$ by (7.10). As $f | \eta(\partial_{w^{-1}}\psi)_w$, one can take ψ such that $f = \eta(\partial_{w^{-1}}\psi)_w$, and the first assertion follows.

Likewise the second, using (7.9) instead of (7.10).

9.3. By going up to $\mathbb{K}[v]$ and then using graded NAK one obtains that (7.3) carries over to the setup over present $\mathbb{K}$. In particular, by [Dem] one has

$$\mathrm{grk}(R(\mathfrak{X}_{\mathcal{W}})) = \mathrm{grk}(R \otimes_{R^{\mathcal{W}}} R) = \sum_{w \in \mathcal{W}} v^{\ell(w)} = \mathrm{grk}(\mathcal{Z}). \tag{1}$$

As in the proof of (9.2), $\forall w \in \mathcal{W}$, by graded NAK one obtains by induction on $\ell(w)$

$$\begin{array}{ccc} R \otimes_{R^{\mathcal{W}}} R & \xrightarrow{\eta} & \mathcal{Z} \\ \uparrow & & \uparrow \\ (R \otimes_{R^{\mathcal{W}}} R)_{\le w} & \twoheadrightarrow & \mathcal{Z}_{\le w}. \end{array}$$

As $\mathcal{Z} \in \mathcal{C}_\nabla$, $\mathcal{Z}$ admits a filtration whose refinement has all its subquotients of the form $\mathcal{Z}^{\mathrm{ou}}_w$, and hence η is surjective. Then η must be bijective by (1). Thus,

Theorem: *η is an isomorphism of graded $\mathbb{K}$-algebras compatible with the structures of R-bimodules.*

9.4. Let $\mathcal{Z}\mathrm{modgr}^{\mathrm{tf}}$ denote the category of graded left $\mathcal{Z}$-modules of finite type that are torsion-free over R. As any object M of $\mathcal{C}$ admits a structure of left $R \otimes_{R^{\mathcal{W}}} R$-module, $M \in \mathcal{Z}\mathrm{modgr}$ via (9.3), and hence by (6.1) one obtains

Corollary: *the functor $F : \mathcal{Z}\mathrm{modgr}^{\mathrm{tf}} \to \mathcal{C}$ is an equivalence.*

9.5. The quotient $\mathcal{Z} \to \mathcal{Z}/\mathcal{Z}_{<w_0} \simeq \mathcal{Z}^{w_0}$ induces a complex $\mathcal{Z}_{<w_0} \to \mathcal{Z} \to \mathcal{Z}^{w_0}$ in $\mathcal{C}_\nabla$ on which (ES) holds; $\forall x \in \mathcal{W}$,

$$(\mathcal{Z}^{w_0})^{\mathrm{ou}}_x \simeq \delta_{x,w_0}\mathcal{Z}^{\mathrm{ou}}_{w_0},$$
$$(\mathcal{Z}_{<w_0})^{\mathrm{ou}}_x \simeq \begin{cases} \mathcal{Z}^{\mathrm{ou}}_x & \text{if } x < w_0, \\ 0 & \text{else.} \end{cases}$$

As $B(w_0) \in \mathcal{C}^{\mathrm{ou}}_P$, one obtains from (1.5.v)

$$\mathcal{C}^\sharp(B(w_0), \mathcal{Z}) \twoheadrightarrow \mathcal{C}^\sharp(B(w_0), \mathcal{Z}^{w_0}) \simeq \mathcal{C}^\sharp(B(w_0)^{w_0}, \mathcal{Z}^{w_0}).$$

One has

$$\begin{aligned} B(w_0)^{w_0} &\simeq R(w_0)(\ell(w_0)) \quad \text{by (5.1)} \\ &\simeq \mathcal{Z}^{w_0}(\ell(w_0)) \quad \text{by the presence of } (1, \dots, 1) \text{ in } \mathcal{Z}. \end{aligned}$$

Let $\varphi \in \mathcal{C}(B(w_0), \mathcal{Z}(\ell(w_0)))$ be a lift of the isomorpshim $B(w_0)^{w_0} \simeq \mathcal{Z}^{w_0}(\ell(w_0))$. Let $f \in B(w_0)$ such that its image in $B(w_0)^{w_0}$ gives a $\mathbb{K}$-linear basis of $(B(w_0)^{w_0})^{-\ell(w_0)}$. By (1) there is $\psi \in \mathcal{C}(\mathcal{Z}(\ell(w_0)), B(w_0))$ such that $1 \mapsto f$. Then $\varphi \circ \psi - \mathrm{id}_{\mathcal{Z}(\ell(w_0))} = 0 \mod \mathcal{Z}_{<w_0}(\ell(w_0))$, and hence $\varphi \circ \psi - \mathrm{id}_{\mathcal{Z}(\ell(w_0))} \notin \mathcal{C}(\mathcal{Z}(\ell(w_0)), \mathcal{Z}(\ell(w_0)))^\times$. As $\mathcal{C}(\mathcal{Z}, \mathcal{Z}) \simeq \mathcal{Z}\mathrm{modgr}(\mathcal{Z}, \mathcal{Z}) \simeq \mathcal{Z}^0 = \mathbb{K}$, $\mathcal{C}(\mathcal{Z}(\ell(w_0)), \mathcal{Z}(\ell(w_0)))$ is local. Thus, $\varphi \circ \psi \in \mathcal{C}(\mathcal{Z}(\ell(w_0)), \mathcal{Z}(\ell(w_0)))^\times$. Then $\mathcal{Z}(\ell(w_0))$ is a direct summand of $B(w_0)$, and hence $\mathcal{Z}(\ell(w_0)) \simeq B(w_0)$. Thus,

Theorem: *There is an isomorphism $B(w_0) \to F(\mathcal{Z})(\ell(w_0))$ in $\mathcal{C}$.*

9.6. Recall from (8.20) that a complex $M_1 \to M_2 \to M_3$ in $\mathcal{C}_\Delta$ on which (ES) holds forms in fact an exact sequence. Then $\mathcal{Z} \in \mathcal{C}^{\mathrm{fe}}_P$ by (9.4.1). Thus, despite the fact that $\mathfrak{S}\mathrm{Bimod} \not\subseteq \mathcal{C}^{\mathrm{fe}}_P$, one has from (9.5) and (8.21)

Corollary: *(i) $B(w_0) \simeq \mathcal{Z}(\ell(w_0)) \in \mathcal{C}^{\mathrm{ou}}_P \cap \mathcal{C}^{\mathrm{fe}}_P$.*

(ii) $\forall B \in \mathfrak{S}\mathrm{Bimod}$, $B(w_0) * B \in \mathcal{C}^{\mathrm{fe}}_P$.

II. Abe's bimodules

Given a root datum $(X, \Delta, X^\vee, \Delta^\vee)$ and a complete DVR $\mathbb{K}$, Abe's bimodules are graded bimodules over the symmetric algebra $S = \mathrm{S}_\mathbb{K}(X^\vee_\mathbb{K})$, $X^\vee_\mathbb{K} = X^\vee \otimes_\mathbb{Z} \mathbb{K}$. They are torsion free over S and are equipped with a "weight space" decomposition over $S^\emptyset = S[\frac{1}{\alpha}|\alpha \in \Delta]$ parametrized by the alcoves in $X \otimes_\mathbb{Z} \mathbb{R}$. They are designed ingeneously to admit a right action by the monoidal category $\mathfrak{SB}$ of Soergel bimodules over S associated to the Coxeter system $(\mathcal{W}, \mathcal{S})$ from I with $\mathcal{W} = \mathcal{W}_f \ltimes \mathbb{Z}\Delta$, $\mathcal{W}_f$ denoting the Weyl group of the root system Δ. The linear representation of $\mathcal{W}$ on $X^\vee_\mathbb{K}$ is given by annihilating $\mathbb{Z}$, in particular, not faithful. To define the action, he prepares another graded $\mathbb{K}$-algebra R isomorphic to S and regards $\mathfrak{SB}$ over R. Thus, Abe's bimodules are graded (S, R)-bimodules M such that $S^\emptyset \otimes_S M = \coprod_{A \in \mathcal{A}} M^\emptyset_A$, $\mathcal{A}$ denoting the set of alcoves. On each $M^\emptyset_A$ the isomorphism between R and S is defined separately depending on A. It is assumed on $\mathbb{K}$ that $2 \in \mathbb{K}^\times$ and that the GKM condition holds, so that the Weyl group $\mathcal{W}_f$ acts faithfully on $X^\vee_\mathbb{K}$. Then the (S, R)-bimodule structure on M gives a decomposition $S^\emptyset \otimes_S M = \coprod_{\Omega \in \mathbb{Z}\Delta \backslash \mathcal{A}} M^\emptyset_\Omega$ with $M^\emptyset_\Omega = \coprod_{A \in \Omega} M^\emptyset_A$. Thus, a morphism of Abe's bimodules from M to N is defined to be a (S, R)-bilinear map φ such that $(S^\emptyset \otimes_S \varphi)(M^\emptyset_A) \subseteq \coprod_{B \in A + \mathbb{Z}\Delta, B \geq A} M^\emptyset_B$ for each $A \in \mathcal{A}$, where $B \geq A$ is the strong linkage/generic Chevalley-Bruhat order on $\mathcal{A}$.

Later on an ideal quotient $\mathcal{K}$ of the category is introduced such that any morphism $\varphi : M \to N$ with $(S^\emptyset \otimes_S \varphi)(M^\emptyset_A) \subseteq \coprod_{B > A} M^\emptyset_B$ $\forall A \in \mathcal{A}$ be annihilated. Then a full subcategory $\mathcal{K}_\Delta$ of $\mathcal{K}$ consisting of those admitting a Δ-flag categorifies Lusztig's periodic module for the 岩堀-Hecke algebra $\mathcal{H}$ of $(\mathcal{W}, \mathcal{S})$, and its subcategory $\mathcal{K}_P$ of "projectives" is equivalent to a certain subcategory $\mathcal{K}_{\mathrm{AJS},P}$ of the combinatorial category of AJS [AJS]. If $\mathbb{K}$ is an algebraically closed field of characteristic $p > h$ the Coxeter number of Δ, $\mathcal{K}_{\mathrm{AJS},P}$ is equivalent to the category of projectives of the principal block of G_1T deformed over the completion $\hat{S}$ of S with respect to the augmentation ideal, where G_1 (resp. T) is the Frobenius kernel (resp. maximal torus) of the reductive algebraic group over $\mathbb{K}$ associated to the root datum. A Δ-flag on M is a filtration of M such that each subquotient associated to an alcove be free over S. To define a filtration and to verify that the $\mathfrak{SB}$-action on $\mathcal{K}$ preserve $\mathcal{K}_\Delta$, the properties (S) and (LE), extracted from [F08a], [F08b], [FL15], play important roles. Likewise, to define a "projective", property (ES) from [F08a], [F08b] is used, and the construction of a projective is done appealing to the structure algebra of the moment graph associated to $\mathcal{W}_f$. Finally, the action of $\mathfrak{SB}$ on the projectives is extended to the whole of the principal block of G_1T corrsponding to the wall-crossing functors. For $p \gg 0$ Lusztig's conjecture on the irreducible G_1T-characters is proved.

1. **Preliminaries**

1.1. Let $(X, \Delta, X^\vee, \Delta^\vee)$ be a root datum [Sp, 7.4.1]. Let $\mathcal{A}$ denote the set of alcoves in $X_\mathbb{R} = X \otimes_\mathbb{Z} \mathbb{R}$, i.e., the set of connected components of $X_\mathbb{R} \setminus \bigcup_{\substack{\alpha \in \Delta \\ n \in \mathbb{Z}}} \{\nu \in X_\mathbb{R} | \langle \nu, \alpha^\vee \rangle = n\}$. Let $\mathcal{W}_f$ be the Weyl group of Δ and $\mathcal{W} = \mathcal{W}_f \ltimes \mathbb{Z}\Delta$ with $\mathbb{Z}\Delta$ acting on $X_\mathbb{R}$ by translations. Thus, $\mathcal{W}$ acts simply transitively on $\mathcal{A}$ [J, II.6.2.4]. Fix a positive system Δ^+ and let $\mathcal{A}^+$ be the set of dominant alcoves in $\mathcal{A}$, i.e., those $A \in \mathcal{A}$ such that, $\forall \nu \in A$, $\forall \alpha \in \Delta^+$, $\langle \nu, \alpha^\vee \rangle > 0$. Let A^+ denote the bottom dominant alcove, and let $\mathcal{S}$ be the set of reflections in the walls of A^+. Thus, $(\mathcal{W}, \mathcal{S})$ forms a Coxeter system with the length function denoted ℓ. Putting $\mathcal{S}_f = \mathcal{S} \cap \mathcal{W}_f$, $(\mathcal{W}_f, \mathcal{S}_f)$ forms a Coxeter subsystem. Through the bijection $\mathcal{W} \to \mathcal{A}$ via $x \mapsto xA^+$, we transport the right action of $\mathcal{W}$ onto $\mathcal{A}$ [S97, p. 92]: $\forall y \in \mathcal{W}$,

$$(1) \qquad (xA^+)y = xyA^+.$$

Let $A = wA^+ = A^+w$, $w \in \mathcal{W}$. If $s \in \mathcal{S}$ is the reflection with respect to a wall H of A^+, $As = wsA^+$ is the alcove adjacent to A over the wall wH of A. $\forall \alpha \in \Delta$, $\forall n \in \mathbb{Z}$, let $s_{\alpha,n} \in \mathcal{W}$ be the reflection with respect to the hyperplane $H_{\alpha,n} = \{\lambda \in X_{\mathbb{R}} | \langle \lambda, \alpha^\vee \rangle = n\}$: $\forall \mu \in X_{\mathbb{R}}$,

$$s_{\alpha,n}\mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha + n\alpha.$$

If $H_{\alpha,n}$ is a wall of A defining s, $wH_{\alpha,n} = H_{w\alpha,n}$ and $As = wsA^+ = wsw^{-1}wA^+ = wsw^{-1}A = ws_{\alpha,n}w^{-1}A = s_{w\alpha,n}A$. As the left and right multiplications on $\mathcal{W}$ are compatible, so they are on $\mathcal{A}$: $\forall x, y \in \mathcal{W}$, $A \in \mathcal{A}$,

$$(2) \quad (xA)y = (x(wA^+))y = ((xw)A^+)y = (xw)yA^+ = x(wy)A^+ = x((wy)A^+) = x(Ay).$$

In particular, letting t_γ, $\gamma \in \mathbb{Z}\Delta$, denote the translation by γ,

$$(3) \qquad (A + \gamma)y = (t_\gamma A)y = t_\gamma(Ay) = Ay + \gamma.$$

More generally, let $\hat{X} = \{\lambda \in X_{\mathbb{R}} | \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ \forall \alpha \in \Delta\}$; X may not contain all the special points in $X_{\mathbb{R}}$ [L80]. Then $\hat{X}$ acts on $\mathcal{A}$ by translation. Note, however, that (3) does not carry over; it may happen that $(A + \lambda)y \neq Ay + \lambda$ if $\lambda \in \hat{X} \setminus \mathbb{Z}\Delta$.

1.2 **Some technicalities:** Abe's bimodules admit an action of Soergel bimodules, which require a linear acton of $\mathcal{W}$. For that let $\Lambda = \{f : \mathcal{A} \to X | f(xA) = \bar{x}f(A) \ \forall A \in \mathcal{A} \ \forall x \in \mathcal{W}\}$, where $\bar{x}$ is the image of x under the projection $\mathcal{W} \to \mathcal{W}_f$. $\forall A \in \mathcal{A}$, there is a bijection $\Lambda \to X$ written $f \mapsto f_A := f(A)$ with inverse $\lambda \mapsto \lambda^A$ such that $\lambda^A(xA) = \bar{x}\lambda \ \forall x \in \mathcal{W}$. Through the bijection we import a structure of abelian group on Λ from X:

$$(f + g)_A = f_A + g_A \quad \forall f, g \in \Lambda,$$

which is independent of the choice of A; we are to check that $(f_A + g_A)^A = (f_B + g_B)^B$, $B \in \mathcal{A}$. If $B = wA$, $w \in \mathcal{W}$,

$$\begin{aligned} f_B + g_B &= f(B) + g(B) = f(wA) + g(wA) = \bar{w}f(A) + \bar{w}g(A) = \bar{w}\{f(A) + g(A)\} \\ &= \bar{w}(f_A + g_A) = \bar{w}\{\{(f_A + g_A)^A\}_A\} = \bar{w}\{(f_A + g_A)^A(A)\} = (f_A + g_A)^A(wA) \\ &= (f_A + g_A)^A(B) = \{(f_A + g_A)^A\}_B, \end{aligned}$$

and hence $(f_B + g_B)^B = (f_A + g_A)^A$, as desired.

If we transport the structure of $\mathcal{W}$-module on X likewise such that $(xf)_A = x(f_A)$, however, the structure on Λ depends on the choice of A:

$$(xf)_B = x(f_B) = x\{f(B)\} = x\{f(wA)\} = x\{\bar{w}(f(A))\},$$

which is not equal in general to $x\{f(A)\} = x(f_A)$. Instead, we define a $\mathcal{W}$-action on Λ such that $\forall x \in \mathcal{W}$, $\forall f \in \Lambda$, $\forall A \in \mathcal{A}$,

$$(1) \qquad (xf)(A) = f(Ax).$$

So-defined xf is indeed an element of Λ thanks to (1.1.2). Also, $\forall y \in \mathcal{W}$,

$$((xy)f)(A) = f(A(xy)) = f((Ax)y) = (yf)(Ax) = (x(yf))(A),$$

and hence $(xy)f = x(yf)$. Thus, the bijection $?_A : \Lambda \to X$ is not $\mathcal{W}$-equivariant.

Likewise, we introduce

$$\Lambda' = \{f : \mathcal{A} \to X^\vee | f(xA) = \bar{x}f(A)\ \forall x \in \mathcal{W}\}.$$

Each $A \in \mathcal{A}$ defines a bijection $\Lambda' \to X^\vee$ via

$$f \mapsto f_A := f(A) \tag{2}$$

with inverse written $\nu \mapsto \nu^A$, under which we transport the structure of abelian group onto Λ': $(f+g)_A = f_A + g_A\ \forall f, g \in \Lambda'$. The structure is independent of the choice of A as for Λ above, and we define a $\mathbb{Z}$-linear $\mathcal{W}$-action on Λ' via

$$(xf)(A) = f(Ax) \quad \forall A \in \mathcal{A}. \tag{3}$$

Now, $\forall f \in \Lambda'$, $\forall g \in \Lambda$, $\forall x \in \mathcal{W}$,

$$\langle g(xA), f(xA)\rangle = \langle \bar{x}g(A), \bar{x}f(A)\rangle = \langle g(A), f(A)\rangle = \langle g_A, f_A\rangle. \tag{4}$$

$\forall f \in \Lambda'$, let now $\tilde{f} \in \Lambda^\vee = \mathrm{Mod}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ such that $\tilde{f}(g) = \langle g_A, f_A\rangle$, which is independent of the choice of A by (4). Then

$$\Lambda' \simeq \Lambda^\vee \quad \text{via} \quad f \mapsto \tilde{f}. \tag{5}$$

For define $\Lambda^\vee \to \Lambda'$ via $\phi \mapsto (\phi')^A$ with $\phi' \in X^\vee$ such that $\phi'(\lambda) = \phi(\lambda^A)\ \forall \lambda \in X$. $\forall g \in \Lambda$,

$$\widetilde{(\phi')^A}(g) = \langle g_A, ((\phi')^A)_A\rangle = \langle g_A, \phi'\rangle = \phi((g_A)^A) = \phi(g),$$

and hence $\widetilde{(\phi')^A} = \phi$. Also,

$$\langle g_A, ((\tilde{f}')^A)_A\rangle = \langle g_A, \tilde{f}'\rangle = \tilde{f}((g_A)^A) = \tilde{f}(g) = \langle g_A, f_A\rangle,$$

and hence $(\tilde{f}')^A = f$.

Thus, we will identify $\Lambda^\vee$ with Λ', and obtain a $\mathbb{Z}$-linear action of $\mathcal{W}$ on $\Lambda^\vee$, and hence a $\mathbb{K}$-linear action on $\Lambda^\vee_{\mathbb{K}} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{K}$, on which Abe's theory of Soergel bimodules in I is applied. $\forall g \in \Lambda$, $\forall f \in \Lambda'$, $\forall x \in \mathcal{W}$,

$$\begin{aligned}
\tilde{f}(xg) &= \langle (xg)_A, f_A\rangle = \langle (xg)(A), f(A)\rangle = \langle g(Ax), f(A)\rangle \\
&= \langle g(yA^+x), f(yA^+)\rangle \quad \text{writing } A = yA^+,\ y \in \mathcal{W} \\
&= \langle g(yxA^+), f(yA^+)\rangle = \langle \overline{yx}g(A^+), \bar{y}f(A^+)\rangle = \langle g(A^+), \overline{x^{-1}}f(A^+)\rangle \\
&= \langle g(A^+), f(x^{-1}A^+)\rangle = \langle g(A^+), f(A^+x^{-1})\rangle = \langle g(A^+), (x^{-1}f)(A^+)\rangle \\
&= \langle g_{A^+}, (x^{-1}f)_{A^+}\rangle = \widetilde{x^{-1}f}(g),
\end{aligned}$$

and hence

(6) $$\langle xg, f\rangle = \langle g, x^{-1}f\rangle.$$

Note that the bijection (2) is not $\mathcal{W}_f$-equivariant:

(7) $$\begin{aligned}(xf)_A &= f(Ax) = f_{Ax} = f(Ax)\\ &= f(wA^+x) \quad \text{if } A = wA^+\\ &= f(wxA^+) \quad \text{by definition}\\ &= f(wxw^{-1}wA^+) = f(wxw^{-1}A) = \overline{wxw^{-1}}\{f(A)\}\\ &\neq x\{f(A)\} = x(f_A).\end{aligned}$$

If we take $A = A^+$, however, $\forall x \in \mathcal{W}_f$, $\forall f \in \Lambda'$, $(xf)_{A^+} = xf_{A^+}$. Thus, the isomorphism between $X^\vee$ and $\Lambda^\vee$ using A^+ gives a $\mathcal{W}_f$-equivariant isomorphism of $\mathbb{K}$-modules

(8) $$X^\vee \otimes_{\mathbb{Z}} \mathbb{K} \xrightarrow{\sim} \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{K}.$$

Lemma: *Let* $f \in \Lambda, \lambda \in X, \gamma \in X^\vee, g \in \Lambda^\vee, x \in \mathcal{W}, A \in \mathcal{A}$.

(i) $(xf)_A = f_{Ax}$.

(ii) $\bar{x}f_A = f_{xA}$.

(iii) $x\lambda^{Ax} = \lambda^A$.

(iv) $(\bar{x}\lambda)^{xA} = \lambda^A$.

(v) $(\bar{x}\gamma)^{xA} = \gamma^A$.

(vi) $xg^{Ax} = g^A$.

Proof: (i) $(xf)_A = (xf)(A) = f(Ax) = f_{Ax}$.

(ii) $\bar{x}f_A = \bar{x}(f(A)) = f(xA) = f_{xA}$.

(iii) One has

$$\begin{aligned}(x\lambda^{Ax})_A &= (x\lambda^{Ax})(A) = \lambda^{Ax}(Ax) \quad \text{by definition (1)}\\ &= (\lambda^{Ax})_{Ax} = \lambda = (\lambda^A)_A.\end{aligned}$$

As $?_A$ is bijective, the assertion follows.

(iv) $\{(\bar{x}\lambda)^{xA}\}_{xA} = \bar{x}\lambda = \bar{x}\{(\lambda^A)_A\} = \bar{x}\{(\lambda^A)(A)\} = (\lambda^A)(xA) = (\lambda^A)_{xA}$.

(v) $\{(\bar{x}\gamma)^{xA}\}_{xA} = \bar{x}\gamma = \bar{x}\{(\gamma^A)_A\} = \bar{x}\gamma^A(A) = \gamma^A(xA) = (\gamma^A)_{xA}$.

(vi) Under the identification of $\Lambda^\vee$ with Λ' one has

$$\begin{aligned}(xg^{Ax})_A &= (xg^{Ax})(A) = g^{Ax}(Ax) \quad \text{by definition (3)}\\ &= g = (g^A)_A,\end{aligned}$$

and hence $xg^{Ax} = g^A$.

1.3. Recall the strong linkage on $\mathcal{A}$ from [J, II.6], which we will denote by $\geq$ after [L80]. Thus, $\forall A \in \mathcal{A}, w \in \mathcal{W}$ with $A \leq wA$, and $\nu \in \bar{A}$,

$$w\nu - \nu \in \mathbb{R}_{\geq 0}\Delta^+. \tag{1}$$

The strong linkage is distinct from the PO on $\mathcal{A}$ induced by the Chevalley-Bruhat order on $\mathcal{W}$; if $s \in \mathcal{S}_f$, $sA^+ = A^+s \not\geq A^+$. For the precise relationship between the two, cf. [S97, claim 4.14, p. 96].

Lemma: *Let $\forall A, A' \in \mathcal{A}$ with $A' = A + \gamma$ for some $\gamma \in \mathbb{Z}\Delta$, $A \leq A'$ iff $\gamma \in \mathbb{N}\Delta^+$.*

Proof: "if" We may assume $\gamma \in \Delta^+$. Take $n \in \mathbb{Z}$ with $n - 1 < \langle \nu, \gamma^\vee \rangle < n \ \forall \nu \in A$. Then $A \leq s_{\gamma,n}A$ by definition. Also, $\langle s_{\gamma,n}\nu, \gamma^\vee \rangle = \langle \nu - \langle \nu, \gamma^\vee \rangle \gamma + n\gamma, \gamma^\vee \rangle = -\langle \nu, \gamma^\vee \rangle + 2n < 1 - n + 2n = n + 1$, and hence $A \leq s_{\gamma,n}A \leq s_{\gamma,n+1}s_{\gamma,n}A = A + \gamma$.

"only if" Take $\nu \in A$. Then $(\nu + \gamma) - \nu \in \mathbb{R}_{\geq 0}\Delta^+$ by (1), and hence $\gamma \in \mathbb{R}_{\geq 0}\Delta^+ \cap \mathbb{Z}\Delta = \mathbb{N}\Delta^+$ [HLA, 10.1].

1.4. We say $J \subseteq \mathcal{A}$ is open iff $\forall A \in J$, $\forall A' \in \mathcal{A}$ with $A' \leq A$, $A' \in J$. This defines a topology on $\mathcal{A}$; $\mathcal{A}$ and $\emptyset$ are both open. If J_ν's are open, so is $\cup_\nu J_\nu$. If J and J' are open, so is $J \cap J'$. Thus, $I \subseteq \mathcal{A}$ is closed iff $\forall A \in I$, $\forall A' \in \mathcal{A}$ with $A' \geq A$, $A' \in I$. For if I is closed, let $A \in I$ and $A' \in \mathcal{A}$ with $A' > A$. If $A' \notin I$, $A' \in \mathcal{A} \setminus I$ open, and hence $A \in \mathcal{A} \setminus I$, absurd. Assume conversely the condition that $\forall A \in I$,$\forall A' \in \mathcal{A}$ with $A' \geq A$, $A' \in I$. Let $A \in \mathcal{A} \setminus I$ and $A' \leq A$. Then $A' \notin I$ by the assumption.

$\forall A, A' \in \mathcal{A}$, let $(\geq A) = \{B \in \mathcal{A} | B \geq A\}$, and define $(> A), (\leq A), (< A)$ likewise. Put also $[A, A'] = (\geq A) \cap (\leq A')$, etc. For $\alpha \in \Delta^+$ take $n \in \mathbb{Z}$ with $n - 1 < \langle \nu, \alpha^\vee \rangle < n \ \forall \nu \in A$, and set $\alpha \uparrow A = s_{\alpha,n}A > A$. Let also $\alpha \downarrow A = s_{\alpha,n-1}A$. Thus, $\alpha \uparrow (\alpha \downarrow A) = A = \alpha \downarrow (\alpha \uparrow A)$.

Lemma: *Let $\Omega \in \mathbb{Z}\Delta \backslash \mathcal{A}$ be a $\mathbb{Z}\Delta$-orbit in $\mathcal{A}$. If I is closed in Ω, so is $Ix = \{Ax | A \in I\}$ in $\Omega x \ \forall x \in \mathcal{W}$.*

Proof: Note first that I is closed in Ω iff $I = \{\cup_{B \in I}(\geq B)\} \cap \Omega = \cup_{B \in I}\{(\geq B) \cap \Omega\}$ iff $\forall B \in I$, $\forall B' \in \Omega$ with $B' \geq B$, $B' \in I$.

Let $B \in I$ and $B' \in \Omega$ with $B'x \geq Bx$. Write $\Omega = A + \mathbb{Z}\Delta$ for some $A \in \mathcal{A}$. Then $B = A + \gamma$ and $B' = A + \gamma'$ for some $\gamma, \gamma' \in \mathbb{Z}\Delta$, and hence

$$\begin{aligned} Ax + \gamma' &= (A + \gamma')x \quad \text{by (1.1.3)} \\ &= B'x \geq Bx = Ax + \gamma. \end{aligned}$$

Then $\gamma' - \gamma \in \mathbb{N}\Delta^+$ by (1.3), and, in turn, $B' \geq B$. Then $B' \in I$, and $B'x \in Ix$.

1.5. Fix a complete DVR $\mathbb{K}$ with maximal ideal $\mathfrak{m}$ throughout the rest of II, so that our categories be Krull-Scmidt. Put $\Lambda^\vee_\mathbb{K} = \Lambda^\vee \otimes_\mathbb{Z} \mathbb{K}$, $X^\vee_\mathbb{K} = X^\vee \otimes_\mathbb{Z} \mathbb{K}$, and $R = \mathrm{S}_\mathbb{K}(\Lambda^\vee_\mathbb{K})$. We endow R with a grading such that $\deg(\Lambda^\vee_\mathbb{K}) = 2$, and consider Soergel bimodules in I over R.

Throughout II we impose on $\mathbb{K}$ the conditions

(A1) $2 \in \mathbb{K}^\times$,

(A2) The GKM condition, cf. [F11, Lem. 9.2]: $\forall \alpha, \beta \in \Delta^+$ distinct, $\forall \mathfrak{m} \in \mathrm{Max}(\mathbb{K})$, $\alpha^\vee$ and $\beta^\vee$ remain linearly independent in $X^\vee_{\mathbb{K}}$ and in $X^\vee \otimes_{\mathbb{Z}} (\mathbb{K}/\mathfrak{m})$,

in addition to the characteristic restrictions on $\mathbb{K}$ from I such that $\mathrm{ch}\mathbb{K}$ is either 0 or above the torsion primes.

Under those assumptions one has

Lemma: *The representation $X^\vee_{\mathbb{K}}$ of $\mathcal{W}_f$ is faithful.*

Proof: Let $w \in \mathcal{W}_f$ be trivial on $X^\vee_{\mathbb{K}}$. As 2 distinct coroots remain distinct in $X^\vee_{\mathbb{K}}$ by the standing assumption, w fixes every coroot over $\mathbb{Z}$ already, and hence $w = e$ [HLA, 10.3].

2. **Abe's bimodules**

2.1. Set $S = \mathrm{S}_{\mathbb{K}}(X^\vee_{\mathbb{K}})$ endowed with a grading such that $\deg(X^\vee_{\mathbb{K}}) = 2$, and let $S^\emptyset = S[\frac{1}{\alpha^\vee} | \alpha \in \Delta]$.

Let S_0 be a commutative flat graded S-algebra. For an S-module M put $M^\emptyset = S^\emptyset \otimes_S M$. In particular, $S_0^\emptyset \simeq S_0[\frac{1}{\alpha^\vee} | \alpha \in \Delta]$. Define a category $\tilde{\mathcal{K}}'(S_0)$ to consist of the graded (S_0, R)-bimodules M such that

(i) M is torsion-free of finite type over S,

(ii) $M^\emptyset$ admits a decomposition $M^\emptyset = \coprod_{A \in \mathcal{A}} M_A^\emptyset$ such that $\forall A \in \mathcal{A}$, $M_A^\emptyset$ is an $(S_0^\emptyset, R)$-bimodule with $mf = f_A m$ $\forall m \in M_A^\emptyset$ $\forall f \in R$; precisely, if $f = \sum_i f_i \otimes a_i$, $f_i \in \Lambda^\vee$ and $a_i \in \mathbb{K}$, $f_A = \sum_i (f_i)_A a_i \in S$, and extend the operation to the whole of R.

A morphism $\varphi \in \tilde{\mathcal{K}}'(S_0)(M, N)$ is a homomorphism of graded (S_0, R)-modules, i.e., of degree 0, such that $\varphi(M_A^\emptyset) \subseteq \coprod_{A' \geq A} N_{A'}^\emptyset$ $\forall A \in \mathcal{A}$. We equip $\tilde{\mathcal{K}}'(S_0)(M, N)$ with a structure of (S_0, R)-bimodule such that $\forall a \in S_0$, $\forall f \in R$, $\forall m \in M$

$$(a\varphi f)(m) = \varphi(amf) = a\varphi(m)f. \tag{1}$$

Thus, $\tilde{\mathcal{K}}'(S_0)$ forms an (S_0, R)-bilinear category [中岡, Def. 3.1.11, p. 124, Def. 3.2.3, p. 130]. Put $\tilde{\mathcal{K}}'(S_0)^\sharp(M, N) = \coprod_{i \in \mathbb{Z}} \tilde{\mathcal{K}}'(S_0)(M, N(i))$, which comes equipped with a structure of graded (S_0, R)-bimodule. Assume that $\varphi \in \tilde{\mathcal{K}}'(S_0)(M, N)$ is invertible with inverse ψ. $\forall A \in \mathcal{A}$, $\mathrm{id}_A = \psi_A^\emptyset \circ \varphi_A^\emptyset$. As each $\psi_B^\emptyset$, $B \in \mathcal{A}$, is injective, we must have

$$\begin{array}{ccc} M^\emptyset & \xrightarrow[\sim]{\varphi^\emptyset} & N^\emptyset \\ \uparrow & & \uparrow \\ M_A^\emptyset & \underset{\varphi_A^\emptyset}{\overset{\sim}{\dashrightarrow}} & N_A^\emptyset. \end{array}$$

Let $\varphi \in \tilde{\mathcal{K}}'(S_0)(M, N)$ in general. If K is the kernel of φ as graded (S_0, R)-bimodules, by flat base change $K^\emptyset = \ker(\varphi^\emptyset) = \coprod_{A\in\mathcal{A}} \ker(\varphi^\emptyset_A)$. Thus,

$$K \hookrightarrow M \text{ gives the kernel of } \varphi \text{ in } \tilde{\mathcal{K}}'(S_0). \tag{2}$$

Assume now that $\varphi \in \tilde{\mathcal{K}}'(S_0)(M,M)$ is an idempotent. Then $M = \ker\varphi \oplus \ker(1-\varphi)$ as (S_0,R)-bimodules with $(\ker\varphi)^\emptyset = \ker(\varphi^\emptyset) = \coprod_{A\in\mathcal{A}} \ker(\varphi^\emptyset_A)$ and $(\ker(1-\varphi))^\emptyset = \ker(1-\varphi^\emptyset) = \coprod_{A\in\mathcal{A}} \ker((1-\varphi)^\emptyset_A) = \coprod_{A\in\mathcal{A}} \ker(1-\varphi^\emptyset_A)$. As $M^\emptyset_A \geq (\ker\varphi)^\emptyset_A \oplus (\ker(1-\varphi))^\emptyset_A$ $\forall A \in \mathcal{A}$,

$$\coprod_{A\in\mathcal{A}} M^\emptyset_A = M^\emptyset \geq \{\coprod_{A\in\mathcal{A}} (\ker\varphi)^\emptyset_A\} \oplus \{\coprod_{A\in\mathcal{A}} (\ker(1-\varphi))^\emptyset_A\} = (\ker\varphi)^\emptyset \oplus (\ker(1-\varphi))^\emptyset = M^\emptyset,$$

and hence $M^\emptyset_A = (\ker\varphi)^\emptyset_A \oplus (\ker(1-\varphi))^\emptyset_A$ $\forall A \in \mathcal{A}$. Thus, the decomposition $M = \ker\varphi \oplus \ker(1-\varphi)$ occurs in $\tilde{\mathcal{K}}'(S_0)$, and

(3) $\tilde{\mathcal{K}}'(S_0)$ is Karoubian/idempotent complete [中岡, Def. 3.3.40, p. 174].

As $\ker(1-\varphi) = \varphi(M)$, φ is the identity on $\ker(1-\varphi)$ and φ vanishes on $\ker\varphi$. Thus, $\forall A \in \mathcal{A}$,

$$\begin{array}{ccc} M^\emptyset & \xrightarrow{\varphi^\emptyset} & M^\emptyset \\ \uparrow & & \uparrow \\ M^\emptyset_A & \dashrightarrow & M^\emptyset_A. \end{array}$$

As $M \in \tilde{\mathcal{K}}'(S_0)$ is torsion-free over S, $M \hookrightarrow M^\emptyset = \coprod_{A\in\mathcal{A}} M^\emptyset_A$. We will denote the $M^\emptyset_A$-component of the image of $m \in M$ by m_A. We define the support of M to be $\operatorname{supp}_{\mathcal{A}}(M) = \{A \in \mathcal{A} | M^\emptyset_A \neq 0\}$. $\forall m \in M$, put $\operatorname{supp}_{\mathcal{A}}(m) = \{A \in \mathcal{A} | m_A \neq 0\}$.

Note that

$$M \in \tilde{\mathcal{K}}'(S_0) \text{ is also torsion-free over } R. \tag{4}$$

For if $ma = 0$, $m \in \tilde{\mathcal{K}}'(S_0)$, $a \in R$, write $m = \sum_{A\in\mathcal{A}} m_A$ in $M^\emptyset = S^\emptyset \otimes_S M$. Then $0 = \sum_A a_A m_A$ $\forall A$. If $m_A \neq 0$, there is $b \in S^\times$ with $bm_A \in M$. Then $ba_A = 0$, and hence $a_A = 0$.

Let $\gamma \in \mathbb{Z}\Delta$. $\forall M \in \tilde{\mathcal{K}}'(S_0)$, let $\mathrm{T}_\gamma(M) = M$ and $\mathrm{T}_\gamma(M)^\emptyset_A = M^\emptyset_{A+\gamma}$ $\forall A \in \mathcal{A}$. $\forall m \in \mathrm{T}_\gamma(M)^\emptyset_A$, $\forall f \in R$,

$$\begin{aligned} mf &= f_{A+\gamma} m = f(A+\gamma)m = f(A)m \quad \text{by definition} \\ &= f_A m, \end{aligned}$$

and hence $\mathrm{T}_\gamma(M)$ comes equipped with a structure of $\tilde{\mathcal{K}}'(S_0)$, and T_γ defines an automorphism of $\tilde{\mathcal{K}}'(S_0)$ with adjoint $\mathrm{T}_{-\gamma}$. If $\lambda \in \hat{X}$, $A \mapsto A+\lambda$ defines a permutation on $\mathcal{A}$. Unless $\lambda \in \mathbb{Z}\Delta$, however, there is $f \in R$ with $f_{A+\lambda} \neq f_A$.

For a graded S_0-module M we let M^i, $i \in \mathbb{Z}$, denote its homogeneous piece of degree i. If $n \in \mathbb{Z}$, we let $M(n)$ denote a graded S_0-module with $M(n)^i = M^{n+i}$, $i \in \mathbb{Z}$. We say that M is graded free over S_0 iff $M \simeq \coprod_{n\in\mathbb{Z}} S_0(n)^{\oplus m(M,n)}$ for some $m(M,n) \in \mathbb{N}$, in which case we call $\sum_{n\in\mathbb{Z}} m(M,n)v^n \in \mathbb{Z}[v,v^{-1}]$, the Laurent polynomial ring in v, the graded rank of M and denote it by $\mathrm{grk}(M)$.

2.2 **Remarks:** Let $M \in \tilde{\mathcal{K}}'(S_0)$.

(i) Let $\Omega \in \mathbb{Z}\Delta\backslash\mathcal{A}$ be a $\mathbb{Z}\Delta$-orbit in $\mathcal{A}$. $\forall m \in \coprod_{A\in\Omega} M_A^\emptyset$, $\forall f \in R$,

(1) $$\exists f_\Omega \in S : mf = f_\Omega m.$$

For if $\Omega = A + \mathbb{Z}\Delta$ and $A' = A+\gamma$, $\gamma \in \mathbb{Z}\Delta$, $\forall f \in \Lambda^\vee$,

$$f_{A'} = f(A') = f(A+\gamma) = f(A) = f_A,$$

and hence $\forall g \in R$, $g_{A'} = g_A$.

As the left action of $\mathcal{W}_f$ is simply transitive on $\mathbb{Z}\Delta\backslash\mathcal{A}$,

(2) $$M^\emptyset = \coprod_{w\in\mathcal{W}_f} (\coprod_{B\in w\Omega} M_B^\emptyset) \quad \text{with} \quad mf = (wf_\Omega)m \ \forall f \in R \ \forall m \in \coprod_{B\in w\Omega} M_B^\emptyset.$$

For if $A \in \Omega$,

$$\begin{aligned} f_{w\Omega} &= f_{wA} \quad \text{by (1)} \\ &= \bar{w} f_A \quad \text{by (1.2.ii)} \\ &= w f_A \\ &= w f_\Omega \quad \text{by (1) again.} \end{aligned}$$

Now, $\mathcal{W}_f$ separates $\mathbb{Z}\Delta\backslash\mathcal{A}$ by the simply transitive action, acts faithfully on $X_{\mathbb{K}}^\vee$ by (1.5), and M is torsion-free over S. It follows that the decomposition of $M^\emptyset$ into the $\coprod_{B\in w\Omega} M_B^\emptyset$, $w \in \mathcal{W}_f$, is determined by the (S_0,R)-bimodule structure on M. Then, $\forall N \in \tilde{\mathcal{K}}'(S_0)$, $\forall \varphi \in (S_0,R)\mathrm{Bimod}(M,N)$, $\forall w \in \mathcal{W}_f$,

$$\varphi^\emptyset(\coprod_{B\in w\Omega} M_B^\emptyset) \subseteq \coprod_{B\in w\Omega} N_B^\emptyset.$$

For let $m \in \coprod_{B\in w\Omega} M_B^\emptyset$ and $A' \in \mathcal{A}$ with $\varphi(m)_{A'} \neq 0$. Let $x \in \mathcal{W}_f$ such that $A' \in x\Omega$. Then, $\forall f \in \Lambda_{\mathbb{K}}^\vee$,

$$\begin{aligned} (f_{w\Omega} - f_{x\Omega})\varphi(m)_{A'} &= \varphi(f_{w\Omega}m)_{A'} - f_{x\Omega}\varphi(m)_{A'} = \varphi(mf)_{A'} - f_{x\Omega}\varphi(m)_{A'} \quad \text{by (1)} \\ &= (\varphi(m)f)_{A'} - f_{x\Omega}\varphi(m)_{A'} = 0 \quad \text{by (1) again.} \end{aligned}$$

As M is torsion-free over S, $f_{w\Omega} - f_{x\Omega} = 0$, and hence $wx^{-1}f_{A'} = f_{wx^{-1}A'} = f_{w\Omega} = f_{x\Omega} = f_{A'}$ by (1.2). Then wx^{-1} is trivial on the whole of $\Lambda_{\mathbb{K}}^\vee$, and $w = x$ by (1.5). Thus, $\forall \Omega' \in \mathbb{Z}\Delta\backslash\mathcal{A}$,

(3) $$\varphi^\emptyset(\coprod_{A\in\Omega'} M_A^\emptyset) \subseteq \coprod_{A\in\Omega'} N_A^\emptyset.$$

For an example of $\phi \in \tilde{\mathcal{K}}'(M,N)$ such that $\phi^\emptyset(M_A^\emptyset) \not\subseteq N_A^\emptyset$ for some A, see i_0^+ in (7.3).

(ii) For $X \subseteq \mathcal{A}$ set $M_{[X]} = M \cap \coprod_{A \in X} M_A^\emptyset$. As $M \leq M^\emptyset$, $(M_{[X]})^\emptyset \leq \coprod_{A \in X} M_A^\emptyset$. If $m \in M_A^\emptyset$, $A \in X$, take $b \in S^\times$ such that $bm \in M$. Then $bm \in M_{[X]}$, and hence $m \in (M_{[X]})^\emptyset$. Thus, $(M_{[X]})^\emptyset = \coprod_{A \in X} M_A^\emptyset$. Then, by setting

$$(M_{[X]})_B^\emptyset = \begin{cases} M_B^\emptyset & \text{if } B \in X, \\ 0 & \text{else,} \end{cases}$$

one obtains that $M_{[X]} \in \tilde{\mathcal{K}}'(S_0)$. Thus, $M \mapsto M_{[X]}$ defines an endofunctor of $\tilde{\mathcal{K}}'(S_0)$. In particular, as M is of finite type over S, we must have $\mathrm{supp}_{\mathcal{A}}(M)$ finite.

(iii) Let $R^\emptyset = R[\frac{1}{(\alpha^\vee)^A} | \alpha^\vee \in \Delta^\vee]$, which is independent of the choice of $A \in \mathcal{A}$; write $A = xA^+$, $x \in \mathcal{W}$. Then $(\alpha^\vee)^{xA^+} = (\bar{x}^{-1}\alpha^\vee)^{A^+}$ by (1.2.v) with $\bar{x}^{-1}\alpha^\vee = (\bar{x}^{-1}\alpha)^\vee \in \Delta^\vee$. Note also that

$$((\alpha^\vee)^{A^+})_A = (\alpha^\vee)^{A^+}(A) = (\alpha^\vee)^{A^+}(xA^+) = \bar{x}\{(\alpha^\vee)^{A^+}(A^+)\} = \bar{x}\alpha^\vee \in \Delta^\vee.$$

As $M \in \tilde{\mathcal{K}}'(S_0)$ has finite support, there is an isomorphism of graded (S_0, R)-bimodules

$$M \otimes_R R^\emptyset \to M^\emptyset \quad \text{via} \quad m \otimes \frac{f}{g} \mapsto \sum_{A \in \mathcal{A}} \frac{f_A}{g_A} m_A \quad \text{with } m = \sum_A m_A \text{ in } M^\emptyset = S^\emptyset \otimes_S M.$$

For denote the map by η. Any element of $M \otimes_R R^\emptyset$ is of the form $m \otimes \frac{1}{f}$ for some $f \in (R^\emptyset)^\times$. If $0 = \eta(m \otimes \frac{1}{f}) = \sum_A \frac{1}{f_A} m_A$, then $0 = (\sum_A \frac{1}{f_A} m_A) f = \sum_A m_A = 1 \otimes m$, and hence $m = 0$. To see the surjectivity, $\forall a \in (S^\emptyset)^\times$,

$$\frac{1}{a} \otimes m = \sum_A \frac{1}{a} m_A = \eta(\sum_A m_A \frac{g}{a^A} \otimes \frac{1}{g})$$

with $g \in (R^\emptyset)^\times$ such that $m_A \frac{g}{a^A} = \frac{g_A}{a} m_A \in M \cap M_A^\emptyset \ \forall A$, which exists as M has finite support.

2.3. A primary example of an object of $\tilde{\mathcal{K}}'(S_0)$ is afforded by S_0 itself. As S is a domain and as S_0 is flat over S, it is torsion-free over S. Let $A \in \mathcal{A}$ and let R act on S_0 from the right such that $gf = f_A g \ \forall g \in S_0, \forall f \in R$, which defines a structure of graded (S_0, R)-bimodule denoted $S_0(A)$. Then $S_0(A)^\emptyset = S_0(A)_A^\emptyset$, and $S_0(A) \in \tilde{\mathcal{K}}'(S_0)$, which we will call the standard module associated to A. Thus, $\mathrm{grk}(S_0(A)) = 1$.

$\forall M \in \tilde{\mathcal{K}}'(S_0)$, one has by (2.2.3)

$$\tilde{\mathcal{K}}'(S_0)^\bullet(S_0(A), M) \simeq \{m \in M | \mathrm{supp}_{\mathcal{A}}(m) \subseteq A + \mathbb{N}\Delta\}.$$

In particular, as $\mathbb{K}$ is a complete DVR, $S_0(A)$ is indecomposable if $(S_0)^0 = \mathbb{K}$.

2.4. Let I be a closed subset of $\mathcal{A}$. $\forall M \in \tilde{\mathcal{K}}'(S_0)$, let $M_I = M_{[I]} = M \cap \coprod_{A \in I} M_A^\emptyset$ as in Rmk. 2.2.ii. One has $M \in \tilde{\mathcal{K}}'(S_0)$ with

$$(M_I)_A^\emptyset = \begin{cases} M_A^\emptyset & \text{if } A \in I. \\ 0 & \text{else,} \end{cases}$$

In particular,

(1) $$M_I = \begin{cases} M & \text{if } I \supseteq \operatorname{supp}_{\mathcal{A}}(M), \\ 0 & \text{if } I \cap \operatorname{supp}_{\mathcal{A}}(M) = \emptyset. \end{cases}$$

Also,

Lemma: *(i) If I' is another closed subset of $\mathcal{A}$, $M_I \cap M_{I'} = M_{I \cap I'} = (M_I)_{I'}$.*

(ii) $\forall A \in \mathcal{A}$, $S_0(A)_I = \begin{cases} S_0(A) & \text{if } A \in I, \\ 0 & \text{else.} \end{cases}$

2.5. **Properties (S) and (LE):** We will argue as in (I.8), but with $\mathcal{W}^\alpha = \langle s_\alpha \rangle \ltimes \mathbb{Z}\alpha \leq \mathcal{W}$ and $S^\alpha = S[\frac{1}{\beta^\vee} | \beta \in \Delta^+ \setminus \{\alpha\}]$ $\forall \alpha \in \Delta^+$. As S is a UFD, under the standing assumptions (1.5), one has

(1) $$\cap_{\alpha \in \Delta^+} S^\alpha = S.$$

For each S-module M put $M^\alpha = S^\alpha \otimes_S M \in \tilde{\mathcal{K}}'((S_0)^\alpha)$. $\forall M \in \tilde{\mathcal{K}}'(S_0)$, we say $M \in \tilde{\mathcal{K}}(S_0)$ iff the following 2 conditions hold on M:

(S) $\forall$ closed I_1 and $I_2 \subseteq \mathcal{A}$, $M_{I_1 \cup I_2} = M_{I_1} + M_{I_2}$,

(LE) $\forall \alpha \in \Delta^+$, $M^\alpha = \coprod_{\Omega \in \mathcal{W}^\alpha \setminus \mathcal{A}} (M^\alpha \cap \coprod_{A \in \Omega} M_A^\emptyset)$.

Arguing as in (I.4.9.iii) shows, $\forall M \in \tilde{\mathcal{K}}'(S_0)$, $\forall I$ closed in $\mathcal{W}$, as M is torsion free over S, that

(2) $$M_I \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \simeq \{M \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_I.$$

Then, one obtains as in (I.8.1.4) that

(3) if $M \in \tilde{\mathcal{K}}(S_0)$, properties (S) and (LE) carry over onto $M \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})$.

Let $\varphi \in \tilde{\mathcal{K}}(S_0)(M, M)$ be an idempotent, let K be the kernel of φ in $\tilde{\mathcal{K}}'(S_0)$, and let N be a direct complement of K in $\tilde{\mathcal{K}}'(S_0)$. Then $M_A^\emptyset = K_A^\emptyset \oplus N_A^\emptyset$ $\forall A \in \mathcal{A}$ by (2.1.3). If I is closed in $\mathcal{A}$, one has in $\tilde{\mathcal{K}}'(S_0)$

$$\begin{aligned} M_I &= (K \oplus N) \cap \coprod_{A \in I} (K \oplus N)_A^\emptyset = (K \oplus N) \cap \{(\coprod_{A \in I} K_A^\emptyset) \oplus (\coprod_{A \in I} N_A^\emptyset)\} \\ &= (K \cap \coprod_{A \in I} K_A^\emptyset) \oplus (N \cap \coprod_{A \in I} N_A^\emptyset) = K_I \oplus N_I. \end{aligned}$$

Then $K_{I_1 \cup I_2} \oplus N_{I_1 \cup I_2} = M_{I_1 \cup I_2} = M_{I_1} + M_{I_2} = (K_{I_1} \oplus N_{I_1}) + (K_{I_2} \oplus N_{I_2}) = (K_{I_1} + K_{I_2}) \oplus$

$(N_{I_1}+N_{I_2})$, and hence $K_{I_1\cup I_2}=K_{I_1}+K_{I_2}$. Also,

$$\begin{aligned}
K^\alpha\oplus N^\alpha = M^\alpha &= \coprod_{\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}}\{M^\alpha\cap(\coprod_{A\in\Omega}M^\emptyset_A)\} = \coprod_{\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}}\{(K\oplus N)^\alpha\cap(\coprod_{A\in\Omega}(K\oplus N)^\emptyset_A)\}\\
&= \coprod_{\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}}\{(K^\alpha\oplus N^\alpha)\cap(\coprod_{A\in\Omega}(K^\emptyset_A\oplus N^\emptyset_A))\}\\
&= \coprod_{\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}}\{(K^\alpha\oplus N^\alpha)\cap((\coprod_{A\in\Omega}K^\emptyset_A)\oplus(\coprod_{A\in\Omega}N^\emptyset_A))\}\\
&= \coprod_{\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}}\{(K^\alpha\cap\coprod_{A\in\Omega}K^\emptyset_A)\oplus(N^\alpha\cap\coprod_{A\in\Omega}N^\emptyset_A)\}\\
&= \{\coprod_{\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}}(K^\alpha\cap\coprod_{A\in\Omega}K^\emptyset_A)\}\oplus\{\coprod_{\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}}(N^\alpha\cap\coprod_{A\in\Omega}N^\emptyset_A)\},
\end{aligned}$$

and hence $K^\alpha=\coprod_{\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}}(K^\alpha\cap\coprod_{A\in\Omega}K^\emptyset_A)$. Thus, $K\in\tilde{\mathcal{K}}(S_0)$, and

(4) $\tilde{\mathcal{K}}(S_0)$ remains Karoubian/idempotent complete [中岡, Def. 3.3.40, p. 174].

Let $\gamma\in\mathbb{Z}\Delta$ and $M\in\tilde{\mathcal{K}}(S_0)$. For closed I_1 and I_2

(5)
$$\begin{aligned}
\mathrm{T}_\gamma(M)_{I_1\cup I_2} &= M\cap\coprod_{A\in I_1\cup I_2}M^\emptyset_{A+\gamma}\\
&= M_{(I_1\cup I_2)+\gamma}\quad\text{as } (I_1\cup I_2)+\gamma:=\{A+\gamma|A\in I_1\cup I_2\}\text{ is closed}\\
&= M_{(I_1+\gamma)\cup(I_2+\gamma)}\quad\text{with } I_j+\gamma=\{A+\gamma|A\in I_j\},\ j=1,2\\
&= M_{I_1+\gamma}+M_{I_2+\gamma}\quad\text{as both } I_1+\gamma \text{ and } I_2+\gamma \text{ remain closed}\\
&= \mathrm{T}_\gamma(M)_{I_1}+\mathrm{T}_\gamma(M)_{I_2}.
\end{aligned}$$

If $\alpha\in\Delta^+$,

(6)
$$\begin{aligned}
\mathrm{T}_\gamma(M)^\alpha = M^\alpha &= \coprod_{\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}}(M^\alpha\cap\coprod_{A\in\Omega}M^\emptyset_A)\\
&= \coprod_{\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}}(M^\alpha\cap\coprod_{A\in\Omega}M^\emptyset_{A+\gamma})\quad\text{as } \Omega+\gamma\in\mathcal{W}^\alpha\backslash\mathcal{W}\\
&= \coprod_{\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}}(M^\alpha\cap\coprod_{A\in\Omega}\mathrm{T}_\gamma(M)^\emptyset_A).
\end{aligned}$$

Thus, T_γ induces an automorphism of $\tilde{\mathcal{K}}(S_0)$.

Lemma: *Let $M\in\tilde{\mathcal{K}}'(S_0)$, $\alpha\in\Delta^+$, and $A\in\mathcal{A}$.*

(i) If $\mathrm{supp}_{\mathcal{A}}(M)\subseteq\mathcal{W}^\alpha A$, (S) holds on M.

(ii) If (LE) holds on M, so does (S) on M^α.

(iii) $S_0(A)\in\tilde{\mathcal{K}}(S_0)$.

Proof: Let I_1, I_2 be two closed subsets of $\mathcal{A}$.

(i) Put $\Omega = \mathcal{W}^\alpha A = \{\dots, \alpha\downarrow(\alpha\downarrow A) = A-\alpha, \alpha\downarrow A, A, \alpha\uparrow A, \alpha\uparrow(\alpha\uparrow A) = A+\alpha, \dots\}$, which is totally ordered under $\geq$. Thus, either $I_1\cap\Omega \subseteq I_2\cap\Omega$ or $I_2\cap\Omega\subseteq I_1\cap\Omega$; assume $B\in(I_1\cap\Omega)\setminus(I_2\cap\Omega)$ and let $B'\in I_2\cap\Omega$. If $B'\leq B$, $B\in I_2\cap\Omega$ as I_2 is closed, absurd, and hence $B\leq B'$ as Ω is totally ordered. Then $B'\in I_1\cap\Omega$ as I_1 is closed.

Thus, we may assume that $I_1\cap\Omega\supseteq I_2\cap\Omega$. Let $I_1' = \overline{I_1\cap\Omega}$ and $I_2' = \overline{I_2\cap\Omega}$. Then $I_1'\supseteq I_2'$, $I_1'\cap\Omega = I_1\cap\Omega$, $I_2'\cap\Omega = I_2\cap\Omega$, and hence

$$\begin{aligned} M_{I_1'} &= M\cap(\coprod_{B\in I_1'} M_B^\emptyset) \quad \text{by definition}\\ &= M\cap(\coprod_{B\in I_1'\cap\Omega} M_B^\emptyset) = M\cap(\coprod_{B\in I_1\cap\Omega} M_B^\emptyset) = M\cap(\coprod_{B\in I_1} M_B^\emptyset) \quad \text{as } \mathrm{supp}_{\mathcal{A}}(M)\subseteq\Omega\\ &= M_{I_1}. \end{aligned}$$

Likewise, $M_{I_2'} = M_{I_2}$, $M_{I_1'\cup I_2'} = M_{I_1\cup I_2}$. Then

$$\begin{aligned} M_{I_1\cup I_2} &= M_{I_1'\cup I_2'} = M_{I_1'} = M_{I_1}\\ &= M_{I_1} + M_{I_2} \quad \text{as } M_{I_2} = M_{I_2'}\subseteq M_{I_1'} = M_{I_1}. \end{aligned}$$

(ii) $\forall\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}$, put $M_\Omega^\alpha = (M^\alpha)_{[\Omega]} = M^\alpha\cap(\coprod_{A\in\Omega} M_A^\emptyset)\in\tilde{\mathcal{K}}'(S_0)$. Thus, $M^\alpha = \coprod_\Omega M_\Omega^\alpha$ by (LE). $\forall I$ closed in $\mathcal{A}$, one has

$$\begin{aligned} (M^\alpha)_I &= M^\alpha\cap\{\coprod_{A\in I}(M^\alpha)_A^\emptyset\} = (\coprod_\Omega M_\Omega^\alpha)\cap\{\coprod_{A\in I}(\coprod_\Omega M_\Omega^\alpha)_A^\emptyset\} = \coprod_\Omega\{M_\Omega^\alpha\cap(\coprod_{A\in I}(M_\Omega^\alpha)_A^\emptyset)\}\\ &= \coprod_\Omega(M_\Omega^\alpha)_I, \end{aligned}$$

and hence

$$\begin{aligned} (M^\alpha)_{I_1\cup I_2} &= \coprod_\Omega (M_\Omega^\alpha)_{I_1\cup I_2}\\ &= \coprod_\Omega\{(M_\Omega^\alpha)_{I_1} + (M_\Omega^\alpha)_{I_2}\} \quad \text{as (S) holds on } M_\Omega^\alpha \text{ by (i)}\\ &= \{\coprod_\Omega(M_\Omega^\alpha)_{I_1}\} + \{\coprod_\Omega(M_\Omega^\alpha)_{I_2}\} = (M^\alpha)_{I_1} + (M^\alpha)_{I_2}. \end{aligned}$$

2.6. Let K be a locally closed subset of $\mathcal{A}$, i.e., $K = I\cap J$ $\exists I$ closed and J open in $\mathcal{A}$. $\forall M\in\tilde{\mathcal{K}}(S_0)$, put $M_K = M_I/M_{I\setminus J}$, which might be denoted M_K^{fe} in the notation of (I.8). Then M_K is torsion-free over over S; if $m\in M_I$ and $a\in S\setminus 0$ with $am\in M_{I\setminus J}$, $m\in\coprod_{A\in I} M_A^\emptyset$ and $am\in\coprod_{A\in I\setminus J} M_A^\emptyset$. Then $m\in\coprod_{A\in I\setminus J} M_A^\emptyset$ already as $M^\emptyset$ is torsion-free over S, and hence

$m \in M_{I\setminus J}$. In particular,

$$(1)\qquad \begin{aligned} M_K \leq (M_K)^{\emptyset} &= (M_I/M_{I\setminus J})^{\emptyset} \simeq (M_I)^{\emptyset}/(M_{I\setminus J})^{\emptyset} \\ &= (\coprod_{A\in I} M_A^{\emptyset})/(\coprod_{A\in I\setminus J} M_A^{\emptyset}) \quad \text{by (2.4)} \\ &\simeq \coprod_{A\in I\cap J} M_A^{\emptyset} = \coprod_{A\in K} M_A^{\emptyset}, \end{aligned}$$

and hence $M_K \in \tilde{\mathcal{K}}'(S_0)$ with

$$(M_K)_A^{\emptyset} = \begin{cases} M_A^{\emptyset} & \text{if } A \in K, \\ 0 & \text{else.} \end{cases}$$

In particular,

$$\operatorname{supp}_{\mathcal{A}}(M_K) = \operatorname{supp}_{\mathcal{A}}(M) \cap K.$$

Also,

$$(2)\qquad M_K \text{ is in } M^{\emptyset} \text{ independent of the choice of } I \text{ and } J \text{ expressing } K.$$

For if $K = I' \cap J'$ with I' closed and J' open,

$$(I \cup I') \cap (J \cap J') = (I \cap J \cap J') \cup (I' \cap J \cap J') = (K \cap J') \cup (K \cap J) = K \cup K = K.$$

Thus it is enough to check that $M_I/M_{I\setminus J} = M_{I\cup I'}/M_{(I\cup I')\setminus(J\cap J')}$. Now,

$$(3)\qquad I \cup I' = I \cup \{(I \cup I') \setminus (J \cap J')\},$$

$$(4)\qquad I \cap \{(I \cup I') \setminus (J \cap J')\} = I \setminus J.$$

For let $A \in I' \setminus I$. As $(I' \setminus I) \cap (J \cap J') \subseteq (I' \cap J') \setminus I = (I \cap J) \setminus I = \emptyset$, $A \notin J \cap J'$. Then $A \in (I\cup I')\setminus(J\cap J')$, and (3) holds. Let next $A \in I\cap\{(I\cup I')\setminus(J\cap J')\} = I\setminus(J\cap J') \supseteq I\setminus J$. Just suppose $A \in J$. Then $A \in I \cap J = I' \cap J' \subseteq J'$, and hence $A \in J \cap J'$, absurd, and hence also (4). Thus

$$\begin{aligned} M_{I\cup I'}/M_{(I\cup I')\setminus(J\cap J')} &= M_{I\cup\{(I\cup I')\setminus(J\cap J')\}}/M_{(I\cup I')\setminus(J\cap J')} \quad \text{by (3)} \\ &= \{M_I + M_{(I\cup I')\setminus(J\cap J')}\}/M_{(I\cup I')\setminus(J\cap J')} \quad \text{by (S)} \\ &\simeq M_I/\{M_I \cap M_{(I\cup I')\setminus(J\cap J')}\} \\ &= M_I/M_{I\cap\{(I\cup I')\setminus(J\cap J')\}} \quad \text{by (2.4.i)} \\ &= M_I/M_{I\setminus J} \quad \text{by (4).} \end{aligned}$$

If $\operatorname{supp}_{\mathcal{A}}(M) \subseteq K$, one has

$$(5)\qquad \begin{aligned} M_K &= M_I/M_{I\setminus J} \\ &= M/0 \quad \text{as } (I \setminus J) \cap \operatorname{supp}_{\mathcal{A}}(M) \subseteq (I \setminus J) \cap (I \cap J) = \emptyset \\ &= M. \end{aligned}$$

$\forall A \in \mathcal{A}$, $\{A\} = (\geq A) \cap (\leq A)$ is locally closed. One has from (2.4.ii)

$$S_0(A)_K \simeq \begin{cases} S_0(A) \simeq S_0(A)_{\{A\}} & \text{if } A \in K, \\ 0 & \text{else.} \end{cases} \tag{6}$$

Warning: Although $M_K \leq M^\emptyset$, that $M_K = M \cap (\coprod_{A \in K} M_A^\emptyset)$ need not hold, cf. (I.2.2.13).

2.7. Let K be a locally closed set in $\mathcal{A}$ and write $K = I \cap J$ with I closed and J open.

Lemma: *$\forall M \in \tilde{\mathcal{K}}(S_0)$, $M_K \in \tilde{\mathcal{K}}(S_0)$.*

Proof: We first show that $\forall I'$ closed in $\mathcal{A}$,

$$(M_K)_{I'} = M_{K \cap I'}. \tag{1}$$

If K is closed, the assertion follows from (2.4.i). If K is open, put $I_1 = \mathcal{A} \setminus K$. Then

$$\begin{aligned} (M_K)_{I'} &= M_K \cap \coprod_{A \in I'} (M_K)_A^\emptyset = M_{\mathcal{A} \cap K} \cap \coprod_{A \in I'} (M_K)_A^\emptyset \\ &= (M/M_{I_1}) \cap \coprod_{A \in I' \cap K} M_A^\emptyset \quad \text{by (2.6.1, 2)} \end{aligned}$$

while

$$\begin{aligned} M_{K \cap I'} &= M_{I'}/M_{I' \setminus K} = M_{I'}/M_{I' \cap I_1} \\ &= M_{I'}/(M_{I'} \cap M_{I_1}) \quad \text{by (2.4.i) again} \\ &\simeq (M_{I'} + M_{I_1})/M_{I_1}. \end{aligned}$$

As $M_{K \cap I'} \leq (M_{K \cap I'})^\emptyset = \coprod_{A \in I' \cap K} M_A^\emptyset$, $M_{K \cap I'} \leq (M_K)_{I'}$. Let $m \in M$ with $m + M_{I_1} \in \coprod_{A \in I' \cap K} M_A^\emptyset$. Then $m_A = 0$ unless $A \in I' \cup I_1$, and hence

$$\begin{aligned} m &\in M_{I' \cup I_1} \\ &= M_{I'} + M_{I_1} \quad \text{as (S) holds on } M. \end{aligned}$$

Thus, $M_{K \cap I'} \simeq (M_K)_{I'}$. In general, one has $M_K = M_{J \cap I} \simeq (M_J)_I$ by above, and hence

$$\begin{aligned} (M_K)_{I'} &\simeq ((M_J)_I)_{I'} = (M_J)_{I \cap I'} \quad \text{by (2.4.i)} \\ &\simeq M_{J \cap I \cap I'} \quad \text{by above} \\ &= M_{K \cap I'}, \quad \text{as desired.} \end{aligned}$$

We show now that (S) holds on M_K. Let I_2, I_3 closed in $\mathcal{A}$. One has

$$\begin{aligned} (M_K)_{I_2 \cup I_3} &= M_{K \cap (I_2 \cup I_3)} \quad \text{by (1)} \\ &= M_{(K \cap I_2) \cup (K \cap I_3)} = M_{(I \cap I_2 \cap J) \cup (I \cap I_3 \cap J)} = M_{\{(I \cap I_2) \cup (I \cap I_3)\} \cap J} \\ &= M_{I \cap (I_2 \cup I_3)}/M_{\{I \cap (I_2 \cup I_3)\} \setminus J} = M_{(I \cap I_2) \cup (I \cap I_3)}/M_{\{I \cap (I_2 \cup I_3)\} \setminus J} \\ &= \{M_{I \cap I_2} + M_{I \cap I_3}\}/M_{\{I \cap (I_2 \cup I_3)\} \setminus J} \quad \text{as (S) holds on } M \\ &\simeq M_{I \cap I_1}/M_{(I \cap I_1) \setminus J} + M_{I \cap I_2}/M_{(I \cap I_2) \setminus J} = M_{K \cap I_2} + M_{K \cap I_3} \\ &= (M_K)_{I_2} + (M_K)_{I_3} \quad \text{by (1) again.} \end{aligned}$$

We show finally that (LE) holds on M_K. Let $\alpha \in \Delta^+$. As $(M_K)^\alpha = (M_I/M_{I\setminus J})^\alpha = (M_I)^\alpha/(M_{I\setminus J})^\alpha$, it is enough to verify that (LE) holds on M_I. Let $m \in (M_I)^\alpha \leq M^\alpha$. As (LE) holds on M, one can write $m = \sum_\Omega m_\Omega$ with $m_\Omega \in M^\alpha \cap \coprod_{A\in\Omega} M^\emptyset_A$. As $m \in (M_I)^\alpha \leq (M_I)^\emptyset = \coprod_{B\in I} M^\emptyset_B$, however, $m_A = 0$ unless $A \in I$. Thus, $m_\Omega \in (M_I)^\alpha \cap \coprod_{A\in\Omega}(M_I)^\emptyset_A$, as desired.

2.8. If $K = I \cap J$ is locally closed in $\mathcal{A}$ with I (resp. J) closed (resp. open) in $\mathcal{A}$, $\forall\varphi \in \tilde{\mathcal{K}}(S_0)(M,N)$,

(1)
$$\begin{array}{ccc}
M & \xrightarrow{\quad\varphi\quad} & N \\
\uparrow & & \uparrow \\
M_I & \dashrightarrow & N_I \\
\downarrow & & \downarrow \\
M_I/M_{I\setminus J} & \dashrightarrow & N_I/N_{I\setminus J} \\
\| & & \| \\
M_K & & N_K,
\end{array}$$

and hence one obtains an endofunctor $?_K$ on $\tilde{\mathcal{K}}(S_0)$.

Lemma: *$\forall M \in \tilde{\mathcal{K}}(S_0)$, $\forall K_1, K_2$ locally closed in $\mathcal{A}$, $(M_{K_1})_{K_2} = M_{K_1\cap K_2}$.*

Proof: Write $K_i = I_i \cap J_i$ with I_i closed and J_i open in $\mathcal{A}$, $i \in \{1,2\}$. Then

$$\begin{aligned}
(M_{K_1})_{K_2} &= (M_{K_1})_{I_2}/(M_{K_1})_{I_2\setminus J_2} \\
&= M_{K_1\cap I_2}/M_{K_1\cap(I_2\setminus J_2)} \quad \text{by (2.7.1)}
\end{aligned}$$

with

$$\begin{aligned}
M_{K_1\cap I_2} &= M_{I_1\cap I_2\cap J_1} = M_{I_1\cap I_2}/M_{(I_1\cap I_2)\setminus J_1}, \\
M_{K_1\cap(I_2\setminus J_2)} &= M_{I_1\cap(I_2\setminus J_2)\cap J_1} = M_{I_1\cap(I_2\setminus J_2)}/M_{\{I_1\cap(I_2\setminus J_2)\}\setminus J_1} = M_{(I_1\cap I_2)\setminus J_2}/M_{(I_1\cap I_2)\setminus J_1\cup J_2)},
\end{aligned}$$

and hence

$$\begin{aligned}
(M_{K_1})_{K_2} &= M_{I_1\cap I_2}/\{M_{(I_1\cap I_2)\setminus J_1} + M_{(I_1\cap I_2)\setminus J_2}\} \\
&= M_{I_1\cap I_2}/M_{(I_1\cap I_2)\setminus J_1\}\cup\{(I_1\cap I_2)\setminus J_2\}} \quad \text{as (S) holds on } M \\
&= M_{I_1\cap I_2}/M_{(I_1\cap I_2)\setminus(J_1\cap J_2)} = M_{I_1\cap I_2\cap J_1\cap J_2} = M_{K_1\cap K_2}.
\end{aligned}$$

2.9. Let $M \in \tilde{\mathcal{K}}(S_0)$. If $I_0 \subset I_1 \subset \cdots \subset I_r$ is a chain of closed subsets of $\mathcal{A}$ with $\mathrm{supp}_{\mathcal{A}}(M)\cap I_0 = \emptyset$ and $\mathrm{supp}_{\mathcal{A}}(M) \subseteq I_r$, one has a filtration of M

$$0 = M_{I_0} \leq M_{I_1} \leq \cdots \leq M_{I_r} = M$$

such that $M_{I_j}/M_{I_{j-1}} \simeq M_{I_j\setminus I_{j-1}}$; $I_j \setminus I_{j-1} = I_j \cap (\mathcal{A}\setminus I_{j-1})$ and $I_j \setminus (\mathcal{A}\setminus I_{j-1}) = I_{j-1}$. One thus obtains exact sequences [中岡, Def. 3.3.29, p. 168]

(1)
$$0 \to M_{I_{j-1}} \to M_{I_j} \to M_{I_j\setminus I_{j-1}} \to 0.$$

Lemma: *Let* $M_1, \dots, M_l \in \tilde{\mathcal{K}}(S_0)$. $\forall I$ *closed in* $\mathcal{A}$, $\forall A \in I$ *with* $I \setminus \{A\}$ *closed, there is a chain of closed subsets* $I_0 \subset I_1 \subset \cdots \subset I_r$ *in* $\mathcal{A}$ *and* $k \in [1, r]$ *such that* $|I_j| = |I_{j-1}| + 1$ $\forall j \in [1, r]$, $I_k \cap \{\cup_i \mathrm{supp}_{\mathcal{A}}(M_i)\} = I \cap \{\cup_i \mathrm{supp}_{\mathcal{A}}(M_i)\}$, $I_{k-1} = I_k \setminus \{A\}$, *and* $\forall i \in [1, l]$, $(M_i)_{I_0} = 0$ *while* $(M_i)_{I_r} = M_i$. *In particular,* $(M_i)_I = (M_i)_{I_k}$ $\forall i \in [1, l]$.

Proof: From [L80, Prop. 3.7] one has $A_0, A_n \in \mathcal{A}$ with $\cup_{i=1}^l \mathrm{supp}_{\mathcal{A}}(M_i) \subseteq [A_0, A_n]$. Put $I_0 = I \setminus (< A_n)$. Enumerate $(I \setminus \{A\}) \cap [A_0, A_n] = \{A_1, \dots, A_{k-1}\}$ and $[A_0, A_n] \setminus I = \{A_{k+1}, \dots, A_r\}$ such that $A_i > A_j$ implies $i < j$ $\forall i, j$. Putting $A_k = A$ and $I_j = I_0 \sqcup \{A_1, \dots, A_j\}$ will do.

2.10 **Δ-flags:** We say that $M \in \tilde{\mathcal{K}}(S_0)$ admits a Δ-flag iff each $M_{\{A\}}$, $A \in \mathcal{A}$, is a graded free S_0-module, i.e., $\forall A \in \mathcal{A}$, $\exists n_i \in \mathbb{N}, i \in \mathbb{Z}$: $M_{\{A\}} \simeq \coprod_{i \in \mathbb{Z}} S_0(A)(i)^{\oplus n_i}$.

As $\mathrm{supp}_{\mathcal{A}}(M)$ is finite, there exist $A_0, A_\infty \in \mathcal{A}$ with $\mathrm{supp}_{\mathcal{A}}(M) \subseteq [A_0, A_\infty]$. One can construct a chain of closed subsets as in (2.9) whose associated filtration is such that its subquotients are all of the form $M_{\{A\}}$. Dually, put $I = (\geq A_0)$, $I_0 = I \setminus (\leq A_\infty)$, choose $A_1 \in \mathcal{A}$ minimal in I_0, and put $I_1 = I_0 \setminus \{A_1\}$. If $B_1 \in I_1$ and $B_2 > B_1$, $B_2 \in I_1$ as I_0 is closed by the minimality of A_1. Take A_2 minimal in I_1 and put $I_2 = I_1 \cup \{A_2\}$. Then $I_2 = I_1 \sqcup \{A_2\}$ is closed likewise. Repeat to get an enumeration $A_1, A_2, \dots, A_n = A_\infty$ of $[A_0, A_\infty]$ such that $I_{j+1} = I_j \setminus \{A_{j+1}\}$ is closed $\forall j \in [0, n[$. Thus, $I_0 \supset I \supset \dots$ form a chain of closed subsets of $\mathcal{A}$ such that $M = M_{I_0} \geq M_{I_1} \geq \cdots \geq M_{I_n} = 0$ with $M_{I_j}/M_{I_j \setminus I_{j+1}} \simeq M_{\{A_{j+1}\}}$ $\forall j$. In particular, M itself is graded free over S_0. A Δ-flag is called a standard filtration in [Ab19b]. Let $\tilde{\mathcal{K}}_\Delta(S_0)$ denote the full subcategory of $\tilde{\mathcal{K}}(S_0)$ consisting of the objects M with a Δ-flag.

Let $M \in \tilde{\mathcal{K}}_\Delta(S_0)$. If $\gamma \in \mathbb{Z}\Delta$, $\forall A \in \mathcal{A}$,

$$\begin{aligned} \mathrm{T}_\gamma(M)_{\{A\}} &= \mathrm{T}_\gamma(M)_{(\geq A) \cap (\leq A)} \simeq \mathrm{T}_\gamma(M)_{\geq A}/\mathrm{T}_\gamma(M)_{(\geq A) \setminus \{A\}} \\ &= M_{\geq A + \lambda}/M_{(\geq A + \gamma) \setminus \{A + \gamma\}} \simeq M_{\{A + \gamma\}}, \end{aligned} \tag{1}$$

and hence T_γ restricts to an automorphism of $\tilde{\mathcal{K}}_\Delta(S_0)$.

$\forall K$ locally closed in $\mathcal{A}$,

$$\begin{aligned} (M_K)_{\{A\}} &= M_{K \cap \{A\}} \quad \text{by (2.8)} \\ &= \begin{cases} M_{\{A\}} & \text{if } A \in K, \\ 0 & \text{else.} \end{cases} \end{aligned} \tag{2}$$

Thus,

Lemma: $\forall M \in \tilde{\mathcal{K}}_\Delta(S_0)$, $\forall K$ *locally closed in* $\mathcal{A}$, $M_K \in \tilde{\mathcal{K}}_\Delta(S_0)$ *and is graded free over* S_0.

Proof: One has $M_K \in \tilde{\mathcal{K}}_\Delta(S_0)$ by (2). We may then assume that $K = \mathcal{A}$, and $M_K = M$ is graded free over S_0 as observed above.

2.11. Note that the category $\tilde{\mathcal{K}}(S_0)$ is not necessarily abelian; a quotient may not be torsion-free.

Definition: We say that property (ES), short for "exact structure", holds on a complex $M_1 \to M_2 \to M_3$ in $\tilde{\mathcal{K}}_\Delta(S_0)$ iff the sequence $0 \to (M_1)_{\{A\}} \to (M_2)_{\{A\}} \to (M_3)_{\{A\}} \to 0$ is exact $\forall A \in \mathcal{A}$, which is just an exact sequence of graded left S_0-modules, cf. [F08b, 4.1], [F08a, 2.5]. We define a category $\tilde{\mathcal{K}}_P(S_0)$ to be the full category of $\tilde{\mathcal{K}}_\Delta(S_0)$ consisting of M such that $\forall$ complex $M_1 \to M_2 \to M_3$ in $\tilde{\mathcal{K}}_\Delta(S_0)$ with (ES), the induced sequence $0 \to \tilde{\mathcal{K}}(M, M_1) \to \tilde{\mathcal{K}}(M, M_2) \to \tilde{\mathcal{K}}(M, M_3) \to 0$ is exact, in which case $0 \to \tilde{\mathcal{K}}(M, M_1(n)) \to \tilde{\mathcal{K}}(M, M_2(n)) \to \tilde{\mathcal{K}}(M, M_3(n)) \to 0$ is exact $\forall n \in \mathbb{Z}$.

One has both $\tilde{\mathcal{K}}_\Delta(S_0)$ and $\tilde{\mathcal{K}}_P(S_0)$ Karoubian/idempotent complete by (2.5.4). As anticipated in (I.1.6/9.20), $S_0(A) \notin \tilde{\mathcal{K}}_P(S_0)$. A first example in $\tilde{\mathcal{K}}_P(S_0)$ will be constructed using a Soergel bimodule in $\mathcal{C}_P^{\mathrm{fe}}$.

Let $M \in \tilde{\mathcal{K}}(S_0)$ and let $M_1 \to M_2 \to M_3$ be a complex with (ES). Let $\gamma \in \mathbb{Z}\Delta$. One has a CD

(1)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{\mathcal{K}}(S_0)(\mathrm{T}_\gamma(M), M_1(n)) & \longrightarrow & \tilde{\mathcal{K}}(S_0)(\mathrm{T}_\gamma(M), M_2(n)) & \longrightarrow & \tilde{\mathcal{K}}(S_0)(\mathrm{T}_\gamma(M), M_3(n)) & \longrightarrow & 0 \\
 & & \sim\Big| & & \sim\Big| & & \sim\Big| & & \\
0 & \longrightarrow & \tilde{\mathcal{K}}(S_0)(M, \mathrm{T}_{-\lambda}(M_1(n))) & \longrightarrow & \tilde{\mathcal{K}}(S_0)(M, \mathrm{T}_{-\lambda}(M_2(n))) & \longrightarrow & \tilde{\mathcal{K}}(S_0)(M, \mathrm{T}_{-\lambda}(M_3(n))) & \longrightarrow & 0.
\end{array}$$

As $\mathrm{T}_{-\lambda}(M_i(n))_{\{A\}} \simeq \mathrm{T}_{-\lambda}(M_i)_{\{A\}}(n) \simeq (M_i)_{\{A-\lambda\}}(n)$ by (2.10.1), $\mathrm{T}_{-\lambda}(M_1) \to \mathrm{T}_{-\lambda}(M_2) \to \mathrm{T}_{-\lambda}(M_3)$ forms a complex with (ES), and hence the bottom sequence of (1) is exact. Thus, T_γ restricts again to an automorphism of $\tilde{\mathcal{K}}_P(S_0)$.

Take now a chain $I_0 \supset I_1 \supset \dots I_r$ of closed subsets of $\mathcal{A}$ with $I_0 \supseteq \cup_{i=1}^3 \mathrm{supp}_{\mathcal{A}}(M_i)$, $(M_i)_{I_r} = 0$ $\forall i$, and $I_j = I_{j+1} \sqcup \{A_{j+1}\}$ for some $A_{j+1} \in \mathcal{A}$, $j \in [0, r[$. Thus, $\forall i$, $(M_i)_{I_j}/(M_i)_{I_{j+1}} \simeq (M_i)_{\{A_{j+1}\}}$. One has a CD

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (M_1)_{I_{r-1}} & \longrightarrow & (M_2)_{I_{r-1}} & \longrightarrow & (M_3)_{I_{r-1}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (M_1)_{I_{r-2}} & \longrightarrow & (M_2)_{I_{r-2}} & \longrightarrow & (M_3)_{I_{r-2}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (M_1)_{\{A_{r-1}\}} & \longrightarrow & (M_2)_{\{A_{r-1}\}} & \longrightarrow & (M_3)_{\{A_{r-1}\}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with the top and the bottom rows exact by (ES) on the complex and inductively. As the columns are all split exact, the middle row must be exact, and hence exactness of $0 \to M_1 \to M_2 \to M_3 \to 0$ follows. More generally,

Lemma: *$\forall K$ locally closed in $\mathcal{A}$, the sequence $0 \to (M_1)_K \to (M_2)_K \to (M_3)_K \to 0$ is exact.*

Proof: As the $(M_1)_K \to (M_2)_K \to (M_3)_K$ forms a complex in $\tilde{\mathcal{K}}_\Delta(S_0)$ with (ES) by (2.10.2), the assertion follows from the above.

2.12. **Lemma:** *$\forall M \in \tilde{\mathcal{K}}_\Delta(S_0)$, $\forall I_1, I_2$ both closed with $I_2 \subseteq I_1$, (ES) holds on complex $M_{I_2} \to M_{I_1} \to M_{I_1}/M_{I_2}$.*

Proof: As $M_{I_1}/M_{I_2} = M_{I_1 \setminus I_2}$ by (2.6), $\forall A \in \mathcal{A}$, sequence

$$0 \to (M_{I_2})_{\{A\}} \to (M_{I_1})_{\{A\}} \to (M_{I_1}/M_{I_2})_{\{A\}} \to 0$$

reads by (2.8)

$$0 \to M_{I_2 \cap \{A\}} \to M_{I_1 \cap \{A\}} \to M_{(I_1 \setminus I_2) \cap \{A\}} \to 0,$$

which is exact.

2.13 **Base change:** Let S_1 be a commutative flat graded S_0-algebra. $\forall M \in \tilde{\mathcal{K}}'(S_0)$, $S_1 \otimes_{S_0} M$ is a graded (S_1, R)-bimodule. Setting $(S_1 \otimes_{S_0} M)^\emptyset = S_1 \otimes_{S_0} M^\emptyset_A$ $\forall A \in \mathcal{A}$ yields $S_1 \otimes_{S_0} M \in \tilde{\mathcal{K}}'(S_1)$: as $S_1 \otimes_{S_0} M$ is torsion-free over S,

$$\begin{aligned}(1) \qquad S_1 \otimes_{S_0} M \hookrightarrow S^\emptyset \otimes_S (S_1 \otimes_{S_0} M) &\simeq S_1^\emptyset \otimes_{S_0} M \simeq S_1 \otimes_{S_0} S_0^\emptyset \otimes_{S_0} M \simeq S_1 \otimes_{S_0} M^\emptyset \\ &= S_1 \otimes_{S_0} \coprod_{A \in \mathcal{A}} M^\emptyset_A \simeq \coprod_{A \in \mathcal{A}} (S_1 \otimes_{S_0} M^\emptyset_A) \\ &= \coprod_{A \in \mathcal{A}} (S_1 \otimes_{S_0} M)^\emptyset_A.\end{aligned}$$

If I is closed in $\mathcal{A}$,

$$\begin{aligned}(2) \qquad (S_1 \otimes_{S_0} M)_I &= (S_1 \otimes_{S_0} M) \cap \coprod_{A \in I} (S_1 \otimes_{S_0} M)^\emptyset_A \\ &= (S_1 \otimes_{S_0} M) \cap \coprod_{A \in I} (S_1 \otimes_{S_0} M^\emptyset_A) \quad \text{by definition} \\ &\simeq (S_1 \otimes_{S_0} M) \cap (S_1 \otimes_{S_0} \coprod_{A \in I} M^\emptyset_A) \\ &\simeq S_1 \otimes_{S_0} (M \cap \coprod_{A \in I} M^\emptyset_A) \quad \text{in } S_1 \otimes_{S_0} M^\emptyset \text{ as } S_1 \text{ is flat over } S_0 \text{ [BCA, Lem. I.2.6.7]} \\ &= S_1 \otimes_{S_0} M_I.\end{aligned}$$

If $K = I \cap J$ with J open in $\mathcal{A}$,

$$\begin{aligned}(3) \qquad (S_1 \otimes_{S_0} M)_K &= (S_1 \otimes_{S_0} M)_I / (S_1 \otimes_{S_0} M)_{I \setminus J} \\ &\simeq S_1 \otimes_{S_0} (M_I / M_{I \setminus J}) \quad \text{by (2)} \\ &= S_1 \otimes_{S_0} M_K.\end{aligned}$$

If $\alpha \in \Delta^+$, $(S_1 \otimes_{S_0} M)^\alpha = S^\alpha \otimes_S (S_1 \otimes_{S_0} M) \simeq S_1 \otimes_{S_0} (S^\alpha \otimes_S M) = S_1 \otimes_{S_0} M^\alpha$, and hence $S_1 \otimes_{S_0} \tilde{\mathcal{K}}(S_0) \subseteq \tilde{\mathcal{K}}(S_1)$ as S_1 is flat over S_0 again. $\forall A \in \mathcal{A}$, $(S_1 \otimes_{S_0} M)_{\{A\}} = S_1 \otimes_{S_0} M_{\{A\}}$ by (3) as $\{A\} = (\geq A) \cap (\leq A)$ is locally closed. Then $S_1 \otimes_{S_0} \tilde{\mathcal{K}}_\Delta(S_0) \subseteq \tilde{\mathcal{K}}_\Delta(S_1)$.

Set $\tilde{\mathcal{K}}' = \tilde{\mathcal{K}}'(S)$, $\tilde{\mathcal{K}} = \tilde{\mathcal{K}}(S)$, $\tilde{\mathcal{K}}_\Delta = \tilde{\mathcal{K}}_\Delta(S)$, $\tilde{\mathcal{K}}_P = \tilde{\mathcal{K}}_P(S)$. $\forall * \in \Delta^+ \sqcup \{\emptyset\}$, put $\tilde{\mathcal{K}}'^* = \tilde{\mathcal{K}}'(S^*)$, $\tilde{\mathcal{K}}^* = \tilde{\mathcal{K}}(S^*)$, $\tilde{\mathcal{K}}^*_\Delta = \tilde{\mathcal{K}}_\Delta(S^*)$, $\tilde{\mathcal{K}}^*_P = \tilde{\mathcal{K}}_P(S^*)$.

$\forall A \in \mathcal{A}$, one has $S_1 \otimes_S S_0(A) \simeq S_1(A)$.

2.14 **The category over $S^\emptyset$:** Let $M \in \tilde{\mathcal{K}}^\emptyset_\Delta = \tilde{\mathcal{K}}_\Delta(S^\emptyset)$. Then

$$M \simeq S^\emptyset \otimes_S M = M^\emptyset = \coprod_{A\in\mathcal{A}} M^\emptyset_A$$

with

$$\begin{aligned} M^\emptyset_A &= (M_{\{A\}})^\emptyset \simeq M_{\{A\}} \quad \text{by (2.6.1)} \\ &\simeq \coprod_i S^\emptyset(A)(n_i) \quad \exists n_i \in \mathbb{Z}. \end{aligned}$$

Proposition: $\forall M \in \tilde{\mathcal{K}}^\emptyset_\Delta$,

$$\begin{aligned} M &\simeq M_{\{A_1\}} \oplus M_{\{A_2\}} \oplus \cdots \oplus M_{\{A_r\}} \quad \exists A_1, \dots A_r \in \mathcal{A} \\ &\simeq \{\coprod_{i_1} S^\emptyset(A_1)(n_{i_1})\} \oplus \cdots \oplus \{\coprod_{i_r} S^\emptyset(A_r)(n_{i_r})\}. \end{aligned}$$

2.15. Let $M \in \tilde{\mathcal{K}}'(S_0)$. Consider $D(M) = S\mathrm{Mod}(M, R)$ equipped with a structure of (S_0, R)-bimodule such that $(a\varphi f)(m) = \varphi(amf)$, $\forall \varphi \in D(M)$, $a \in S$, $f \in R$, with gradation such that $D(M)^i = \{\varphi \in D(M) | \varphi(M^j) \subseteq R^{i+j}\ \forall j\}\ \forall i \in \mathbb{Z}$. As M is of finite type over S, one has $D(M) = \coprod_i D(M)^i$. Also,

$$\begin{aligned} S^\emptyset \otimes_S D(M) &\simeq S\mathrm{Mod}(M, S^\emptyset) \simeq S^\emptyset\mathrm{Mod}(M^\emptyset, S^\emptyset) \simeq S^\emptyset\mathrm{Mod}(\coprod_{A\in\mathcal{A}} M^\emptyset_A, S^\emptyset) \\ &\simeq \coprod_{A\in\mathcal{A}} S^\emptyset\mathrm{Mod}(M^\emptyset_A, S^\emptyset) \quad \text{as } \mathrm{supp}_{\mathcal{A}}(M) \text{ is finite.} \end{aligned}$$

If $\varphi \in S^\emptyset\mathrm{Mod}(M^\emptyset_A, S^\emptyset)$, $\forall f \in R$, $\forall m \in M^\emptyset_A$,

$$(\varphi f)(m) = \varphi(mf) = \varphi(f_A m) = f_A \varphi(m),$$

and hence $\varphi f = f_A \varphi$. If $a \in S \setminus 0$ and $a\varphi = 0$, then $\forall m \in M$,

$$0 = (a\varphi)(m) = \varphi(am) = a\varphi(m),$$

$\varphi(m) = 0$ as S is a domain. Then $\varphi = 0$, and hence $D(M)$ is torsion-free of finite type over S, and admits a structure of $\tilde{\mathcal{K}}'(S_0)$ with $D(M)^\emptyset_A = S^\emptyset\mathrm{Mod}(M^\emptyset_A, S^\emptyset)\ \forall A \in \mathcal{A}$. If $M \in \tilde{\mathcal{K}}(S_0)$, however, the conditions (S) and (LE) may be hard to verify on $D(M)$.

3. **Action of the Soergel bimodules**

3.1. To define Soergel bimodules of the Coxeter system $(\mathcal{W}, \mathcal{S})$ over $R = \mathrm{S}_{\mathbb{K}}(\Lambda^\vee_{\mathbb{K}})$ as in (I.2), we need $\alpha_s \in \Lambda_{\mathbb{K}}$ and $\alpha^\vee_s \in \Lambda^\vee_{\mathbb{K}}\ \forall s \in \mathcal{S}$. Fix $A \in \mathcal{A}$. There is $\alpha \in \Delta^+$ and $n \in \mathbb{Z}$ such that $s_{\alpha,n}A = As$. Put $\alpha_s = \alpha^A \in \Lambda$ and $\alpha^\vee_s = (\alpha^\vee)^A \in \Lambda^\vee$. If $t = wsw^{-1}$, $w \in \mathcal{W}$, write $Aw = w'A$. Then

$$At = Awsw^{-1} = w'Asw^{-1} = w's_{\alpha,n}Aw^{-1} = w's_{\alpha,n}(w')^{-1}A = s_{\overline{w'}\alpha,n'}A \quad \exists n' \in \mathbb{Z},$$

and we put $\alpha_t = (\overline{w'}\alpha)^A$, $\alpha^\vee_t = (\overline{w'}\alpha^\vee)^A$.

Lemma: *(i) The pair $(\alpha_s, \alpha_s^\vee)$ is independent of the choice of A up to sign.*

(ii) $\alpha_t = w\alpha_s$, $\alpha_t^\vee = w\alpha_s^\vee$, the pair of which is independent of the choice of A up to sign.

Proof: (i) Let $A' \in \mathcal{A}$ and $\beta \in \Delta^+$, $m \in \mathbb{Z}$ with $A's = s_{\beta,m}A'$. Take $x \in \mathcal{W}$ with $A' = xA$. Writing $x = t_x\bar{x}$ with $t_x \in \mathbb{Z}\Delta$, $\forall \nu \in X$,

$$\begin{aligned}(xs_{\alpha,n}x^{-1})\nu &= t_x\bar{x}s_{\alpha,n}(t_x\bar{x})^{-1}\nu = t_x\bar{x}s_{\alpha,n}\bar{x}^{-1}(-t_x)\nu \\ &= t_x\bar{x}\{\bar{x}^{-1}(\nu - t_x) - \langle\bar{x}^{-1}(\nu - t_x), \alpha^\vee\rangle\alpha + n\alpha\} \\ &= t_x(\nu - t_x - \langle\nu - t_x, \bar{x}\alpha^\vee\rangle\bar{x}\alpha + n\bar{x}\alpha) = \nu - \langle\nu, \bar{x}\alpha^\vee\rangle\bar{x}\alpha + n\bar{x}\alpha + \langle t_x, \bar{x}\alpha^\vee\rangle\bar{x}\alpha \\ &= s_{\bar{x}\alpha, n+\langle t_x, \bar{x}\alpha^\vee\rangle}\nu.\end{aligned}$$

and hence

(1) $$xs_{\alpha,n}x^{-1} = s_{\bar{x}\alpha, n+\langle t_x, \bar{x}\alpha^\vee\rangle}.$$

Then $A's = xs_{\alpha,n}A = xs_{\alpha,n}x^{-1}xA = s_{\bar{x}\alpha,n+\langle t_x,\bar{x}\alpha^\vee\rangle}xA = s_{\bar{x}\alpha,n+\langle t_x,\bar{x}\alpha^\vee\rangle}A'$. Thus, $\beta = \varepsilon\bar{x}\alpha, \varepsilon \in \{\pm 1\}$,

$$\begin{aligned}\beta^{A'} &= (\varepsilon\bar{x}\alpha)^{xA} = \varepsilon(\bar{x}\alpha)^{xA} = \varepsilon\alpha^A = \varepsilon\alpha_s \quad \text{by (1.2.iv)}, \\ (\beta^\vee)^{A'} &= (\varepsilon\bar{x}\alpha^\vee)^{xA} = \varepsilon(\bar{x}\alpha^\vee)^{xA} = \varepsilon(\alpha^\vee)^A = \varepsilon\alpha_s^\vee \quad \text{by (1.2.v)}.\end{aligned}$$

(ii) One has

$$(w\alpha_s)_A = (w(\alpha^A))_A = (w(\alpha^A))(A) = \alpha^A(Aw) = \alpha^A(w'A) = \overline{w'}(\alpha^A(A)) = \overline{w'}\alpha = ((\overline{w'}\alpha)^A)_A,$$

and hence $w\alpha_s = (\overline{w'}\alpha)^A = \alpha_t$. Likewise,

$$\begin{aligned}(w\alpha_s^\vee)_A &= (w(\alpha^\vee)^A)_A = (w(\alpha^\vee)^A)(A) = (\alpha^\vee)^A(Aw) \\ &= (\alpha^\vee)^A(w'A) = \overline{w'}(\alpha^\vee)^A(A) = \overline{w'}\alpha^\vee = ((\overline{w'}\alpha^\vee)^A)_A,\end{aligned}$$

and hence $w\alpha_s^\vee = (\overline{w'}\alpha^\vee)^A = \alpha_t^\vee$.

3.2. One has, $\forall s \in \mathcal{S}$, $\forall \lambda \in \Lambda$, $\forall \nu \in \Lambda^\vee$,

(1) $$s\lambda = \lambda - \langle\lambda, \alpha_s^\vee\rangle\alpha_s, \quad \text{and} \quad s\nu = \nu - \langle\alpha_s, \nu\rangle\alpha_s^\vee.$$

For if $A \in \mathcal{A}$,

$$\begin{aligned}(s\lambda)_A &= (s\lambda)(A) = \lambda(As) = \lambda(s_{\alpha,n}A) = s_\alpha\lambda(A) = s_\alpha\lambda_A = \lambda_A - \langle\lambda_A, \alpha^\vee\rangle\alpha \\ &= \lambda_A - \langle\lambda_A, ((\alpha^\vee)^A)_A\rangle(\alpha^A)_A = (\lambda - \langle\lambda_A, (\alpha_s^\vee)_A\rangle\alpha_s)_A \\ &= (\lambda - \langle\lambda, \alpha_s^\vee\rangle\alpha_s)_A \quad \text{by (1.2.5)},\end{aligned}$$

and likewise the second.

Also,

(2) $$\begin{aligned}\langle\alpha_s, \alpha_s^\vee\rangle &= \langle(\alpha_s)_A, (\alpha_s^\vee)_A\rangle \quad \text{from (1.2.5)} \\ &= \langle(\alpha^A)_A, ((\alpha^\vee)^A)_A\rangle \quad \text{in the notation of (3.1)} \\ &= \langle\alpha, \alpha^\vee\rangle = 2.\end{aligned}$$

As $2 \in \mathbb{K}^\times$ by (1.5.A1), $\alpha_s^\vee \neq 0$ in $\Lambda_{\mathbb{K}}^\vee$ and $\langle \alpha_s, ? \rangle : \Lambda_{\mathbb{K}}^\vee \to \mathbb{K}$ is surjective. Thus, the assumptions in (I.1.1) are fulfilled with $V = \Lambda_{\mathbb{K}}^\vee$.

We add another

Assumption: *The assumption (I.3.3) holds.*

For a sufficient condition under which the assumption holds see (I.3.4); if the fundamental weights exist in $\Lambda_{\mathbb{K}}$, the sufficient condition may fail. In type G_2 let α_1, α_2 be the simple roots with α_1 short. In characteristic 3 there is no $\lambda \in \Lambda_{\mathbb{K}}$ such that $\langle \lambda, \alpha_1^\vee \rangle = 0$ while that $\langle \lambda, 2\alpha_1^\vee + 3\alpha_2^\vee \rangle = 1$. On the other hand, let $\Delta^s = \{\alpha_1, \dots, \alpha_l\}$ be the set of simple roots and assume that $\forall i \in [1, l]$, $\exists \varpi_i \in X_{\mathbb{K}} : \forall j \in [1, l]$, $\langle \varpi, \alpha_j^\vee \rangle = \delta_{i,j}$. Given two reflections t_1, t_2 we may assume $\alpha_{t_1} = \alpha_i^\vee$ for some $i \in [1, l]$. Thus we are after $\sum_{j \neq i} c_j \varpi_j \in X_{\mathbb{K}}$ with $\langle \sum_{j \neq i} c_j \varpi_j, \alpha_{t_2}^\vee \rangle = 1$. If $\mathrm{ch}\mathbb{K} > 3$, writing $\alpha_{t_2}^\vee = \sum_{k=1}^l b_k \alpha_k^\vee$, $\gcd(b_k | 1 \leq k \leq l) \in \mathbb{K}^\times$, and hence such $\sum c_j \varpi_j$ exists. If $\det[\![\langle \alpha_i, \alpha_j^\vee \rangle]\!] \in \mathbb{K}^\times$, those ϖ_i's exist in $\sum_{k=1}^l \mathbb{K}\alpha_j$: if $M[\![\langle \alpha_i, \alpha_j^\vee \rangle]\!] = \mathrm{id}$, $\langle \sum_k M_{ik} \alpha_k, \alpha_j^\vee \rangle = \delta_{i,j}$.

3.3. Let now $\mathfrak{SB}$ denote the monoidal category of graded R-bimodules defined in (I.2), denoted $\mathfrak{S}$Bimod there, for the present Coxeter system $(\mathcal{W}, \mathcal{S})$ and the representation $\Lambda_{\mathbb{K}}^\vee$ with $\{\alpha_s, \alpha_s^\vee | s \in \mathcal{S}\}$ from (3.1). Recall $R^\emptyset = R[\frac{1}{(\alpha^\vee)^A} | \alpha \in \Delta]$ for $A \in \mathcal{A}$ from Rmk. 2.2.(iii) and $Q = \mathrm{Frac}(R)$.

$\forall B \in \mathfrak{SB}$, put $B^\emptyset = R^\emptyset \otimes_R B$. $\forall s \in \mathcal{S}$, recall from (I.2.2) an object $B(s) = R \otimes_{R^s} R(1)$ with $R^s = \{a \in R | sa = a\}$. From (I.2.2.16) one has $B(s)^\emptyset = \coprod_{w \in \mathcal{W}} B(s)_w^\emptyset$ with

$$(1) \qquad B(s)_w^\emptyset = \begin{cases} R^\emptyset(\delta_s \otimes 1 - 1 \otimes s\delta_s) & \text{if } w = e, \\ R^\emptyset(\delta_s \otimes 1 - 1 \otimes \delta_s) & \text{if } w = s, , \\ 0 & \text{else} \end{cases}$$

where $\delta_s \in \Lambda_{\mathbb{K}}^\vee$ with $\langle \alpha_s, \delta_s \rangle = 1$. Also, from (I.2.2.8)

$$B(s) \hookrightarrow Q \otimes_R B(s) = Q(e) \oplus Q(s) \quad \text{via} \quad a \otimes b \mapsto (ab, a(sb)).$$

Lemma: *$\forall B \in \mathfrak{SB}$, $B^\emptyset \simeq \coprod_{x \in \mathcal{W}} B_x^\emptyset$ as $R^\emptyset$-bimodules with $amb = a(xb)m$ $\forall a, b \in R^\emptyset$, $\forall m \in B_x^\emptyset$ such that $Q \otimes_R B_x^\emptyset \simeq Q \otimes_{R^\emptyset} B_x^\emptyset \simeq B_x^Q$ as Q-bimodules.*

Proof: Let $M \in R$Bimod with $M^\emptyset = \coprod_{x \in \mathcal{W}} M_x^\emptyset$ as $R^\emptyset$-bimodules such that

$$(2) \qquad amb = a(xb)m \quad \forall a, b \in R^\emptyset, \forall m \in M_x^\emptyset.$$

Then

$$\begin{aligned}(B(s) \otimes_R M)^\emptyset &= R^\emptyset \otimes_R B(s) \otimes_R M \simeq (R^\emptyset \otimes_R B(s)) \otimes_{R^\emptyset} (R^\emptyset \otimes_R M) = B(s)^\emptyset \otimes_{R^\emptyset} M^\emptyset \\ &= (R_e^\emptyset \oplus R_s^\emptyset) \otimes_{R^\emptyset} \coprod_{x \in \mathcal{W}} M_x^\emptyset = \coprod_x \{(R_e^\emptyset \otimes_{R^\emptyset} M_x^\emptyset) \oplus (R_s^\emptyset \otimes_{R^\emptyset} M_x^\emptyset)\},\end{aligned}$$

and hence $(B(s) \otimes_R M)^\emptyset = \coprod_{x \in \mathcal{W}} (B(s) \otimes_R M)_x^\emptyset$ with (2) holding on $(B(s) \otimes_R M)_x^\emptyset = (R_e^\emptyset \otimes_{R^\emptyset} M_x^\emptyset) \oplus (R_s^\emptyset \otimes_{R^\emptyset} M_{sx}^\emptyset)$. The assertion follows inductively.

3.4 **The action:** $\forall M \in \tilde{\mathcal{K}}'(S_0)$, $\forall B \in \mathfrak{SB}$, we define $M * B \in \tilde{\mathcal{K}}'(S_0)$ to be the (S_0, R)-bimodule $M \otimes_R B$ with

$$(M * B)^{\emptyset}_A = \coprod_{x \in \mathcal{W}} M^{\emptyset}_{Ax^{-1}} \otimes_{R^{\emptyset}} B^{\emptyset}_x \simeq \coprod_{x \in \mathcal{W}} M^{\emptyset}_{Ax^{-1}} \otimes_R B^{\emptyset}_x. \tag{1}$$

As B is a free left R-module, $M \otimes_R B$ remains torsion-free over S. $\forall m \in M^{\emptyset}_{Ax^{-1}}$, $\forall b \in B^{\emptyset}_x$, $\forall a \in R^{\emptyset}$, $(m \otimes b)a = m \otimes (xa)b = m(xa) \otimes b = (xa)_{Ax^{-1}} m \otimes b$ with

$$\begin{aligned} (xa)_{Ax^{-1}} &= a_{Ax^{-1}x} \quad \text{by (1.2.i)} \\ &= a_A, \end{aligned}$$

and hence (1) is well-defined. Let $\varphi \in \tilde{\mathcal{K}}'(S_0)(M, N)$. By (2.2.3)

$$\varphi(M^{\emptyset}_{Ax^{-1}}) \subseteq \coprod_{\substack{A' \in Ax^{-1} + \mathbb{Z}\Delta \\ A' \geq Ax^{-1}}} N^{\emptyset}_{A'}.$$

If $A' = Ax^{-1} + \gamma$ with $\gamma \in \mathbb{Z}\Delta$, $A' \geq Ax^{-1}$ iff $\gamma \in \mathbb{N}\Delta^+$ by (1.3). Then $A'x = A + \gamma$ as the right $\mathcal{W}$-action commute with the left $\mathcal{W}$-action, and hence $A'x \geq A$ by (1.3) again. Then

$$\coprod_{\substack{A' \in Ax^{-1} + \mathbb{Z}\Delta \\ A' \geq Ax^{-1}}} N^{\emptyset}_{A'} = \coprod_{\substack{A'x^{-1} \in Ax^{-1} + \mathbb{Z}\Delta \\ A'x^{-1} \geq Ax^{-1}}} N^{\emptyset}_{A'x^{-1}} = \coprod_{\substack{A' \in A + \mathbb{Z}\Delta \\ A' \geq A}} N^{\emptyset}_{A'x^{-1}},$$

and hence

$$(\varphi \otimes_R B^{\emptyset}_x)(M^{\emptyset}_{Ax^{-1}} \otimes_{R^{\emptyset}} B^{\emptyset}_x) \subseteq \coprod_{\substack{A' \in A + \mathbb{Z}\Delta \\ A' \geq A}} N^{\emptyset}_{A'x^{-1}} \otimes_R B^{\emptyset}_x \subseteq \coprod_{A' \geq A} (N * B)^{\emptyset}_{A'}.$$

Thus, $(\varphi \otimes_R B)^{\emptyset}((M * B)^{\emptyset}_A) \subseteq \coprod_{A' \geq A} (N * B)^{\emptyset}_{A'}$, and $\varphi \otimes_R B \in \tilde{\mathcal{K}}'(S_0)$. Likewise, $\forall \psi \in \mathfrak{SB}(B, B')$, $M \otimes \psi \in \tilde{\mathcal{K}}'(S_0)(M * B, M * B')$. Thus, $*$ is bi-functorial, and defines a right action of the monoidal category $\mathfrak{SB}$ of Soergel bimodules on $\tilde{\mathcal{K}}'(S_0)$. We will denote $\varphi \otimes_R B$ (resp. $M \otimes_R \psi$) by $\varphi * B$ (resp. $M * \psi$).

Let $\gamma \in \mathbb{Z}\Delta$. $\forall A \in \mathcal{A}$,

$$\begin{aligned} \mathrm{T}_\gamma(M * B)^{\emptyset}_A = (M * B)^{\emptyset}_{A+\gamma} &= \coprod_{x \in \mathcal{W}} M^{\emptyset}_{(A+\gamma)x^{-1}} \otimes_R B^{\emptyset}_x \\ &= \coprod_{x \in \mathcal{W}} M^{\emptyset}_{Ax^{-1}+\gamma} \otimes_R B^{\emptyset}_x \quad \text{by (1.1.3), which may fail for } \gamma \in \hat{X} \text{ in general} \\ &= \coprod_{x \in \mathcal{W}} \mathrm{T}_\gamma(M)^{\emptyset}_{Ax^{-1}} \otimes_R B^{\emptyset}_x = \{\mathrm{T}_\gamma(M) * B\}^{\emptyset}_A, \end{aligned}$$

and hence

$$\mathrm{T}_\gamma(M * B) = \mathrm{T}_\gamma(M) * B. \tag{2}$$

3.5 **Lemma:** $\forall M \in \tilde{\mathcal{K}}'(S_0)$, $\mathrm{supp}_{\mathcal{A}}(M * B) = \{Ax | A \in \mathrm{supp}_{\mathcal{A}}(M), x \in \mathrm{supp}_{\mathcal{W}}(B)\}$.

Proof: One has

$$\begin{aligned}\operatorname{supp}_{\mathcal{A}}(M * B) &= \{A | Ax^{-1} \in \operatorname{supp}_{\mathcal{A}}(M)\ \exists x \in \operatorname{supp}_{\mathcal{W}}(B)\} \\ &= \{Ax | A \in \operatorname{supp}_{\mathcal{A}}(M), x \in \operatorname{supp}_{\mathcal{W}}(B)\}.\end{aligned}$$

3.6. For $M \in \tilde{\mathcal{K}}'(S_0)$ let us regard $M * B(s)$ as $M \otimes_{R^s} R = (M \otimes_{R^s} R^s) \oplus (M \otimes_{R^s} R^s\delta_s)$, using that $R = R^s \oplus \delta_s R^s$ with $\langle \alpha_s, \delta_s \rangle = 1$ from (I.2.1). Accordingly, $(M * B(s))^{\emptyset} = (M^{\emptyset} \otimes_{R^s} R^s) \oplus (M^{\emptyset} \otimes_{R^s} R^s\delta_s)$. Then

(1)
$$\begin{aligned}(M * B(s))^{\emptyset}_A &= \coprod_{x \in \mathcal{W}} M^{\emptyset}_{Ax^{-1}} \otimes_{R^{\emptyset}} B(s)^{\emptyset}_x \quad \text{by definition (3.4.1)} \\ &= M^{\emptyset}_A \otimes_{R^{\emptyset}} B(s)^{\emptyset}_e \oplus M^{\emptyset}_{As} \otimes_{R^{\emptyset}} B(s)^{\emptyset}_s \\ &= M^{\emptyset}_A \otimes_{R^{\emptyset}} R^{\emptyset}(\delta_s \otimes 1 - 1 \otimes s\delta_s) \oplus M^{\emptyset}_{As} \otimes_{R^{\emptyset}} R^{\emptyset}(\delta_s \otimes 1 - 1 \otimes \delta_s) \quad \text{by (3.3.1)} \\ &= \{m\delta_s \otimes 1 - m \otimes s\delta_s | m \in M^{\emptyset}_A\} \oplus \{m'\delta_s \otimes 1 - m' \otimes \delta_s | m' \in M^{\emptyset}_{As}\} \\ &\simeq M^{\emptyset}_A \oplus M^{\emptyset}_{As}\end{aligned}$$

under $(m \otimes f, m' \otimes g) \mapsto (mf + m'g, m(sf) + m'(sg))$ as left $S^{\emptyset}_0$-modules;

$$\begin{aligned}(m\delta_s \otimes 1 - m \otimes s\delta_s, &m'\delta_s \otimes 1 - m' \otimes \delta_s) \mapsto \\ &(m\delta_s - m(s\delta_s) + m'\delta_s - m'\delta_s, m\delta_s - m\delta_s + m'\delta_s - m's\delta_s) \\ &= (m(\delta_s - s\delta_s), m'(\delta_s - s\delta_s)) = (m\alpha_s^{\vee}, m'\alpha_s^{\vee}) = ((\alpha_s^{\vee})_A m, (\alpha_s^{\vee})_{As} m') \\ &= (\alpha^{\vee} m, -\alpha^{\vee} m')\end{aligned}$$

as $(\alpha_s^{\vee})_{As} = (\alpha_s^{\vee})(As) = (\alpha_s^{\vee})(s_{\alpha,n}A) = s_\alpha(\alpha_s^{\vee}(A)) = s_\alpha((\alpha^{\vee})^A(A)) = s_\alpha((\alpha^{\vee})^A)_A = s_\alpha \alpha^{\vee} = -\alpha^{\vee}$, with $\alpha^{\vee} \in (S^{\emptyset})^{\times}$. The isomorphism is, however, only left $S^{\emptyset}_0$-linear. For recall from (3.3.1) the right R-module structure on $B(s)$, which reads, $\forall a \in R$, $(\delta_s \otimes 1 - 1 \otimes s\delta_s)a = a(\delta_s \otimes 1 - 1 \otimes s\delta_s) = a\delta_s \otimes 1 - a \otimes s\delta_s$ while $(\delta_s \otimes 1 - 1 \otimes \delta_s)a = (sa)(\delta_s \otimes 1 - 1 \otimes \delta_s) = (sa)\delta_s \otimes 1 - (sa) \otimes s\delta_s$, and hence

$$\begin{aligned}(m\delta_s \otimes 1 - m \otimes s\delta_s, &m'\delta_s \otimes 1 - m' \otimes \delta_s)f = (mf\delta_s \otimes 1 - mf \otimes s\delta_s, m'(sf)\delta_s \otimes 1 - m'(sf) \otimes \delta_s) \\ &\mapsto (mf\delta_s - mf(s\delta_s), m'(sf)\delta_s - m'(sf)s\delta_s) = (mf\alpha_s^{\vee}, m'(sf)\alpha_s^{\vee}) \\ &= (m\alpha_s^{\vee} f, m'\alpha_s^{\vee}(sf)).\end{aligned}$$

Thus, the right action on $M^{\emptyset}_{As}$ must be twisted by s. The projection

$$\begin{array}{ccc} M * B(s) & \longrightarrow & (M * B(s))^{\emptyset}_A \\ & \searrow & \wr\| \\ & & M^{\emptyset}_A \oplus M^{\emptyset}_{As} \end{array}$$

now reads

(2)
$$M \otimes_{R^s} R \ni m \otimes f \mapsto (m_A f, m_{As} sf).$$

Proposition: $\forall M, N \in \tilde{\mathcal{K}}'(S_0)$, $\forall n \in \mathbb{Z}$,

$$\tilde{\mathcal{K}}'(S_0)(M * B(s), N(n)) \simeq \tilde{\mathcal{K}}'(S_0)(M, (N * B(s))(n)).$$

Proof: As (S_0, R)-bimodules we regard $M * B(s) = M \otimes_{R^s} R$ and $N * B(s) = N \otimes_{R^s} R$. Put $\delta = \delta_s \in \Lambda^\vee_{\mathbb{K}}$ above. Recall from [Lib, Lem. 3.3]/(I.2.6.ii) a bijection $(S_0, R)\mathrm{Bimodgr}(M \otimes_{R^s} R, N) \to (S_0, R)\mathrm{Bimodgr}(M, N \otimes_{R^s} R)$ via $\varphi \mapsto \psi$ such that $\psi(m) = \varphi(m\delta \otimes 1) \otimes 1 - \varphi(m \otimes 1) \otimes s\delta$. Thus, it is enough to verify that $\varphi \in \tilde{\mathcal{K}}'(S_0)$ iff $\psi \in \tilde{\mathcal{K}}'(S_0)$.

$\forall m \in M$ with $1 \otimes m \in M^\emptyset = S^\emptyset \otimes_S M$, $\forall A' \in \mathcal{A}$, one has in $N^\emptyset_{A'} \oplus N^\emptyset_{A's}$

$$\begin{aligned}(3)\qquad \psi^Q(1 \otimes m)_{A'} &= \{\varphi(m\delta \otimes 1) \otimes 1 - \varphi(m \otimes 1) \otimes s\delta\}_{A'} \\ &= (\varphi(m\delta \otimes 1)_{A'}, \varphi(m\delta \otimes 1)_{A's}) - (\varphi(m \otimes 1)_{A'} s\delta, \varphi(m \otimes 1)_{A's}\delta) \quad \text{by (2)} \\ &= (\varphi(m\delta \otimes 1 - m \otimes s\delta))_{A'}, \varphi(m\delta \otimes 1 - m \otimes \delta)_{A's}).\end{aligned}$$

Thus, if $m \in M^\emptyset_A$,

$$(4)\qquad \psi^Q(m)_{A'} = (\varphi^Q(m\delta \otimes 1 - m \otimes s\delta))_{A'}, \varphi^Q(m\delta \otimes 1 - m \otimes \delta)_{A's}).$$

As $m\delta \otimes 1 - m \otimes s\delta \in (M * B(s))^\emptyset_A$ and as $m\delta \otimes 1 - m \otimes \delta \in (M * B(s))^\emptyset_{As}$, one has from (2.2.3)

$$\varphi^Q(m\delta \otimes 1 - m \otimes s\delta) \in \coprod_{\substack{A' \in A + \mathbb{Z}\Delta \\ A' \geq A}} N^\emptyset_{A'},$$

$$\varphi^Q(m\delta \otimes 1 - m \otimes \delta) \in \coprod_{\substack{A' \in As + \mathbb{Z}\Delta \\ A' \geq As}} N^\emptyset_{A'},$$

and hence $\psi^Q(m)_{A'} = 0$ unless either $A' \in A + \mathbb{Z}\Delta$ and $A' \geq A$ or $A's \in As + \mathbb{Z}\Delta$ and $A's \geq As$. In the 2nd case write $A's = As + \gamma$, $\gamma \in \mathbb{Z}\Delta$. Then $\gamma \in \mathbb{N}\Delta$ by (1.3). As $A' = A + \gamma$, $A' \geq A$ by (1.3) again. Thus, $\psi^Q(m) \in \coprod_{A' \geq A} (N * B(s))_{A'}$, as desired.

Conversely, assume $\psi \in \tilde{\mathcal{K}}'(S_0)$. One has from (1)

$$(M * B(s))^\emptyset_A = \{(m\delta \otimes 1 - m \otimes s\delta, m'\delta \otimes 1 - m' \otimes \delta | m \in M^\emptyset_A, m' \in M^\emptyset_{As}\}.$$

If $m \in M^\emptyset_A$,

$$\begin{aligned}\coprod_{A'} (\varphi^Q(m\delta \otimes 1 - m \otimes s\delta)_{A'}, \varphi^Q(m\delta \otimes 1 - m \otimes \delta)_{A's}) &= \psi^Q(m) \quad \text{by (4)} \\ &\in \coprod_{\substack{A' \in A + \mathbb{Z} \\ A' \geq A}} (N * B(s))^\emptyset_{A'} = \coprod_{\substack{A' \in A + \mathbb{Z}\Delta \\ A' \geq A}} (N^\emptyset_{A'} \oplus N^\emptyset_{A's}),\end{aligned}$$

and hence $\varphi^Q(m\delta \otimes 1 - m \otimes s\delta) \in \coprod_{A' \geq A} N^\emptyset_{A'}$. If $m' \in M^\emptyset_{As}$,

$$\psi^Q(m') \in \coprod_{\substack{A' \in As + \mathbb{Z} \\ A' \geq As}} (N * B(s))^\emptyset_{A'} = \coprod_{\substack{A' \in As + \mathbb{Z} \\ A' \geq As}} (N^\emptyset_{A'} \oplus N^\emptyset_{A's}),$$

and hence $\varphi^Q(m'\delta \otimes 1 - m' \otimes \delta) \in \coprod_{\substack{A' \in As + \mathbb{Z} \\ A' \geq As}} N^\emptyset_{A's}$ by (4) again, in which case writing $A' = As + \gamma$, $\gamma \in \mathbb{Z}\Delta$, $\gamma \in \mathbb{N}\Delta$ by (1.3) again and $A's = A + \gamma$. Then $A's \geq A$ by (1.3) again, and $\varphi^Q(m'\delta \otimes 1 - m' \otimes \delta) \in \coprod_{A's \geq A} N^\emptyset_{A's} = \coprod_{A' \geq A} N^\emptyset_{A'}$. Thus, $\varphi^Q((M * B(s))^\emptyset_A) \subseteq \coprod_{A' \geq A} N^\emptyset_{A'}$, as desired.

3.7. We will show that $\tilde{\mathcal{K}}_\Delta(S_0)*\mathfrak{S}\mathfrak{B}=\tilde{\mathcal{K}}_\Delta(S_0)$ and that $\tilde{\mathcal{K}}_P(S_0)*\mathfrak{S}\mathfrak{B}=\tilde{\mathcal{K}}_P(S_0)$. To start with, recall $\mathcal{W}^\alpha=\{e,s_\alpha\}\ltimes\mathbb{Z}\alpha$, $\alpha\in\Delta^+$, from (2.5). Let $M\in\tilde{\mathcal{K}}'(S_0)$ and put $M^\Omega=M^\alpha\cap\coprod_{A\in\Omega}M^\emptyset_A$ $\forall\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}$.

Lemma: *Let $s\in\mathcal{S}$ and $\delta_s\in\Lambda^\vee_{\mathbb{K}}$ with $\langle\alpha_s,\delta_s\rangle=1$, e.g., $\frac{1}{2}\alpha_s^\vee$.*

(i) If $\Omega s=\Omega$, $(M*B(s))^\Omega\simeq M^\Omega*B(s)$.

(ii) If $\Omega s\neq\Omega$, the right action by $\alpha_s^\vee$ on M^Ω is invertible and

$$(M*B(s))^\Omega\simeq M^\Omega\otimes_R R(\delta_s\otimes 1-1\otimes s\delta_s)\oplus M^{\Omega s}\otimes_R R(\delta_s\otimes 1-1\otimes\delta_s).$$

(iii) If (LE) holds on M, so does it on $M*B$ $\forall B\in\mathfrak{S}\mathfrak{B}$.

Proof: (i) One has

$$\begin{aligned}
(M*B(s))^\Omega&=(M*B(s))^\alpha\cap\coprod_{A\in\Omega}(M*B(s))^\emptyset_A\\
&=(M*B(s))^\alpha\cap\coprod_{A\in\Omega}\{M^\emptyset_A\otimes_R B(s)^\emptyset_e\oplus M^\emptyset_{As}\otimes_R B(s)^\emptyset_s\}\\
&=(M*B(s))^\alpha\cap\{(\coprod_{A\in\Omega}M^\emptyset_A\otimes_R B(s)^\emptyset_e)\oplus(\coprod_{A\in\Omega}M^\emptyset_A\otimes_R B(s)^\emptyset_s)\}\quad\text{as }\Omega s=\Omega\\
&\simeq(M^\alpha\otimes_R B(s))\cap\coprod_{A\in\Omega}(M^\emptyset_A\otimes_R B(s)^\emptyset)\\
&\simeq(M^\alpha\cap\coprod_{A\in\Omega}M^\emptyset_A)\otimes_R B(s)\quad\text{as }B(s)\text{ is left }R\text{-free}\\
&=M^\Omega*B(s).
\end{aligned}$$

(ii) Let $A\in\Omega$ and put $\beta^\vee=(\alpha_s^\vee)_A$; $As=s_{\beta,r}A$ $\exists r\in\mathbb{Z}$. As $\Omega=\mathcal{W}^\alpha A=(A+\mathbb{Z}\alpha)\cup(s_\alpha A+\mathbb{Z}\alpha)$ and $\Omega s=\mathcal{W}^\alpha As=(As+\mathbb{Z}\alpha)\cup(s_\alpha As+\mathbb{Z}\alpha)$, $As\neq s_{\alpha,n}A$ $\forall n\in\mathbb{Z}$, and hence $\beta\neq\pm\alpha$ and $s_\alpha\beta\neq\pm\alpha$. Thus, $\beta^\vee,s_\alpha\beta^\vee\in(S^\alpha)^\times$. Take $\delta\in X^\vee_{\mathbb{K}}$ with $\langle\alpha,\delta\rangle=1$. $\forall m\in M^\Omega$, there are $m_1\in\coprod_{A'\in A+\mathbb{Z}\alpha}M^\emptyset_{A'}$ and $m_2\in\coprod_{A'\in s_\alpha A+\mathbb{Z}\alpha}M^\emptyset_{A'}$ such that $m=m_1+m_2$. $\forall f\in R$, one has from (2.2.1) that $m_1f=f_Am_1$ and $m_2f=s_\alpha(f_A)m_2$, and hence

$$\begin{aligned}
&m_1\alpha_s^\vee=(\alpha_s^\vee)_Am_1=\beta^\vee m_1,\quad m_2\alpha_s^\vee=s_\alpha((\alpha_s^\vee)_A)m_2=s_\alpha\beta^\vee m_2,\\
&m_1\delta^A=(\delta^A)_Am_1=\delta m_1,\quad m_2\delta^A=s_\alpha((\delta^A)_A)m_2=(s_\alpha\delta)m_2=(\delta-\alpha^\vee)m_2.
\end{aligned}$$

Then $m\alpha_s^\vee=\beta^\vee m_1+s_\alpha\beta^\vee m_2$, $m\delta^A=\delta m_1+(\delta-\alpha^\vee)m_2$, and hence

$$\begin{aligned}
&\{\frac{1}{\beta^\vee}m+\frac{\langle\alpha,\beta^\vee\rangle}{\beta^\vee s_\alpha(\beta^\vee)}(\delta m-m\delta^A)\}\alpha_s^\vee\\
&=(\frac{1}{\beta^\vee}+\frac{\langle\alpha,\beta^\vee\rangle\delta}{\beta^\vee s_\alpha(\beta^\vee)})\{\beta^\vee m_1+s_\alpha(\beta^\vee)m_2\}-\frac{\langle\alpha,\beta^\vee\rangle}{\beta^\vee s_\alpha(\beta^\vee)}\{\delta\beta^\vee m_1+(\delta-\alpha^\vee)s_\alpha(\beta^\vee)m_2\}\\
&=m_1+m_2=m.
\end{aligned}$$

Thus, $M^\Omega \alpha_s^\vee = M^\Omega$. Also, if $m \in M^\Omega$ with $m\alpha_s^\vee = 0$, $0 == \beta^\vee m_1 + s_\alpha(\beta^\vee)m_2$. As $\beta^\vee, s_\alpha(\beta^\vee) \in (S^\emptyset)^\times$, we must have $m_1 = m_2 = 0$, and hence $m = 0$. It now follows that the right multiplication by $\alpha_s^\vee$ on M^Ω is invertible, and also on $(M * B(s))^\Omega$. Then

$$(M * B(s))^\Omega \simeq (M * B(s))^\Omega \otimes_R R[\frac{1}{\alpha_s^\vee}].$$

Put $B(s)[\frac{1}{\alpha_s^\vee}] = B(s) \otimes_R R[\frac{1}{\alpha_s^\vee}]$. As $(\delta \otimes 1 - 1 \otimes s\delta)\alpha_s^\vee = \alpha_s^\vee(\delta \otimes 1 - 1 \otimes s\delta)$ and as $(\delta \otimes 1 - 1 \otimes \delta)\alpha_s^\vee = s(\alpha_s^\vee)(\delta \otimes 1 - 1 \otimes \delta) = -\alpha_s^\vee(\delta \otimes 1 - 1 \otimes \delta)$ by (3.2.2), one has from (3.3.1)

$$B(s)[\frac{1}{\alpha_s^\vee}] = R[\frac{1}{\alpha_s^\vee}](\delta \otimes 1 - 1 \otimes s\delta) \oplus R[\frac{1}{\alpha_s^\vee}](\delta \otimes 1 - 1 \otimes \delta)$$

with $R[\frac{1}{\alpha_s^\vee}](\delta \otimes 1 - 1 \otimes s\delta) \subseteq B(s)_e^\emptyset$ while $R[\frac{1}{\alpha_s^\vee}](\delta \otimes 1 - 1 \otimes \delta) \subseteq B(s)_s^\emptyset$. Thus,

$$\begin{aligned}
&(M * B(s))^\Omega \otimes_R R[\frac{1}{\alpha_s^\vee}] \simeq (M * B(s)[\frac{1}{\alpha_s^\vee}])^\alpha \cap \coprod_{A \in \Omega} (M * B(s))_A^\emptyset \\
&= (M^\alpha \otimes_R B(s)[\frac{1}{\alpha_s^\vee}]) \cap \coprod_{A \in \Omega} \{(M_A^\emptyset \otimes_R B(s)_e^\emptyset) \oplus (M_{As}^\emptyset \otimes_R B(s)_s^\emptyset)\} \\
&= \{(M^\alpha \otimes_R R[\frac{1}{\alpha_s^\vee}](\delta \otimes 1 - 1 \otimes s\delta)) \oplus (M^\alpha \otimes_R R[\frac{1}{\alpha_s^\vee}](\delta \otimes 1 - 1 \otimes \delta))\} \\
&\qquad \cap \{\coprod_{A \in \Omega} M_A^\emptyset \otimes_R R[\frac{1}{\alpha_s^\vee}](\delta \otimes 1 - 1 \otimes s\delta) \oplus \coprod_{A \in \Omega s} M_A^\emptyset \otimes_R R[\frac{1}{\alpha_s^\vee}](\delta \otimes 1 - 1 \otimes \delta)\} \\
&= (M^\alpha \cap \coprod_{A \in \Omega} M_A^\emptyset) \otimes_R R[\frac{1}{\alpha_s^\vee}](\delta \otimes 1 - 1 \otimes s\delta) \oplus (M^\alpha \cap \coprod_{A \in \Omega s} M_A^\emptyset) \otimes_R R[\frac{1}{\alpha_s^\vee}](\delta \otimes 1 - 1 \otimes \delta) \\
&\qquad \text{as } \alpha_s^\vee \text{ is invertible on } M^\Omega \text{ and on } M^{\Omega s} \\
&= M^\Omega \otimes_R R(\delta \otimes 1 - 1 \otimes s\delta) \oplus M^{\Omega s} \otimes_R R(\delta \otimes 1 - 1 \otimes \delta).
\end{aligned}$$

(iii) It is enough to show that $(M * B(s))^\alpha = \coprod_{\Omega \in \mathcal{W}^\alpha \backslash \mathcal{A}} (M * B(s))^\Omega$. Let $\{\Omega_1, \dots, \Omega_r\}$ be a complete set of representatives of $\{\Omega \in \mathcal{W}^\alpha \backslash \mathcal{A} | \Omega s \neq \Omega\}/\{e, s\}$. Then $\{\Omega \in \mathcal{W}^\alpha \backslash \mathcal{A} | \Omega s \neq \Omega\} =$

$\{\Omega_i, \Omega_i s | i \in [1, r]\}$, and hence

$$\begin{aligned}
\coprod_{\Omega \in \mathcal{W}^\alpha \backslash \mathcal{A}} (M * B(s))^\Omega &= \{\coprod_{\Omega s = \Omega} (M * B(s))^\Omega\} \oplus \coprod_{i=1}^r \{(M * B(s))^{\Omega_i} \oplus (M * B(s))^{\Omega_i s}\} \\
&= \{\coprod_{\Omega s = \Omega} (M^\Omega * B(s))\} \oplus \coprod_{i=1}^r \{(M * B(s))^{\Omega_i} \oplus (M * B(s))^{\Omega_i s}\} \quad \text{by (i)} \\
&= \{\coprod_{\Omega s = \Omega} (M^\Omega * B(s))\} \oplus \coprod_{i=1}^r \{M^{\Omega_i} \otimes_R R(\delta \otimes 1 - 1 \otimes s\delta) \oplus M^{\Omega_i s} \otimes_R R(\delta \otimes 1 - 1 \otimes \delta) \\
&\qquad \oplus M^{\Omega_i s} \otimes_R R(\delta \otimes 1 - 1 \otimes s\delta) \oplus M^{\Omega_i} \otimes_R R(\delta \otimes 1 - 1 \otimes \delta)\} \quad \text{by (ii)} \\
&= \{\coprod_{\Omega s = \Omega} (M^\Omega * B(s))\} \oplus \coprod_{i=1}^r \{(M^{\Omega_i} \otimes_{R[\frac{1}{\alpha_s^\vee}]} B(s)[\frac{1}{\alpha_s^\vee}]) \oplus (M^{\Omega_i s} \otimes_{R[\frac{1}{\alpha_s^\vee}]} B(s)[\frac{1}{\alpha_s^\vee}])\} \\
&\qquad\qquad \text{by (ii) again} \\
&= \{\coprod_{\Omega s = \Omega} (M^\Omega * B(s))\} \oplus \coprod_{i=1}^r \{(M^{\Omega_i} * B(s)) \oplus (M^{\Omega_i s} * B(s))\} \\
&= (\coprod_{\Omega \in \mathcal{W}^\alpha \backslash \mathcal{A}} M^\Omega) * B(s) \\
&= M^\alpha * B(s) \quad \text{as (LE) holds on } M \\
&= (M * B(s))^\alpha.
\end{aligned}$$

3.8. Let $M \in \tilde{\mathcal{K}}(S_0)$ and $s \in \mathcal{S}$. As (LE) holds on $M * B(s)$ by (3.7), so does (S) on $(M * B(s))^\alpha$ $\forall \alpha \in \Delta^+$ by (2.5).

Lemma: *If I is closed in $\mathcal{A}$ with $Is = I$, $(M * B(s))_I = M_I * B(s)$.*

Proof: One has

$$\begin{aligned}
((M * B(s))_I)^\emptyset &= \coprod_{A \in I} (M * B(s))^\emptyset_A = \coprod_{A \in I} \{M^\emptyset_A \otimes_R B(s)^\emptyset_e \oplus M^\emptyset_{As} \otimes_R B(s)^\emptyset_s\} \\
&= \coprod_{A \in \mathcal{A}} \{(M_I)^\emptyset_A \otimes_R B(s)^\emptyset_e \oplus (M_I)^\emptyset_A \otimes_R B(s)^\emptyset_s\} \quad \text{by (2.4) as } Is = I \\
&= \coprod_{A \in \mathcal{A}} \{(M_I)^\emptyset_A \otimes_R B(s)\} = (M_I * B(s))^\emptyset,
\end{aligned}$$

and hence

$$\begin{aligned}
(M * B(s))_I &= (M * B(s)) \cap (M_I * B(s))^\emptyset = (M \otimes_{R^s} R(1)) \cap (M_I)^\emptyset \otimes_{R^s} R(1) \\
&\simeq (M \cap (M_I)^\emptyset) \otimes_{R^s} R(1) \quad \text{as } R \text{ is free over } R^s \\
&= M_I \otimes_{R^s} R(1) = M_I * B(s).
\end{aligned}$$

3.9. Let $M \in \tilde{\mathcal{K}}(S_0)$, $s \in \mathcal{S}$, and put $N = M * B(s)$. Let $A \in \mathcal{A}$ with $As < A$. We know that $\{A, As\} = (\geq As) \cap (\leq A)$ [L80, 1.4.1] is locally closed.

Lemma: *Let $I = Is$ (resp. J) closed (resp. open) in $\mathcal{A}$ with $I \cap J = \{A, As\}$. One has isomorphisms of graded left S_0-modules*

$$N_{I\setminus\{As\}}/N_{I\setminus\{A,As\}} \simeq M_{\{A,As\}}(-1), \quad N_I/N_{I\setminus\{As\}} \simeq M_{\{A,As\}}(1).$$

Proof: Note first that $I \setminus \{A, As\} = I \setminus J$ and $I \setminus \{As\} = (I \setminus J) \cup (\geq A)$ are both closed, and hence that $N_I, N_{I\setminus\{As\}}, N_{I\setminus\{A,As\}} \in \tilde{\mathcal{K}}'(S_0)$.

Consider a short exact sequence

(1) $$0 \to N_{I\setminus\{As\}}/N_{I\setminus\{A,As\}} \to N_I/N_{I\setminus\{A,As\}} \to N_I/N_{I\setminus\{As\}} \to 0.$$

Put $L_1 = N_{I\setminus\{As\}}/N_{I\setminus\{A,As\}}, L = N_I/N_{I\setminus\{A,As\}}, \bar{L} = N_I/N_{I\setminus\{As\}}$. By flat base change (1) yields a CD of exact sequences

(2) $$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1^{\emptyset} & \longrightarrow & L^{\emptyset} & \longrightarrow & \bar{L}^{\emptyset} & \longrightarrow & 0 \\ & & \sim\big| & & \big|\sim & & \big|\sim & & \\ 0 & \longrightarrow & N_A^{\emptyset} & \longrightarrow & N_A^{\emptyset} \oplus N_{As}^{\emptyset} & \longrightarrow & N_{As}^{\emptyset} & \longrightarrow & 0. \end{array}$$

By (2.6) all $L_1, L, \bar{L} \in \tilde{\mathcal{K}}'(S_0)$, and hence $L_1 = L_1 \cap (L_1)^{\emptyset} = L_1 \cap L_A^{\emptyset} = L \cap L_A^{\emptyset}$; if $x \in L \cap L_A^{\emptyset}$, $x = 0$ in $\bar{L} \leq \bar{L}^{\emptyset}$, and hence $x \in L_1$ from (1). We are then to show that $L \cap L_A^{\emptyset} \simeq M_{\{A,As\}}(-1)$ and $\bar{L} \simeq M_{\{A,As\}}(1)$ as graded left S_0-modules. One has

(3) $$\begin{aligned} L &= (M_I * B(s))/(M_{I\setminus\{A,As\}} * B(s)) \quad \text{by (3.8)} \\ &\simeq \{M_I \otimes_{R^s} R(1)\}/\{M_{I\setminus\{A,As\}} \otimes_{R^s} R(1)\} \\ &\simeq (M_I/M_{I\setminus\{A,As\}}) \otimes_{R^s} R(1) \quad \text{as } R \text{ is flat over } R^s \\ &= M_{\{A,As\}} \otimes_{R^s} R(1) \quad \text{as } M \in \tilde{\mathcal{K}}(S_0) \\ &= M_{\{A,As\}} * B(s). \end{aligned}$$

By (3.6.2) one has

$$\begin{array}{ccccc} L(-1) & & \longrightarrow & & L_A^{\emptyset} \simeq (M_{\{A,As\}})_A^{\emptyset} \oplus (M_{\{A,As\}})_{As}^{\emptyset} \\ \| & & & & \\ M_{\{A,As\}} \otimes_{R^s} R^s \oplus M_{\{A,As\}} \otimes_{R^s} R^s\delta & & (m_1 \otimes 1, m_2 \otimes \delta) & \longmapsto & (m_{1,A} + m_{2,A}\delta, m_{1,As} + m_{2,As}s\delta) \\ \downarrow & & \Big\downarrow & & \\ L_{As}^{\emptyset} & & & & \\ \sim\big| & & & & \\ (M_{\{A,As\}})_{As}^{\emptyset} \oplus (M_{\{A,As\}})_A^{\emptyset} & & (m_{1,As} + m_{2,As}\delta, m_{1,A} + m_{2,A}s\delta). & & \end{array}$$

Then

(4) $$\begin{aligned} (m_1 \otimes 1, m_2 \otimes \delta) \in L_A^{\emptyset} \quad &\text{iff} \quad m_{1,As} + m_{2,As}\delta = 0 = m_{1,A} + m_{2,A}s\delta \\ &\text{iff} \quad \begin{cases} m_{1,A} = -m_{2,A}s\delta = -(s\delta)_A m_{2,A} \\ m_{1,As} = -\delta_{As}m_{2,As} = -(s\delta)_A m_{2,As} \quad \text{by (1.2.i)} \end{cases} \\ &\text{iff} \quad m_1 = -(s\delta)_A m_2 \quad \text{as } \mathrm{supp}_{\mathcal{A}}(m_1) \text{ and } \mathrm{supp}_{\mathcal{A}}(m_2) \subseteq \{A, As\}. \end{aligned}$$

Thus,

$$L \cap L_A^{\emptyset} = \{(-(s\delta)_A m \otimes 1, m \otimes \delta) | m \in M_{\{A,As\}}\}(1)$$
$$\simeq M_{\{A,As\}}(-1) \quad \text{as } \deg(\delta) = 2 = \deg((s\delta)_A).$$

Consider next an epi of graded left S_0-modules

$$L \simeq M_{\{A,As\}} \otimes_{R^s} R(1) \to M_{\{A,As\}}(1) \quad \text{via} \quad m \otimes f \mapsto (sf)_A m.$$

As $(m_1 \otimes 1, m_2 \otimes \delta) \mapsto m_1 + (s\delta)_A m_2$, its kernel is $L \cap L_A^{\emptyset}$ by (4), and hence $M_{\{A,As\}}(1) \simeq L/(L \cap L_A^{\emptyset}) \simeq L/L_1 \simeq \bar{L}$.

3.10. Let $A \in \mathcal{A}$ with $As < A$. Recall from [L80, Prop. 3.2] that

(1) $$\forall B \in \mathcal{A} \text{ with } B \leq A, Bs \leq A.$$

Then

(2) $$\leq A = (\leq A)s.$$

For if $B \leq A$, $Bs \leq A$, and hence $(\leq A)s \subseteq (\leq A)$. In turn, hitting s from the right yields $(\leq A) \subseteq (\leq A)s$.

Let I (resp. J) be closed (resp. open) in $\mathcal{A}$. Then

(3) $$I \cup Is \text{ is closed.}$$

For let $B \in Is$ and $B' > B$. Then $Bs \in I$. Assume first $B' < B's$. Then $Bs \leq B's$ by (1), and hence $B's \in I$, and $B' \in Is$. If $B's < B'$, $Bs \leq B'$ by (1) again, and hence $B' \in I$, as desired.

As we do not know yet if $M * B(s) \in \tilde{\mathcal{K}}(S_0)$, it is not appropriate to express $(M * B(s))_I/(M * B(s))_{I \setminus J}$ as $(M * B(s))_{I \cup J}$.

Lemma: *Let $M \in \tilde{\mathcal{K}}_\Delta(S_0)$.*

(i) If $I \cap J = \{As\}$, $(M * B(s))_I/(M * B(s))_{I \setminus J} \simeq M_{\{A,As\}}(1)$ as graded left S_0-modules.

(ii) If $I \cap J = \{A\}$, $(M * B(s))_I/(M * B(s))_{I \setminus J} \simeq M_{\{A,As\}}(-1)$ as graded left S_0-modules.

Proof: Put $N = M * B(s) \in \tilde{\mathcal{K}}'(S_0)$.

(i) Put $I_1 = (\geq As)$. Then $I_1 = I_1 s$ by [L80, Prop. 3.2]. As I is closed with $As \in I$, $I_1 \subseteq I$. Thus

$$N_{I_1}/N_{I_1 \setminus \{As\}} \hookrightarrow N_I/N_{I \setminus \{As\}} = N_I/N_{I \setminus J}.$$

As $I_1 \cap (\leq A) = \{A, As\}$ by [L80, 1.4.1], $N_{I_1}/N_{I_1 \setminus \{As\}} \simeq M_{\{A,As\}}(1)$ by (3.9), and hence $M_{\{A,As\}}(1) \leq N_I/N_{I \setminus J}$.

$\forall \alpha \in \Delta^+$, $M^\alpha \in \tilde{\mathcal{K}}(S_0^\alpha)$ by (2.13). Then (LE) holds on $N^\alpha \simeq M^\alpha * B(s)$ by (3.7), and hence (S) holds on $N^\alpha = (N^\alpha)^\alpha$ by (2.5). Thus, $N^\alpha \in \tilde{\mathcal{K}}(S_0^\alpha)$. Then $(N^\alpha)_{I \setminus J}$ does not depend on the choice of I and J by (2.6.2), and hence

$$\begin{aligned}(N^\alpha)_I/(N^\alpha)_{I\setminus J} &\simeq (N^\alpha)_{I_1}/(N^\alpha)_{I_1\setminus\{As\}}\\ &\simeq M^\alpha_{\{A,As\}}(1) \quad \text{by (3.9) again.}\end{aligned}$$

As M admits a Δ-flag, $M_{\{A,As\}}$ is graded free over S_0 by (2.10). Then

$$\begin{aligned}M_{\{A,As\}}(1) &= \cap_{\alpha\in\Delta}(M^\alpha_{\{A,As\}}(1)) = \cap_{\alpha\in\Delta}\{(N^\alpha)_I/(N^\alpha)_{I\setminus J}\}\\ &\geq N_I/N_{I\setminus J} \quad \text{as } N_I/N_{I\setminus J}\in\tilde{\mathcal{K}}'(S_0) \text{ is torsion-free over } S.\end{aligned}$$

Thus, $N_I/N_{I\setminus J}\simeq M^\alpha_{\{A,As\}}(1)$.

(ii) We first show that

$$N_I/N_{I\setminus J}\hookrightarrow M_{\{A,As\}}(-1). \tag{4}$$

As $I\setminus J = I\setminus(\leq A)$, we may assume $J=(\leq A)$. Then $J=Js$ by (2). Put $I_2' = I\cup Is$, which is right s-invariant closed in $\mathcal{A}$ by (3). As $I_2'\cap J = (I\cap J)\cup(Is\cap J) = (I\cap J)\cup(I\cap J)s = \{A,As\}$, $I_2'\setminus\{A,As\} = I_2'\setminus J$ and $I_2'\setminus\{As\} = I_2'\setminus(\leq As)$ are both closed. Also, $I_2'\setminus\{As\}\supseteq I$; if $I\ni As$, $I\supseteq\{A,As\}$ implying $I\cap J\supseteq\{A,As\}$, absurd. As $I\not\ni As$ again, $I\setminus\{A,As\} = I\setminus\{A\} = I\setminus J$, and hence $N_I/N_{I\setminus J}\hookrightarrow N_{I_2'\setminus\{As\}}/N_{I_2'\setminus\{A,As\}}\simeq M_{\{A,As\}}(-1)$ by (3.9).

Take now a sequence of closed subsets $I_0\subset\cdots\subset I_r$ with $|I_{i+1}| = |I_i|+1$ $\forall i$ such that $I_0 = I_0 s$ and $I_r = I_r s$, $N_{I_0}=0$, $N_{I_r}=N$, $I_k = I$ and $I_{k-1} = I\setminus\{A\}$ for some $k\in[1,r]$. Write $I_i = I_{i-1}\sqcup\{A_i\}$.

Assume for the moment that $\mathbb{K}$ is a field. Thus, letting $?^d$ denote the d-th homogeneous piece, $\dim_{\mathbb{K}} N^d = \sum_j \dim_{\mathbb{K}}(N_{I_j}/N_{I_{j-1}})^d$. By (i) and (4) one has

$$\dim_{\mathbb{K}}(N_{I_j}/N_{I_{j-1}})^d \leq \dim_{\mathbb{K}} M^{d+\varepsilon(A_j)}_{\{A_j,A_js\}} \quad\text{with}\quad \varepsilon(A_j) = \begin{cases}-1 & \text{if } A_js<A_j,\\ 1 & \text{else.}\end{cases}$$

Then

$$\begin{aligned}\sum_{j=1}^r \dim_{\mathbb{K}} M^{d+\varepsilon(A_j)}_{\{A_j,A_js\}} &= \sum_{j=1}^r\{\dim_{\mathbb{K}} M^{d+\varepsilon(A_j)}_{\{A_j\}} + \dim_{\mathbb{K}} M^{d+\varepsilon(A_j)}_{\{A_js\}}\}\\ &= \sum_{A_js>A_j}\dim_{\mathbb{K}} M^{d+1}_{\{A_j\}} + \sum_{A_js>A_j}\dim_{\mathbb{K}} M^{d+1}_{\{A_js\}} + \sum_{A_js<A_j}\dim_{\mathbb{K}} M^{d-1}_{\{A_j\}} + \sum_{A_js<A_j}\dim_{\mathbb{K}} M^{d-1}_{\{A_js\}}\\ &= \sum_{A_js>A_j}\dim_{\mathbb{K}} M^{d+1}_{\{A_j\}} + \sum_{A_js<A_j}\dim_{\mathbb{K}} M^{d+1}_{\{A_j\}} + \sum_{A_js<A_j}\dim_{\mathbb{K}} M^{d-1}_{\{A_j\}} + \sum_{A_js<A_j}\dim_{\mathbb{K}} M^{d-1}_{\{A_j\}}\\ &= \sum_j \dim_{\mathbb{K}} M^{d+1}_{\{A_j\}} + \sum_j \dim_{\mathbb{K}} M^{d-1}_{\{A_j\}} = \dim_{\mathbb{K}} M^{d+1} + \dim_{\mathbb{K}} M^{d-1}.\end{aligned}$$

On the other hand, if $\langle\alpha_s,\delta\rangle = 1$, $N = M\otimes_{R^s}R(1) = M(1)\otimes_{R^s}R^s\oplus M(1)\otimes_{R^s}R^s\delta$, and hence

$$\begin{aligned}\dim_{\mathbb{K}} N^d &= \dim_{\mathbb{K}} M(1)^d + \dim_{\mathbb{K}} M(1)^{d-2} \quad \text{as } \deg\delta = 2\\ &= \dim_{\mathbb{K}} M^{d+1} + \dim_{\mathbb{K}} M^{d-1}.\end{aligned}$$

Then

$$\dim_{\mathbb{K}} N^d = \sum_j \dim_{\mathbb{K}} (N_{I_j}/N_{I_{j-1}})^d \leq \sum_j \dim_{\mathbb{K}} M^{d+\varepsilon(A_j)}_{\{A_j, A_j s\}} = \dim_{\mathbb{K}} N^d.$$

It follows that we must have in (4) an isomorphism $N_I/N_{I\setminus J} \xrightarrow{\sim} M_{\{A,As\}}(1)$.

Back to the general complete DVR $\mathbb{K}$, we have from (2.5.2) that

$$N_{I_j} \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \simeq \{N \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})\}_{I_j},$$

and hence $(N_{I_j} \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}))_j$ gives a filtration of $N \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m})$. Then $N_I/N_{I\setminus J} \otimes_{\mathbb{K}} (\mathbb{K}/\mathfrak{m}) \xrightarrow{\sim} M_{\{A,As\}}(-1) \otimes_{\mathbb{K}} \mathbb{K}/\mathfrak{m}$ as left $S_{\mathbb{K}/\mathfrak{m}}$-modules by the case considered above. Then by (1) and graded NAK

$$(N_I/N_{I\setminus J}) \otimes_{\mathbb{K}} \mathbb{K}_{\mathfrak{m}} \xrightarrow{\sim} M_{\{A,As\}}(-1) \otimes_{\mathbb{K}} \mathbb{K}_{\mathfrak{m}},$$

and hence $N_I/N_{I\setminus J} \xrightarrow{\sim} M_{\{A,As\}}(-1)$.

3.11. Let $M \in \mathcal{K}_\Delta(S_0)$, $s \in \mathcal{S}$, and $N = M * B(s)$.

Lemma: *$\forall I_1$ and I_2 closed with $I_1 \supseteq I_2$, N_{I_1}/N_{I_2} is graded free over S_0.*

Proof: Take a sequence $I_2 = I'_0 \subset I'_1 \subset \cdots \subset I'_r = I_1$ of closed subsets in $\mathcal{A}$ with $|I'_i| = |I'_{i-1}|+1$ $\forall i \in [1,r]$, and write $I'_i = I'_{i-1} \sqcup \{A_i\}$. As $\{A_i\} = I_i \setminus I_{i-1} = I_i \cap (\mathcal{A} \setminus I_{i-1})$, one has from (3.10)

$$N_{I'_i}/N_{I'_{i-1}} \simeq M_{\{A_i, A_i s\}}(\varepsilon(A_i)) \quad \exists \varepsilon(A_i) \in \{\pm 1\},$$

which is graded free over S_0 by (2.10); if $A_i s < A_i$, $\{A_i, A_i s\} = (\geq A_i s) \cap (\leq A_i)$ by [L80, 1.4.1]. Then $N_{I_1}/N_{I_2} = N_{I'_r}/N_{I'_0}$ is graded free over S_0.

3.12. We are now ready to show

Proposition: $\tilde{\mathcal{K}}_\Delta(S_0) * \mathfrak{S}\mathfrak{B} = \tilde{\mathcal{K}}_\Delta(S_0)$.

Proof: Put $N = M * B(s)$, $M \in \tilde{\mathcal{K}}_\Delta(S_0)$, $s \in \mathcal{S}$. We know from (3.7) that (LE) holds on N. We show next that (S) holds on N, so $N \in \tilde{\mathcal{K}}$. Given I_1 and I_2 both closed in $\mathcal{A}$, consider $N_{I_1}/N_{I_1 \cap I_2} \hookrightarrow N_{I_1 \cup I_2}/N_{I_2}$. Both terms of the imbedding are graded free over S_0 by (3.11). $\forall \alpha \in \Delta^+$, (S) holds on N^α by (2.5), and hence the imbedding turns invertible upon base extension to S_0^α by $S^\alpha \otimes_S ?$. Then

$$N_{I_1 \cup I_2}/N_{I_2} = \cap_\alpha (N^\alpha_{I_1 \cup I_2}/N^\alpha_{I_2}) \simeq \cap_\alpha (N^\alpha_{I_1}/N^\alpha_{I_1 \cap I_2}) = N_{I_1}/N_{I_1 \cap I_2},$$

and hence $N_{I_1 \cup I_2} = N_{I_1} + N_{I_2}$.

Finally, $\forall A \in \mathcal{A}$, $N_{\{A\}} \simeq M_{\{A,As\}}(\pm 1)$ by (3.10), which is graded free over S by (2.10), and hence $N \in \tilde{\mathcal{K}}_\Delta(S_0)$.

3.13. **Corollary:** *$\forall M \in \tilde{\mathcal{K}}_\Delta(S_0)$, $\forall A \in \mathcal{A}$, $\forall s \in \mathcal{S}$,*

$$\operatorname{grk}((M * B(s))_{\{A\}}) = \begin{cases} v^{-1}\{\operatorname{grk}(M_{\{A\}}) + \operatorname{grk}(M_{\{As\}})\} & \text{if } As < A, \\ v\{\operatorname{grk}(M_{\{A\}}) + \operatorname{grk}(M_{\{As\}})\} & \text{else.} \end{cases}$$

Proof: One has $M * B(s) \in \tilde{\mathcal{K}}_\Delta$ by (3.12), and by (3.10)

$$(M * B(s))_{\{A\}} \simeq \begin{cases} M_{\{A,As\}}(-1) & \text{if } As < A, \\ M_{\{A,As\}}(1) & \text{else.} \end{cases}$$

Thus, if $As < A$,

$$\begin{aligned} \operatorname{grk}((M * B(s))_{\{A\}}) &= v^{-1}\operatorname{grk}(M_{\{A,As\}}) \quad \text{by convention (I.7.2)} \\ &= v^{-1}\{\operatorname{grk}(M_{\{A\}}) + \operatorname{grk}(M_{\{As\}})\}, \end{aligned}$$

and likewise if $As > A$.

3.14 **Proposition:** $\tilde{\mathcal{K}}_P(S_0) * \mathfrak{SB} \subseteq \tilde{\mathcal{K}}_P(S_0)$.

Proof: We have only to show that $M * B(s) \in \tilde{\mathcal{K}}_P(S_0)\ \forall M \in \tilde{\mathcal{K}}_P(S_0)\ \forall s \in \mathcal{S}$. As $M * B(s) \in \tilde{\mathcal{K}}_\Delta$ by (3.12), we are left to show that $\forall$ complex $M_1 \to M_2 \to M_3$ in $\tilde{\mathcal{K}}_\Delta(S_0)$ with (ES),

(1) $\quad 0 \to \tilde{\mathcal{K}}_\Delta(S_0)(M * B(s), M_1) \to \mathcal{K}_\Delta(S_0)(M * B(s), M_2) \to \mathcal{K}_\Delta(S_0)(M * B(s), M_3) \to 0$

is exact.

By (3.6) the sequence (1) reads

$$0 \to \tilde{\mathcal{K}}_\Delta(S_0)(M, M_1 * B(s)) \to \mathcal{K}_\Delta(S_0)(M, M_2 * B(s)) \to \mathcal{K}_\Delta(S_0)(M, M_3 * B(s)) \to 0.$$

As $M \in \tilde{\mathcal{K}}_P(S_0)$, it is enough to show that (ES) holds on the complex $M_1 * B(s) \to M_2 * B(s) \to M_3 * B(s)$, i.e., $\forall A \in \mathcal{A}$,

$$0 \to (M_1 * B(s))_{\{A\}} \to (M_2 * B(s))_{\{A\}} \to (M_3 * B(s))_{\{A\}} \to 0$$

is exact. By (3.13) the sequence reads

$$0 \to (M_1)_{\{A,As\}}(\pm 1) \to (M_2)_{\{A,As\}}(\pm 1) \to (M_3)_{\{A,As\}}(\pm 1) \to 0$$

with ± 1 varying simultaneously, which is exact by (2.11).

3.15. Recall from (1.2.5), fixing $A \in \mathcal{A}$, an isomororphism of graded $\mathbb{K}$-algebras $S = S_{\mathbb{K}}(X_{\mathbb{K}}^\vee) \simeq S_{\mathbb{K}}(\Lambda_{\mathbb{K}}^\vee) = R$ via $a \mapsto a^A\ \forall a \in S$. Under the identification one has, $\forall B \in \mathfrak{SB}$, $\forall x \in \mathcal{W}$, an isomorphism of $R^\emptyset$-bimodules $S(A) * B = S(A) \otimes_R B \to B$ via $a \otimes m \mapsto a^A m$ such that

$$(1) \qquad \begin{array}{ccc} S(A) * B & \xrightarrow{\qquad\qquad\qquad} & B \\ \big\downarrow & & \big\downarrow \\ (S(A) * B)^\emptyset & \dashrightarrow & B^\emptyset \\ \big\uparrow & & \big\uparrow \\ (S(A)_* B)^\emptyset_{Ax} = S(A)^\emptyset_A \otimes_{R^\emptyset} B^\emptyset_x = S^\emptyset \otimes_{R^\emptyset} B^\emptyset_x & \xrightarrow{\sim} & B^\emptyset_x. \end{array}$$

One has, $\forall m \in B^\emptyset_x$, $\forall a \in S$,

$$(1 \otimes m)a^{Ax} = a(1 \otimes m) = a \otimes m = 1 \otimes a^A m \mapsto a^A m = m x^{-1}(a^A)$$

with $x^{-1}(a^A) = x^{-1}(a^{Axx^{-1}}) = a^{Ax}$ by (1.2.vi). The following justifies (I.8.17)

Proposition: *$S(A)*?$ imbeds $\mathfrak{SB}$ into $\tilde{\mathcal{K}}_\Delta$.*

Proof: As $S(A) \in \tilde{\mathcal{K}}_\Delta$, the assertion follows from (3.12).

4. **Projectives**

$\forall M, N \in \tilde{\mathcal{K}}'$, $S\mathrm{Modgr}^\sharp(M,N) = S\mathrm{Mod}(M,N)$ [AJS, E.1] is of finite type over S as both M and N are. Then $(\tilde{\mathcal{K}}')^\sharp(M,N)$ is of finite type over S, and hence $\tilde{\mathcal{K}}'(M,N)$ is finite dimensional over $\mathbb{K}$. It follows that $\tilde{\mathcal{K}}'$ is Krull-Schmidt [CR, pf of 16.10, p. 126], and so is $\tilde{\mathcal{K}}_P$. In this section we will study $\tilde{\mathcal{K}}_P$.

4.1. Recall from (I.9.5) that $B(w_0) \in \mathcal{C}_P^{\mathrm{fe}}$. Let $A^- = A^+ w_0 = w_0 A^+$ and set $Q(A^-) = S(A^-) * B(w_0)(-\ell(w_0))$. As $S(A^-) \in \tilde{\mathcal{K}}_\Delta$, one has $Q(A^-) \in \tilde{\mathcal{K}}_\Delta$ by (3.12). Specifically, recall from (I.9.4) an isomorphism $B(w_0)(-\ell(w_0)) \simeq F(\mathcal{Z})$ in $\mathcal{C}$. We will denote $\mathcal{Z}$ by $\mathcal{Z}_f$ in present Chap. II and suppress F. In particular, $\mathrm{supp}_{\mathcal{W}}(B(w_0)) = \mathcal{W}_f$, and hence $\mathrm{supp}_{\mathcal{A}}(Q(A^-)) = A^-\mathcal{W}_f = \mathcal{W}_f A^-$. Recall from (I.9.2) that, $\forall w \in \mathcal{W}_f$,

$$B(w_0)(-\ell(w_0))^{\mathrm{fe}}_{\{w\}} \simeq (\mathcal{Z}_f)^{\mathrm{fe}}_{\{w\}} \simeq R(w)(-2\ell(w)).$$

Let $d : \mathcal{A} \times \mathcal{A} \to \mathbb{Z}$ be a function from [L80, 1.4]. It follows that

(1)
$$\begin{aligned} Q(A^-)_{\{A^- w\}} &\simeq S(A^- w)(-2\ell(w)) = S(A^- w)(2d(A^- w, A^-)) \\ &= S(w_0 w w_0 A^-)(2d(w_0 w w_0 A^-, A^-)). \end{aligned}$$

Recall also from (I.9.3) an isomorphism $R \otimes_{R^{\mathcal{W}_f}} R \to \mathcal{Z}_f$ of graded $\mathbb{K}$-algebras compatible with their structure of R-bimodules.

Lemma: *$\forall M \in \tilde{\mathcal{K}}'(S_0)$, $\tilde{\mathcal{K}}'(S_0)^\sharp(S_0 \otimes_S Q(A^-), M) \simeq M_{\geq A^-}$.*

Proof: Define a structure of right R-module on S using an isomorphism $S \simeq R$ via $f_{A^-} \hookleftarrow f$. One then obtains an isomorphism of (S,R)-bimodules

$$Q(A_-) \simeq S(A^-) * (R \otimes_{R^{\mathcal{W}_f}} R) \simeq S \otimes_R (R \otimes_{R^{\mathcal{W}_f}} R) \simeq S \otimes_{R^{\mathcal{W}_f}} R,$$

and hence $S_0 \otimes_S Q(A_-) \simeq S_0 \otimes_{R^{\mathcal{W}_f}} R$.

$\forall A \in \mathcal{A}$, $\forall m \in M_A^\emptyset$, $\forall f \in R^{\mathcal{W}_f}$, one has $mf = f_A m = f(A)m$ with

$$\begin{aligned} f(A) &= f(xA^-) \quad \text{if } A = xA^-,\ x \in \mathcal{W} \\ &= \bar{x} f(A^-) = f(\bar{x}A^-) \quad \text{by definition (1.2)} \\ &= f(\bar{x} w_0 A^+) = f(A^+ \bar{x} w_0) = f(A^- w_0 \bar{x} w_0) = (w_0 \bar{x} w_0 f)(A^-) \quad \text{by definition (1.2.3)} \\ &= f(A^-) \quad \text{as } f \in R^{\mathcal{W}_f} \\ &= f_{A^-}, \end{aligned}$$

and hence M admits a structure of $S_0 \otimes_{R^{\mathcal{W}_f}} R$-module. Then

$$\tilde{\mathcal{K}}'(S_0)^\sharp(S_0 \otimes_S Q(A_-), M) \leq (S_0 \otimes_{R^{\mathcal{W}_f}} R)\mathrm{Mod}(S_0 \otimes_{R^{\mathcal{W}_f}} R, M) \simeq M.$$

Moreover, $\forall\varphi \in \tilde{\mathcal{K}}'(S_0)^\sharp(S_0 \otimes_S Q(A^-), M)$, $\varphi(S_0 \otimes_S Q(A^-)) = \varphi((S_0 \otimes_S Q(A^-))_{\geq A^-}) \leq M_{\geq A^-}$, and hence $\tilde{\mathcal{K}}'(S_0)^\sharp(S_0 \otimes_S Q(A_-), M) \leq M_{\geq A^-}$.

Now, given $m \in M_{\geq A^-}$, let $\psi \in (S_0 \otimes_{R^{\mathcal{W}_f}} R)\mathrm{Mod}(S_0 \otimes_{R^{\mathcal{W}_f}} R, M)$ such that $1_{S_0 \otimes_{R^{\mathcal{W}_f}} R} \mapsto m$. To see that $\psi \in \tilde{\mathcal{K}}'(S_0)^\sharp(S_0 \otimes_S Q(A^-), M)$, one has to check that $\forall A \in \mathcal{W}_f A^-$, $\psi^\emptyset((S_0 \otimes_S Q(A^-))^\emptyset_A) \subseteq M^\emptyset_{\geq A}$. By Rmk. 2.2.(i) one has $\psi^\emptyset((S_0 \otimes_S Q(A^-))^\emptyset_A) \subseteq \coprod_{B \in A+\mathbb{Z}\Delta} M^\emptyset_B$. Then the assertion will follow from the following lemma.

4.2. $\forall\lambda \in \hat{X}$. Let $A^-_\lambda = A^- + \lambda$, and $\mathcal{W}_\lambda = \mathrm{C}_{\mathcal{W}}(\lambda) = \{x \in \mathcal{W} | x\lambda = \lambda\} = t_\lambda \mathcal{W}_f t_{-\lambda}$.

Lemma: $\forall A \in \mathcal{W}_\lambda A^-_\lambda$, $(A + \mathbb{Z}\Delta) \cap (\geq A^-_\lambda) = \{A' \in A + \mathbb{Z}\Delta | A' \geq A\}$.

Proof: We may assume that $\lambda = 0$. It is enough to show that LHS $\subseteq$ RHS. Let $A' \in$ LHS. Write $A = wA^-$, $w \in \mathcal{W}_f$, and $A' = A + \gamma$, $\gamma \in \mathbb{Z}\Delta$. By (1.3) one has only to check that $\gamma \in \mathbb{N}\Delta^+$. Write $A' = xA$, $x \in \mathcal{W}$. By definition, $\forall\nu \in A$, $x\nu - \nu \in \mathbb{N}\Delta^+$. The assertion follows.

4.3. Given a complex $M \to M' \to M''$ in $\tilde{\mathcal{K}}_\Delta(S_0)$ with (ES), one has an exact sequence $0 \to M_{\geq A^-} \to M'_{\geq A^-} \to M''_{\geq A^-} \to 0$ by (2.11), and hence a sequence

$$0 \to \tilde{\mathcal{K}}(S_0)^\sharp(Q(A^-), M) \to \tilde{\mathcal{K}}(S_0)^\sharp(Q(A^-), M') \to \tilde{\mathcal{K}}(S_0)^\sharp(Q(A^-), M'') \to 0.$$

is exact by (4.1). Thus, $S_0 \otimes_S Q(A^-) \in \tilde{\mathcal{K}}_P(S_0)$. Also, from (4.1) one has, as $\mathbb{K}$-modules,

$$\tilde{\mathcal{K}}'(Q(A^-), Q(A^-)) \simeq \{Q(A^-)_{\geq A^-}\}^0 \simeq Q(A^-)^0 \simeq \mathcal{Z}^0_f \simeq \mathbb{K},$$

and hence, together with (4.1.1),

Proposition: *$S_0 \otimes_S Q(A^-)$ is an object of $\tilde{\mathcal{K}}_P(S_0)$ with* $\mathrm{supp}_{\mathcal{A}}(S_0 \otimes_S Q(A^-)) = A^-\mathcal{W}_f$ *such that $\{S_0 \otimes_S Q(A^-)\}_{\{A\}} \simeq S_0(A)(2d(A, A^-))$ $\forall A \in A^-\mathcal{W}_f$. In $\tilde{\mathcal{K}}'$, $Q(A^-)$ is indecomposable.*

4.4. Put $\mathcal{A}^- = w_0\mathcal{A}^+$. Let $A \in \mathcal{A}^-$ and write $A = A^-x$, $x \in \mathcal{W}$. Then, $\forall y \in \mathcal{W}$ with $y < x$, $\forall w \in \mathcal{W}_f$,

$$wA^-y > A. \tag{1}$$

For $d(A, A^-) = \ell(x)$ by [L80, Lem. 3.6], and hence $A^-y > A$ by [L80, Cor. 3.4]. If $wA^-y \notin \mathcal{A}^-$, take $w' \in \mathcal{W}_f$ such that $w'wA^-y \in \mathcal{A}^-$. Then $w'wA^-y \in \mathcal{A}^- < wA^-y$ by [J, II.6.4.5], and hence we may assume $wA^-y \in \mathcal{A}^-$. Write $y = zy'$ with $z \in \mathcal{W}_f$ and $y' \in \mathcal{W}$ with $A^-y' \in \mathcal{A}^-$ such that $\ell(y) = \ell(z) + \ell(y')$ [L80, Lem. 3.6]. Then

$$\begin{aligned} wA^-y &= wA^-zy' = ww_0A^+zy' = ww_0zA^+y' = ww_0zw_0A^-y' \geq A^-y' \\ &> A^-x \quad \text{by [L80, Cor. 3.4] again as } y' \leq y < x. \end{aligned}$$

Lemma: *Let $A \in \mathcal{A}^-$ and write $A = A^-x$, $x \in \mathcal{W}$. Let $\underline{x} = (s_1, \ldots, s_r)$ be a reduced expression of x.*

(i) $(Q(A^-) * B(\underline{x}))_{\{A\}} \simeq S(A)(r)$.

(ii) $\mathrm{supp}_{\mathcal{A}}(Q(A^-) * B(\underline{x})) \subseteq (\geq A)$.

Proof: (ii) $\forall M \in \tilde{\mathcal{K}}'$, $\forall s \in \mathcal{S}$, one has $\mathrm{supp}_{\mathcal{A}}(M * B(s)) = \mathrm{supp}_{\mathcal{A}}(M) \cup \mathrm{supp}_{\mathcal{A}}(M)s$ by (3.5). As $\mathrm{supp}_{\mathcal{W}}(B(\underline{x})) = (\leq x)$ by (I.2.4),

$$\begin{aligned}\mathrm{supp}_{\mathcal{A}}(Q(A^-) * B(\underline{x})) &= \bigcup_{\substack{y \in \mathcal{W} \\ y \leq x}} \mathrm{supp}_{\mathcal{A}}(Q(A^-))y = \{wA^-y | w \in \mathcal{W}_f, y \leq x\} \\ &\subseteq (\geq A) \quad \text{by (1).}\end{aligned}$$

(i) Induction on r. If $r = 0$, $Q(A^-)_{\{A^-\}} \simeq S(A^-)$ by (4.1.1). Put $Q = Q(A^-) * B(s_1, \dots, s_{r-1})$ and $s = s_r$. Then $As > A$. As $Q \in \tilde{\mathcal{K}}_\Delta$ by (3.12), one has by (3.10) an isomorphism of graded S-modules $(Q * B(s))_{\{A\}} \simeq Q_{\{A,As\}}(1)$ with

$$\begin{aligned} Q_{\{A,As\}} &= (Q_{\{A,As\}})_{\geq As} \quad \text{as } \mathrm{supp}_{\mathcal{A}}(Q_{\{A,As\}}) \subseteq \mathrm{supp}_{\mathcal{A}}(Q) \subseteq (\geq As) \text{ by (ii)} \\ &= Q_{\{A,As\} \cap (\geq As)} \quad \text{by (2.8)} \\ &= Q_{\{As\}} \\ &\simeq S(r-1) \quad \text{by the induction hypothesis.} \end{aligned}$$

Thus, $(Q(A^-) * B(\underline{x}))_{\{A\}} = (Q * B(s))_{\{A\}} \simeq S(A)(r-1)(1) = S(A)(r)$.

4.5. Let $A \in \mathcal{A}^-$, write $A = A^-x$, and let $\underline{x}$ be a reduced expression of $x \in \mathcal{W}$. By (4.4) there is an indecomposable direct summand $Q(A)(\ell(x))$ of $Q(A^-) * B(\underline{x})$ such that $\mathrm{supp}_{\mathcal{A}}(Q(A)) \subseteq (\geq A)$ and that $Q(A)_{\{A\}} \simeq S(A)$. For $A \in \mathcal{A}$ in general, take $\gamma \in \mathbb{Z}\Delta$ such that $A \in \mathcal{A}^- + \gamma = \{B + \gamma | B \in \mathcal{A}^-\}$, and set $Q(A) = \mathrm{T}_\gamma(Q(A - \gamma))$, which belongs to $\tilde{\mathcal{K}}_P$ by (2.11) with $\tilde{\mathcal{K}}'(Q(A), Q(A)) \simeq \mathbb{K}$.

We show next that any object of $\tilde{\mathcal{K}}_P$ is a direct sum of $Q(A)(n)$'s, $A \in \mathcal{A}$, $n \in \mathbb{Z}$. Thus, let $M \in \tilde{\mathcal{K}}_P$. Let $A \in \mathcal{A}$ be minimal in $\mathrm{supp}_{\mathcal{A}}(M)$. Then $M_{\{A\}} = M_{\geq A}/M_{>A} \neq 0$, which is graded free over S, and hence there is $n \in \mathbb{Z}$ such that $Q(A)(n)_{\{A\}}$ is a direct summand of $M_{\{A\}}$. Let

$$Q(A)(n)_{\{A\}} \underset{\pi}{\overset{i}{\rightleftarrows}} M_{\{A\}}$$

be the corresponding imbedding and the projection, resp., of degree 0. Let I be a closed subset of $\mathcal{A}$ with $I \supseteq \mathrm{supp}_{\mathcal{A}}(M)$ and $I \setminus \{A\}$ is closed. Then $I \supseteq (\geq A) \supseteq \mathrm{supp}_{\mathcal{A}}(Q(A))$. By (2.12) the property (ES) holds on both complexes $M_{I \setminus \{A\}} \to M_I = M \to M_I/M_{I \setminus \{A\}} = M_{\{A\}}$ and $Q(A)(n)_{I \setminus \{A\}} \to Q(A)(n)_I = Q(A)(n) \to Q(A)(n)_{\{A\}}$. As $Q(A)(n)$ and $M \in \tilde{\mathcal{K}}_P$, one has

$$\begin{array}{ccccc} M & \longrightarrow & M_{\{A\}} & \longleftarrow & M \\ {\scriptstyle \hat{i}} \uparrow & & {\scriptstyle i} \uparrow \downarrow {\scriptstyle \pi} & & \downarrow {\scriptstyle \hat{\pi}} \\ Q(A)(n) & \longrightarrow & Q(A)(n)_{\{A\}} & \longleftarrow & Q(A)(n) \end{array}$$

such that $\hat{\pi} \circ \hat{i} \in \tilde{\mathcal{K}}(Q(A)(n), Q(A)(n)) \simeq \mathbb{K}$ inducing the identity on $Q(A)(n)_{\{A\}}$. Then, $\mathrm{id} - \hat{\pi} \circ \hat{i} \notin \tilde{\mathcal{K}}(Q(A)(n), Q(A)(n))^\times$. As $\mathbb{K}$ is local, we must have $\hat{\pi} \circ \hat{i} \in \tilde{\mathcal{K}}(Q(A)(n), Q(A)(n))^\times$, and hence $Q(A)(n)$ is a direct summand of M in $\tilde{\mathcal{K}}_P$. We have now obtained

Theorem: *(i) $\forall A \in \mathcal{A}$, there is a unique, up to isomorphism, $Q(A) \in \tilde{\mathcal{K}}_P$ indecomposable in $\tilde{\mathcal{K}}'$ such that* $\mathrm{supp}_{\mathcal{A}}(Q(A)) \subseteq (\geq A)$ *and* $Q(A)_{\{A\}} \simeq S(A)$.

(ii) Any object of $\tilde{\mathcal{K}}_P$ is a direct sum of $Q(A)(n)$, $A \in \mathcal{A}$, $n \in \mathbb{Z}$.

4.6. $\forall \gamma \in \mathbb{Z}\Delta$, put $A^-_\gamma = A^- + \gamma$. Then $Q(A^-_\gamma) \simeq \mathrm{T}_{-\gamma}(Q(A^-))$ by the unicity (4.5). In particular,

(1) $$\mathrm{supp}_{\mathcal{A}}(Q(A^-_\gamma)) = A^-_\gamma \mathcal{W}_f,$$

and

(2) any object of $\tilde{\mathcal{K}}_P$ is a direct summand of a direct sum of some
$$Q(A^-_\gamma) * B(\underline{x})(n),\ \gamma \in \mathbb{Z}\Delta,\ \underline{x} \in \mathcal{S}^r,\ r \in \mathbb{N},\ n \in \mathbb{Z}.$$

Also, $\forall M \in \tilde{\mathcal{K}}'(S_0)$,

(3) $$\begin{aligned}\tilde{\mathcal{K}}'(S_0)^\sharp(S_0 \otimes_S Q(A^-_\gamma), M) &\simeq \tilde{\mathcal{K}}'(S_0)^\sharp(\mathrm{T}_{-\gamma}(S_0 \otimes_S Q(A^-)), M)\\ &\simeq \tilde{\mathcal{K}}'(S_0)^\sharp(S_0 \otimes_S Q(A^-), \mathrm{T}_\gamma(M))\\ &\simeq \{\mathrm{T}_\gamma(M)\}_{\geq A^-} \quad \text{by (4.1)}\\ &= M_{\geq A^- + \gamma} = M_{\geq A^-_\gamma}.\end{aligned}$$

Corollary: *Let $M, N \in \tilde{\mathcal{K}}_P$.*

(i) $\tilde{\mathcal{K}}^\sharp(M, N)$ is graded free of finite rank over S.

(ii) $S_0 \otimes_S \tilde{\mathcal{K}}^\sharp(M, N) \simeq \tilde{\mathcal{K}}(S_0)^\sharp(S_0 \otimes_S M, S_0 \otimes_S N)$.

Proof: (i) By (2) we may assume $M = Q(A^-_\gamma) * B(s_1, \dots, s_r)(n)$ for some $\gamma \in \mathbb{Z}\Delta$, $n \in \mathbb{Z}$, $s_1, \dots, s_r \in \mathcal{S}$. Then
$$\begin{aligned}\tilde{\mathcal{K}}^\sharp(M, N) &\simeq \tilde{\mathcal{K}}^\sharp_P(Q(A^-_\gamma), N * B(s_r) * \cdots * B(s_1)(-n)) \quad \text{by (3.6)}\\ &\simeq (N * B(s_r, \dots, s_1)(-n))_{\geq A^-_\gamma} \quad \text{by (3)},\end{aligned}$$
which is graded free of finite rank over S by (3.12) and (2.10).

(ii) By (3.6) again we may assume that $M = Q(A^-_\gamma)$ for some $\gamma \in \mathbb{Z}\Delta$. Then
$$\begin{aligned}S_0 \otimes_S \tilde{\mathcal{K}}^\sharp(M, N) &\simeq S_0 \otimes_S N_{\geq A^-_\gamma} \quad \text{by (4.6.3)}\\ &\simeq (S_0 \otimes_S N)_{\geq A^-_\gamma}\\ &\simeq \tilde{\mathcal{K}}(S_0)^\sharp(S_0 \otimes_S M, S_0 \otimes_S N) \quad \text{by (4.6.3) again.}\end{aligned}$$

4.7. $\forall \lambda \in \hat{X}$, let $A^-_\lambda = A^- + \lambda$. Recall $\mathcal{W}_\lambda = \mathrm{C}_{\mathcal{W}}(\lambda) = t_\lambda \mathcal{W}_f t_{-\lambda}$. $\forall w \in \mathcal{W}_f$, $t_\lambda w t_{-\lambda} = t_{\lambda - w\lambda} w$ with $\lambda - w\lambda \in \mathbb{Z}\Delta$. In particular, $\forall \alpha \in \Delta^+$, $t_\lambda s_\alpha t_{-\lambda} = s_{\alpha, \langle \lambda, \alpha^\vee \rangle} = t_{\langle \lambda, \alpha^\vee \rangle \alpha} s_\alpha$. Thus, $\mathcal{W}_\lambda A^-_\lambda = \{t_\lambda w t_{-\lambda} A^-_\lambda = wA^- + \lambda | w \in \mathcal{W}_f\}$.

Proposition: *One has* $\mathrm{supp}_{\mathcal{A}}(Q(A_\lambda^-)) = \mathcal{W}_\lambda A_\lambda^-$ *with*

$$Q(A_\lambda^-)_{\{wA^-+\lambda\}} \simeq S(wA^- + \lambda)(-2\ell(w)) = S(wA^- + \lambda)(2d(wA^- + \lambda, A_\lambda^-)).$$

$\forall M \in \tilde{\mathcal{K}}'(S_0)$, $\tilde{\mathcal{K}}'(S_0)^\sharp(S_0 \otimes_S Q(A_\lambda^-), M) \simeq M_{\geq A_\lambda^-}$.

Proof: Let $Q = \{a \in \prod_{\mathcal{W}_\lambda A_\lambda^-} S | a_A \equiv a_{t_\lambda s_\alpha t_{-\lambda} A} \mod \alpha^\vee \ \forall A \in \mathcal{W}_\lambda A_\lambda^- \ \forall \alpha \in \Delta^+\}$ with a_B, $B \in \mathcal{A}$, denoting the B-th component of a, not to be confused with $\mathrm{Frac}(R)$ in I. We equip Q with a structure of (S,R)-bimodule such that, $\forall a \in Q$, $\forall b \in S$, $\forall g \in R$, $\forall A \in \mathcal{W}_\lambda A_\lambda^-$, $(ba)_A = ba_A$ while $(ag)_A = g_A a_A$. As $t_\lambda s_\alpha t_{-\lambda} = s_{\alpha,\langle\lambda,\alpha^\vee\rangle} = t_{\langle\lambda,\alpha^\vee\rangle\alpha} s_\alpha$, $g_{t_\lambda s_\alpha t_{-\lambda} A} = s_\alpha g_A$ by (1.2.ii), and Q is well-defined. As in (I.6.1.2), $Q^\emptyset \simeq \prod_{\mathcal{W}_\lambda A_\lambda^-} S^\emptyset$ with, $\forall B \in \mathcal{A}$,

$$Q_B^\emptyset \simeq \begin{cases} S(B)^\emptyset & \text{if } B \in \mathcal{W}_\lambda A_\lambda^-, \\ 0 & \text{else,} \end{cases}$$

and hence $Q \in \tilde{\mathcal{K}}'$ with support $\mathcal{W}_\lambda A_\lambda^-$. $\forall \alpha \in \Delta^+$,

$$\begin{aligned} Q^\alpha &= \{a \in \prod_{\mathcal{W}_\lambda A_\lambda^-} S^\alpha | a_A \equiv a_{s_{\alpha,\langle\lambda,\alpha^\vee\rangle} A} \mod \alpha^\vee \ \forall A \in \mathcal{W}_\lambda A_\lambda^-\} \\ &= \coprod_{\substack{A \in \mathcal{W}_\lambda A_\lambda^- \\ A < s_{\alpha,\langle\lambda,\alpha^\vee\rangle} A}} \{(0,\dots,0,a,0,\dots,0,a',0\dots,0) | a,a' \in S^\alpha \text{ with } a \equiv a' \mod \alpha^\vee\} \\ &\qquad \text{with } a \text{ (resp. } a'\text{) placed at the } A\text{-th (resp. } s_{\alpha,\langle\lambda,\alpha^\vee\rangle}A\text{-th)} \\ &= \coprod_{\Omega \in \mathcal{W}^\alpha \backslash \mathcal{A}} (Q^\alpha \cap \coprod_{A \in \Omega} Q_A^\emptyset), \end{aligned}$$

and hence (LE) holds on Q. To check (S) on Q, let I_1 and I_2 be 2 closed subsets of $\mathcal{A}$. As $Q_{I_j} = \{(a_A) \in Q | a_A = 0 \ \forall A \notin I_j\}$, $j \in [1,2]$, $Q_{I_1} + Q_{I_2} \subseteq Q_{I_1 \cup I_2}$. By (2.5) and (LE) on Q^α, $\alpha \in \Delta^+$, the inclusion turns to an equality upon base extension to S^α. Then

$$\begin{aligned} Q_{I_1} + Q_{I_2} &= \cap_{\alpha \in \Delta^+} (Q_{I_1} + Q_{I_2})^\alpha \quad \text{as } \cap_{\alpha \in \Delta^+} S^\alpha = S \text{ in each coomponent} \\ &= \cap_{\alpha \in \Delta^+} (Q_{I_1}^\alpha + Q_{I_2}^\alpha) = \cap_{\alpha \in \Delta^+} Q_{I_1 \cup I_2}^\alpha = Q_{I_1 \cup I_2}, \end{aligned}$$

and hence $Q \in \tilde{\mathcal{K}}$.

Let now $\mathcal{Z}' = \{z \in \prod_{\mathcal{W}_f} S | z_{s_\alpha w} \equiv z_w \mod \alpha^\vee \ \forall w \in \mathcal{W}_f \ \forall \alpha \in \Delta^+\}$ equipped with a structure of (S,R)-bimodule such that, $\forall z \in \mathcal{Z}'$, $\forall a \in S$, $\forall g \in R$, $\forall w \in \mathcal{W}_f$, $(az)_w = az_w$ while

$(zg)_w = g_{wA^-} z_w$. Under the identification $S \simeq R$ via $a \mapsto a^{A^-}$,

$$\begin{aligned}
S(A^-) * \mathcal{Z}_f &= S(A^-) \otimes_R \{z \in \prod_{\mathcal{W}_f} R | z_{tw} \equiv z_w \mod (\alpha_t^\vee)^{A^+} \ \forall w \in \mathcal{W}_f \ \forall t \in \mathcal{T}\} \\
&\qquad\qquad\qquad \text{with } A^+ t = s_{\alpha_t} A^+ \text{ after (3.1)} \\
&\simeq \{z \in \prod_{\mathcal{W}_f} S | (z_{tw})^{A^-} \equiv (z_w)^{A^-} \mod (\alpha_t^\vee)^{A^+} \ \forall w \in \mathcal{W}_f \ \forall t \in \mathcal{T}\} \\
&= \{z \in \prod_{\mathcal{W}_f} S | z_{tw} \equiv z_w \mod ((\alpha_t^\vee)^{A^+})_{A^-} \ \forall w \in \mathcal{W}_f \ \forall t \in \mathcal{T}\} \quad \text{with} \\
&\qquad ((\alpha_t^\vee)^{A^+})_{A^-} = ((\alpha_t^\vee)^{A^+})(A^-) = ((\alpha_t^\vee)^{A^+})(w_0 A^+) = w_0((\alpha_t^\vee)^{A^+}(A^+)) = w_0 \alpha_t^\vee \\
&= \{z \in \prod_{\mathcal{W}_f} S | z_{tw} \equiv z_w \mod w_0 \alpha_t^\vee \ \forall w \in \mathcal{W}_f \ \forall t \in \mathcal{T}\},
\end{aligned}$$

and hence

$$\begin{aligned}
Q(A^-) = S(A^-) * \mathcal{Z}_f &\simeq \{a \in \prod_{A^-\mathcal{W}_f} S | a_{A^- tw} \equiv a_{A^- w} \mod w_0 \alpha_t^\vee \ \forall w \in \mathcal{W}_f \ \forall t \in \mathcal{T}\} \quad \text{with} \\
&\qquad A^- w = w_0 w w_0 A^- \quad \text{and} \quad A^- tw = w_0 t w_0 w_0 w w_0 A^- \\
&= \{a \in \prod_{\mathcal{W}_f A^-} S | a_{twA^-} \equiv a_{wA^-} \mod \alpha_t^\vee \ \forall w \in \mathcal{W}_f \ \forall t \in \mathcal{T}\} \\
&\simeq \mathcal{Z}' \quad \text{by setting } z_w = a_{wA^-}, \ w \in \mathcal{W}_f,
\end{aligned}$$

which equippes $\mathcal{Z}'$ with a structure of $\tilde{\mathcal{K}}_P$. Then $\eta : \mathcal{Z}' \to Q$ via $z \mapsto a$ with $a_{wA^- + \lambda} = z_w$ $\forall w \in \mathcal{W}_f$ is an isomorphism of graded left S-modules, though not compatible with the structure of right R-modules unless $\lambda \in \mathbb{Z}\Delta$. Nonetheless, under η one obtains an isomorphism of graded left S-modules

$$\begin{aligned}
Q_{\{wA^- + \lambda\}} &\simeq \mathcal{Z}'_{\{w\}} \simeq Q(A^-)_{\{wA^-\}} \\
&\simeq S(-2\ell(w)) = S(2d(wA^- + \lambda, wA^-)) \quad \text{by (4.1.1).}
\end{aligned}$$

Thus, $Q \in \tilde{\mathcal{K}}_\Delta$.

Recall that, $\forall M \in \tilde{\mathcal{K}}'$, $\forall m \in M$, $\forall g \in R^{\mathcal{W}_f}$,

(1) $$mg = g_{A^-} m.$$

For we may assume that $m \in M_A^\emptyset$ for some $A \in \mathcal{A}$. If $A = wA^- + \gamma$, $w \in \mathcal{W}_f$, $\gamma \in \mathbb{Z}\Delta$, $mg = g_A m$ with

$$\begin{aligned}
g_A = g(A) &= g(wA^- + \gamma) = w(g_{A^-}) \\
&= g(wA^-) = g(ww_0A^+) = g(A^+ww_0) = g(A^-w_0ww_0) = (w_0ww_0g)(A^-) \\
&= g(A^-) \quad \text{as } g \in R^{\mathcal{W}_f} \\
&= g_{A^-}.
\end{aligned}$$

Thus, the action by $S \otimes_{\mathbb{K}} R$ on M factors through $S \otimes_{R^{\mathcal{W}_f}} R$.

Consider finally a graded homomorphism of (S,R)-modules $\xi : S\otimes_{R^{\mathcal{W}_f}} R \to Q$ via $a\otimes g \mapsto (ag_{wA^-+\lambda}|w\in\mathcal{W}_f)$, which is well-defined by (1) as $(ag_{wA^-+\lambda}|w\in\mathcal{W}_f) = a(1|w\in\mathcal{W}_f)g$. Writing $A^-_\lambda = A^-+\lambda = xA^-+\gamma$ for some $x\in\mathcal{W}_f$ and $\gamma\in\mathbb{Z}\Delta$, one has, $\forall g\in R$, $w\in\mathcal{W}_f$,

$$\begin{aligned} g_{wA^-+\lambda} &= g(wA^-+\lambda) = g(w(xA^-+\gamma-\lambda)+\lambda) = g(wxA^-+w\gamma-w\lambda+\lambda) \\ &= g(wxA^-) \quad \text{as } w\gamma-w\lambda+\lambda\in\mathbb{Z}\Delta \\ &= g(wxw_0A^+) = g(wA^+xw_0) = g(wA^-w_0xw_0) = (w_0xw_0g)(wA^-) = (w_0xw_0g)_{wA^-}, \end{aligned}$$

and hence obtains a CD

$$\begin{array}{ccc} S\otimes_{R^{\mathcal{W}_f}} R & \xrightarrow{\quad\xi\quad} & Q \\ {\scriptstyle S\otimes_{R^{\mathcal{W}_f}} w_0xw_0?}\big\downarrow\wr & & \big\uparrow\eta \\ S\otimes_{R^{\mathcal{W}_f}} R & \longrightarrow & \mathcal{Z}' \\ a\otimes g & \longmapsto & (ag_{wA^-}|w\in\mathcal{W}_f). \end{array}$$

As $S\otimes_{\mathcal{W}_f} w_0xw_0?$ and η are both bijective as well as the bottom map by (4.1), so is ξ. Then $\forall M\in\tilde{\mathcal{K}}'(S_0)$,

$$\tilde{\mathcal{K}}'(S_0)^\sharp(S_0\otimes_S Q, M) \leq (S_0\otimes_{R^{\mathcal{W}_f}} R)\mathrm{Mod}(S_0\otimes_{R^{\mathcal{W}_f}} R, M)\simeq M.$$

As $\mathrm{supp}_{\mathcal{A}}(Q) = \mathcal{W}_fA^-_\lambda$, $\tilde{\mathcal{K}}'(S_0)^\sharp(S_0\otimes_S Q, M)\leq M_{\geq A^-_\lambda}$, which is an equality by (4.2) as in (4.1). Thus, $Q\in\tilde{\mathcal{K}}_P$, and by unicity $Q\simeq Q(A^-_\lambda)$.

4.8. Keep the notation of (4.7). Under the isomorphism $S\simeq R$ via $a\mapsto a^{A^+}$, one has

$$\begin{aligned} S(A^+)*\mathcal{Z}_f &\simeq \{z\in\prod_{\mathcal{W}_f} S|z_{tw}\equiv z_w \mod ((\alpha_t^\vee)^{A^+})_{A^+} = \alpha_t^\vee\ \forall w\in\mathcal{W}_f\ \forall t\in\mathcal{T}\} \\ &= \mathcal{Z}'\simeq Q(A^-) = S(A^-)*B(w_0)(-\ell(w_0)), \end{aligned}$$

and hence

$$S(A^+)*B(w_0)(-\ell(w_0))\simeq S(A^-)*B(w_0)(-\ell(w_0)).$$

Let $\lambda\in\hat{X}$ and $\mathcal{W}_\lambda = \mathrm{C}_{\mathcal{W}}(\lambda) = t_\lambda\mathcal{W}_ft_{-\lambda}$ as in (4.7). Let $\mathcal{W}'_\lambda = \{w\in\mathcal{W}|A^+_\lambda w\in\mathcal{W}_\lambda A^+_\lambda\}$. If $A^+_\lambda x' = xA^+_\lambda$ and $A^+_\lambda y' = yA^+_\lambda$, $x',y'\in\mathcal{W}'_\lambda$, $A^+_\lambda(x'y') = (xA^+_\lambda)y' = x(A^+_\lambda y') = x(yA^+_\lambda) = (xy)A^+_\lambda$, and hence one has isomorphisms of groups

$$\begin{array}{ccccc} \mathcal{W}'_\lambda & \xrightarrow{\ \sim\ } & \mathcal{W}_\lambda & \xrightarrow{\ \sim\ } & \mathcal{W}_f \\ x' & \longmapsto & x & \longmapsto & \overline{x} \\ & & t_\lambda s_\alpha t_{-\lambda} = s_{\alpha,\langle\lambda,\alpha^\vee\rangle} & \longmapsto & s_\alpha. \end{array}$$

Thus, for each $\alpha\in\Delta^+$, let $\hat{s}'_\alpha\in\mathcal{W}'_\lambda$ and $\hat{s}_\alpha\in\mathcal{W}_\lambda$ denote the elements corresponding to s_α under the isomorphisms. Let $w'_\lambda\in\mathcal{W}'_\lambda$ with $A^+_\lambda w'_\lambda = A^-_\lambda = w_\lambda A^+_\lambda$. Then by (I.9.4)

$$B(w'_\lambda)(-\ell(w'_\lambda))\simeq\mathcal{Z}_\lambda := \{z\in\prod_{\mathcal{W}'_\lambda} R|z_{\hat{s}'_\alpha x'}\equiv z_{x'} \mod \alpha^\vee_{\hat{s}'_\alpha}\ \forall x'\in\mathcal{W}'_\lambda\ \forall\alpha\in\Delta^+\},$$

and hence, under the isomorphism $R \simeq S$ via $g \mapsto g_{A^+_\lambda}$,

$$S(A^+_\lambda) * B(w'_\lambda)(-\ell(w'_\lambda)) \simeq \{a \in \prod_{\mathcal{W}'_\lambda} S | a_{A^+_\lambda \hat{s}'_\alpha x'} \equiv a_{A^+_\lambda x'} \mod (\alpha^\vee_{\hat{s}'_\alpha})_{A^+_\lambda}\ \forall x' \in \mathcal{W}'_\lambda\ \forall \alpha \in \Delta^+\}$$
$$= \{a \in \prod_{\mathcal{W}_\lambda} S | a_{\hat{s}_\alpha x A^+_\lambda} \equiv a_{x A^+_\lambda} \mod (\alpha^\vee_{\hat{s}'_\alpha})_{A^+_\lambda}\ \forall x \in \mathcal{W}_\lambda\ \forall \alpha \in \Delta^+\}.$$

If $A^+_\lambda = A^+ x$, $x \in \mathcal{W}$, $\hat{s}_\alpha A^+_\lambda = \hat{s}_\alpha A^+ x = A^+ \hat{s}_\alpha x = A^+ x x^{-1} \hat{s}_\alpha x = A^+_\lambda s_{\bar{x}^{-1}\alpha, n}$ for some $n \in \mathbb{Z}$ as $\hat{s}_\alpha = t_\lambda s_\alpha t_{-\lambda} = t_{\lambda - s_\alpha \lambda} s_\alpha = s_{\alpha, \langle \lambda, \alpha \rangle}$. Then $A^+ \hat{s}'_\alpha = A^+ s_{\bar{x}^{-1}\alpha, n} = s_{\bar{x}^{-1}\alpha, n} A^+$, and hence $\alpha^\vee_{\hat{s}'_\alpha} = ((\bar{x}^{-1}\alpha)^\vee)^{A^+} = (\bar{x}^{-1}\alpha^\vee)^{A^+}$ by definition (3.1). Thus,

$$(\alpha^\vee_{\hat{s}'_\alpha})_{A^+_\lambda} = ((\bar{x}^{-1}\alpha^\vee)^{A^+})_{A^+_\lambda} = (\bar{x}^{-1}\alpha^\vee)^{A^+}(A^+ x) = (\bar{x}^{-1}\alpha^\vee)^{A^+}(x A^+) = \bar{x}\{(\bar{x}^{-1}\alpha^\vee)^{A^+}(A^+)\}$$
$$= \bar{x}(\bar{x}^{-1}\alpha^\vee) = \alpha^\vee.$$

It follows that

$$S(A^+_\lambda) * B(w'_\lambda)(-\ell(w'_\lambda)) = \{a \in \prod_{\mathcal{W}_\lambda A^+_\lambda} S | a_{\hat{s}_\alpha A} \equiv a_A \mod \alpha^\vee\ \forall A \in \mathcal{W}_\lambda A^+_\lambda\ \forall \alpha \in \Delta^+\}$$
$$= Q \quad \text{as } \mathcal{W}_\lambda A^+_\lambda = \{wA^+ + \lambda | w \in \mathcal{W}_f\} = \{wA^- + \lambda | w \in \mathcal{W}_f\} = \mathcal{W}_\lambda A^-_\lambda$$
$$\simeq S(A^-_\lambda) * B(w'_\lambda)(-\ell(w'_\lambda)) \quad \text{likewise.}$$

We have obtained

Corollary: $S(A^+_\lambda) * B(w'_\lambda)(-\ell(w_0)) \simeq Q(A^-_\lambda) \simeq S(A^-_\lambda) * B(w'_\lambda)(-\ell(w_0))$.

5. **Categorification**

5.1. Recall from (I.3.1) the 岩堀-Hecke algebra $\mathcal{H}$ over $\mathbb{Z}[v, v^{-1}]$ associated to $(\mathcal{W}, \mathcal{S})$. The periodic module $\mathcal{P} = \coprod_{A \in \mathcal{A}} \mathbb{Z}[v, v^{-1}]A$ is a right $\mathcal{H}$-module [S97, Lem. 4.1] such that

$$AH_s = \begin{cases} As & \text{if } As > A, \\ As + (v^{-1} - v)A & \text{else,} \end{cases} \tag{1}$$

i.e.,

$$A\underline{H}_s = A(H_s + v) = \begin{cases} As + vA & \text{if } As > A, \\ As + v^{-1}A & \text{else.} \end{cases}$$

For an additive category $\mathcal{C}$ let $[\mathcal{C}]$ denote its split Grothendieck group. Recall from (I.5.3) a $\mathbb{Z}[v, v^{-1}]$-algebra isomorphism

$$[\mathfrak{S}\mathfrak{B}] \xrightarrow{\sim} \mathcal{H} \quad \text{via} \quad [B(s)] \mapsto \underline{H}_s = H_s + v \quad \forall s \in \mathcal{S}. \tag{2}$$

By (3.12) the abelian group $[\tilde{\mathcal{K}}_\Delta]$ admits a structure of right $[\mathfrak{S}\mathfrak{B}]$-module such that $[M][B] = [M * B]$ $\forall M \in \tilde{\mathcal{K}}_\Delta$ $\forall B \in \mathfrak{S}\mathfrak{B}$. Fix a length function $\ell : \mathcal{A} \to \mathbb{Z}$ in the sense of [L80, 1.11]: $\forall A, B \in \mathcal{A}$, $d(A, B) = \ell(B) - \ell(A)$. Define $\mathrm{ch} : [\tilde{\mathcal{K}}_\Delta] \to \mathcal{P}$ via

$$\mathrm{ch}[M] = \sum_{A \in \mathcal{A}} v^{\ell(A)} \mathrm{grk}(M_{\{A\}})A,$$

which is $[\mathfrak{SB}] \simeq \mathcal{H}$-linear: $\forall M \in \tilde{\mathcal{K}}_\Delta$, $\forall s \in \mathcal{S}$,

$$\begin{aligned}
\mathrm{ch}[M * B(s)] &= \sum_{A\in\mathcal{A}} v^{\ell(A)}\mathrm{grk}((M * B(s))_{\{A\}})A \\
&= \sum_{A\in\mathcal{A}} v^{\ell(A)} \begin{cases} \{v^{-1}\mathrm{grk}(M_{\{A\}}) + v^{-1}\mathrm{grk}(M_{\{As\}})\}A & \text{if } As < A, \\ \{v\mathrm{grk}(M_{\{A\}}) + v\mathrm{grk}(M_{\{As\}})\}A & \text{else} \end{cases} \\
&\qquad\qquad \text{by (3.13)} \\
&= \sum_{A>As} \{v^{\ell(A)-1}\mathrm{grk}(M_{\{A\}}) + v^{\ell(A)-1}\mathrm{grk}(M_{\{As\}})\}A \\
&\qquad\qquad + \sum_{A<As} \{v^{\ell(A)+1}\mathrm{grk}(M_{\{A\}}) + v^{\ell(A)+1}\mathrm{grk}(M_{\{As\}})\}A,
\end{aligned}$$

in which the coefficient of A is

$$\begin{cases} v^{\ell(A)-1}\mathrm{grk}(M_{\{A\}}) + v^{\ell(As)}\mathrm{grk}(M_{\{As\}}) & \text{if } As < A, \\ v^{\ell(A)+1}\mathrm{grk}(M_{\{A\}}) + v^{\ell(As)}\mathrm{grk}(M_{\{As\}}) & \text{else} \end{cases}$$

as $\ell(A) - \ell(As) = d(As, A) = \begin{cases} 1 & \text{if } As < A, \\ -1 & \text{else.} \end{cases}$

On the other hand,

$$\begin{aligned}
(\mathrm{ch}[M])(H_s + v) &= \sum_{A\in\mathcal{A}} v^{\ell(A)}\mathrm{grk}(M_{\{A\}}) \begin{cases} As + vA & \text{if } As > A, \\ As + v^{-1}A & \text{else} \end{cases} \\
&= \sum_{A>As} \{v^{\ell(A)}\mathrm{grk}(M_{\{A\}})As + v^{\ell(A)-1}\mathrm{grk}(M_{\{A\}})A\} \\
&\qquad\qquad + \sum_{A<As} \{v^{\ell(A)}\mathrm{grk}(M_{\{A\}})As + v^{\ell(A)+1}\mathrm{grk}(M_{\{A\}})A\}.
\end{aligned}$$

Thus, $\mathrm{ch}[M * B(s)] = (\mathrm{ch}[M])(H_s + v) = (\mathrm{ch}[M])[B(s)]$ under the identification $[\mathfrak{SB}] \simeq \mathcal{H}$.

As the $[S(A)]$, $A \in \mathcal{A}$, form a $\mathbb{Z}[v, v^{-1}]$-linear basis of $[\tilde{\mathcal{K}}_\Delta]$ and as $\mathrm{ch}[S(A)] = v^{\ell(A)}A$, we have obtained a categorification of the periodic modules:

Theorem: $\mathrm{ch}\colon [\tilde{\mathcal{K}}_\Delta] \to \mathcal{P}$ *is an* $\mathcal{H}$*-linear isomorphism.*

5.2. By (3.14) the $\mathcal{H}$-linear isomorphism $\mathrm{ch}\colon [\tilde{\mathcal{K}}_\Delta] \to \mathcal{P}$ restricts to an $\mathcal{H}$-linear map on $[\tilde{\mathcal{K}}_P]$. $\forall \lambda \in \hat{X}$, put $e_\lambda = \sum_{w\in\mathcal{W}_\lambda} v^{-\ell(wA_\lambda^-)}wA_\lambda^-$, which is distinct from one in [L80, 1.7, p. 125] but agrees with E_λ in [S97, p. 93] up to a power of v;

$$\begin{aligned}
(1) \qquad e_\lambda &= \sum_{A\in\mathcal{W}_f A^+} v^{-\ell(A+\lambda)}(A + \lambda) = \sum_{w\in\mathcal{W}_f} v^{-\ell(wA^+ + \lambda)}(wA^+ + \lambda) \\
&= \sum_{w\in\mathcal{W}_f} v^{\ell(w)-\ell(A^+ + \lambda)}(wA^+ + \lambda) = \sum_{w\in\mathcal{W}_f} v^{\ell(w)-\ell(A_\lambda^+)}(wA^+ + \lambda) \\
&\qquad \text{as } \ell(A_\lambda^+) - \ell(wA^+ + \lambda) = d(wA^+ + \lambda, A^+ + \lambda) = d(wA^+, A^+) = \ell(w) \\
&= v^{-\ell(A_\lambda^+)}E_\lambda.
\end{aligned}$$

Set $\mathcal{P}^0 = \sum_{\lambda \in \hat{X}} e_\lambda \mathcal{H} \subseteq \mathcal{P} = \coprod_{A \in \mathcal{A}} \mathbb{Z}[v, v^{-1}]A$.

Lemma: $\forall \lambda \in \hat{X}$, $\mathrm{ch}[Q(A_\lambda^-)] = v^{2\ell(A_\lambda^-)} e_\lambda = v^{\ell(A_\lambda^-) - \ell(w_0)} E_\lambda$.

Proof: By (4.7)

$$\begin{aligned}
\mathrm{ch}[Q(A_\lambda^-)] &= \sum_{A \in \mathcal{A}} v^{\ell(A)} \mathrm{grk}(Q(A_\lambda^-)_{\{A\}}) A = \sum_{w \in \mathcal{W}_\lambda} v^{\ell(wA_\lambda^-)} \mathrm{grk}(Q(A_\lambda^-)_{\{wA_\lambda^-\}}) wA_\lambda^- \\
&= \sum_{w \in \mathcal{W}_\lambda} v^{\ell(wA_\lambda^-)} \mathrm{grk}(S(2d(wA_\lambda^-, A_\lambda^-))) wA_\lambda^- = \sum_{w \in \mathcal{W}_\lambda} v^{\ell(wA_\lambda^-) + 2d(wA_\lambda^-, A_\lambda^-)} wA_\lambda^- \\
&= \sum_{w \in \mathcal{W}_\lambda} v^{-\ell(wA_\lambda^-) + 2\ell(A_\lambda^-)} wA_\lambda^- = v^{2\ell(A_\lambda^-)} e_\lambda.
\end{aligned}$$

5.3. Identify $[\mathfrak{S}\mathfrak{B}]$ with $\mathcal{H}$ by (5.1.2). $\forall \lambda \in \hat{X}$, $e_\lambda = v^{-2\ell(A_\lambda^-)} \mathrm{ch}[Q(A_\lambda^-)]$ by (5.2). As $\mathrm{ch} : [\tilde{\mathcal{K}}_P] \to \mathcal{P}^0$ is $\mathcal{H}$-equivariant, the map is surjective. On the other hand, by (4.5),

$$[\tilde{\mathcal{K}}_P] = \coprod_{A \in \mathcal{A}} \mathbb{Z}[v, v^{-1}][Q(A)] \quad \text{with} \quad \mathrm{ch}[Q(A)] \in v^{\ell(A)} A + \sum_{A' > A} \mathbb{Z}[v, v^{-1}]A',$$

and hence the $\mathrm{ch}[Q(A)]$, $A \in \mathcal{A}$, remain $\mathbb{Z}[v, v^{-1}]$-linearly independent. Thus,

Corollary: $\mathrm{ch} : [\tilde{\mathcal{K}}_P] \to \mathcal{P}^0$ *is an isomorphism of right $\mathcal{H}$-modules.*

5.4. Let $\lambda \in \hat{X}$. From (4.8) recall $\mathcal{W}_\lambda = \mathrm{C}_{\mathcal{W}}(\lambda)$, $\mathcal{W}'_\lambda = \{x \in \mathcal{W} | A_\lambda^+ x \in \mathcal{W}_\lambda A_\lambda^+\}$, $w'_\lambda \in \mathcal{W}'_\lambda$ such that $A_\lambda^+ w'_\lambda = A_\lambda^-$, and $B(w'_\lambda) \simeq \mathcal{Z}_\lambda(\ell(w_0))$. $\forall x \in \mathcal{W}$,

$$\begin{aligned}
D(B(w'_\lambda)^x) &\simeq B(w'_\lambda)_x \quad \text{by (I.2.8 and 4.5)} \\
&\simeq \mathcal{Z}_\lambda(\ell(w_0))_x \\
&\simeq \begin{cases} (\prod_{\alpha \in \Delta^+} \alpha^\vee_{\hat{s}'_\alpha}) R(x)(\ell(w_0)) \simeq R(x)(-\ell(w_0)) & \text{if } x \in \mathcal{W}'_\lambda, \\ 0 & \text{else,} \end{cases}
\end{aligned}$$

and hence by (I.2.9)

(1) $$B(w'_\lambda)^x \simeq \begin{cases} R(x)(\ell(w_0)) & \text{if } x \in \mathcal{W}'_\lambda, \\ 0 & \text{else.} \end{cases}$$

In particular,

(2) $$\mathrm{ch}[B(w'_\lambda)] = \sum_{x \in \mathcal{W}'_\lambda} v^{-\ell(x)} \mathrm{grk}(B(w'_\lambda)^x) H_x = v^{\ell(w_0)} \sum_{x \in \mathcal{W}'_\lambda} v^{-\ell(x)} H_x.$$

5.5. Keep the notation of (5.4). Let $\mathcal{S}_\lambda = t_\lambda \mathcal{S}_f t_{-\lambda}$ and $\mathcal{S}'_\lambda = \{x \in \mathcal{W}'_\lambda | A_\lambda^+ x \in \mathcal{S}_\lambda\}$. Thus, one has isomorphisms of Coxeter systems $(\mathcal{W}_f, \mathcal{S}_f) \simeq (\mathcal{W}_\lambda, \mathcal{S}_\lambda) \simeq (\mathcal{W}'_\lambda, \mathcal{S}'_\lambda)$. Let Π_λ^- be the set of alcoves in the box $\{\nu \in X_\mathbb{R} | \langle \lambda, \alpha^\vee \rangle - 1 < \langle \nu, \alpha^\vee \rangle < \langle \lambda, \alpha^\vee \rangle \ \forall \alpha \in \Delta^s\}$ and put $\Pi_\lambda = \Pi_\lambda^- w'_\lambda$. Thus, A_λ^- (resp. A_λ^+) is the top (resp. bottom) alcove of Π_λ^- (resp. Π_λ^-).

Lemma: *$\forall w \in \mathcal{W}$ with $A_\lambda^+ w \subseteq \Pi_\lambda$ (resp. $A_\lambda^- w \subseteq \Pi_\lambda^-$), $B(w'_\lambda w)$ is a direct summand of $B(w'_\lambda) * B(w'_\lambda w)(\ell(w_0))$. In particular,*

$$S(A_\lambda^-) * B(w'_\lambda w) \ \textit{(resp. } S(A_\lambda^+) * B(w'_\lambda w)\textit{)} \ \in \tilde{\mathcal{K}}_P.$$

Proof: $\forall x \in \mathcal{W}$, one has $\mathrm{ch}[B(\underline{x})] = \underline{H}_{\underline{x}}$ by (I.5.2). As $sw'_\lambda w < w'_\lambda w \ \forall s \in \mathcal{S}'_\lambda$, $\underline{H}_s \mathrm{ch}[B(w'_\lambda w)] = (v+v^{-1})\mathrm{ch}[B(w'_\lambda w)]$ by (I.5.10.ii), and hence $H_s \mathrm{ch}[B(w'_\lambda w)] = v^{-1}\mathrm{ch}[B(w'_\lambda w)]$. Thus, $\forall x \in \mathcal{W}'_\lambda$,

$$H_x \mathrm{ch}[B(w'_\lambda w)] = v^{-\ell(x)}\mathrm{ch}[B(w'_\lambda w)]. \tag{1}$$

Put $l = \ell(w'_\lambda) = \ell(w_0)$. Then

$$\begin{aligned}
\mathrm{ch}([B(w'_\lambda) * B(w'_\lambda w)]) &= \mathrm{ch}[B(w'_\lambda)]\mathrm{ch}[B(w'_\lambda w)] \\
&= \sum_{y \in \mathcal{W}'_\lambda} v^{-\ell(y)+l} H_y \mathrm{ch}[B(w'_\lambda w)] \quad \text{by (5.4.2)} \\
&= \sum_{y \in \mathcal{W}'_\lambda} v^{-2\ell(y)+l} \mathrm{ch}[B(w'_\lambda w)] \quad \text{by (1).}
\end{aligned}$$

It follows from the isomorphism $[\mathfrak{S}\mathfrak{B}] \simeq \mathcal{H}$ that $B(w'_\lambda) * B(w'_\lambda w) \simeq \coprod_{y \in \mathcal{W}'_\lambda} B(w'_\lambda w)(l - 2\ell(y))$. Thus, $S(A_\lambda^-) * B(w'_\lambda w)$ is a direct summand of $S(A_\lambda^-) * B(w'_\lambda) * B(w'_\lambda w)(-l) \simeq Q(A_\lambda^-) * B(w'_\lambda w)$ by (4.8). Then $S(A_\lambda^-) * B(w'_\lambda w) \in \tilde{\mathcal{K}}_P$ by (3.14). Likewise $S(A_\lambda^+) * B(w'_\lambda w) \in \tilde{\mathcal{K}}_P$ for w with $A_\lambda^- w \in \Pi_\lambda^-$.

5.6. Keep the notation of (5.5). Recall $\underline{P}_A \in \mathcal{P}^0$ from [S97, Th. 4.3] and let $(\underline{H}_x | x \in \mathcal{W})$ be the KL-basis of $\mathcal{H}$.

Lemma: *$\forall w \in \mathcal{W}$ with $A_\lambda^+ w \in \Pi_\lambda$, $A_\lambda^- \underline{H}_{w'_\lambda w} = \underline{P}_{A_\lambda^+ w}$.*

Proof: We make use of results from [L80]. The action of $\mathcal{H}$ on $\mathcal{A}$ in loc. cit. (resp. in the present setup after [S97]) is from the left (resp. right) with respect to the Coxeter system $(\mathcal{W}, \mathcal{S})$ with $\mathcal{S}$ associated to the faces of an arbitrary alcove, i.e., the orbits of an alcove, and hence $\mathcal{S}$ remains the same as the one in [L80]. Thus, our AT_y is $v^{\ell(A)} T_{y^{-1}} A_L$ for , $A \in \mathcal{A}$, $y \in \mathcal{W}$ and A_L denoting the element of $\mathcal{M}$ in [L80, 1.6], corresponding to A [S97, Rmk. 4.2]; $As_1 \dots s_r$

in [S97] is $v^{\ell(A)}s_r \dots s_1 A_L$ in [L80]. One has

$$\begin{aligned}
A_\lambda^- \underline{H}_{w'_\lambda w} &= A_\lambda^- \sum_{y\in\mathcal{W}} h_{y,w'_\lambda w} H_y \quad \text{after [S97, Def. 2.5]} \\
&= A_\lambda^- \sum_{y\in\mathcal{W}} v^{\ell(w'_\lambda w)-\ell(y)} P_{y,w'_\lambda w} v^{\ell(y)} T_y \quad \text{by [S97, Rmk. 2.5 and a remark on p. 84]} \\
&= \sum_{y\in\mathcal{W}} v^{\ell(w'_\lambda w)} P_{y,w'_\lambda w} T_{y^{-1}} v^{\ell(A_\lambda^-)} (A_\lambda^-)_L \\
&= v^{\ell(w'_\lambda w)+\ell(A_\lambda^-)} \sum_{y\in\mathcal{W}} P_{y^{-1},w^{-1}w'^{-1}_\lambda} T_{y^{-1}} (A_\lambda^-)_L \quad \text{[K88, 1.6.6]} \\
&= v^{\ell(w'_\lambda w)+\ell(A_\lambda^-)} \sum_{y\in\mathcal{W}} P_{y,w^{-1}w'_\lambda} T_y (A_\lambda^-)_L = v^{\ell(w'_\lambda w)+\ell(A_\lambda^-)} C^*_{w^{-1}w'_\lambda} (A_\lambda^-)_L \\
&\qquad \text{with } C^*_{w^{-1}w'_\lambda} \text{ a Kazhdan-Lusztig basis element of } \mathcal{H} \text{ [L80, 5.1]} \\
&= v^{\ell(w'_\lambda w)+\ell(A_\lambda^-)} D_{w^{-1}(A_\lambda^+)_L} \quad \text{by [L80, proof of Th. 5.2, p. 136]} \\
&= v^{\ell(w'_\lambda w)+\ell(A_\lambda^-)} \sum_{B\in\mathcal{A}} Q_{B,w^{-1}A_\lambda^+} B_L \quad \text{by definition [L80, Th. 5.2]} \\
&= v^{\ell(w'_\lambda w)+\ell(A_\lambda^-)} \sum_{B\in\mathcal{A}} v^{d(A_\lambda^+,B)} p_{B,A_\lambda^+ w} B_L \quad \text{by [S97, Rmk. 4.4]; there is an error in sign} \\
&\qquad \text{loc. cit.} \\
&= v^{\ell(w'_\lambda w)+\ell(A_\lambda^-)} \sum_{B\in\mathcal{A}} v^{d(A_\lambda^+ w,B)} p_{B,A_\lambda^+ w} v^{-\ell(B)} B \quad \text{by [S97, Rmk. 4.2] again} \\
&= v^{\ell(w'_\lambda w)+\ell(A_\lambda^-)-\ell(A_\lambda^+ w)} \sum_{B\in\mathcal{A}} p_{B,A_\lambda^+ w} B \\
&= \sum_{B\in\mathcal{A}} p_{B,A_\lambda^+ w} B \\
&= \underline{P}_{A_\lambda^+ w} \quad \text{by definition [S97, Rmk. 4.4]} \\
&= \underline{P}_{A_\lambda^- w'_\lambda w}.
\end{aligned}$$

6. **Quotient categories**

In order to relate our categories to the combinatorial category of [AJS], we have to introduce ideal quotients of the categories.

6.1. $\forall M, N \in \tilde{\mathcal{K}}'(S_0)$, put

$$\mathcal{I}(M,N) = \{\varphi \in \tilde{\mathcal{K}}'(S_0)(M,N) | \varphi^\emptyset(M_A^\emptyset) \subseteq \coprod_{A'>A} N_{A'}^\emptyset\}.$$

As $\varphi^\emptyset(M_A^\emptyset) \subseteq \coprod_{A'\geq A} N_{A'}^\emptyset$, $\mathcal{I}$ forms an ideal of the set of morphisms of $\tilde{\mathcal{K}}'(S_0)$ [中岡, Def. 3.2.41, p. 146]. Define $\mathcal{K}'(S_0)$ to be the ideal quotient $\tilde{\mathcal{K}}'(S_0)/\mathcal{I}$ [中岡, Def. 3.2.43, p. 147], and likewise $\mathcal{K}(S_0)$ and $\mathcal{K}_\Delta(S_0)$. $\forall A \in \mathcal{A}$, recall from (2.8.1) the endofunctor $?_{\{A\}}$ on $\tilde{\mathcal{K}}(S_0)$;

$$\begin{array}{ccc} M_{\{A\}} & \xrightarrow{\varphi_{\{A\}}} & N_{\{A\}} \\ \| & & \| \\ M_{\geq A}/M_{>A} & & N_{\geq A}/N_{>A}. \end{array}$$

$\forall \varphi \in \tilde{\mathcal{K}}(S_0)(M,N)$, $\forall A \in \mathcal{A}$,

$$\varphi^\emptyset(M_A^\emptyset) \subseteq \coprod_{A'>A} N_{A'}^\emptyset \quad \text{iff} \quad \varphi_{\{A\}} = 0, \tag{1}$$

and hence $?_{\{A\}}$ induces a functor $\mathcal{K}(S_0) \to S_0\mathrm{Modgr}$ via $M \mapsto M_{\{A\}}$. Thus, one can define that (ES) holds on a complex $M_1 \to M_2 \to M_3$ in $\mathcal{K}(S_0)$ iff $0 \to M_{1,\{A\}} \to M_{2,\{A\}} \to M_{3,\{A\}} \to 0$ is exact as left S_0-modules $\forall A \in \mathcal{A}$. Note, however, that a sequence $M_1 \to M_2 \to M_3$ in $\tilde{\mathcal{K}}(S_0)$ may form a complex only in $\mathcal{K}(S_0)$ but not in $\tilde{\mathcal{K}}(S_0)$. For the time being we define $\mathcal{K}_P(S_0)$ to be the full subcategory of $\mathcal{K}_\Delta(S_0)$ consisting of $M \in \mathcal{K}_\Delta(S_0)$ such that $\forall$ complex $M_1 \to M_2 \to M_3$ in $\mathcal{K}_\Delta(S_0)$ with (ES) holding, sequence

$$0 \to \mathcal{K}_\Delta(S_0)(M,M_1) \to \mathcal{K}_\Delta(S_0)(M,M_2) \to \mathcal{K}_\Delta(S_0)(M,M_3) \to 0$$

is exact. We will see below that $\mathcal{K}_P(S_0)$ is, in fact, the ideal quotient of $\tilde{\mathcal{K}}_P(S_0)$.

Lemma: *Let $M,N \in \tilde{\mathcal{K}}'(S_0)$, $\varphi \in \tilde{\mathcal{K}}'(S_0)(M,N)$, and $B \in \mathfrak{SB}$. If $\varphi(M_A^\emptyset) \subseteq \coprod_{A'>A} N_{A'}^\emptyset$ $\forall A \in \mathcal{A}$,*

$$(\varphi * B)^\emptyset((M*B)_A^\emptyset) \subseteq \coprod_{A'>A} (N*B)_{A'}^\emptyset.$$

Proof: By definition $(M*B)_A^\emptyset = \coprod_{x\in\mathcal{W}} M_{Ax^{-1}}^\emptyset \otimes_R B_x^\emptyset$. By Rmk. 2.2.(i) under the hypothesis

$$\varphi^\emptyset(M_{Ax^{-1}}^\emptyset) \otimes_R B_x^\emptyset \subseteq \coprod_{\substack{A'>A \\ A'\in Ax^{-1}+\mathbb{Z}\Delta}} N_{A'}^\emptyset \otimes_R B_x^\emptyset = \coprod_{\substack{A'x^{-1}>Ax^{-1} \\ A'x^{-1}\in Ax^{-1}+\mathbb{Z}\Delta}} N_{A'x^{-1}}^\emptyset \otimes_R B_x^\emptyset.$$

Write $A'x^{-1} = Ax^{-1} + \gamma$ $\exists \gamma \in \mathbb{Z}\Delta$. By (1.3) one has that $A'x^{-1} \geq Ax^{-1}$ iff $\gamma \in \mathbb{N}\Delta^+$, in which case $A' = (Ax^{-1}+\gamma)x = (t_\gamma Ax^{-1})x = t_\gamma A = A + \gamma$, and hence $A' \geq A$. Thus,

$$(\varphi * B)^\emptyset((M*B)_A^\emptyset) \subseteq \coprod_{A'>A} N_{A'x^{-1}}^\emptyset \otimes_R B_x^\emptyset = \coprod_{A'>A} (N*B)_{A'}^\emptyset.$$

6.2. From the lemma above one has obtained a bifunctor $\mathcal{K}'(S_0) \times \mathfrak{SB} \to \mathcal{K}'(S_0)$ via $(M,B) \mapsto M*B$, and by (3.12) a bifunctor $\mathcal{K}_\Delta(S_0) \times \mathfrak{SB} \to \mathcal{K}_\Delta(S_0)$.

Proposition: $\forall M,N \in \mathcal{K}'(S_0)$, $\forall s \in \mathcal{S}$,

$$\mathcal{K}'(S_0)(M*B(s),N) \simeq \mathcal{K}'(S_0)(M,N*B(s)).$$

Proof: Take $\delta \in \Lambda_{\mathbb{K}}^{\vee}$ with $\langle \alpha_s, \delta \rangle = 1$. Recall from the proof of Prop. 3.6 a bijection

$$(S_0, R)\mathrm{Bimod}(M \otimes_{R^s} R, N) \xrightarrow{\sim} (S_0, R)\mathrm{Bimod}(M, N \otimes_{R^s} R) \quad \text{via} \quad \varphi \mapsto \psi$$

with $\psi(m) = \varphi(m\delta \otimes 1) \otimes 1 - \varphi(m \otimes 1) \otimes s\delta$, and that $\varphi \in \tilde{\mathcal{K}}'(S_0)$ iff $\psi \in \tilde{\mathcal{K}}'(S_0)$. The argument also shows that $\forall A \in \mathcal{A}$, $\varphi^{\emptyset}((M * B(s))_A^{\emptyset}) \subseteq \coprod_{A' > A} N_{A'}^{\emptyset}$ iff $\psi^{\emptyset}(M_A^{\emptyset}) \subseteq \coprod_{A' > A} (N * B(s))_{A'}^{\emptyset}$. Thus, $\varphi \in \mathcal{K}'(S_0)$ iff $\psi \in \mathcal{K}'(S_0)$.

6.3. Any quotient of a local ring remains local [AF, 15.15, p. 170], and hence

(1) any indecomposable in $\tilde{\mathcal{K}}'(S_0)$ remains so in $\mathcal{K}'(S_0)$.

Lemma: *Let K be a locally closed subset of $\mathcal{A}$ such that $\forall A \in K$, $(A + \mathbb{Z}\Delta) \cap K = \{A\}$.*

(i) $\forall \varphi \in \tilde{\mathcal{K}}(S_0)(M, N)$ vanishing in $\mathcal{K}(S_0)$, $\varphi_K : M_K \to N_K$ vanishes in $\tilde{\mathcal{K}}(S_0)$.

(ii) Let $M_1 \to M_2 \to M_3$ be a sequence in $\tilde{\mathcal{K}}_{\Delta}(S_0)$. If (ES) holds on the sequence in $\mathcal{K}(S_0)$, (ES) holds on the sequence $(M_1)_K \to (M_2)_K \to (M_3)_K$ in $\tilde{\mathcal{K}}(S_0)$. In particular, $0 \to (M_1)_K \to (M_2)_K \to (M_3)_K \to 0$ is exact in $(S_0, R)\mathrm{Bimodgr}$ by (2.11).

Proof: (i) $\forall A \in \mathcal{A}$,

$$\begin{aligned}\varphi^{\emptyset}(M_A^{\emptyset}) &\subseteq \coprod_{\substack{A' > A \\ A' \in A + \mathbb{Z}\Delta}} N_{A'}^{\emptyset} \quad \text{by Rmk. 2.2.(i)} \\ &= 0 \quad \text{in } (N_K)^{\emptyset} = \coprod_{A' \in K} N_{A'}^{\emptyset} \text{ as } (A + \mathbb{Z}\Delta) \cap K = \{A\},\end{aligned}$$

and hence $\varphi_K = 0$ in $\tilde{\mathcal{K}}(S_0)$.

(ii) The composite $M_1 \to M_3$ vanishes in $\mathcal{K}(S_0)$ by the hypothesis. Then so does $(M_1)_K \to (M_3)_K$ in $\tilde{\mathcal{K}}(S_0)$ by (i), and hence (ES) holds on $(M_1)_K \to (M_2)_K \to (M_3)_K$ in $\mathcal{K}(S_0)$.

6.4 **Lemma:** *If (ES) holds on a complex $M_1 \to M_2 \to M_3$ in $\mathcal{K}_{\Delta}(S_0)$, so does it on $M_1 * B \to M_2 * B \to M_3 * B$ in $\mathcal{K}_{\Delta}(S_0)$ $\forall B \in \mathfrak{S}\mathfrak{B}$. Thus, $\mathcal{K}_P(S_0) * \mathfrak{S}\mathfrak{B} \subseteq \mathcal{K}_P(S_0)$.*

Proof: We may assume $B = B(s)$ for some $s \in \mathcal{S}$. From (3.12) we know that each $M_i * B(s) \in \tilde{\mathcal{K}}_{\Delta}(S_0)$, and from (3.10) that, $\forall A \in \mathcal{A}$, $(M_i * B(s))_{\{A\}} \simeq (M_i)_{\{A, As\}}(\varepsilon(A))$ with $\varepsilon(A) = \pm 1$ depending on whether or not $As > A$. By (6.3.ii) one has an exact sequence

$$0 \to (M_1)_{\{A,As\}} \to (M_2)_{\{A,As\}} \to (M_3)_{\{A,As\}} \to 0$$

in $(S_0, R)\mathrm{Bimodgr}$. Thus,

$$0 \to (M_1 * B(s))_{\{A\}} \to (M_2 * B(s))_{\{A\}} \to (M_3 * B(s))_{\{A\}} \to 0$$

is exact in $(S_0, R)\mathrm{Bimodgr}$.

If $M \in \mathcal{K}_P(S_0)$, one has a CD by (6.2)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}(S_0)(M * B(s), M_1) & \longrightarrow & \mathcal{K}(S_0)(M * B(s), M_2) & \longrightarrow & \mathcal{K}(S_0)(M * B(s), M_3) & \longrightarrow & 0 \\ & & \wr | & & \wr | & & \wr | & & \\ 0 & \longrightarrow & \mathcal{K}(S_0)(M, M_1 * B(s)) & \longrightarrow & \mathcal{K}(S_0)(M, M_2 * B(s)) & \longrightarrow & \mathcal{K}(S_0)(M, M_3 * B(s)) & \longrightarrow & 0 \end{array}$$

with the bottom row exact by above. Thus, $M * B(s) \in \mathcal{K}_P(S_0)$.

6.5. Let $\lambda \in \hat{X}$ and put $I = (\geq A_\lambda^-)$. Then

(1) $$\mathcal{W}_\lambda A_\lambda^- \text{ is open in } I, \text{ and hence locally closed in } \mathcal{A}.$$

For let $A_1 \in \mathcal{W}_\lambda A_\lambda^-$ and $A_2 \in I$ with $A_2 \leq A_1$. We are to check that $A_2 \in \mathcal{W}_\lambda A_\lambda^-$. As $\mathcal{W} = \mathcal{W}_\lambda \ltimes \mathbb{Z}\Delta$ is transitive on $\mathcal{A}$, there is $A_3 \in \mathcal{W}_\lambda A_\lambda^-$ such that $A_2 \in A_3 + \mathbb{Z}\Delta$. Then $A_2 \geq A_3$ by (4.2). Write $A_1 = xA_3$, $x \in \mathcal{W}_\lambda$, and $A_2 = A_3 + \gamma$ for some $\gamma \in \mathbb{Z}\Delta$. Thus, $\gamma \in \mathbb{N}\Delta^+$ by (1.3). As $xA_3 = A_1 \geq A_2 = A_3 + \gamma$, $\lambda + \gamma \uparrow\uparrow x\lambda$. Then $\mathbb{N}\Delta^+ \ni x\lambda - (\lambda + \gamma) = -\gamma$, and hence $\gamma = 0$. Thus, $A_2 = A_3 \in \mathcal{W}_\lambda A_\lambda^-$.

Also,

(2) $$\forall A \in \mathcal{W}_\lambda A_\lambda^-, (A + \mathbb{Z}\Delta) \cap \mathcal{W}_\lambda A_\lambda^- = \{A\}.$$

For LHS $\subsetneq (A + \mathbb{Z}\Delta) \cap I = (A + \mathbb{Z}\Delta) \cap (\geq A)$ by (4.2). Thus, if $A' \in$ LHS, $A' = A + \gamma$ for some $\gamma \in \mathbb{N}\Delta^+$ by (1.3). As $\lambda \in \overline{A'} \cap \overline{A}$, we must have $\gamma = 0$, and hence $A' = A$.

Lemma: *$\forall M \in \mathcal{K}(S_0)$, $\mathcal{K}(S_0)^\sharp(S_0 \otimes_S Q(A_\lambda^-), M) \simeq M_{\mathcal{W}_\lambda A_\lambda^-}$ as graded (S_0, R)-bimodules.*

Proof: Recall from (4.7) that $\tilde{\mathcal{K}}(S_0)^\sharp(S_0 \otimes_S Q(A_\lambda^-), M) \simeq M_I$ via $\varphi \mapsto \varphi(q)$. Let $\varphi \in \tilde{\mathcal{K}}(S_0)^\sharp(S_0 \otimes_S Q(A_\lambda^-), M)$ with $\varphi^\emptyset((S_0 \otimes Q(A_\lambda^-))_A^\emptyset) \subseteq \coprod_{A'>A} M_{A'}^\emptyset$ $\forall A \in \mathcal{A}$. As $\mathrm{supp}_\mathcal{A}(S_0 \otimes_S Q(A_\lambda^-)) = \mathcal{W}_\lambda A_\lambda^-$, $(\mathrm{im}\varphi)_A = 0$ $\forall A \in \mathcal{W}_\lambda A_\lambda^-$ by (2) and Rmk. 2.2.(i). As $\mathcal{W}_\lambda A_\lambda^-$ is open in I, $\mathrm{im}\varphi \subseteq M_{I \setminus \mathcal{W}_\lambda A_\lambda^-}$, and hence $\varphi(q) \in M_{I \setminus \mathcal{W}_\lambda A_\lambda^-}$. On the other hand, $S_0 \otimes_S Q(A_\lambda^-) = S_0 qR$. It follows that $\{\varphi | \varphi^\emptyset((S_0 \otimes_S Q(A_\lambda^-))_A^\emptyset) \subseteq \coprod_{A'>A} M_{A'}^\emptyset\ \forall A \in \mathcal{A}\}$ is sent under the isomorphism onto $\{m \in M_I | m_A = 0\ \forall A \in \mathcal{W}_\lambda A_\lambda^-\}$. Thus,

$$\mathcal{K}(S_0)(S_0 \otimes_S Q(A_\lambda^-), M) \xrightarrow{\sim} M_I / M_{I \setminus \mathcal{W}_\lambda A_\lambda^-} = M_I / M_{I \setminus (\mathcal{A} \setminus (I \setminus \mathcal{W}_\lambda A_\lambda^-))} = M_{I \cap (\mathcal{A} \setminus (I \setminus \mathcal{W}_\lambda A_\lambda^-))} = M_{\mathcal{W}_\lambda A_\lambda^-}.$$

6.6. Recall that $\mathcal{K}_P$ is defined as the full subcategory of $\mathcal{K}_\Delta$ consisting of those $M \in \mathrm{Ob}(\mathcal{K}_\Delta) = \mathrm{Ob}(\tilde{\mathcal{K}}_\Delta)$ such that $\forall$ complex $M_1 \to M_2 \to M_3$ in $\mathcal{K}_\Delta$ with (ES) holding, i.e., $0 \to (M_1)_{\{A\}} \to (M_2)_{\{A\}} \to (M_3)_{\{A\}} \to 0$ is exact as left S-modules/(S, R)-bimodules $\forall A \in \mathcal{A}$,

$$0 \to \mathcal{K}_\Delta(M, M_1) \to \mathcal{K}_\Delta(M, M_2) \to \mathcal{K}_\Delta(M, M_3) \to 0$$

is exact.

Proposition: $\mathrm{Ob}(\mathcal{K}_P) = \mathrm{Ob}(\tilde{\mathcal{K}}_P)$, *and hence $\mathcal{K}_P$ is the ideal quotient of $\tilde{\mathcal{K}}_P$. $\forall \gamma \in \mathbb{Z}\Delta$, the automorphism T_γ on $\tilde{\mathcal{K}}_P$ induces an automorphism of $\mathcal{K}_P$ denoted by the same letter.*

Proof: We show first that $\mathrm{Ob}(\tilde{\mathcal{K}}_P) \subseteq \mathrm{Ob}(\mathcal{K}_P)$. Let $M \in \mathrm{Ob}(\tilde{\mathcal{K}}_P)$. By (4.6.2) we may assume $M = Q(A_\lambda^-) * B(s_1, \dots, s_r)$ for some $\lambda \in \hat{X}$, $s_1, \dots, s_r \in \mathcal{S}$. By (6.4) we may further assume $M = Q(A_\lambda^-)$.

Put $K = \mathcal{W}_\lambda A_\lambda^-$. Let $M_1 \to M_2 \to M_3$ be a complex in $\mathcal{K}_\Delta$ with (ES) holding. By (6.5) one has K locally closed and $\mathcal{K}_\Delta^\sharp(M, M_i) \simeq (M_i)_K$ $\forall i$. As $K \cap (A + \mathbb{Z}\Delta) = \{A\}$ $\forall A \in K$ by (6.5.2),

$$0 \to (M_1)_K \to (M_2)_K \to (M_3)_K \to 0$$

is exact as (S,R)-bimodules by (6.3.ii). Thus, $M \in \mathcal{K}_P$.

Let now $M' \in \mathrm{Ob}(\mathcal{K}_P)$. As $Q(A) \in \tilde{\mathcal{K}}_P$ remains indecomposable in $\mathcal{K}_\Delta$ $\forall A \in \mathcal{A}$, the proof of (4.5) carries over to $\mathcal{K}_P$ to yield that M' is a direct sum of $Q(A)(n)$'s, $A \in \mathcal{A}$, $n \in \mathbb{Z}$, in $\tilde{\mathcal{K}}_\Delta$; there exist $i' \in \mathcal{K}(Q(A)(n), M')$ and $p' \in \mathcal{K}(M', Q(A)(n))$ such that $p' \circ i' \in \mathcal{K}(Q(A)(n), Q(A)(n))^\times$. If $\hat{i}'$ (resp. $\hat{p}'$) is a lift in $\tilde{\mathcal{K}}$ of i' (resp. p'), we may assume $\hat{p}' \circ \hat{i}' + \varphi = \mathrm{id}$ for some $\varphi \in \tilde{\mathcal{K}}(Q(A)(n), Q(A)(n))$ with $\varphi^\emptyset(Q(A)(n)^\emptyset_{A'}) \subseteq \coprod_{A''>A'} Q(A)(n)^\emptyset_{A''}$ $\forall A' \in \mathcal{A}$. As φ is nilpotent, $\hat{p}' \circ \hat{i}' \in \tilde{\mathcal{K}}(Q(A)(n), Q(A)(n))^\times$. Thus, $M' \in \tilde{\mathcal{K}}_P$.

6.7. Recall that S_0 denotes a flat commutative graded S-algebra.

Corollary: *Let $M \in \mathcal{K}_P$, $N \in \mathcal{K}_\Delta$.*

(i) $S_0 \otimes_S \mathcal{K}(M,N) \xrightarrow{\sim} \mathcal{K}(S_0)(S_0 \otimes_S M, S_0 \otimes_S N)$ *via* $a \otimes \varphi \mapsto a(S_0 \otimes_S \varphi)$.

(ii) $S_0 \otimes_S M \in \mathcal{K}_P(S_0)$.

Proof: By (4.6.2) we may assume $M = Q(A^-_\lambda) * B(\underline{x})(n)$ for some $\lambda \in \hat{X}$, $n \in \mathbb{Z}$, $\underline{x} = (s_1, \dots, s_r) \in \mathcal{S}^r$.

(i) By (6.2) we may further assume that $M = Q(A^-_\lambda)$. By (6.5) one has a CD

$$\begin{array}{ccc} S_0 \otimes_S \mathcal{K}^\sharp(Q(A^-_\lambda), N) & \longrightarrow & \mathcal{K}(S_0)^\sharp(S_0 \otimes_S Q(A^-_\lambda), S_0 \otimes_S N) \\ \sim\big| & & \big|\sim \\ S_0 \otimes_S N_{\mathcal{W}_\lambda A^-_\lambda} & \longrightarrow & (S_0 \otimes_S N)_{\mathcal{W}_\lambda A^-_\lambda} \end{array}$$

with the bottom row invertible by (2.13.3).

(ii) Let $M_1 \to M_2 \to M_3$ be a complex in $\mathcal{K}_\Delta(S_0)$ with (ES) holding. By (6.2) and (6.5) one has a CD

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}(S_0)^\sharp(S_0 \otimes_S M, M_1) & \xrightarrow{\sim} & \mathcal{K}(S_0)^\sharp(S_0 \otimes_S Q(A^-_\lambda), M_1 * B(\underline{x})(-n)) & \xrightarrow{\sim} & \{M_1 * B(\underline{x})(-n)\}_{\mathcal{W}_\lambda A^-_\lambda} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}(S_0)^\sharp(S_0 \otimes_S M, M_2) & \xrightarrow{\sim} & \mathcal{K}(S_0)^\sharp(S_0 \otimes_S Q(A^-_\lambda), M_2 * B(\underline{x})(-n)) & \xrightarrow{\sim} & \{M_2 * B(\underline{x})(-n)\}_{\mathcal{W}_\lambda A^-_\lambda} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}(S_0)^\sharp(S_0 \otimes_S M, M_3) & \xrightarrow{\sim} & \mathcal{K}(S_0)^\sharp(S_0 \otimes_S Q(A^-_\lambda), M_3 * B(\underline{x})(-n)) & \xrightarrow{\sim} & \{M_3 * B(\underline{x})(-n)\}_{\mathcal{W}_\lambda A^-_\lambda} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

with the right column exact by (6.4), (6.5.2) and (6.3.ii).

6.8. By (6.6) one may define $\mathrm{ch} : [\mathcal{K}_P] \to \mathcal{P}^0$ by the same formula as on $[\tilde{\mathcal{K}}_P]$. Thus, under the identification $[\mathfrak{S}\mathfrak{B}] \simeq \mathcal{H}$, one obtains from (5.3)

Theorem: $\mathrm{ch} : [\mathcal{K}_P] \to \mathcal{P}^0$ *is an isomorphism of right $\mathcal{H}$-modules.*

6.9. Recall from (I.5.4)/[S97, p. 84] a ring endomorphism $\bar{?}$ of $\mathcal{H}$ such that $\bar{v} = v^{-1}$ and $\bar{H}_x = (H_{x^{-1}})^{-1}$ $\forall x \in \mathcal{W}$. Recall also from [S97, Th. 4.1] an $\mathcal{H}$-skew linear involution $\bar{?}$ on $\mathcal{P}^0$ such that $\forall m \in \mathcal{P}^0$, $\forall h \in \mathcal{H}$, $\overline{mh} = \bar{m}\bar{h}$.

$\forall \lambda \in \hat{X}$, one has

$$
\begin{aligned}
(1) \qquad \overline{\mathrm{ch}(Q(A_\lambda^-)(\ell(w_0) - \ell(A_\lambda^-))} &= \overline{v^{\ell(w_0)-\ell(A_\lambda^-)}\mathrm{ch}(Q(A_\lambda^-))} \\
&= \overline{v^{\ell(w_0)-\ell(A_\lambda^-)+\ell(A_\lambda^-)-\ell(w_0)}E_\lambda} \quad \text{by (5.2)} \\
&= E_\lambda \quad \text{by [S97, Th. 4.3]} \\
&= v^{\ell(w_0)-\ell(A_\lambda^-)}\mathrm{ch}(Q(A_\lambda^-)) \quad \text{by (5.2) again} \\
&= \mathrm{ch}(Q(A_\lambda^-)(\ell(w_0) - \ell(A_\lambda^-))).
\end{aligned}
$$

$\forall m' \in \mathcal{P}$, writing $\bar{m} = \sum_{A\in\mathcal{A}} c_A A$ and $m' = \sum_{A\in\mathcal{A}} d_A A$, $c_A, d_A \in \mathbb{Z}[v, v^{-1}]$, set $(m, m')_{\mathcal{P}} = \sum_{A\in\mathcal{A}} c_A d_A$. Recall from (I.5.4) an anti-involution $\omega : \mathcal{H} \to \mathcal{H}$ via $\sum_{x\in\mathcal{W}} a_x H_x \mapsto \sum_{x\in\mathcal{W}} \overline{a_x} H_x^{-1} = \sum_{x\in\mathcal{W}} a_x(v^{-1}) H_x^{-1}$. In particular, $\omega(\underline{H}_s) = \underline{H}_s$ $\forall s \in \mathcal{S}$ by (I.5.5).

Lemma: *$\forall m \in \mathcal{P}^0$, $\forall m' \in \mathcal{P}$, $\forall h \in \mathcal{H}$,*

$$(mh, m')_{\mathcal{P}} = (m, m'\omega(h))_{\mathcal{P}}.$$

Proof: If the assertion holds for $h, h' \in \mathcal{H}$,

$$(mhh', m')_{\mathcal{P}} = (mh, m'\omega(h'))_{\mathcal{P}} = (m, m'\omega(h')\omega(h))_{\mathcal{P}} = (m, m'\omega(hh'))_{\mathcal{P}}.$$

Also, as $\overline{mv} = \bar{m}\bar{v} = \bar{m}v^{-1}$,

$$(mv, m')_{\mathcal{P}} = v^{-1}(m, m')_{\mathcal{P}} = (m, m'v^{-1})_{\mathcal{P}} = (m, m'\omega(v))_{\mathcal{P}}.$$

Thus, we may assume that $h = \underline{H}_s$, $s \in \mathcal{S}$, $m = E_\lambda$, $\lambda \in \hat{X}$, $m' = A'$, $A' \in \mathcal{A}$.

One has

$$
\begin{aligned}
\overline{E_\lambda \underline{H}_s} = E_\lambda \underline{H}_s = v^{\ell(A_\lambda^+)} e_\lambda \underline{H}_s &= v^{\ell(A_\lambda^+)} \sum_{A\in\mathcal{W}_f A^+} v^{-\ell(A+\lambda)}(A+\lambda)\underline{H}_s \quad \text{by (5.2.1)} \\
&= \sum_{A\in\mathcal{W}_f A^+} v^{d(A,A^+)}(A\underline{H}_s + \lambda) \quad \text{by (5.1.1)},
\end{aligned}
$$

and hence both sides vanish unless $A' - \lambda \in (\mathcal{W}_f A^+) \cup (\mathcal{W}_f A^+ s)$. Thus, we may assume that $A' \in \{A + \lambda, As + \lambda | A \in \mathcal{W}_f A^+\}$.

Assume first that $A' = A + \lambda$, $A \in \mathcal{W}_f A^+$, and $As > A$. Then

$$\overline{E_\lambda \underline{H}_s} = E_\lambda \underline{H}_s = \sum_{\substack{B\in\mathcal{W}_f A^+\\ Bs>B}} v^{d(B,A^+)}\{(Bs+vB)+\lambda\} + \sum_{\substack{B\in\mathcal{W}_f A^+\\ Bs<B}} v^{d(B,A^+)}\{(Bs+v^{-1}B)+\lambda\},$$

and hence

$$(E_\lambda \underline{H}_s, A')_{\mathcal{P}} = \begin{cases} v^{d(A,A^+)+1} + v^{d(As,A^+)} & \text{if } As \in \mathcal{W}_f A^+,\\ v^{d(A,A^+)+1} & \text{else}\end{cases}$$

while

$$\begin{aligned}(E_\lambda, A'\omega(\underline{H}_s))_{\mathcal{P}} &= (E_\lambda, A'\underline{H}_s)_{\mathcal{P}} = (E_\lambda, (A+\lambda)\underline{H}_s)_{\mathcal{P}}\\ &= (E_\lambda, A\underline{H}_s + \lambda)_{\mathcal{P}} \quad \text{by (5.1.1) again}\\ &= (E_\lambda, (As+vA)+\lambda)_{\mathcal{P}} = \begin{cases} v^{d(A,A^+)+1} + v^{d(As,A^+)} & \text{if } As \in \mathcal{W}_f A^+,\\ v^{d(A,A^+)+1} & \text{else.}\end{cases}\end{aligned}$$

Assume next that $A' = A + \lambda$, $A \in \mathcal{W}_f A^+$, and $As < A$. Then

$$(E_\lambda \underline{H}_s, A')_{\mathcal{P}} = \begin{cases} v^{d(A,A^+)-1} + v^{d(As,A^+)} & \text{if } As \in \mathcal{W}_f A^+,\\ v^{d(A,A^+)-1} & \text{else}\end{cases}$$

while

$$\begin{aligned}(E_\lambda, A'\omega(\underline{H}_s))_{\mathcal{P}} &= (E_\lambda, A'\underline{H}_s)_{\mathcal{P}} = (E_\lambda, (As+v^{-1}A)+\lambda)_{\mathcal{P}}\\ &= \begin{cases} v^{d(As,A^+)} + v^{d(A,A^+)-1} & \text{if } As \in \mathcal{W}_f A^+,\\ v^{d(A,A^+)-1} & \text{else.}\end{cases}\end{aligned}$$

6.10 **Formula for the mophism space:** As $\mathrm{Ob}(\mathcal{K}_\Delta) = \mathrm{Ob}(\tilde{\mathcal{K}}_\Delta)$, one has from (5.1) an $\mathcal{H}$-linear map $\mathrm{ch} : [\mathcal{K}_\Delta] \to \mathcal{P}$.

Theorem: *$\forall P \in \mathcal{K}_P$, $\forall M \in \mathcal{K}_\Delta$, $\mathcal{K}^\sharp(P, M)$ is left graded free over S with*

$$\mathrm{grk}(\mathcal{K}^\sharp(P,M)) = v^{-2\ell(w_0)}(\mathrm{ch}(P), \mathrm{ch}(M))_{\mathcal{P}}.$$

Proof: As $[\mathcal{K}_P] \xrightarrow[\sim]{\mathrm{ch}} \mathcal{P}^0$ with $\mathrm{ch}(Q(A_\lambda^-) * B(s_1) * \cdots * B(s_r)) = v^{2\ell(A_\lambda^-)} e_\lambda \underline{H}_{s_1} \dots \underline{H}_{s_r}$, $\lambda \in \hat{X}$, $s_1, \dots, s_r \in \mathcal{S}$, by (5.2) and (5.3), and as

$$\mathcal{P}^0 = \sum_{\substack{\lambda\in\hat{X}\\ s_1,\dots,s_r\in\mathcal{S}, r\in\mathbb{N}}} \mathbb{Z}[v,v^{-1}] e_\lambda \underline{H}_{s_1} \dots \underline{H}_{s_r}$$

by definition (5.2), one has

$$[\mathcal{K}_P] = \sum_{\substack{\lambda\in\hat{X}\\ s_1,\dots,s_r\in\mathcal{S}, r\in\mathbb{N}}} \mathbb{Z}[v,v^{-1}][Q(A_\lambda^-) * B(s_1) * \cdots * B(s_r)].$$

Then, as ω is an anti-involution, one may assume by (6.2) and (6.9) that $P = Q(A_\lambda^-)$. Let $(?,?)'$ be a $\mathbb{Z}[v, v^{-1}]$-bilinear pairing on $\mathcal{P}$ such that $(A, A') = \delta_{A,A'}$ $\forall A, A' \in \mathcal{A}$. Recall from (6.9.1) that

$$\overline{\mathrm{ch}(Q(A_\lambda^-))} = v^{-\ell(A_\lambda^-)+\ell(w_0)}E_\lambda = v^{-\ell(A_\lambda^-)+\ell(w_0)+\ell(A_\lambda^+)}e_\lambda = v^{2\ell(w_0)}e_\lambda.$$

Then

$$\begin{aligned}(\mathrm{ch}(Q(A_\lambda^-)), \mathrm{ch}(M))_{\mathcal{P}} &= v^{2\ell(w_0)}(e_\lambda, \mathrm{ch}(M))' = v^{2\ell(w_0)}(\sum_{A\in\mathcal{W}_\lambda A_\lambda^-} v^{-\ell(A)}A, \sum_{A\in\mathcal{A}} v^{\ell(A)}\mathrm{grk}(M_{\{A\}})A)' \\ &= v^{2\ell(w_0)} \sum_{A\in\mathcal{W}_\lambda A_\lambda^-} \mathrm{grk}(M_{\{A\}}) = v^{2\ell(w_0)}\mathrm{grk}(M_{\mathcal{W}_\lambda A_\lambda^-}) \\ &= v^{2\ell(w_0)}\mathrm{grk}(\mathcal{K}^\sharp(Q(A_\lambda^-), M)) \quad \text{by (6.5).}\end{aligned}$$

6.11 **The category $\mathcal{K}_P^\alpha = \mathcal{K}_P(S^\alpha)$:** Fix $\alpha \in \Delta^+$. $\forall A \in \mathcal{A}$, let $Q_\alpha(A) = \{(a, b) \in S^2 | a \equiv b \mod \alpha^\vee\} = \{(a, a + b\alpha^\vee) | a, b \in S\}$ with the left daiagonal S-action and a right action of R given by $(a, b)f = (f_A a, (s_\alpha f_A)b$ $\forall f \in R$. Thus, $Q_\alpha(A)^\emptyset = S^\emptyset \oplus S^\emptyset$. Recall from (1.4) that $\alpha \uparrow A = s_{\alpha,n}A > A$ with $n \in \mathbb{Z}$ such that $\forall \nu \in A$, $n - 1 < \langle \nu, \alpha^\vee\rangle < n$. By (1.2)

$$s_\alpha(f_A) = s_\alpha(f(A)) = f(s_\alpha A) = f(s_{\alpha,n}A) = f_{\alpha\uparrow A}. \tag{1}$$

Define $\forall A' \in \mathcal{A}$,

$$Q_\alpha(A)^\emptyset_{A'} = \begin{cases} S^\emptyset \oplus 0 & \text{if } A' = A, \\ 0 \oplus S^\emptyset & \text{if } A' = \alpha \uparrow A, \\ 0 & \text{else.} \end{cases}$$

Thus, $Q_\alpha(A) \in \tilde{\mathcal{K}}'$. As $\mathrm{supp}_{\mathcal{A}}(Q_\alpha(A)) \subseteq \mathcal{W}^\alpha A = \{\dots, A-\alpha, (\alpha \uparrow A)-\alpha, A, \alpha \uparrow A, A+\alpha, \dots\}$, (S) holds on $\alpha \uparrow A$ by (2.5.i). Also,

$$Q_\alpha(A)^\alpha = Q_\alpha(A)^\alpha \cap (S^\emptyset \oplus S^\emptyset) = Q_\alpha(A)^\alpha \cap \{Q_\alpha(A)^\emptyset_A \oplus Q_\alpha(A)^\emptyset_{\alpha\uparrow A}\}.$$

If $\beta \in \Delta^+ \setminus \{\pm\alpha\}$,

$$Q_\alpha(A)^\beta = S^\beta \oplus S^\beta = \{Q_\alpha(A)^\beta \cap Q_\alpha(A)^\emptyset_A\} \oplus \{Q_\alpha(A)^\beta \cap Q_\alpha(A)^\emptyset_{\alpha\uparrow A}\}.$$

Thus, (LE) holds on $Q_\alpha(A)$, and hence $Q_\alpha(A) \in \tilde{\mathcal{K}}$. One has

$$\begin{aligned} Q_\alpha(A)_{\{A\}} &= Q_\alpha(A)_{\geq A}/Q_\alpha(A)_{>A} = Q_\alpha(A)/Q_\alpha(A)_{>A} \\ &= \{(a, b) \in S^2 | a \equiv b \mod \alpha^\vee\}/\{(0, b)|\alpha^\vee|b\} \xrightarrow{\sim} S \quad \text{via} \quad (a, b) \mapsto a \\ &\qquad \text{as } Q_\alpha(A) = Q_\alpha(A)_{>A} + \{(a, a)|a \in S\}, \\ Q_\alpha(A)_{\{\alpha\uparrow A\}} &= Q_\alpha(A)_{>A} \simeq S(-2), \end{aligned}$$

and hence $Q_\alpha(A) \in \tilde{\mathcal{K}}_\Delta$.

Consider a graded (S,R)-bimodule homomorphism $\xi : S \otimes_{\mathbb{K}} R \to Q_\alpha(A)$ via $a \otimes f \mapsto (af_A, af_{\alpha\uparrow A})$. Let $S^{s_\alpha} = \{a \in S | s_\alpha a = a\}$. $\forall a \in S^{s_\alpha}$,

$$\begin{aligned}(a^A)_{\alpha\uparrow A} &= s_\alpha((a^A)_A) \quad \text{by (1)} \\ &= s_\alpha a = a,\end{aligned}$$

and hence

$$\begin{array}{ccc} S\otimes_{\mathbb{K}} R & \xrightarrow{\xi} & Q_\alpha(A). \\ \downarrow & \overset{\bar{\xi}}{\nearrow} & \\ S\otimes_{S^{s_\alpha}} R & & \end{array}$$

If $\delta \in X_{\mathbb{K}}^\vee$ with $\langle \alpha, \delta\rangle = 1$, one has as in (I.2.1).

(2) $$S = S^{s_\alpha} \oplus \delta S^{s_\alpha}.$$

By (1) again

$$\begin{aligned} \xi(1\otimes 1) &= (1, s_\alpha(1_A)) = (1,1), \\ \xi(1\otimes \delta^A) &= ((\delta^A)_A, (\delta^A)_{\alpha\uparrow A}) = (\delta, s_\alpha\delta) = (\delta, \delta - \alpha^\vee), \end{aligned}$$

and hence ξ is surjective. $\forall a, b \in S$ with $0 = \xi(1\otimes a^A + \delta\otimes b^A) = ((a^A)_A, s_\alpha((a^A)_A)) + (\delta(b^A)_A, \delta s_\alpha((a^A)_A)) = (a+\delta b, s_\alpha a + \delta s_\alpha b)), 0 = -s_\alpha(\delta b) + \delta s_\alpha b = -(\delta-\alpha^\vee)s_\alpha b + \delta s_\alpha b = \alpha^\vee s_\alpha b$, and hence $b = 0$. Then $a = 0$, and $\bar{\xi}$ is bijective. Thus,

(3) $$S\otimes_{S^{s_\alpha}} R \simeq Q_\alpha(A).$$

Then

$$\begin{aligned} \tilde{\mathcal{K}}'(Q_\alpha(A), Q_\alpha(A)) &\leq (S\otimes_{S^{s_\alpha}} R)\mathrm{Modgr}(S\otimes_{S^{s_\alpha}} R, Q_\alpha(A)) \simeq M \\ &\simeq Q_\alpha(A)^0 \quad \text{as } \deg(1\otimes 1) = 0 \\ &= \mathbb{K}(1\otimes 1), \end{aligned}$$

and hence

(4) $$Q_\alpha(A) \text{ is indecomposable in } \tilde{\mathcal{K}}'.$$

Likewise,

(5) $$Q_\alpha(A)^\alpha \text{ is indecomposable in } (\tilde{\mathcal{K}}')^\alpha.$$

Let now that $M \in \tilde{\mathcal{K}}'$ with $\mathrm{supp}_{\mathcal{A}}(M) \subseteq \mathcal{W}^\alpha A$. $\forall a \in S^{s_\alpha}$, $\forall r \in \mathbb{Z}$,

$$(a^A)_{s_{\alpha,r}A} = a^A(s_{\alpha,r}A) = s_\alpha(a^A(A)) = s_\alpha a = a,$$

and hence M admits a structure of left graded $S\otimes_{S^{s_\alpha}} R$-module. Then $\tilde{\mathcal{K}}'(Q_\alpha(A), M) \leq (S\otimes_{S^{s_\alpha}} R)\mathrm{Mod}(S\otimes_{S^{s_\alpha}} R, M) \simeq M$. As $\mathrm{supp}_{\mathcal{A}}(Q_\alpha(A)) = \{A, \alpha\uparrow A\}$ with $A < (\alpha \uparrow A)$, $\tilde{\mathcal{K}}'(Q_\alpha(A), M) \subseteq M_{\geq A}$. Given $m \in M_{\geq A}$, take $\varphi \in (S\otimes_{S^{s_\alpha}} R)\mathrm{Mod}(Q_{A,\alpha}, M)$ with $\varphi(1,1) = m$. As $M^\emptyset_{\geq A}$ is an (S,R)-bisubmodule of $M^\emptyset$, $\mathrm{im}(\varphi^\emptyset) \subseteq M^\emptyset_{\geq A}$. In particular,

$$\begin{aligned} \varphi^\emptyset(Q_\alpha(A)^\emptyset_{\alpha\uparrow A}) &\subseteq \coprod_{A' \in \{(\alpha\uparrow A)+\mathbb{Z}\Delta\}\cap \mathcal{W}^\alpha A \cap (\geq A)} M^\emptyset_{A'} \quad \text{by Rmk. 2.2.i}, \\ &\subseteq \coprod_{A' \geq \alpha\uparrow A} M^\emptyset_{A'} \quad \text{as } A \notin \{(\alpha\uparrow A)+\mathbb{Z}\Delta\} \cap \mathcal{W}^\alpha A, \end{aligned}$$

and hence $\varphi \in \tilde{\mathcal{K}}'(Q_\alpha(A), M)$, and

$$\tilde{\mathcal{K}}'^\sharp(Q_\alpha(A), M) \simeq M_{\geq A}. \tag{6}$$

Under the isomorphism $\{\varphi \in \tilde{\mathcal{K}}'^\sharp(Q_\alpha(A), M) | \varphi^\emptyset(Q_\alpha(A)^\emptyset_{A'}) \subseteq \coprod_{A''>A'} M^\emptyset_{A''} \ \forall A' \in \{A, \alpha \uparrow A\}\}$ is mapped onto $\{m \in M_{\geq A} | m_{A'} = 0 \ \forall A' \in \{A, \alpha \uparrow A\}\}$ as $\alpha \uparrow A \notin (> A) \cap (A + \mathbb{Z}\Delta) \cap \mathcal{W}^\alpha A$. Also, $\{A, \alpha \uparrow A\}$ is open in $(\geq A)$; if $A' \in \{A, \alpha \uparrow A\}$ and $A \leq A'' < A'$, $A' = \alpha \uparrow A$. As $d(A, \alpha \uparrow A) = 1$ by [L80, Lem. 2.5], we must have $A'' = A$. Thus,

$$\mathcal{K}'^\sharp(Q_\alpha(A), , M) \simeq M_{\geq A}/M_{(\geq A)\setminus\{A,\alpha\uparrow A\}} = M_{\{A,\alpha\uparrow A\}}. \tag{7}$$

Likewise, $\forall M \in \mathcal{K}^\alpha$ with $\mathrm{supp}_{\mathcal{A}}(M) \subseteq \mathcal{W}^\alpha A$,

$$(\mathcal{K}^\alpha)^\sharp(Q_\alpha(A)^\alpha, M) \simeq M_{\{A,\alpha\uparrow A\}}. \tag{8}$$

Lemma: $Q_\alpha(A)^\alpha \in \mathcal{K}^\alpha_P$.

Proof: Let $M \in \mathcal{K}^\alpha_\Delta$. As (LE) holds on M, $M = M^\alpha = \coprod_i M_i$ with $\mathrm{supp}_{\mathcal{A}}(M_i) \subseteq \mathcal{W}^\alpha A_i$ for some $A_i \in \mathcal{A}$. As $\{A, \alpha \uparrow A\} \subseteq \mathcal{W}^\alpha A$, one has by (8)

$$(\mathcal{K}^\alpha)^\sharp(Q_\alpha(A)^\alpha, M) \simeq \coprod_i (M_i)_{\{A,\alpha\uparrow A\}} = M_{\{A,\alpha\uparrow A\}}. \tag{9}$$

Given a complex $M' \to M \to M''$ in $\mathcal{K}^\alpha_\Delta$ with (ES) holding, one has from (9) a CD

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathcal{K}^\alpha)^\sharp(Q^\alpha_{A,\alpha}, M') & \longrightarrow & (\mathcal{K}^\alpha)^\sharp(Q^\alpha_{A,\alpha}, M) & \longrightarrow & (\mathcal{K}^\alpha)^\sharp(Q^\alpha_{A,\alpha}, M'') & \longrightarrow & 0 \\
 & & \wr\| & & \wr\| & & \wr\| & & \\
0 & \longrightarrow & M'_{\{A,\alpha\uparrow A\}} & \longrightarrow & M_{\{A,\alpha\uparrow A\}} & \longrightarrow & M''_{\{A,\alpha\uparrow A\}} & \longrightarrow & 0
\end{array}$$

with the bottom row exactby (6.3.ii); $(A+\mathbb{Z}\Delta) \cap \{A, \alpha \uparrow A\} = \{A\}, (\alpha \uparrow A+\mathbb{Z}\Delta) \cap \{A, \alpha \uparrow A\} = \{\alpha \uparrow A\}$.

6.12. One can now argue as in (6.6) to obtain

Proposition: *Any object of $\mathcal{K}^\alpha_P$ is a direct sum of some $Q_\alpha(A)^\alpha(n)$, $A \in \mathcal{A}, n \in \mathbb{Z}$.*

7. **The combinatorial category of AJS**

We recall the combinatorial category of AJS after a version by Fiebig [F11], which we denote by $\mathcal{K}_{\mathrm{AJS}}$. We construct a functor $\mathcal{F} : \mathcal{K}_\Delta \to \mathcal{K}_{\mathrm{AJS}}$, and show that $\mathcal{F}$ is fully faithful on $\mathcal{K}_P$. Let S_0 be a flat commutative graded S-algebra.

7.1. The category $\mathcal{K}_{\mathrm{AJS}}(S_0)$ is defined as follows [F11, Defs. 5.2, 5.3]. An object of $\mathcal{K}_{\mathrm{AJS}}(S_0)$ is $\mathcal{M} = ((\mathcal{M}(A)|A \in \mathcal{A}), (\mathcal{M}(A, \alpha)|A \in \mathcal{A}, \alpha \in \Delta^+))$, where $\mathcal{M}(A)$ is a graded $(S_0)^\emptyset$-module while $\mathcal{M}(A, \alpha)$ is a graded $(S_0)^\alpha$-submodule of $\mathcal{M}(A) \oplus \mathcal{M}(\alpha \uparrow A)$. A morphism $f \in \mathcal{K}_{\mathrm{AJS}}(S_0)(\mathcal{M}, \mathcal{N})$ is a collection of $f(A) \in (S_0)^\emptyset\mathrm{Modgr}(\mathcal{M}(A), \mathcal{N}(A))$, $A \in \mathcal{A}$, sending each

$\mathcal{M}(A,\alpha)$ into $\mathcal{N}(A,\alpha)$, $\alpha\in\Delta^+$. Put $\mathcal{K}_{\mathrm{AJS}}=\mathcal{K}_{\mathrm{AJS}}(S)$ and $\mathcal{K}^*_{\mathrm{AJS}}=\mathcal{K}_{\mathrm{AJS}}(S^*)$ for $*\in\{\emptyset\}\sqcup\Delta^+$. $\forall s\in\mathcal{S}$, the wall-crossing translation endofunctor Θ_s on $\mathcal{K}_{\mathrm{AJS}}$ is defined as

$$\text{(1)}\qquad (\Theta_s\mathcal{M})(A)=\mathcal{M}(A)\oplus\mathcal{M}(As),$$

$$(\Theta_s\mathcal{M})(A,\alpha)=\begin{cases}\mathcal{M}(A,\alpha)\oplus\mathcal{M}(As,\alpha) & \text{if } As\notin\mathcal{W}^\alpha A,\\ \{(x,y)\in\mathcal{M}(A,\alpha)^2 | x-y\in\alpha^\vee\mathcal{M}(A,\alpha)\} & \text{if } As=\alpha\uparrow A,\\ \alpha^\vee\mathcal{M}(As,\alpha)\oplus\mathcal{M}(\alpha\uparrow A,\alpha) & \text{if } As=\alpha\downarrow A.\end{cases}$$

Define a functor $\mathcal{F}(S_0):\mathcal{K}_\Delta(S_0)\to\mathcal{K}_{\mathrm{AJS}}(S_0)$ by setting

$$\{\mathcal{F}(S_0)(M)\}(A)=M_A^\emptyset \quad\text{and}\quad \{\mathcal{F}(S_0)(M)\}(A,\alpha)=\mathrm{im}(M^\alpha_{[A,\alpha\uparrow A]}\to M_A^\emptyset\oplus M^\emptyset_{\alpha\uparrow A}).$$

Recall from (2.13.3) that $(M_{[A,\alpha\uparrow A]})^\alpha\simeq(M^\alpha)_{[A,\alpha\uparrow A]}$. Put $\mathcal{F}=\mathcal{F}(S)$ and $\mathcal{F}^*=\mathcal{F}(S^*)$ for $*\in\{\emptyset\}\sqcup\Delta$. As $M\in\mathcal{K}_\Delta(S_0)$, one has $M^\alpha=\coprod_{\Omega\in\mathcal{W}^\alpha\backslash\mathcal{A}}M^\Omega$ with $\mathrm{supp}_{\mathcal{A}}(M^\Omega)\subseteq\Omega$ by (LE). Then

$$\text{(2)}\qquad \{\mathcal{F}(S_0)(M)\}(A,\alpha)=\mathrm{im}((M^{\mathcal{W}^\alpha A})_{[A,\alpha\uparrow A]}\to M_A^\emptyset\oplus M^\emptyset_{\alpha\uparrow A})\quad\text{as } \{A,\alpha\uparrow A\}\subseteq\mathcal{W}^\alpha A$$
$$\simeq (M^{\mathcal{W}^\alpha A})_{[A,\alpha\uparrow A]}$$

as $(M^{\mathcal{W}^\alpha A})_{[A,\alpha\uparrow A]}\subseteq\coprod_{A'\in[A,\alpha\uparrow A]\cap\Omega}M^\emptyset_{A'}=M^\emptyset_A\oplus M^\emptyset_{\alpha\uparrow A}$.

7.2. Let $M\in\mathcal{K}_\Delta$ and $s\in\mathcal{S}$. Take $\delta\in\Lambda^\vee_{\mathbb{K}}$ with $\langle\alpha_s,\delta\rangle=1$. Recall from (3.3) that $B(s)^\emptyset=B(s)^\emptyset_e\oplus B(s)^\emptyset_s$ with $B(s)^\emptyset_e$ (resp. $B(s)^\emptyset_s$) free over $R^\emptyset$ of basis $b_e=\frac{1}{\alpha_s^\vee}(\delta\otimes1-1\otimes s\delta)$ (resp. $b_s=\frac{1}{\alpha_s^\vee}(\delta\otimes1-1\otimes\delta)$). $\forall A\in\mathcal{A}$,

$$\text{(1)}\qquad \{\mathcal{F}(M*B(s))\}(A)=(M*B(s))^\emptyset_A$$
$$\simeq(M^\emptyset_A\otimes_R Rb_e)\oplus(M^\emptyset_{As}\otimes_R Rb_s)\simeq M^\emptyset_A\oplus M^\emptyset_{As}\quad\text{by (3.6.1)}$$
$$=(\mathcal{F}M)(A)\oplus(\mathcal{F}M)(As)=\{\Theta_s(\mathcal{F}M)\}(A).$$

Proposition: *$\forall M\in\mathcal{K}_\Delta$, $\forall s\in\mathcal{S}$, $\mathcal{F}(M*B(s))\simeq\Theta_s(\mathcal{F}(M))$.*

Proof: Let $\alpha\in\Delta^+$. We verify under (1) that

$$\text{(2)}\qquad \mathcal{F}(M*B(s))(A,\alpha)\simeq\{\Theta_s(\mathcal{F}M)\}(A,\alpha).$$

Put $\Omega=\mathcal{W}^\alpha A$. By (7.1.2) one has LHS $\simeq\{(M*B(s))^\Omega\}_{[A,\alpha\uparrow A]}$.

Assume first that $As\notin\Omega$. As $As\in\Omega s\setminus\Omega$,

$$(M*B(s))^\Omega\simeq(M^\Omega\otimes_R Rb_e)\oplus(M^{\Omega s}\otimes_R Rb_s)\qquad\text{by (3.7.ii)}$$

with, $\forall A'\in\mathcal{A}$,

$$(M^\Omega\otimes_R Rb_e)^\emptyset_{A'}=(M^\Omega)^\emptyset_{A'}\otimes_R Rb_e\quad\text{as } b_e\in B(s)^\emptyset_e,$$
$$(M^{\Omega s}\otimes_R Rb_s)^\emptyset_{A'}=(M^{(\Omega s)})^\emptyset_{A's}\otimes_R Rb_s\quad\text{as } b_s\in B(s)^\emptyset_s.$$

Then

$$\begin{aligned}(M^{\Omega}\otimes_R Rb_e)_{[A,\alpha\uparrow A]} &= (M^{\Omega}\otimes_R Rb_e)_{\geq A}/(M^{\Omega}\otimes_R Rb_e)_{(\geq A)\setminus(\leq \alpha\uparrow A)}\\ &\simeq (M^{\Omega})_{[A,\alpha\uparrow A]}\otimes_R Rb_e,\\ (M^{\Omega s}\otimes_R Rb_s)_{[A,\alpha\uparrow A]} &= (M^{\Omega s})_{[As,\alpha\uparrow As]}\otimes_R Rb_s\\ &\qquad\text{as } \Omega s\cap[A,\alpha\uparrow(A)]s=\Omega s\cap[As,\alpha\uparrow(As)],\end{aligned}$$

and hence

$$\begin{aligned}\{(M*B(s))^{\Omega}\}_{[A,\alpha\uparrow A]} &\simeq (M^{\Omega})_{[A,\alpha\uparrow A]}\otimes_R Rb_e\oplus(M^{(\Omega s)})_{[As,\alpha\uparrow As]}\otimes_R Rb_s\\ &\simeq (M^{\Omega})_{[A,\alpha\uparrow A]}\oplus(M^{(\Omega s)})_{[As,\alpha\uparrow As]}\quad\text{as graded left } S^{\alpha}\text{-modules}\\ &= (\mathcal{F}M)(A,\alpha)\oplus(\mathcal{F}M)(As,\alpha)\quad\text{by (7.1.2) again}\\ &= \{\Theta_s(\mathcal{F}M)\}(A,\alpha).\end{aligned}$$

Assume next that $As=\alpha\uparrow A$. Then $\Omega s=\Omega$, $[A,\alpha\uparrow A]=[A,As]=\{A,As\}=(\geq A)\cap(\leq As)$ with $(\geq A)=(\geq A)s$ [L80, Prop. 3.2], and by (3.1)

$$(\alpha_s^{\vee})_A=\pm\alpha^{\vee}. \tag{3}$$

Then

$$\begin{aligned}\{(M*B(s))^{\Omega}\}_{[A,\alpha\uparrow A]} &= (M^{\Omega}*B(s))_{[A,\alpha\uparrow A]}\quad\text{by (3.7.i)}\\ &= (M^{\Omega})_{[A,\alpha\uparrow A]}*B(s)\quad\text{by (3.9.3)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\{\Theta_s(\mathcal{F}M)\}(A,\alpha) &= \{(x,y)\in(\mathcal{F}M)(A,\alpha)^2|x-y\in\alpha^{\vee}(\mathcal{F}M)(A,\alpha)\}\\ &= \{(x,y)\in\{(M^{\Omega})_{\{A,As\}}\}^2|x-y\in\alpha^{\vee}(M^{\Omega})_{\{A,As\}}\}\quad\text{by (7.1.2)}.\end{aligned}$$

Put $N=M^{\Omega}$ for simplicity. We are to show that $N_{\{A,As\}}\otimes_R B(s)$ and $\{(x,y)\in N^2|x-y\in\alpha^{\vee}N\}$ coincide in

$$(M_{\{A,As\}}*B(s))^{\emptyset}=(M_{\{A,As\}}*B(s))^{\emptyset}_A\oplus(M_{\{A,As\}}*B(s))^{\emptyset}_{As}\simeq(M^{\emptyset}_A\oplus M^{\emptyset}_{As})\oplus(M^{\emptyset}_{As}\oplus M^{\emptyset}_A).$$

We let m_B, $m\in M$, $B\in\mathcal{A}$, denote the B-component of m in $M^{\emptyset}$. Regarding $N_{\{A,As\}}\otimes_R B(s)$ as $N_{\{A,As\}}\otimes_{R^s}R=(N_{\{A,As\}}\otimes_R R)\oplus(N_{\{A,As\}}\otimes_R R\delta)$, the image of $m_1\otimes 1+m_2\otimes\delta$, $m_1,m_2\in N_{\{A,As\}}$, in $(M^{\emptyset}_A\oplus M^{\emptyset}_{As})\oplus(M^{\emptyset}_{As}\oplus M^{\emptyset}_A)$ is

$$\begin{aligned}&(m_{1,A},\,m_{1,As},m_{1,As},m_{1,A})+(m_{2,A}\delta,m_{2,As}s\delta,m_{2,As}\delta,m_{2,A}s\delta)\quad\text{by (3.6.1)}\\ &\quad=(m_{1,A}+\delta_A m_{2,A},m_{1,As}+(s\delta)_{As}m_{2,As},m_{1,As}+\delta_{As}m_{2,As},m_{1,A}+(s\delta)_A m_{2,A})\\ &\quad=(m_{1,A}+\delta_A m_{2,A},m_{1,As}+\delta_A m_{2,As},m_{1,As}+(s\delta)_A m_{2,As},m_{1,A}+(s\delta)_A m_{2,A})\quad\text{by (1.2.i)}\end{aligned}$$

with

$$\begin{aligned}&(m_{1,A}+\delta_A m_{2,A},m_{1,As}+\delta_A m_{2,As})-(m_{1,A}+(s\delta)_A m_{2,A},m_{1,As}+(s\delta)_A m_{2,As})\\ &\quad=(\delta_A-(s\delta)_A)(m_{2,A},,m_{2,As})=(\delta-s\delta)_A(m_{2,A},,m_{2,As})=(\alpha_s^{\vee})_A(m_{2,A},,m_{2,As})\\ &\quad=\pm\alpha^{\vee}(m_{2,A},,m_{2,As})\quad\text{by (3)},\end{aligned}$$

and hence $N_{\{A,As\}} \otimes_R B(s) \subseteq (\Theta_s(\mathcal{F}M))(A,\alpha)$. Given $(x, x-(\alpha_s^\vee)_A y) \in \mathrm{RHS}$ for $x, y \in N_{\{A,As\}}$, take $m_1 = x - \delta_A y, m_2 = y \in N_{\{A,As\}}$. Then $m_1 \otimes 1 + m_2 \otimes \delta$ realizes $(x, x-(\alpha_s^\vee)_A y)$.

Assume finally that $As = \alpha \downarrow A$. Then $\Omega s = \Omega$ and $As < A < \alpha \uparrow A < (\alpha \uparrow A)s = A + \alpha$ and $(\alpha_s^\vee)_A = \pm\alpha^\vee$ again. One has

$$\begin{aligned}
\{(M * B(s))^\Omega\}_{[A,\alpha\uparrow A]} &= (N * B(s))_{[A,\alpha\uparrow A]} \quad \text{by (3.7.i)}\\
&= (N * B(s))_{\geq A}/(N * B(s))_{(\geq A)\setminus(\leq \alpha\uparrow A)}\\
&= (N * B(s))_{> As}/(N * B(s))_{(> As)\setminus(\leq \alpha\uparrow A)} \quad \text{as } \mathrm{supp}_{\mathcal{A}}(N * B(s)) \subseteq \Omega\\
&= (N * B(s))_{]As,\alpha\uparrow A]}
\end{aligned}$$

with

$$\begin{array}{ccc}
(N * B(s))_{>As} & \hookrightarrow & \coprod_{A' \geq A} (N * B(s))^\emptyset_{A'}\\
\downarrow & & \downarrow\\
(N * B(s))_{]As,\alpha\uparrow A]} & \hookrightarrow & (N * B(s))^\emptyset_A \oplus (N * B(s))^\emptyset_{\alpha\uparrow A}\\
 & & \shortparallel\\
 & & (N^\emptyset_A \oplus N^\emptyset_{As}) \oplus (N^\emptyset_{\alpha\uparrow A} \oplus N^\emptyset_{A+\alpha}).
\end{array} \tag{4}$$

As $(\geq As) = (\geq As)s$ by [L80, Prop. 3.2], $(N * B(s))_{>As} \leq (N * B(s))_{\geq As} = N_{\geq As} * B(s)$ by (3.8). Consider

$$\begin{array}{ccc}
N_{\geq As} * B(s) & \hookrightarrow & \coprod_{A' \geq As} (N * B(s))^\emptyset_{A'}\\
\shortparallel & & \downarrow \mathrm{proj}\\
N_{\geq As} \otimes_{R^s} R & \cdots\cdots\cdots\rightarrow & (N * B(s))^\emptyset_{As} \oplus (N * B(s))^\emptyset_A \oplus (N * B(s))^\emptyset_{\alpha\uparrow A}\\
m \otimes f & & \downarrow \wr\\
\downarrow & & \\
(m_{As}f, m_A sf, m_A f, m_{As} sf, m_{\alpha\uparrow A} f, m_{A+\alpha} sf) & & (N^\emptyset_{As} \oplus N^\emptyset_A) \oplus (N^\emptyset_A \oplus N^\emptyset_{As}) \oplus (N^\emptyset_{\alpha\uparrow A} \oplus N^\emptyset_{A+\alpha})
\end{array}$$

from (3.6.1). Any element of $N_{\geq As} \otimes_{R^s} R = N_{\geq As} \otimes_{R^s} (R^s \oplus R^s\delta)$ is of the form $m_1 \otimes 1 + m_2 \otimes \delta$ for some $m_1, m_2 \in N_{\geq As}$, and $m_1 \otimes 1 + m_2 \otimes \delta \in (N * B(s))_{>As}$ iff its As-component in $(N * B(s))^\emptyset$ vanishes. Writing $(N * B(s))^\emptyset_{As} \simeq N^\emptyset_{As} \oplus N^\emptyset_A$,

$$\begin{aligned}
(m_1 \otimes 1 + m_2 \otimes \delta)_{As} &= (m_{1,As} + m_{2,As}\delta, m_{1,A} + m_{2,A}s\delta) \quad \text{by (3.6.1)}\\
&= (m_{1,As} + (s\delta)_A m_{2,As}, m_{1,A} + (s\delta)_A m_{2,A}) \quad \text{by (1.2.i)}.
\end{aligned}$$

Thus, it suffices to show that

$$\{m_1 \otimes 1 + m_2 \otimes \delta \in N_{\geq As} \otimes_{R^s} R | m_{1,A} + (s\delta)_A m_{2,A} = 0 = m_{1,As} + (s\delta)_A m_{2,As}\} \tag{5}$$

under (4) coincides in $(N * B(s))^\emptyset_A \oplus (N * B(s))^\emptyset_{\alpha\uparrow A} = (N^\emptyset_A \oplus N^\emptyset_{As}) \oplus (N^\emptyset_{\alpha\uparrow A} \oplus N^\emptyset_{A+\alpha})$ with

$$\alpha^\vee(\mathcal{F}M)(As,\alpha) \oplus (\mathcal{F}M)(\alpha\uparrow A,\alpha) = \alpha^\vee N_{[As,\alpha\uparrow(As)]} \oplus N_{[\alpha\uparrow A, A+\alpha]} = \alpha^\vee N_{[As,A]} \oplus N_{[\alpha\uparrow A, A+\alpha]}$$

from (7.1.2). The image of $m_1 \otimes 1 + m_2 \otimes \delta$ in (5) under (4) is

$$\begin{aligned}
&(m_{1,A} + m_{2,A}\delta, m_{1,As} + m_{2,As}s\delta, m_{1,\alpha\uparrow A} + m_{2,\alpha\uparrow A}\delta, m_{1,A+\alpha} + m_{2,A+\alpha}s\delta)\\
&\quad = (m_{1,A} + \delta_A m_{2,A}, m_{1,As} + (s\delta)_{As} m_{2,As}, m_{1,\alpha\uparrow A} + \delta_{\alpha\uparrow A} m_{2,\alpha\uparrow A}, m_{1,A+\alpha} + (s\delta)_{A+\alpha} m_{2,A+\alpha})\\
&\quad = (m_{1,A} + \delta_A m_{2,A}, m_{1,As} + \delta_A m_{2,As}, m_{1,\alpha\uparrow A} + (s\delta)_A m_{2,\alpha\uparrow A}, m_{1,A+\alpha} + (s\delta)_A m_{2,A+\alpha})\\
&\qquad\quad \text{by (1.2.i)}
\end{aligned}$$

with $m_{1,A}+\delta_A m_{2,A} = -(s\delta)_A m_{2,A}+\delta_A m_{2,A} = (\alpha_s^\vee)_A m_{2,A}$ and $m_{1,As}+\delta_A m_{2,As} = -(s\delta)_A m_{2,As}+\delta_A m_{2,As} = (\alpha_s^\vee)_A m_{2,As}$. Thus, by (3) again, the images of the elements of (5) are contained in $\alpha^\vee N_{[As,A]} \oplus N_{[\alpha\uparrow A, A+\alpha]}$.

Let finally $m_1' \in N_{[As,A]} = N_{\geq As}/N_{(\geq As)\setminus(\leq A)}$ and $m_2' \in N_{[\alpha\uparrow A,A+\alpha]} = N_{\geq \alpha\uparrow A}/N_{(\alpha\uparrow A)\setminus(\leq A+\alpha)}$. Take a lift $m_1 \in N_{\geq As}$ and $m_2 \in N_{\geq\alpha\uparrow A}$, resp. Put $m = m_2 \otimes 1 + m_1 \otimes \delta - (s\delta)_A m_1 \otimes 1 = \{m_2 - (s\delta)_A m_1\} \otimes 1 + m_1 \otimes \delta \in N_{\geq As} \otimes_{R^s} R$. As $m_2 \in N_{\geq \alpha\uparrow A}$, $m_{2,A} = 0 = m_{2,As}$. Then

$$m_{2,A} - (s\delta)_A m_{1,A} + (s\delta)_A m_{1,A} = 0 = m_{2,As} - (s\delta)_A m_{1,As} + (s\delta)_A m_{1,As},$$

and hence m belongs to (5). As the image of m in $(N_A^\emptyset \oplus N_{As}^\emptyset) \oplus (N_{\alpha\uparrow A}^\emptyset \oplus N_{A+\alpha}^\emptyset)$ is

$$\begin{aligned}((\alpha_s^\vee)_A m_{1,A}, (\alpha_s^\vee)_A m_{1,As}, \{m_2-(s\delta)_A m_1\}_{\alpha\uparrow A}+\delta_{\alpha\uparrow A} m_{1,\alpha\uparrow A}, \{m_2-(s\delta)_A m_1\}_{A+\alpha}+(s\delta)_{A+\alpha} m_{1,A+\alpha}) \\ = ((\alpha_s^\vee)_A m_{1,A}, (\alpha_s^\vee)_A m_{1,As}, m_{2,\alpha\uparrow A}, m_{2,A+\alpha})\end{aligned}$$

as $\delta_{\alpha\uparrow A} = \delta_{As+\alpha} = \delta_{As} = (s\delta)_A$ and $(s\delta)_{A+\alpha} = (s\delta)_A$, realizing $((\alpha_s^\vee)_A m_1', m_2')$. The assertion follows.

7.3. We now start a task of showing that $\mathcal{F}$ is fully faithful on $\mathcal{K}_P$. Recall from (6.12) that the objects of $\mathcal{K}_P^\alpha$ are easy to describe. Let $A \in \mathcal{A}$ and $\alpha \in \Delta^+$. Recall from (6.11) that $Q_\alpha(A) = \{(a,b) \in S^2 | a \equiv b \mod \alpha^\vee\} \in \tilde{\mathcal{K}}_\Delta$ with the right R-action $(a,b)f = (f_A a, (s_\alpha f_A)b)$ and $\forall A' \in \mathcal{A}$,

$$Q_\alpha(A)_{A'}^\emptyset = \begin{cases} S^\emptyset \oplus 0 & \text{if } A' = A, \\ 0 \oplus S^\emptyset & \text{if } A' = \alpha\uparrow A, \\ 0 & \text{else,} \end{cases}$$

$$Q_\alpha(\alpha\uparrow A)_{A'}^\emptyset = \begin{cases} S^\emptyset \oplus 0 & \text{if } A' = \alpha\uparrow A, \\ 0 \oplus S^\emptyset & \text{if } A' = \alpha\uparrow(\alpha\uparrow A) = A+\alpha, \\ 0 & \text{else,} \end{cases}$$

$$Q_\alpha(\alpha\downarrow A)_{A'}^\emptyset = \begin{cases} S^\emptyset \oplus 0 & \text{if } A' = \alpha\downarrow A, \\ 0 \oplus S^\emptyset & \text{if } A' = A, \\ 0 & \text{else.} \end{cases}$$

Define

$$\begin{aligned} i_0 &\in \tilde{\mathcal{K}}(Q_\alpha(A), Q_\alpha(A)(2)) \quad \text{via} \quad (a,b) \mapsto (0, \alpha^\vee b), \\ i_0^+ &\in \tilde{\mathcal{K}}(Q_\alpha(A), Q_\alpha(\alpha\uparrow A)) \quad \text{via} \quad (a,b) \mapsto (b,a), \\ i_0^- &\in \tilde{\mathcal{K}}(Q_\alpha(A), Q_\alpha(\alpha\downarrow A)(2)) \quad \text{via} \quad (a,b) \mapsto (0, \alpha^\vee a); \end{aligned}$$

We will denote their images in $\mathcal{K}$ by the same letters.

7.4. Let S_0 be a flat commutative graded S-algebra.

Lemma: *Let $A, A' \in \mathcal{A}$.*

(i) $\mathcal{K}(S_0)^\sharp(S_0 \otimes_S Q_\alpha(A), S_0 \otimes_S Q_\alpha(A)) = \tilde{\mathcal{K}}(S_0)^\sharp(S_0 \otimes_S Q_\alpha(A), S_0 \otimes_S Q_\alpha(A)) = S_0\mathrm{id} \oplus S_0 i_0$. In particular, $\mathcal{K}(S_0)(S_0 \otimes_S Q_\alpha(A), S_0 \otimes_S Q_\alpha(A)) = \tilde{\mathcal{K}}(S_0)(S_0 \otimes_S Q_\alpha(A), S_0 \otimes_S Q_\alpha(A)) = \mathbb{K}\mathrm{id}$, and $S_0 \otimes_S Q_\alpha(A)$ remains indecomposable in both $\mathcal{K}(S_0)$ and $\tilde{\mathcal{K}}(S_0)$.

(ii) $\mathcal{K}(S_0)^\sharp(S_0\otimes_S Q_\alpha(A), S_0\otimes_S Q_\alpha(\alpha\uparrow A)) = S_0 i_0^+$.

(iii) $\mathcal{K}(S_0)^\sharp(S_0\otimes_S Q_\alpha(A), S_0\otimes_S Q_\alpha(\alpha\downarrow A)) = S_0 i_0^-$.

(iv) If $A\not\in\{\alpha\downarrow A', A', \alpha\uparrow A'\}$, $\mathcal{K}(S_0)^\sharp(S_0\otimes_S Q_\alpha(A), S_0\otimes_S Q_\alpha(A')) = 0$.

Proof: Put $M = S_0\otimes_S Q_\alpha(A)$. Thus, $\mathrm{supp}_{\mathcal{A}}(M) = \{A, \alpha\uparrow A\}$.

(i) Let $\varphi\in\tilde{\mathcal{K}}(S_0)^\sharp(M,M)$. Then

$$\begin{aligned}\varphi^\emptyset(M_A^\emptyset) &\subseteq \coprod_{\substack{A'\geq A\\ A'\in A+\mathbb{Z}\Delta}} M_{A'}^\emptyset \quad \text{by Rmk. 2.2.(i)},\\ &= M_A^\emptyset,\\ \varphi^\emptyset(M_{\alpha\uparrow A}^\emptyset) &\subseteq M_{\alpha\uparrow A}^\emptyset \quad \text{likewise},\end{aligned}$$

and hence

(1) $$\varphi^\emptyset(M_{A'}^\emptyset) \subseteq M_{A'}^\emptyset \quad \forall A\in\mathcal{A}.$$

Thus, $\tilde{\mathcal{K}}(S_0)^\sharp(M,M) = \mathcal{K}(S_0)^\sharp(M,M)$.

We show next that $\phi\in S_0\mathrm{id}\oplus S_0 i_0$. By (1) we must have $\varphi^\emptyset = (\varphi_1,\varphi_2)$ for some $\varphi_1,\varphi_2\in S_0^\emptyset\mathrm{Mod}(S_0^\emptyset, S_0^\emptyset)$. Then $\varphi_1 = a\mathrm{id}_{S_0^\emptyset}$ for some $a\in S_0^\emptyset$. Put $\psi = \varphi - a\mathrm{id}$. As $(\psi^\emptyset)_1 = 0$, $\mathrm{im}\psi\subseteq 0\oplus\alpha^\vee S_0$, and hence $\psi = b i_0$ for some $b\in S_0$.

(ii) Put $N = S_0\otimes_S Q_\alpha(\alpha\uparrow A)$, and $\varphi\in\tilde{\mathcal{K}}(S_0)^\sharp(M,N)$. $\forall A'\in\mathcal{A}$,

$$N_{A'}^\emptyset = \begin{cases} S^\emptyset & \text{if } A'\in\{\alpha\uparrow A, A+\alpha\},\\ 0 & \text{else},\end{cases}$$

and hence by Rmk. 2.2.(i)

$$\varphi^\emptyset(M_A^\emptyset)\subseteq \coprod_{\substack{A'\geq A\\ A'\in A+\mathbb{Z}\Delta}} N_{A'}^\emptyset = N_{A+\alpha}^\emptyset, \quad \varphi^\emptyset(M_{\alpha\uparrow A}^\emptyset)\subseteq \coprod_{\substack{A'\geq \alpha\uparrow A\\ A'\in \alpha\uparrow A+\mathbb{Z}\Delta}} N_{A'}^\emptyset = N_{\alpha\uparrow A}^\emptyset.$$

Thus, there are $\varphi_1,\varphi_2\in S_0\mathrm{Mod}(S_0,S_0)$ such that $\forall a,b\in S_0$, $\varphi(a,b) = (\varphi_1(b),\varphi_2(a))$. Write $\varphi_1 = c\mathrm{id}$ for some $c\in S_0$. Then $(\varphi - c i_0^+)^\emptyset(M_{\alpha\uparrow A}^\emptyset) = 0$, and hence $\varphi - c i_0^+ = 0$ in $\mathcal{K}(S_0)$.

(iii) Put $N = S_0\otimes_S Q_\alpha(\alpha\downarrow A)$, and $\varphi\in\tilde{\mathcal{K}}(S_0)^\sharp(M,N)$. $\forall A'\in\mathcal{A}$,

$$N_{A'}^\emptyset = \begin{cases} S^\emptyset & \text{if } A'\in\{\alpha\downarrow A, A\},\\ 0 & \text{else},\end{cases}$$

and hence

$$\varphi^\emptyset(M_A^\emptyset)\subseteq \coprod_{A'\geq A} N_{A'}^\emptyset = N_A^\emptyset, \quad \varphi^\emptyset(M_{\alpha\uparrow A}^\emptyset)\subseteq \coprod_{A'\geq\alpha\uparrow A} N_{A'}^\emptyset = 0.$$

Thus, there is $\varphi_1 \in S_0\mathrm{Mod}(S_0, S_0)$ such that $\forall a, b \in S_0$, $\varphi(a, b) = (0, \varphi_1(a))$. As $\varphi_1 = c\mathrm{id}$ for some $c \in S_0$, $\varphi(1, 1) = (0, c)$, and hence $\alpha^\vee | c$. Thus, $\varphi \in S_0 i_0^-$.

(iv) Let $\varphi \in \tilde{\mathcal{K}}(S_0)^\sharp(S_0 \otimes_S Q_\alpha(A), S_0 \otimes_S Q_\alpha(\alpha \downarrow A'))$. As $\mathrm{supp}_{\mathcal{A}}(Q_\alpha(A)) = \{A, \alpha \uparrow A\}$ is disjoint from $\mathrm{supp}_{\mathcal{A}}(Q_\alpha(A')) = \{A', \alpha \uparrow A'\}$, $\varphi = 0$ in $\mathcal{K}(S_0)$.

7.5. We calculate next in $\mathcal{K}_{\mathrm{AJS}}$. Let $A \in \mathcal{A}$, $\alpha \in \Delta^+$, and put $\mathcal{Q}_{A,\alpha} = \mathcal{F}(Q_\alpha(A))$. Thus, $\forall A' \in \mathcal{A}$, $\forall \beta \in \Delta^+$,

$$\mathcal{Q}_{A,\alpha}(A') = \mathcal{F}(Q_\alpha(A))(A') = Q_\alpha(A)^\emptyset_{A'} = \begin{cases} S^\emptyset & \text{if } A' \in \{A, \alpha \uparrow A\}, \\ 0 & \text{else,} \end{cases}$$

$$\mathcal{Q}_{A,\alpha}(A', \beta) \simeq (Q_\alpha(A)^{\mathcal{W}^\beta A'})_{[A', \beta \uparrow A']} \subseteq Q_\alpha(A)^\emptyset_{A'} \oplus Q_\alpha(A)^\emptyset_{\beta \uparrow A'} \quad \text{by (7.1.2).}$$

Lemma: *In $Q_\alpha(A)^\emptyset_{A'} \oplus Q_\alpha(A)^\emptyset_{\beta \uparrow A'}$ one has*

$$\mathcal{Q}_{\alpha,A}(A', \beta) = \begin{cases} S^\beta \oplus 0 & \text{if } A' \in \{A, \alpha \uparrow A\} \text{ and } \beta \neq \alpha, \\ 0 \oplus S^\beta & \text{if } A' \in \{\beta \downarrow A, \beta \downarrow (\alpha \uparrow A)\} \text{ and } \beta \neq \alpha, \\ \alpha^\vee S^\alpha \oplus 0 & \text{if } A' = \alpha \uparrow A \text{ and } \beta = \alpha, \\ \{Q_\alpha(A)\}^\alpha & \text{if } A' = A \text{ and } \beta = \alpha, \\ 0 \oplus S^\alpha & \text{if } A' = \alpha \downarrow A \text{ and } \beta = \alpha, \\ 0 & \text{else.} \end{cases}$$

Proof: Assume first that $\beta \neq \alpha$. One has $\{Q_\alpha(A)\}^\beta = S^\beta \otimes_S Q_\alpha(A) = S^\beta(A) \oplus S^\beta(\alpha \uparrow A)$ as $\alpha \in (S^\beta)^\times$, and hence in $Q_\alpha(A)^\emptyset_{A'} \oplus Q_\alpha(A)^\emptyset_{\beta \uparrow A'}$

$$\{(Q_\alpha(A))^\beta\}_{[A', \beta \uparrow A']} = \begin{cases} S^\beta(A') \oplus 0 & \text{if } A' = A, \\ S^\beta(A') \oplus 0 & \text{if } A' = \alpha \uparrow A, \\ 0 \oplus S^\beta(\beta \uparrow A') & \text{if } A' = \beta \downarrow A, \\ 0 \oplus S^\beta(\beta \uparrow A') & \text{if } A' = \beta \downarrow \alpha \uparrow A, \\ 0 & \text{else.} \end{cases}$$

Assume next that $\beta = \alpha$. As $Q_\alpha(A)^\alpha = \{(a, b) \in S^\alpha(A) \oplus S^\alpha(\alpha \uparrow A) | a \equiv b \mod \alpha^\vee\}$,

$$\begin{aligned} \{(Q_\alpha(A))^\beta\}_{[A', \alpha \uparrow A']} &= \{(Q_\alpha(A))^\alpha\}_{\geq A'} / \{(Q_\alpha(A))^\alpha\}_{(\geq A') \setminus (\leq \alpha \uparrow A')} \\ &= \begin{cases} \alpha^\vee S^\alpha(A') \oplus 0 & \text{if } A' = \alpha \uparrow A, \\ Q_\alpha(A)^\alpha & \text{if } A' = A, \\ 0 \oplus \alpha^\vee S^\alpha(\alpha \uparrow A') & \text{if } A' = \alpha \downarrow A, \\ 0 & \text{else.} \end{cases} \end{aligned}$$

7.6 . Put $\iota_0 = \mathcal{F}(i_0), \iota_0^+ = \mathcal{F}(i_0^+), \iota_0^- = \mathcal{F}(i_0^-)$. Thus,

$$\begin{array}{ccc} \mathcal{Q}_{A,\alpha}(A) & \xrightarrow{\iota_0(A)} & \mathcal{Q}_{A,\alpha}(A)(2) \\ \| & & \| \\ S^\emptyset & \xrightarrow[0]{} & S^\emptyset(2), \end{array} \qquad \begin{array}{ccc} \mathcal{Q}_{A,\alpha}(\alpha \uparrow A) & \xrightarrow{\iota_0(\alpha \uparrow A)} & \mathcal{Q}_{A,\alpha}(\alpha \uparrow A)(2) \\ \| & & \| \\ S^\emptyset & \xrightarrow[\alpha^\vee \mathrm{id}]{} & S^\emptyset(2), \end{array}$$

$$\begin{array}{ccc} \mathcal{Q}_{A,\alpha}(A) & \overset{\iota_0^+(A)}{\cdots\rightarrow} & \mathcal{Q}_{\alpha\uparrow A,\alpha}(A) \\ \| & & \| \\ S^{\emptyset} & \xrightarrow[0]{} & 0, \end{array} \qquad \begin{array}{ccc} \mathcal{Q}_{A,\alpha}(\alpha\uparrow A) & \overset{\iota_0^+(\alpha\uparrow A)}{\cdots\cdots\rightarrow} & \mathcal{Q}_{\alpha\uparrow A,\alpha}(\alpha\uparrow A) \\ \| & & \| \\ S^{\emptyset} & \xrightarrow[\mathrm{id}]{} & S^{\emptyset}, \end{array}$$

$$\begin{array}{ccc} \mathcal{Q}_{A,\alpha}(A) & \overset{\iota_0^-(A)}{\cdots\rightarrow} & \mathcal{Q}_{\alpha\downarrow A,\alpha}(A) \\ \| & & \| \\ S^{\emptyset} & \xrightarrow[\alpha^\vee \mathrm{id}]{} & S^{\emptyset}(2), \end{array} \qquad \begin{array}{ccc} \mathcal{Q}_{A,\alpha}(\alpha\uparrow A) & \overset{\iota_0^-(\alpha\uparrow A)}{\cdots\cdots\rightarrow} & \mathcal{Q}_{\alpha\downarrow A,\alpha}(\alpha\uparrow A) \\ \| & & \| \\ S^{\emptyset} & \longrightarrow & 0. \end{array}$$

Lemma: *(i)* $\mathcal{K}_{\mathrm{AJS}}(S_0)^\sharp(S_0\otimes_S \mathcal{Q}_{A,\alpha}, S_0\otimes_S \mathcal{Q}_{A,\alpha}) = S_0\mathrm{id}\oplus S_0\iota_0$.

(ii) $\mathcal{K}_{\mathrm{AJS}}(S_0)^\sharp(S_0\otimes_S \mathcal{Q}_{A,\alpha}, S_0\otimes_S \mathcal{Q}_{\alpha\uparrow A,\alpha}) = S_0\iota_0^+$.

(iii) $\mathcal{K}_{\mathrm{AJS}}(S_0)^\sharp(S_0\otimes_S \mathcal{Q}_{A,\alpha}, S_0\otimes_S \mathcal{Q}_{\alpha\downarrow A,\alpha}) = S_0\iota_0^-$.

(iv) If $A\notin\{\alpha\downarrow A', A', \alpha\uparrow A'\}$, $\mathcal{K}_{\mathrm{AJS}}(S_0)^\sharp(S_0\otimes_S \mathcal{Q}_{A,\alpha}, S_0\otimes_S \mathcal{Q}_{A',\alpha}) = 0$.

Proof: Put $\mathcal{M} = S_0\otimes_S \mathcal{Q}_{A,\alpha}$.

(i) Let $\varphi\in\mathcal{K}_{\mathrm{AJS}}(S_0)^\sharp(\mathcal{M},\mathcal{M})$. As $\mathcal{M}(A') = 0$ unless $A'\in\{A,\alpha\uparrow A\}$, $\varphi(A') = 0$ unless $A'\in\{A,\alpha\uparrow A\}$. By (7.5) one has, $\forall\beta\in\Delta^+$, a CD

$$\begin{array}{ccc} \mathcal{M}(\beta\downarrow A)\oplus\mathcal{M}(A) & \xrightarrow{\varphi(\beta\downarrow A)\oplus\varphi(A)} & \mathcal{M}(\beta\downarrow A)\oplus\mathcal{M}(A) \\ \cup\!\uparrow & & \cup\!\uparrow \\ \mathcal{M}(\beta\downarrow A,\beta) & & \mathcal{M}(\beta\downarrow A,\beta) \\ \| & & \| \\ 0\oplus S_0^\beta & \xrightarrow[\varphi(\beta\downarrow A,\beta)]{} & 0\oplus S_0^\beta. \end{array}$$

Then $\varphi(A)(S_0^\beta)\subseteq S_0^\beta$, and hence $\varphi(A)(S_0) = \varphi(A)(\cap_\beta S_0^\beta)\subseteq\cap_\beta S_0^\beta = S_0$. As $\varphi(A)$ is S_0-linear, $\varphi(A) = c\mathrm{id}$ for some $c\in S_0$. If $\beta\neq\alpha$, one has a CD

$$\begin{array}{ccc} \mathcal{M}(\alpha\uparrow A)\oplus\mathcal{M}(\beta\uparrow\alpha\uparrow A) & \xrightarrow{\varphi(\alpha\uparrow A)\oplus\varphi(\beta\uparrow\alpha\uparrow A)} & \mathcal{M}(\alpha\uparrow A)\oplus\mathcal{M}(\beta\uparrow\alpha\uparrow A) \\ \cup\!\uparrow & & \cup\!\uparrow \\ \mathcal{M}(\alpha\uparrow A,\beta) & & \mathcal{M}(\alpha\uparrow A,\beta) \\ \| & & \| \\ S_0^\beta\oplus 0 & \xrightarrow[\varphi(\alpha\uparrow A,\beta)]{} & S_0^\beta\oplus 0, \end{array}$$

and hence $\varphi(\alpha\uparrow A)(S_0^\beta)\subseteq S_0^\beta$. If $\beta = \alpha$, one has a CD

$$\begin{array}{ccc} \mathcal{M}(\alpha\uparrow A)\oplus\mathcal{M}(A+\alpha) & \xrightarrow{\varphi(\alpha\uparrow A)\oplus\varphi(A+\alpha)} & \mathcal{M}(\alpha\uparrow A)\oplus\mathcal{M}(A+\alpha) \\ \cup\!\uparrow & & \cup\!\uparrow \\ \mathcal{M}(\alpha\uparrow A,\alpha) & & \mathcal{M}(\alpha\uparrow A,\alpha) \\ \| & & \| \\ \alpha^\vee S_0^\alpha\oplus 0 & \xrightarrow[\varphi(\alpha\uparrow A,\alpha)]{} & \alpha^\vee S_0^\alpha\oplus 0, \end{array}$$

and hence $\alpha^\vee\varphi(\alpha\uparrow A)(S_0^\alpha)=\varphi(\alpha\uparrow A)(\alpha^\vee S_0^\alpha)\subseteq\alpha^\vee S_0^\alpha$. As S_0 is flat over S,

$$\begin{array}{ccc} S_0\otimes_S S & \xrightarrow{S_0\otimes_S\alpha^\vee} & S_0\otimes_S S \\ \sim\big| & & \big|\sim \\ S_0 & \dashrightarrow & S_0, \end{array}$$

and hence $\varphi(\alpha\uparrow A)(S_0^\alpha)\subseteq S_0^\alpha$. Then $\varphi(\alpha\uparrow A)(S_0)=\varphi(\alpha\uparrow A)(\cap_{\beta\in\Delta^+}S_0)\subseteq\cap_{\beta\in\Delta^+}S_0=S_0$, and $\varphi(\alpha\uparrow A)=d\mathrm{id}_{S_0^\emptyset}$ for some $d\in S_0$. Put $\psi=\varphi-c\mathrm{id}$. Then $\psi(A)=0$, and hence $\mathcal{M}(A,\alpha)\ni(\psi(A)(1_{S_0^\emptyset}),\psi(\alpha\uparrow A)(1_{S_0^\emptyset}))=(0,d-c)$. Thus, $\alpha^\vee|d-c$ and $\psi=\frac{d-c}{\alpha^\vee}\iota_0$.

(ii) Put $\mathcal{N}=S_0\otimes_S\mathcal{Q}_{\alpha\uparrow A,\alpha}$, and let $\varphi\in\mathcal{K}_{\mathrm{AJS}}(\mathcal{M},\mathcal{N})$. As $\{A,\alpha\uparrow A\}\cap\{\alpha\uparrow A,A+\alpha\}=\{\alpha\uparrow A\}$, $\varphi(A')=0$ unless $A'=\alpha\uparrow A$ by (7.5). $\forall\beta\in\Delta^+\setminus\{\alpha\}$, one has by (7.5) a CD

$$\begin{array}{ccc} \mathcal{M}(\alpha\uparrow A)\oplus\mathcal{M}(\beta\uparrow\alpha\uparrow A) & \xrightarrow{\varphi(\alpha\uparrow A)\oplus\varphi(\beta\uparrow\alpha\uparrow A)} & \mathcal{N}(\alpha\uparrow A)\oplus\mathcal{N}(\beta\uparrow\alpha\uparrow A) \\ \uparrow & & \uparrow \\ \mathcal{M}(\alpha\uparrow A,\beta) & & \mathcal{N}(\alpha\uparrow A,\beta) \\ \| & & \| \\ S_0^\beta\oplus 0 & \xrightarrow[\varphi(\alpha\uparrow A,\beta)]{} & S_0^\beta\oplus 0, \end{array}$$

and hence $\varphi(\alpha\uparrow A)(S_0^\beta)\subseteq S_0^\beta$. Also, there is a CD

$$\begin{array}{ccc} \mathcal{M}(A)\oplus\mathcal{M}(\alpha\uparrow A) & \xrightarrow{\varphi(A)\oplus\varphi(\alpha\uparrow A)} & \mathcal{N}(A)\oplus\mathcal{N}(\alpha\uparrow A) \\ \uparrow & & \uparrow \\ \mathcal{M}(A,\alpha) & & \mathcal{N}(A,\alpha) \\ \| & & \| \\ \{(a,b)\in(S_0^\alpha)^2|a\equiv b\mod\alpha^\vee\} & \xrightarrow[\varphi(A,\alpha)]{} & 0\oplus S_0^\alpha. \end{array}$$

As $(a,a)\in\mathcal{M}(A,\alpha)\ \forall a\in S_0^\alpha$, $\varphi(\alpha\uparrow A)(S_0^\alpha)\subseteq S_0^\alpha$. Then $\varphi(\alpha\uparrow A)(S_0)=\varphi(\alpha\uparrow A)(\cap_{\beta\in\Delta^+}S_0^\beta)\subseteq\cap_{\beta\in\Delta^+}\varphi(\alpha\uparrow A)(S_0^\beta)\subseteq\cap_{\beta\in\Delta^+}S_0^\beta=S_0$, and hence $\varphi(\alpha\uparrow A)\in S_0\mathrm{id}$. Thus, $\varphi\in S_0\iota_0^+$.

(iii) Put $\mathcal{N}=S_0\otimes_S\mathcal{Q}_{\alpha\downarrow A,\alpha}$, and let $\varphi\in\mathcal{K}_{\mathrm{AJS}}(S_0)^\sharp(\mathcal{M},\mathcal{N})$. As $\{A,\alpha\uparrow A\}\cap\{\alpha\downarrow A,A\}=\{A\}$, $\varphi(A')=0$ unless $A'=A$ by (7.5). $\forall\beta\in\Delta^+\setminus\{\alpha\}$, one has by (7.5) a CD

$$\begin{array}{ccc} \mathcal{M}(A)\oplus\mathcal{M}(\beta\uparrow A) & \xrightarrow{\varphi(A)\oplus\varphi(\beta\uparrow A)} & \mathcal{N}(A)\oplus\mathcal{N}(\beta\uparrow A) \\ \uparrow & & \uparrow \\ \mathcal{M}(A,\beta) & & \mathcal{N}(A,\beta) \\ \| & & \| \\ S_0^\beta\oplus 0 & \xrightarrow[\varphi(A,\beta)]{} & S_0^\beta\oplus 0, \end{array}$$

and hence $\varphi(A)(S_0^\beta) \subseteq S_0^\beta$. Also, there is a CD

$$\begin{array}{ccc}
\mathcal{M}(A) \oplus \mathcal{M}(\alpha \uparrow A) & \xrightarrow{\varphi(A)\oplus\varphi(\alpha\uparrow A)} & \mathcal{N}(A) \oplus \mathcal{N}(\alpha \uparrow A) \\
\uparrow & & \uparrow \\
\mathcal{M}(A,\alpha) & & \mathcal{N}(A,\alpha) \\
\parallel & & \parallel \\
\{(a,b) \in (S_0^\alpha)^2 | a \equiv b \mod \alpha^\vee\} & \xrightarrow[\varphi(A,\alpha)]{} & \alpha^\vee S_0^\alpha \oplus 0,
\end{array}$$

and hence $\varphi(A)(S_0^\alpha) \subseteq \alpha^\vee S_0^\alpha$. As $S_0^\beta = \alpha^\vee S_0^\beta$ $\forall \beta \neq \alpha$, $\varphi(A)(S_0) = \varphi(A)(\cap_{\beta\in\Delta^+} S_0^\beta) \subseteq \cap_{\beta\in\Delta^+}\alpha^\vee S_0^\beta = \alpha^\vee S_0$. Thus, $\varphi(A) \in \alpha^\vee S_0\mathrm{id}$, and $\varphi \in S_0\iota_0^-$.

(iv) Let $\varphi \in \mathcal{K}_{\mathrm{AJS}}(S_0)^\sharp(S_0 \otimes_S \mathcal{Q}_{A,\alpha}, S_0 \otimes_S \mathcal{Q}_{A',\alpha})$. As $\{A, \alpha \uparrow A\} \cap \{A', \alpha \uparrow A'\} = \emptyset$, $\varphi(A'') = 0$ $\forall A'' \in \mathcal{A}$, and hence $\varphi = 0$. $\square$

7.7. Putting together (7.4) and (7.6) yields

Lemma: $\forall A, A' \in \mathcal{A}$, $\forall \alpha \in \Delta^+$,

$$\begin{aligned}
\mathcal{K}(S_0)^\sharp(S_0 \otimes_S Q_\alpha(A), S_0 \otimes_S Q_\alpha(A')) &\xrightarrow[\sim]{\mathcal{F}(S_0)} \mathcal{K}_{\mathrm{AJS}}(S_0)^\sharp(S_0 \otimes_S \mathcal{F}(Q_\alpha(A)), S_0 \otimes_S \mathcal{F}(Q_\alpha(A'))) \\
&= \mathcal{K}_{\mathrm{AJS}}(S_0)^\sharp(S_0 \otimes_S \mathcal{Q}_{A,\alpha}, S_0 \otimes_S \mathcal{Q}_{A',\alpha}).
\end{aligned}$$

7.8. Let $\alpha \in \Delta^+$. $\forall A, A' \in \mathcal{A}$,

$$\begin{aligned}
\mathcal{K}_P^\alpha(Q_\alpha(A)^\alpha, Q_\alpha(A')^\alpha) &= \mathcal{K}_P(S^\alpha)(S^\alpha \otimes_S Q_\alpha(A), S^\alpha \otimes_S Q_\alpha(A)) \\
&\simeq \mathcal{K}_{\mathrm{AJS}}(S^\alpha)(S^\alpha \otimes_S \mathcal{Q}_{A,\alpha}, S^\alpha \otimes_S \mathcal{Q}_{A',\alpha}) \quad \text{by (7.7)} \\
&= \mathcal{K}_{\mathrm{AJS}}^\alpha(\mathcal{F}^\alpha(Q_\alpha(A)^\alpha), \mathcal{F}^\alpha(Q_\alpha(A')^\alpha)).
\end{aligned} \tag{1}$$

Then by (6.12) one has $\mathcal{K}_P^\alpha(M,N) \simeq \mathcal{K}_{\mathrm{AJS}}^\alpha(\mathcal{F}^\alpha(M), \mathcal{F}^\alpha(N))$ $\forall M, N \in \mathcal{K}_P^\alpha$. Thus,

Lemma: *$\forall \alpha \in \Delta^+$, the functor $\mathcal{F}^\alpha : \mathcal{K}_P^\alpha \to \mathcal{K}_{\mathrm{AJS}}^\alpha$ is fully faithful.*

7.9. $\forall M, N \in \mathcal{K}_P$, one has $\mathcal{K}_P^\sharp(M,N)$ graded free over S by (6.10). Then

$$\begin{aligned}
\mathcal{K}_P^\sharp(M,N) &= \cap_{\alpha\in\Delta^+} S^\alpha \otimes_S \mathcal{K}_P^\sharp(M,N) \\
&= \cap_{\alpha\in\Delta^+} (\mathcal{K}_P^\alpha)^\sharp(S^\alpha \otimes_S M, S^\alpha \otimes_S N) \quad \text{by (6.7)} \\
&= \cap_{\alpha\in\Delta^+} (\mathcal{K}_{\mathrm{AJS}}^\alpha)^\sharp(\mathcal{F}^\alpha(S^\alpha \otimes_S M), \mathcal{F}^\alpha(S^\alpha \otimes_S N)) \quad \text{by (7.8)} \\
&= \cap_{\alpha\in\Delta^+} (\mathcal{K}_{\mathrm{AJS}}^\alpha)^\sharp(S^\alpha \otimes_S \mathcal{F}(M), S^\alpha \otimes_S \mathcal{F}(N)) \\
&\geq \mathcal{K}_{\mathrm{AJS}}^\sharp(\mathcal{F}(M), \mathcal{F}(N)) \quad \text{as it is torsion-free over } S.
\end{aligned} \tag{1}$$

Proposition: *The functor $\mathcal{F} : \mathcal{K}_P \to \mathcal{K}_{\mathrm{AJS}}$ is fully faithful.*

Proof: We are to show that $\forall M, N \in \mathcal{K}_P$, $\mathcal{K}_P(M,N) \simeq \mathcal{K}_{\text{AJS}}(\mathcal{F}(M), \mathcal{F}(N))$. For that it is enough to show that $\forall M, N \in \mathcal{K}_P$, $\mathcal{K}_P^\sharp(M,N) \simeq \mathcal{K}_{\text{AJS}}^\sharp(\mathcal{F}(M), \mathcal{F}(N))$. By the CD

$$\begin{array}{ccc} \mathcal{K}_P^\sharp(M,N) & \xrightarrow{\mathcal{F}(M,N)^\sharp} & \mathcal{K}_{\text{AJS}}^\sharp(\mathcal{F}(M), \mathcal{F}(N)) \\ & \searrow \quad \swarrow & \\ & \prod_{A \in \mathcal{A}} S^\emptyset \text{Mod}(M_A^\emptyset, N_A^\emptyset) & \end{array}$$

$\mathcal{F}(M,N)^\sharp$ is injective, and hence bijective by (1).

7.10. Put $\mathcal{Q}_\lambda = \mathcal{F}(Q(A_\lambda^-))$ $\forall \lambda \in \hat{X}$. Let $\mathcal{K}_{\text{AJS},P}$ be the full subcategory of $\mathcal{K}_{\text{AJS}}$ consisting of the direct summands of direct sums of objects of the form $(\Theta_{s_1} \circ \cdots \circ \Theta_{s_r})(\mathcal{Q}_\lambda)(n)$, $\lambda \in \hat{X}, s_1, \ldots, s_r \in \mathcal{S}, n \in \mathbb{Z}$. From (7.2) and (7.9) one obtains

Theorem: *$\mathcal{K}_P \simeq \mathcal{K}_{\text{AJS},P}$. In particular, $\mathcal{K}_{\text{AJS},P}$ admits a right action of $\mathfrak{SB}$.*

8. G_1T-representations

Assume from now on throughout the rest of the paper that $\mathbb{K}$ is an algebraically closed field of characteristic $p > h$ the Coxeter number of Δ [J, II.6.2.9]; for the characteristic requirement see also [RW18, 4.2]. Let G be a simply connected semisimple algebraic group over $\mathbb{K}$ with the root datum $(X, \delta, X^\vee, \Delta^\vee)$, T a maximal torus of G, $\mathfrak{g} = \text{Lie}(G), \mathfrak{h} = \text{Lie}(T)$. In particular, $\hat{X} = X$. Let $\hat{S}$ be the completion of $S = \text{S}_\mathbb{K}(X_\mathbb{K}^\vee)$ at the maximal ideal $(X_\mathbb{K}^\vee)$. For $S' \in \{\hat{S}, \mathbb{K}\}$ let $\mathcal{C}_{S'}$ denote the category of [AJS, 2.3]; $\hat{S}$ is flat over S [AM, 10.14]. Thus $\mathcal{C}_\mathbb{K}$ is equivalent to the category of finite dimensional G_1T-modules, G_1 the Frobenius kernel of G. $\forall \lambda \in \hat{X}$ let $S'(\lambda) \in \mathcal{C}_{S'}$ denote the Verma module of highest weight λ and $P_{S'}(\lambda) \in \mathcal{C}_{S'}$ an indecomposable projective such that $\mathbb{K} \otimes_{S'} P_{S'}(\lambda)$ is the projective cover of the irreducible of highest weight λ; such exsists over $\hat{S}$ by [AJS, 4.19]. Let $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$. $\forall w \in \mathcal{W}$, set $w \bullet_p 0 = pw(\frac{1}{p}\rho - \rho)$. Let $\mathcal{C}_{S',0}$ denote the full subcategory of $\mathcal{C}_{S'}$ consisting of the quotients of $\coprod_{w \in \mathcal{W}} P_{S'}(w \bullet_p 0)^{\oplus n_w}$, $n_w \in \mathbb{N}$. Thus, $\mathcal{C}_{S',0}$ is a direct summand of $\mathcal{C}_{S'}$ [AJS, 6.13]. If b denotes the principal block of $\mathcal{C}_{S'}$ over S' [AJS, 6.9], the category $\mathcal{D}_{S'}(b)$ from [AJS, 6.9, 6.10] is a full subcategory of $\mathcal{C}_{S',0}$. Set $\text{Proj}(\mathcal{C}_{S',0}) = \{P \in \mathcal{C}_{S',0} | P \text{ projective}\}$. The category $\mathcal{C}_{S',0}$ is equipped with the wall-crossing functors Θ_s, $s \in \mathcal{S}$, [AJS, 16.3].

8.1. Let $\mathcal{K}_{\text{AJS}}(\hat{S})$ denote the category $\mathcal{K}_{\text{AJS}}$ over $\hat{S}$ in place of S, denoted $\mathcal{K}_k(0)$ in [F11, Def. 5.2, p. 156], consisting of objects $\mathcal{M} = ((\mathcal{M}(A) | A \in \mathcal{A}), (\mathcal{M}(A, \alpha) | A \in \mathcal{A}, \alpha \in \Delta^+))$ with $\mathcal{M}(A)$ an $\hat{S}^\emptyset$-module and $\mathcal{M}(A, \alpha)$ an $\hat{S}^\alpha$-submodule of $\mathcal{M}(A) \oplus \mathcal{M}(\alpha \uparrow A)$, equipped with wall-crossing functors Θ_s, $s \in \mathcal{S}$; in particular, the morphisms in $\mathcal{K}_{\text{AJS}}(\hat{S})$ is ungraded. Let $\mathcal{K}_{\text{AJS},P}(\hat{S})$ denote the full subcategory of $\mathcal{K}_{\text{AJS}}(\hat{S})$ consisting of the direct summands of direct sums of some $(\Theta_{s_1} \circ \cdots \circ \Theta_{s_r})(\hat{S} \otimes_S \mathcal{Q}_\lambda)(n)$, $\lambda \in X, s_1, \ldots, s_r \in \mathcal{S}, n \in \mathbb{Z}$.

A main theorem of [AJS] may be phrased as

Theorem: *There is an equivalence of categories $\mathcal{V} : \text{Proj}(\mathcal{C}_{\hat{S},0}) \to \mathcal{K}_{AJS,P}(\hat{S})$ compatible with Θ_s and Θ_s $\forall s \in \mathcal{S}$.*

Proof: By [AJS, 9.4, 14.14.6] there is a fully faithful functor $\mathcal{V}_b : \mathcal{D}_{\hat{S}}(b) \to \mathcal{K}_{\mathrm{AJS}}(\hat{S})$ compatible with all Θ_s and $\mathit{\Theta}_s$, $s \in \mathcal{S}$.

Let $A \in \Pi_\lambda^-$, $\lambda \in X$, and let $x, y \in \mathcal{W}$ such that $A = xA_\lambda^+$ and $A_\lambda^+ = yA^+$. If $(s_1, \dots, s_r)$ is a reduced expression of x, $P_{\hat{S}}(x \bullet_p 0)$ is a direct summand of $\Theta_{s_1} \circ \cdots \circ \Theta_{s_r} Z_{\hat{S}}(y \bullet_p 0)$ with $Z_{\hat{S}}(0)$ denoting the deformed G_1T-Verma module over $\hat{S}$ of highest weight $y \bullet_p 0$ [F11, Prop. 8.3]. Then expanding [F11, Th. 8.5] and restricting to $\mathrm{Proj}(\mathcal{C}_{\hat{S},0})$ yields the assertion.

8.2. Define a category $\mathbb{K} \otimes_{\hat{S}} \mathrm{Proj}(\mathcal{C}_{\hat{S},0})$ whose objects are the same as those of $\mathrm{Proj}(\mathcal{C}_{\hat{S},0})$ with

$$\{\mathbb{K} \otimes_{\hat{S}} \mathrm{Proj}(\mathcal{C}_{\hat{S},0})\}(M,N) = \mathbb{K} \otimes_{\hat{S}} \mathrm{Proj}(\mathcal{C}_{\hat{S},0})(M,N) \quad \forall M, N \in \mathrm{Ob}(\mathrm{Proj}(\mathcal{C}_{\hat{S},0})).$$

Lemma: $\mathbb{K} \otimes_{\hat{S}} \mathrm{Proj}(\mathcal{C}_{\hat{S},0}) \simeq \mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$.

Proof: Define a functor $\mathbb{K} \otimes_{\hat{S}} \mathrm{Proj}(\mathcal{C}_{\hat{S},0}) \to \mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$ via $P \mapsto \mathbb{K} \otimes_{\hat{S}} P$, which is well-defined and dense by [AJS, 4.19]. Also,

$$\begin{aligned}\{\mathbb{K} \otimes_{\hat{S}} \mathrm{Proj}(\mathcal{C}_{\hat{S},0})\}(M,N) &= \mathbb{K} \otimes_{\hat{S}} \mathrm{Proj}(\mathcal{C}_{\hat{S},0})(M,N) \quad \text{by definition} \\ &\simeq \mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})(\mathbb{K} \otimes_{\hat{S}} M, \mathbb{K} \otimes_{\hat{S}} N) \quad \text{by [AJS, 3.3].}\end{aligned}$$

8.3. Let $\mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}}$ denote the degraded category of $\mathcal{K}_{\mathrm{AJS},P}$. Define $\hat{S} \otimes_S \mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}}$ as in (8.2); the objects are the same as those of $\mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}}$ with $\{\hat{S} \otimes_S \mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}}\}(M,N) = \hat{S} \otimes_S \mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}}(M,N)$ $\forall M, N \in \mathrm{Ob}(\mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}})$. There is a fully faithful functor $F : \hat{S} \otimes_S \mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}} \to \mathcal{K}_{\mathrm{AJS},P}(\hat{S})$ [AJS, 14.8]. In particular, the indecomposables are preserved under F. The indecomposables of $\mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}}$ are those of $\mathcal{K}_{\mathrm{AJS},P}$ by [GG, Th. 3.1], and hence correspond to $Q(A)$'s, $A \in \mathcal{A}$, under (7.10). On the other hand, the indecomposables of $\mathcal{K}_{\mathrm{AJS},P}(\hat{S}) \simeq \mathrm{Proj}(\mathcal{C}_{S_0,0})$ are also parametrized by $\mathcal{A}$ by (8.1). Thus, F is dense, and we have obtained an equivalence

$$\mathcal{K}_{\mathrm{AJS},P}(\hat{S}) \simeq \hat{S} \otimes_S \mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}}. \tag{1}$$

Define now categories $\mathbb{K} \otimes_{\hat{S}} \mathcal{K}_{\mathrm{AJS},P}(\hat{S})$ and $\mathbb{K} \otimes_S \mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}}$ as in (8.2) likewise. The objects of those may now be identified by (1). Then, $\forall M, N \in \mathrm{Ob}(\mathcal{K}_{\mathrm{AJS},P}(\hat{S}))$,

$$\begin{aligned}\{\mathbb{K} \otimes_{\hat{S}} \mathcal{K}_{\mathrm{AJS},P}(\hat{S})\}(M,N) &= \mathbb{K} \otimes_{\hat{S}} \mathcal{K}_{\mathrm{AJS},P}(\hat{S})(M,N) \quad \text{by definition} \\ &\simeq \mathbb{K} \otimes_{\hat{S}} (\hat{S} \otimes_S \mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}})(M,N) \quad \text{by (1)} \\ &\simeq \mathbb{K} \otimes_S \mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}}(M,N).\end{aligned}$$

Thus, we have obtained another equivalence

Lemma: $\mathbb{K} \otimes_{\hat{S}} \mathcal{K}_{\mathrm{AJS},P}(\hat{S}) \simeq \mathbb{K} \otimes_S \mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}}$.

8.4. Let $\mathcal{K}_P^{\mathrm{degr}}$ denote the degradation of $\mathcal{K}_P$. One has obtained

$$\begin{aligned}
(1)\qquad \mathrm{Proj}(\mathcal{C}_{\mathbb{K},0}) &\simeq \mathbb{K}\otimes_{\hat{S}} \mathrm{Proj}(\mathcal{C}_{\hat{S},0}) \quad \text{by } (8.2)\\
&\simeq \mathbb{K}\otimes_{\hat{S}} \mathcal{K}_{\mathrm{AJS},P}(\hat{S}) \quad \text{by } (8.1)\\
&\simeq \mathbb{K}\otimes_{S} \mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}} \quad \text{by } (8.3)\\
&\simeq \mathbb{K}\otimes_{S} \mathcal{K}_{P}^{\mathrm{degr}} \quad \text{by } (7.10).
\end{aligned}$$

As the action of $\mathfrak{S}\mathfrak{B}$ on $\mathcal{K}_P$ is S-linear, it induces an action on $\mathbb{K}\otimes_S \mathcal{K}_P^{\mathrm{degr}}$, and hence on $\mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$, which we write as $(M,B)\mapsto M*B$. Under this action each $B(s)$, $s\in\mathcal{S}$, acts as the wall-crossing translation functor Θ_s on $\mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$ by (7.2) and (8.1). We now start showing that the action extends onto the whole of $\mathcal{C}_{\mathbb{K},0}$.

Recall first auto-equivalence T_γ on $\mathcal{K}_P$ from (6.6), and define an auto-equivalence $\mathrm{T}_{\mathrm{AJS},\gamma}$ on $\mathcal{K}_{\mathrm{AJS},P}$ via $\mathrm{T}_{\mathrm{AJS},\gamma}(\mathcal{M})(A)=\mathcal{M}(A+\gamma)$ and $\mathrm{T}_{\mathrm{AJS},\gamma}(\mathcal{M})(A,\alpha)=\mathcal{M}(A+\gamma,\alpha)$ $\forall A\in\mathcal{A}$ $\forall\alpha\in\Delta^+$. As T_γ (resp. $\mathrm{T}_{\mathrm{AJS},\gamma}$) is S-linear, $\mathbb{K}\otimes_S\mathrm{T}_\gamma$ (resp. $\mathbb{K}\otimes_S\mathrm{T}_{\mathrm{AJS},\gamma}$) defines an auto-equivalence on $\mathbb{K}\otimes_S\mathcal{K}_P$ (resp. $\mathbb{K}\otimes_S\mathcal{K}_{\mathrm{AJS},P}$) equipping it with a structure of $\mathbb{Z}\Delta$-category [AJS, E.1]. Then the equivalences $\mathbb{K}\otimes_S\mathcal{K}_P^{\mathrm{degr}}\simeq\mathbb{K}\otimes_S\mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}}\simeq\mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$ from (1) are those of $\mathbb{Z}\Delta$-categories.

Recall also that $\mathcal{C}_{\mathbb{K},0}$ is equipped with a structure of $\mathbb{Z}\Delta$-category such that $M\mapsto M\otimes p\gamma$, $\gamma\in\mathbb{Z}\Delta$, and so is $\mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$. Fix a projective $\mathbb{Z}\Delta$-generator P of $\mathcal{C}_{\mathbb{K},0}$ and set $E=\mathcal{C}_{\mathbb{K},0}^\sharp(P,P)=\coprod_{\gamma\in\mathbb{Z}\Delta}\mathcal{C}_{\mathbb{K},0}(P,P\otimes_{\mathbb{K}}p\gamma)$, which is a $\mathbb{Z}\Delta$-graded algebra. Let $\mathrm{mod}_{\mathbb{Z}\Delta}E$ denote the category of $\mathbb{Z}\Delta$-graded right E-modules of finite type. By [AJS, E.4] there is an equivalences of categories

$$(2)\qquad \mathcal{C}_{\mathbb{K},0}\to\mathrm{mod}_{\mathbb{Z}\Delta}E \quad\text{via}\quad M\mapsto\mathcal{C}_{\mathbb{K},0}^\sharp(P,M)=\coprod_{\gamma\mathbb{Z}\Delta}\mathcal{C}_{\mathbb{K},0}(P,M\otimes_{\mathbb{K}}p\gamma),$$

where the structure of graded right E-module on $\mathcal{C}_{\mathbb{K},0}^\sharp(P,M)$ is given by setting $f\varphi=f\circ\varphi$, $f\in\mathcal{C}_{\mathbb{K},0}^\sharp(P,M),\varphi\in E$. Let $\mathrm{Proj}_{\mathbb{Z}\Delta}(E)$ denote the full subcategory of $\mathrm{mod}_{\mathbb{Z}\Delta}E$ consisting of its projectives.

Lemma: *$\forall Q\in\mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$, $\forall B\in\mathfrak{S}\mathfrak{B}$, $\forall\gamma\in\mathbb{Z}\Delta$, $(Q*B)\otimes_{\mathbb{K}}p\gamma\simeq(Q\otimes_{\mathbb{K}}p\gamma)*B$.*

Proof: By the equivalences of $\mathbb{Z}\Delta$-categories $\mathbb{K}\otimes_S\mathcal{K}_P^{\mathrm{degr}}\simeq\mathbb{K}\otimes_S\mathcal{K}_{\mathrm{AJS},P}^{\mathrm{degr}}\simeq\mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$ it suffices to check that $\mathrm{T}_\gamma(M*B)\simeq\mathrm{T}_\gamma(M)*B$ $\forall M\in\mathcal{K}_P$, $\forall B\in\mathfrak{S}\mathfrak{B}$, which holds by (3.4.2).

8.5. Now that the action of $\mathfrak{S}\mathfrak{B}$ on $\mathrm{Proj}(\mathcal{C}_{\mathbb{K},0})$ is compatible with its structure of $\mathbb{Z}\Delta$-category, there is induced an action of $\mathfrak{S}\mathfrak{B}$ on $\mathrm{Proj}_{\mathbb{Z}\Delta}(E)$ under (8.4.2), which we denote by $(M,B)\mapsto M*B$. In particular, $\forall B\in\mathfrak{S}\mathfrak{B}$, let $E(B)=\mathcal{C}_{\mathbb{K},0}^\sharp(P,P*B)$. Recall that B is a left graded free R-module by (I.2.2.1) and by graded Quillen-Suslin [Lam, Cor. II.5.4.7, p. 79]. Then

$$(1)\qquad \mathcal{C}_{\mathbb{K},0}^\sharp(P,P*B)\simeq E*B\in\mathrm{Proj}_{\mathbb{Z}\Delta}(E)\quad\text{via}\quad \varphi(?)\otimes_R b\leftarrow\!\shortmid\varphi*b.$$

Lemma: *$\forall Q\in\mathrm{Proj}_{\mathbb{Z}\Delta}(E)$, $\forall B\in\mathfrak{S}\mathfrak{B}$, $Q\otimes_E E(B)\simeq Q*B$.*

Proof: $\forall\nu\in\mathbb{Z}\Delta$, let Q^ν denote the ν-th homogeneous part of Q. $\forall x\in Q^\nu$, let $\varphi_x\in\mathrm{mod}_{\mathbb{Z}\Delta}E(E,Q(\nu))$ via $1\mapsto x$. Under the $\mathbb{Z}\Delta$-graded equivalence $\mathrm{Proj}_{\mathbb{Z}\Delta}(E)\simeq\mathbb{K}\otimes_S\mathcal{K}_P^{\mathrm{degr}}$

(8.4.1) one obtains

$$\begin{aligned}\varphi_x * B &\in \mathrm{Proj}_{\mathbb{Z}\Delta}(E)(E*B, Q(\nu)*B)\\ &\simeq \mathrm{Proj}_{\mathbb{Z}\Delta}(E)(E*B, (Q*B)(\nu)) \quad \text{by (8.4)},\end{aligned}$$

which in turn induces a morphism of $\mathrm{Proj}_{\mathbb{Z}\Delta}(E)$

$$\begin{array}{ccccc} Q\otimes_E E(B) & \dashrightarrow & Q*B & & (\varphi_x * B)(m). \\ \wr\Vert & \nearrow & & \nearrow & \\ Q\otimes_E (E*B) & & x\otimes m & \mapsto & \end{array}$$

If $x\otimes m \in Q^\nu \otimes E(B)^\mu$, $(\varphi_x * B)(m) \in \{(Q*B)(\nu)\}^\mu = (Q*B)^{\nu+\mu}$. This is an isomorphism if $Q = E$, and hence in general by the 5-lemma.

8.6. $\forall M \in \mathrm{mod}_{\mathbb{Z}\Delta} E$, $\forall B \in \mathfrak{S}\mathfrak{B}$, set $M*B = M\otimes_E E(B)$. $\forall B' \in \mathfrak{S}\mathfrak{B}$,

$$\begin{aligned}E(B)\otimes_E E(B') &\simeq E(B)*B' \simeq (E*B)*B' \quad \text{by (8.5)}\\ &= E*(B*B') \quad \text{as } \mathrm{Proj}_{\mathbb{Z}\Delta}(E) \text{ admits a right } \mathfrak{S}\mathfrak{B}\text{-action}\\ &\simeq E(B*B'),\end{aligned}$$

and hence

$$\begin{aligned}(M*B)*B' &= \{M\otimes_E E(B)\}\otimes_E E(B') \simeq M\otimes_E (E(B)\otimes_E E(B'))\\ &\simeq M\otimes_E E(B*B') = M*(B*B').\end{aligned}$$

Thus, $\mathrm{mod}_{\mathbb{Z}\Delta} E$ comes equipped with a right action by $\mathfrak{S}\mathfrak{B}$, and so therefore does $\mathcal{C}_{\mathbb{K},0}$ under (8.4.2). One has obtained

Theorem: *There is a right action of $\mathfrak{S}\mathfrak{B}$ on the whole of $\mathcal{C}_{\mathbb{K},0}$ such that each $B(s)$, $s \in \mathcal{S}$, acts by the wall-crossing translation functor Θ_s.*

Proof: To see the last assertion, let $M \in \mathcal{C}_{\mathbb{K},0}$ and let $P' \to P \to M \to 0$ be a projective resolution. As both $*B(s)$ and Θ_s are exact, one has a CD of exact sequences

$$\begin{array}{ccccccc} \Theta_s(P') & \longrightarrow & \Theta_s(P) & \longrightarrow & \Theta_s(M) & \longrightarrow & 0 \\ \sim\downarrow & & \downarrow\sim & & & & \\ P'*B(s) & \longrightarrow & P*B(s) & \longrightarrow & M*B(s) & \longrightarrow & 0, \end{array}$$

and hence $\Theta_s(M) \simeq M*B(s)$.

8.7 **Characters:** Each $P \in \mathrm{Proj}(\mathcal{C}_{S,0})$ admits a Verma flag [AJS, 2.16]. Let $(P : Z_S(w\bullet_p 0))$, $w \in \mathcal{W}$, denote the multiplicity of $Z_S(w\bullet_p 0)$ in the flag, and likewise $(\hat{S}\otimes_S P : \hat{S}\otimes_S Z_S(w\bullet_p 0))$.

Lemma: *$\forall P \in \mathrm{Proj}(\mathcal{C}_{S,0})$, $\forall M \in \mathcal{K}_P$ with $\mathcal{V}_S(P) \simeq \mathcal{F}(M)$ in $\mathcal{K}_{\mathrm{AJS}}$,*

$$(P : Z_S(w\bullet_p 0)) = \mathrm{rk}\,_S(M_{\{wA^+\}}).$$

Proof: Let $\mathcal{V}_{\hat{S}}(\hat{S}\otimes_S P) = ((\mathcal{M}(A)|A\in\mathcal{A}), (\mathcal{M}(A,\alpha)|A\in\mathcal{A}, \alpha\in\Delta^+))$. Then

$$\begin{aligned}(P : Z_S(w\bullet_p 0)) &= (\hat{S}\otimes_S P : \hat{S}\otimes_S Z_S(w\bullet_p 0)) \\ &= \mathrm{rk}_{\hat{S}^\emptyset}\mathcal{M}(wA^+) \quad \text{by [AJS, 14.10]} \\ &= \mathrm{rk}_{\hat{S}^\emptyset}((\hat{S}\otimes_S M)^\emptyset_{wA^+}) = \mathrm{rk}_{S^\emptyset}(M^\emptyset_{wA^+}) \\ &= \mathrm{rk}_S(M_{\{wA^+\}}).\end{aligned}$$

8.8. Now, indecomposable $P_S(w\bullet_p 0)$, $w\in\mathcal{W}$, is characterized in $\mathrm{Proj}(\mathcal{C}_{S,0})$ by the properties that $(P_S(w\bullet_p 0) : Z_S(w\bullet_p 0)) = 1$ and that $(P_S(w\bullet_p 0) : Z_S(x\bullet_p 0)) = 0$ unless $xA^+\geq wA^+$, and hence

Proposition: $\forall w\in\mathcal{W}$, $\mathcal{V}_S(P_S(w\bullet_p 0))\simeq\mathcal{F}(Q(wA^+))$.

8.9. From (8.7) and (8.8) follows

Corollary: $\forall x,y\in\mathcal{W}$, $(P_{\mathbb{K}}(x\bullet_p 0) : Z_{\mathbb{K}}(y\bullet_p 0)) = \mathrm{rk}_S(Q(xA^+)_{\{yA^+\}})$.

8.10. Let $\lambda\in X$, $w_\lambda, w'_\lambda, w\in\mathcal{W}$ such that $A^-_\lambda w'_\lambda = A^+_\lambda = w_\lambda A^-_\lambda$ and $A^+_\lambda w\subseteq\Pi_\lambda$. Soergel's conjecture on $B(w'_\lambda w)$ states that $\mathrm{ch}[B(w'_\lambda w)] = \underline{H}_{w'_\lambda w}$. This holds for large p by [EW14], transferring to the Elias-Williamson diagrammatic category from $\mathfrak{SB}$ by an equivalence [Ab19a, Th. 5.9], but fails in general [W]; $\mathrm{ch}[B(x)], x\in\mathcal{W}$, can be computed in terms of the ranks of the local intersection forms [JW17] as in (5.13). The computations may be done in principle, using only the diagrammatic relations of [EW16], independent of the ambient spaces of the realizations of $(\mathcal{W},\mathcal{S})$.

Theorem: *If Soergel's conjecture holds on $B(w'_\lambda w)$,*

$$S(A^-_\lambda)*B(w'_\lambda w)\simeq Q(A^-_\lambda w)(\ell(w_0)-\ell(w)).$$

Proof: We know from (5.5) that $S(A^-_\lambda)*B(w'_\lambda w)\in\tilde{\mathcal{K}}_P$, and hence belong to $\mathcal{K}_P$ by (6.6). One has

$$\begin{aligned}(1)\qquad \mathrm{ch}[S(A^-_\lambda)*B(w'_\lambda w)] &= \mathrm{ch}[S(A^-_\lambda)]\underline{H}_{w_\lambda w} \quad \text{by (5.1) under the hypothesis} \\ &= v^{-\ell(A^-_\lambda)}A^-_\lambda\underline{H}_{w'_\lambda w} \\ &= v^{-\ell(A^-_\lambda)}\underline{P}_{A^+_\lambda w} \quad \text{by (5.6),}\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{ch}[S(A^-_\lambda)(\ell(A^-_\lambda))*B(w'_\lambda w)] &= \underline{P}_{A^+_\lambda w} \\ &= \overline{\mathrm{ch}[S(A^-_\lambda)(\ell(A^-_\lambda))*B(w'_\lambda w)]} \quad \text{by [S97, Th. 4.3].}\end{aligned}$$

On the other hand, as $w_\lambda A_\lambda^+ w = A_\lambda^- w$ is the minimal alcove appearing in $\underline{P}_{A_\lambda^+ w}$, one has from [S97, Lem. 4.21]

$$\underline{P}_{A_\lambda^+ w} \in v^{\ell(w_0)}(A_\lambda^- w + \sum_{\substack{B\in\mathcal{A}\\ B>A_\lambda^- w}} v^{-1}\mathbb{Z}[v^{-1}]B). \tag{2}$$

Then by (6.10)

$$\operatorname{grk}(\mathcal{K}^\sharp(S(A_\lambda^-)(\ell(A_\lambda^-)) * B(w'_\lambda w), S(A_\lambda^-)(\ell(A_\lambda^-)) * B(w'_\lambda w)) \in v^{-2\ell(w_0)}v^{2\ell(w_0)}(1+v^{-2}\mathbb{Z}[v^{-1}]),$$

and hence $\mathcal{K}(S(A_\lambda^-)(\ell(A_\lambda^-)) * B(w'_\lambda w), S(A_\lambda^-)(\ell(A_\lambda^-)) * B(w'_\lambda w)) = \mathbb{K}$ and $S(A_\lambda^-)(\ell(A_\lambda^-)) * B(w'_\lambda w)$ is indecomposable.

Meanwhile, $\operatorname{ch}[Q(A_\lambda^- w)] \in v^{-\ell(A_\lambda^- w)}A_\lambda^- w + \sum_{B>A_\lambda^- w}\mathbb{Z}[v,v^{-1}]B$ by (4.5). It follows from (2) and (5.3) that $S(A_\lambda^-)(\ell(A_\lambda^-)) * B(w'_\lambda w)(-\ell(w_0)) \simeq Q(A_\lambda^- w)(\ell(A_\lambda^- w))$, and hence

$$S(A_\lambda^-) * B(w'_\lambda w) \simeq Q(A_\lambda^- w)(\ell(w_0)+\ell(A_\lambda^- w)-\ell(A_\lambda^-)) = Q(A_\lambda^- w)(\ell(w_0)-\ell(w)).$$

8.11. Let $w \in \mathcal{W}$ with $A^+w \subseteq \Pi$. Let $p_{A,B}$ denote the periodic KL-polynomials from [S97, Rmk. 4.4].

Corollary: *If Soergel's conjecture holds on $B(w_0w)$, $\operatorname{rk}_S(Q(A^-w)_{\{A\}}) = p_{A,A^+w}(1)$ $\forall A \in \mathcal{A}$, and hence $\forall x \in \mathcal{W}$,*

$$(P_{\mathbb{K}}(w_0w \bullet_p 0) : Z_{\mathbb{K}}(x \bullet_p 0)) = p_{A^+x,A^+w}(1).$$

Proof: Put $l = \ell(w_0)$. One has

$$\begin{aligned} v^{l-\ell(w)}\sum_{B\in\mathcal{A}} v^{\ell(B)}\operatorname{grk}(Q(A^-w)_{\{B\}})B &= \operatorname{ch}[Q(A^-w)(l-\ell(w))] \\ &= \operatorname{ch}[S(A^-) * B(w_0w)] \quad \text{by (8.10)} \\ &= v^{-\ell(A^-)}\underline{P}_{A^+w} \quad \text{by (8.10.1)} \\ &= v^{-\ell(A^-)}\sum_B p_{B,A^+w}B \quad \text{by definition [S97, Rmk. 4.4].} \end{aligned}$$

Thus, $\operatorname{rk}_S(Q(A^-w)_{\{B\}}) = p_{B,A^+w}(1)$. Then

$$\begin{aligned} (P_{\mathbb{K}}(w_0w \bullet_p 0) : Z_{\mathbb{K}}(x \bullet_p 0)) &= \operatorname{rk}_S(Q(w_0wA^+)_{\{xA^+\}}) \quad \text{by (8.9)} \\ &= \operatorname{rk}_S(Q(A^-w)_{\{A^+x\}}) = p_{A^+x,A^+w}(1). \end{aligned}$$

8.12. One has

$$\begin{aligned} p_{A^+x,A^+w}(1) &= Q_{A^+x,A^+w}(1) \quad \text{with } Q \text{ as in [L80] by [S97, Rmk. 4.4]} \\ &= (P_{\mathbb{K}}(w_0w \bullet_p 0) : Z_{\mathbb{K}}(x \bullet_p 0)) \quad \text{cf. [K88, 5.1.1],} \end{aligned}$$

which is consistent with (8.11). Also,

$$\begin{aligned}[Z_{\mathbb{K}}(x \bullet_p 0) : L_{\mathbb{K}}(w_0 w \bullet_p 0)] &= p_{w_0 x A^+, wA^+}(1) \quad \text{after [F10, 3.4]} \\ &= Q_{w_0 x A^+, wA^+}(1) \\ &= Q_{xA^+, wA^+}(1) \quad \text{by [L80, Cor. 8.4]} \\ &= p_{xA^+, wA^+}(1) = p_{A^+ x, A^+ w}(1) = (P_{\mathbb{K}}(w_0 w \bullet_p 0) : Z_{\mathbb{K}}(x \bullet_p 0)),\end{aligned}$$

which is again consistent with (8.11).

References

[Ab19a] Abe, N., *On Soergel bimodules*, arXiv:1901.02336

[Ab19b] Abe, N., *A Hecke action on G_1T-modules*, arXiv:1904.11350

[Ab20] Abe, N., *On singular Soergel bimodules*, arXiv:2004.09014

[AMRW] Achar, P.N., Makisumi, S., Riche, S. and Williamson, G., *Koszul duality for Kac-Moody groups and characters of tilting modules*, JAMS **32** No. 1 (2019), 261-310

[AJS] Andersen, H.H., Jantzen, J.C. and Soergel, W., Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p : independence of p, Astérisque **220**, 1994 SMF

[AF] Anderson, F. and Fuller, K., Rings and Categories of Modules, 2nd. ed., GTM **13**, 1992 Springer

[AM] Atiyah, M.F. and MacDonald, I.G. , Introduction to Commutative Algebra, Addison-Wesley (1994)

[BB] Björner, A. and Brenti, F., Combinatorics of Coxeter Groups, GTM **231**, 2005 Springer

[BCA] Bourbaki, N., Algèbre commutative, Paris 1961 (ch. I/II), 1962 (ch. III/IV), 1964 (ch. V/VI), 1965 (ch. VII) (Hermann)

[BH] Bruns, W. and Herzog, J. , Cohen-Macaulay rings, Camb. studies in adv. math. **39** 1998

[CR] Curtis, C.W. and Reiner, I., Methods of Representation Theory I, Wiley Interscience, NewYork, 1981

[Dem] Demazure, M., *Invariants symétriques entiers des groupes de Weyl et torsion*, Invent. Math. **21** (1973), 287-301

[EW14] Elias, B. and Williamson, G, *The Hodge theory of Soergel bimodules*, Ann. of Math. **180** (2014), 1089-1136

[EW16] Elias, B. and Williamson, G., *Soergel calculus*, Rep. Th., **20** (2016), 295-374

[F08a] P. Fiebig, *The combinatorics of Coxeter categories*, Trans. Amer. Math. Soc. **360** (2008), no. 8, 4211-4233

[F08b] Fiebig, P., *Sheaves on moment graphs and a localization of Verma flags*, Adv. Math., **217** (2008), 683-712

[F10] Fiebig, P., *Lusztig's conjecture as a moment graph problem*, Bull. Lond. Math. Soc., **42** (2010), 957-972

[F11] Fiebig, P., *Sheaves on affine Schubert varieties, modular representations, and Lusztig's conjecture*, J. Amer. Math. Soc., **24** (2011), 133-181

[FL15] Fiebig, P. and Lanini, M., *Sheaves on the alcoves I: Projectivity and wall crossing functors*, arXiv:1504.01699

[GG] Gordon, R. and Green, E. L., *Graded Artin algebras*, J. Algebra **76** (1982), 111-137

[GKM] Goresky, M., Kottwitz, R., and MacPherson, R., *Equivariant cohomology, Koszul duality, and the localization theorem*, Invt. Math. **131** (1998), no. 1, 25-83.

[HLA] Humphreys, J.E., Introduction to Lie Algebras and Representation Theory. Springer-Verlag, New York (1972)

[HRC] Humphreys, J., Reflection Groups and Coxeter Groups (Cambridge Studies in Advanced Math. 29), Cambridge etc. 1990 (Cambridge Univ.)

[J] Jantzen, J. C., Representations of Algebraic Groups, Math. Surveys and Monographs **107**, 2003 AMS

[JGr] Jantzen, J. C., *Moment graphs and representations*, Geometric methods in representation theory. I, 249-341, Sémin. Congr., 24-I, Soc. Math. France, Paris, 2012.

[JW17] Jensen, L. T. and Williamson, G.,*The p-canonical basis for Hecke algebras*, in Categorification and higher representation theory, Contemp. Math., **683**, Amer. Math. Soc., Providence, RI, 2017, 333-361.

[JMW] Juteau, D., Mautner, C. and Williamson, G., *Parity sheaves*, JAMS **27** (2014), 1169-1212

[K88] Kaneda, M., *The Kazhdan-Lusztig polynomials arising in the modular representation theory of reductive algebraic groups*, RIMS 講究録 **670** (1988), 129-162.

[K19] Kaneda, M., *Representation theory of the general linear groups after Riche and Williamson*, OCAMI Preprint Series 19-19

[Lam] Lam, T. Y., Serre's Problem on Projective Modules, Springer Monographs in Math., 2000 Springer

[Lib] Libedinsky, N., *Sur la catégorie des bimodules de Soergel*, J. Algebra **320** (2008), no. 7, 2675-2694

[L] Lusztig, G., *Some problems in the representation theory of finite Chevalley groups*, In: The Santa Cruz Conference on Finite Groups, Univ. California, Santa Cruz, CA, 1979, Proc. Sympos. Pure Math., 37, Amer. Math. Soc., Providence, RI, 1980, pp. 313-317

[L80] Lusztig, G., *Hecke algebras and Jantzen's generic decomposition patterns*, Adv. Math. **37** (1980), no. 2, 121-164

[L85] Lusztig, G., *Cells in affine Weyl groups*, in Algebraic groups and related topics, Adv. Studies in Pure Math. **6**, pp. 255-287, North-Holland 1985

[中岡] 中岡宏行, 圏論の技法, 2015 日本評論社

[NvO] Năstăsescu, C. and Van Oystaeyen, F., Methods of Graded Rings, LNM **1836**, 2004 Springer

[RW18] Riche, S. and Williamson, G., Tilting Modules and the p-Canonical Basis, Astérisque **397**, 2018 SMF

[RW19] Riche, S. and Williamson, G., *A simple character formula*, arXiv:1904.08085v1

[Rot] Rotman, J.J., An Introduction to Homological Algebra, 2nd ed., UTX, New York etc. 2009 (Springer)

[Sob] Sobaje, P., *On character formulas for simple and tilting modules*, Adv. Math. **369** (2020), 1- 8

[S92] Soergel, W., *The combinatorics of Harish-Chandra bimodules*, J. Reine Angew. Math. **429** (1992), 49- 74

[S97] Soergel, W., *Kazhdan-Lusztig polynomials and a combinatoric for tilting modules*, Rep. Th. **1**, (1997), 83-114

[S07] Soergel, W., *Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen*, JIM. Jussieu **6** (2007), p. 501-525

[Sp] Springer, T. A., Linear Algebraic Groups (Progress in Math. 9), 2nd ed., Boston etc. 1998 (Birkhauser)

[W] Williamson, G., *Schubert calculus and torsion explosion*, JAMS **30** (2017), 1023-1046