Parallel Kähler submanifolds and R-spaces

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This article is dedicated to the memory of Professor Tadashi Nagano.

Abstract. *Parallel Kähler submanifolds* here mean complex submanifolds immersed in complex projective spaces with parallel second fundamental form. Such submanifolds were classified by Hisao Nakagawa and Ryoichi Takagi (1976), Masaru Takeuchi (1978, 1984) by two different methods of unitary representation theory and Jordan triple systems. In this article we briefly survey such related submanifold theory and give the third proof for their classification theorem, based on the differential geometric characterization of R-spaces due to Carlos Olmos and Cristián U. Sánchez (1991).

1. Introduction

Let M^m be a complex m-dimensional complex submanifold immersed in a complex n-dimensional complex projective space $\mathbb{C}P^n$. Here $\mathbb{C}P^n$ is endowed with the Fubini-Study metric of constant holomorphic sectional curvature 4. Then M becomes intrinsically a Kähler manifold with respect to the metric g_M and complex structure J induced from $\mathbb{C}P^n$, thus M is also called a *Kähler submanifold* of $\mathbb{C}P^n$. When M is not contained in any proper totally geodesic complex submanifold $\mathbb{C}P^k$ $(0 \leq k \leq n-1)$ of $\mathbb{C}P^n$, we say that M is *fully* immersed in $\mathbb{C}P^n$. We denote by α^M the second fundamental form of M in $\mathbb{C}P^n$. The covariant derivative $\nabla^*\alpha^M$ of α^M in terms of the normal connection $\nabla^\perp$ and the Levi-Civita connection ∇^M is defined as

$$(\nabla^*\alpha^M)_X(Y,Z) := \nabla^\perp_X(\alpha^M(Y,Z)) - \alpha^M(\nabla^M_X Y, Z) - \alpha^M(Y, \nabla^M_X Z) \tag{1.1}$$

for any smooth vector fields X, Y, Z on M. If α^M satisfies the equation

$$\nabla^*\alpha^M = 0, \tag{1.2}$$

then we say that the submanifold M has *parallel second fundamental form.* A complex submanifold of $\mathbb{C}P^n$ with parallel second fundamental form is called simply a *parallel Kähler submanifold.* From the Gauss equation it is well-known that any

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Kähler submanifold M of $\mathbb{C}P^n$ with parallel second fundamental form is a locally Hermitian symmetric space. In 1976-77 Nakagawa and Takagi [**11**] and Takeuchi [**19**] have classified parallel Kähler submanifolds of $\mathbb{C}P^n$ as follows:

Theorem 1.1 (Nakagawa and Takagi [**11**], Takeuchi [**19**]). *A complex submanifold M fully immersed in $\mathbf{C}P^n$ has parallel second fundamental form if and only if it is congruent to the open part of one of the following seven complex submanifolds:*

$$M_1 = \mathbb{C}P^m(4) \subset \mathbb{C}P^m(4) \text{ totally geodesic}$$
$$M_2 = \mathbb{C}P^m(2) \subset \mathbb{C}P^{n+\frac{1}{2}n(n+1)}(4)$$
$$M_3 = \mathbb{C}P^{n-s}(4) \times \mathbb{C}P^s(4) \longrightarrow \mathbb{C}P^{n+s(n-s)}(4)$$
$$M_4 = Q_n(\mathbb{C}) \longrightarrow \mathbb{C}P^{n+1}(4)\ (n \geq 3)$$
$$M_5 = SU(s+2)/S(U(2)\times U(s)) \longrightarrow \mathbb{C}P^{2s+\frac{1}{2}s(s+1)}(4)\ (s \geq 3)$$
$$M_6 = SO(10)/U(5) \longrightarrow \mathbb{C}P^{15}(4)$$
$$M_7 = E_6/((U(1)\times Spin(10))/\mathbb{Z}_4) \longrightarrow \mathbb{C}P^{26}(4)$$

Note that they are Hermitian symmetric spaces of compact type and rank at most 2. Several beautiful curvature characterizations related to those seven Kähler submanifolds are known as [**16**] etc., inspired by Ogiue's conjectures ([**12**]).

The first proof of Theorem 1.1 was given by Nakagawa-Takagi ([**11**]) and Takeuchi ([**19**]) by the method of the unitary representation theory for compact Lie groups. For a Kähler immersion $\varphi : M \to \mathbb{C}P^n$, the degree $d(\varphi)$ of φ (cf. [**11**]) is defined in terms of the higher order holomorphic osculating spaces along φ, and by definition $d(\varphi) = 1$ or 2 if and only if φ has parallel second fundamental form. By Calabi's rigidity and extension theorem ([**2**]), we may assume that M is a compact Hermitian symmetric space and φ is full, and then φ is an equivariant holomorphic map with respect to a unitary representation ρ of a maximal connected isometry group G on M into $SU(n+1)$. Moreover, if we decompose M into a direct product of irreducible compact Hermitian symmetric spaces M_i $(1 \leq i \leq \ell)$, then there is the p_i-th standard embedding $\varphi_i : M_i \to \mathbb{C}P^{n_i}$ for each i such that φ is expressed as a tensor product map of those equivariant holomorphic maps φ_i. If we denote by r_i the rank of each M_i, then they showed the degree formula $d(\varphi) = \sum_{i=1}^{\ell} p_i r_i$ ([**11**], [**19**]). By determining all Kähler immersions φ with $d(\varphi) = 1$ or 2, they obtained Theorem 1.1.

The second proof of Theorem 1.1 was given by Takeuchi ([**20**]) in 1984 by the algebraic method of Jordan triple systems. It is based on the correspondence between positive definite Hermitian Jordan triple systems and irreducible symmetric bounded domains, which is due to M. Koecher ([**9**]), see also I. Satake ([**17**]). A crucial point of [**20**] is to construct a positive definite Hermitian Jordan triple system with a Jordan product defined from the second fundamental form of a given parallel Kähler submanifold of $\mathbb{C}P^n$. The corresponding symmetric bounded domain is an irreducible Hermitian symmetric space G^*/K of noncompact type. Let $\mathfrak{g}^* = \mathfrak{k} + \mathfrak{p}^*$ be the Cartan decomposition of $\mathfrak{g}^*$ and we have an identification $\mathfrak{p}^{1,0} \cong \mathbb{C}^{n+1}$, where $\mathfrak{p}^{\mathbb{C}} = \mathfrak{p}^{1,0} + \mathfrak{p}^{0,1}$ is the eigenspace decomposition of the complexification $\mathfrak{p}^{\mathbb{C}} = (\mathfrak{p}^*)^{\mathbb{C}}$ with respect to the complex structure tensor of the Hermitian symmetric space G^*/K. We take the highest root vector $E^+(\neq 0) \in \mathfrak{p}^{1,0}$ relative to the maximal abelian subalgebra of $\mathfrak{k}$. Through the Hopf fibration $\pi : S^{2n+1}(1) \to \mathbb{C}P^n$, he showed that the original parallel Kähler submaifold M^m is congruent to $\pi(\mathrm{Ad}(K)E^+) \subset \mathbb{C}P^n$, which is the projection of an *R-space* $\mathrm{Ad}(K)E^+$. In general an *R-space* is by definition a compact homogeneous space obtained as an orbit of the isotropy representation of a Riemannian symmetric space, i.e. an *s-representation* (see Section 2). In this way he obtained

THEOREM 1.2 (Takeuchi [**20**]). *Any parallel Kähler submanifold of $\mathbb{C}P^n$ can be obtained by the projection of an R-space obtained as an orbit of the isotropy representation of an irreducible Hermitian symmetric space.*

In 1991 Olmos and Sánchez ([**13**]) gave a differential geometric characterization of R-spaces standardly embedded in Euclidean spaces. They showed that a submanifold N immersed in a Euclidean space $\mathbb{R}^{m+k}$ is a standardly embedded R-space if and only if there exists a *canonical connection* ∇^c (see Section 2 for the definition) on the tangent vector bundle TN such that the second fundamental form α^N of N satisfies the equation

$$(1.3) \qquad (\nabla^c \alpha^N)_X(Y, Z) := \nabla^\perp_X(\alpha^N(Y, Z)) - \alpha^N(\nabla^c_X Y, Z) - \alpha^N(Y, \nabla^c_X Z) = 0$$

for each smooth vector field X, Y, Z on M. It successfully generalizes the classification theorem due to Ferus ([**7**],[**8**], see also [**4**]) for parallel submanifolds in Euclidean spaces by means of *symmetric R-spaces* in the case when ∇^c is the Levi-Civita connection ∇^N of N. Differential geometry of symmetric R-spaces has a long and fruitful history and it was first studied by a pioneering work of Tadashi Nagano [**10**] in 1965 from the viewpoint of transformation groups.

In this article we shall give the third proof of Theorems 1.1 and 1.2 based on the differential geometric characterization of R-spaces due to Olmos-Sánchez. The main results of this article (Theorem 3.2) are the explicit construction of a canonical connection (different from the Levi-Civita connection!) on the inverse image $\hat{M}$ of any complex submanifold M of $\mathbb{C}P^n$ under the Hopf fibration and that $\hat{M}$ satisfies the Olmos-Sánchez's condition (1.3) with respect to this canonical connection if and only if M is a parallel Kähler submanifold of $\mathbb{C}P^n$. Moreover by Olmos-Sánchez's theorem and some elementary arguments it will be shown that such an inverse image is a standardly embedded R-space obtained as an orbit of the isotropy representation of an irreducible Hermitian symmetric space. It implies the classification theorem of parallel Kähler submanifolds of $\mathbb{C}P^n$.

This article is organized as follows: In Section 2 we recall the definition of R-spaces and the standard embeddings constructed from an arbitrary given compact symmetric space G/K. And we explain the precise definition of a canonical connection on a Riemannian manifold and Olmos-Sánchez's theorem of differential geometric characterizations for R-spaces. In Section 3 we show our main results. Our main tool is the classical technique of Riemannian submersions ([**14**]) for the Hopf fibration restricted to a Kähler submanifold of a complex projective space. Moreover we discuss related properties and the classification theorem of parallel Kähler submanifolds.

As other related topics, totally complex submanifolds in quaternionic projective spaces with parallel second fundamental form were classified by K. Tsukada in 1985 ([**21**]). More recently we have also obtained a similar result for such submanifolds. It will be described in detail in the forthcoming joint paper with Kaname Hashimoto (OCAMI) and Jong Taek Cho (Chonnam National University).

Throughout this article any manifold is smooth, connected and second countable.

2. R-spaces and Olmos-Sánchez's characterization

Let (G, K) be a compact symmetric pair associated with a compact symmetric space G/K. Here G is a connected compact Lie group with Lie algebra $\mathfrak{g}$ and K is a compact Lie subgroup of G with Lie algebra $\mathfrak{k}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{g}$ as a symmetric Lie algebra. The vector space $\mathfrak{p}$ can be regarded as a Euclidean space by the restriction of an $\mathrm{Ad}(G)$-invariant inner product $\langle \ , \ \rangle$ of $\mathfrak{g}$ to $\mathfrak{p}$. The isotropy representation of a compact symmetric space G/K is given by an orthogonal representation of K on the vector space $\mathfrak{p}$

$$\mathrm{Ad}_{\mathfrak{p}} : K \ni a \longmapsto \mathrm{Ad}(a)|_{\mathfrak{p}} \in O(\mathfrak{p}),$$

which is also called an *s-representation.* It is well-known to be a polar representation. For any non-zero $H \in \mathfrak{p}$, we define a compact homogeneous space K/K_H diffeomorphic to an orbit of the isotropy representation of K through H by

$$\Phi_H : K/K_H \ni aK_H \longmapsto \mathrm{Ad}(a)H \in \mathrm{Ad}(K)H \subset \mathfrak{p}$$

where $K_H := \{a \in K \mid \mathrm{Ad}(a)H = H\}$. Then so obtained compact homogeneous space K/K_H is called an *R-space* and the embedding $\Phi_H : K/K_H \to \mathfrak{p}$ is called *the standard embedding* of R-space K/K_H. When K/K_H is a compact symmetric space, $K/K_H$is called a *symmetric R-space.* Then the standard embedding $\Phi_H : K/K_H \to \mathfrak{p}$ has parallel second fundamental form and symmetric R-spaces give all submanifolds of Euclidean spaces with parallel second fundamental form (Ferus [**7**]). When $H \in \mathfrak{p}$ is a regular element (by definition $\mathrm{Ad}(K)H$ is of maximal dimension), K/K_H is called a *regular R-space.* Then $\Phi_H : K/K_H \to \mathfrak{p}$ is a homogeneous isoparametric submanifold of a Euclidean space and regular R-spaces give all homogeneous isoparametric submanifolds of Euclidean spaces ([**15**] and [**3**]).

Olmos and Sánchez ([**13**]) have showed that a general R-space standardly embedded into a Euclidean spaces can be characterized by the parallelism of the second fundamental form with respect to the normal connection and a *canonical connection* (not necessary the Levi-Civita connection!) on a given submanifold of a Euclidean space. Next we shall describe their results.

Let N be a connected submanifold immersed in the Euclidean space $\mathbb{R}^l$. Let g_N be a Riemannian metric on N induced from $\mathbb{R}^l$ and let ∇^N denote the Levi-Civita connection of a Riemannian manifold (N, g_N). An affine connection $\tilde{\nabla}$ on N is called a *metric connection* with respect to g if $\tilde{\nabla}$ satisfies the condition

$$\tilde{\nabla} g_N = 0. \tag{2.1}$$

Let D be a tensor field on N of type $(1, 2)$ defined by

$$D := \nabla^N - \tilde{\nabla}. \tag{2.2}$$

The metric condition (2.1) is equivalent to the condition that for each vector $X \in TN$ the linear endomorphism D_X is skew-symmetric with respect to g_N, that is,

$$g_N(D_X Y, Z) + g_N(Y, D_X Z) = 0 \quad (\forall Y, Z \in TN). \tag{2.3}$$

A metric connection ∇^c is called a *canonical connection* of a Riemannian manifold (N, g_N) if ∇^c satisfies the condition

$$\nabla^c D^c = 0, \tag{2.4}$$

where $D^c := \nabla^N - \nabla^c$ is a tensor field on N of type $(1, 2)$. Note that the Levi-Civita connection itself is a *trivial* example of a canonical connection of (N, g_N) as $D^c = 0$ in this case.

The covariant derivative $\nabla^c\alpha^N$ of the second fundamental form α^N in terms of the normal connection $\nabla^\perp$ and a *canonical connection* ∇^c is defined by

$$(\nabla^c_X\alpha^N)(Y,Z) := \nabla^\perp_X(\alpha^N(Y,Z)) - \alpha^N(\nabla^c_X Y, Z) - \alpha^N(Y, \nabla^c_X Z) \tag{2.5}$$

for any smooth vector fields X, Y, Z on M. Then

THEOREM 2.1 (Olmos and Sánchez [**13**]). *Let N be a connected compact submanifold fully embedded in the Euclidean space $\mathbb{R}^l$. Then the following three conditions are equivalent each other:*

(1) *There is a canonical connection ∇^c on N such that*

$$\nabla^c\alpha^N = 0. \tag{2.6}$$

(2) *N is a homogeneous submanifold with constant principal curvatures (see* [**13**, p.127, Definition 1.2] *for the definition).*

(3) *N is an orbit of an s-representation, that is, an R-space standardly embedded in the Euclidean space.*

In this case we call a tensor field D^c on N of type $(1,2)$ defined defined by $D^c := \nabla^N - \nabla^c$ a *homogeneous structure tensor field* on a submanifold N. Notice that the argument of [**13**] also works to have the local version of this theorem. The proof of the implication (2) $\Rightarrow$ (3) uses the classification theorem of polar representations by Dadok [**3**], which clams that any orthogonal polar representation is orbit-equivalent to an s-representation. See also [**5**], [**6**], [**1**] for a conceptional proof of Dadok's result subject to some restriction and further works.

3. Homogeneous structure on the inverse images of parallel Kähler submanifolds under the Hopf fibration

Let $\mathbb{C}^{n+1}$ be an $n+1$-dimensional complex Euclidean space with the standard Hermitian inner product $\langle\langle \mathbf{x}, \mathbf{y}\rangle\rangle := \sum_{i=1}^{n+1} x_i\bar{y}_i$ for each $\mathbf{x} = (x_1, \cdots, x_{n+1}), \mathbf{y} = (y_1, \cdots, y_{n+1}) \in \mathbb{C}^{n+1}$. Let $\langle \mathbf{x}, \mathbf{y}\rangle := \mathrm{Re}\langle\langle \mathbf{x}, \mathbf{y}\rangle\rangle$ denote the standard real inner product of $\mathbb{C}^{n+1}$. Let $S^{2n+1}(1) := \{\mathbf{x} \in \mathbb{C}^{n+1} \mid \langle \mathbf{x}, \mathbf{x}\rangle = 1\}$ be the unit standard hypersphere of $\mathbb{C}^{n+1}$ and $\pi : S^{2n+1}(1) \longrightarrow \mathbb{C}P^n$ be the Hopf fibration over an n-dimensional complex projective space $\mathbb{C}P^n$. We endow $\mathbb{C}P^n$ with the Fubini-Study metric of constant holomorphic sectional curvature 4 so that $\pi : S^{2n+1}(1) \longrightarrow \mathbb{C}P^n$ is a Riemannian submersion. Then we have an orthogonal direct sum decomposition of the tangent vector bundle of $S^{2n+1}(1)$ into vertical and horizontal subbundles:

$$TS^{2n+1}(1) = \mathcal{V}S^{2n+1}(1) \oplus \mathcal{H}S^{2n+1}(1).$$

For a tangent vector or a vector field X on $\mathbb{C}P^n$, we denote by $\tilde{X}$ the horizontal lift of X to $S^{2n+1}(1)$. A horizontal vector field on $\mathbb{C}P^n$ is called *basis* if it is given as a horizontal lift of a vector field on $S^{2n+1}(1)$.

We denote also by $\mathbf{x}$ the position vector of a point in $\mathbb{C}^{n+1}$. The vertical subspace at each point $\mathbf{x} \in S^{2n+1}(1)$ is expressed as

$$\mathcal{V}_{\mathbf{x}}S^{2n+1}(1) = \mathbb{R}\sqrt{-1}\mathbf{x}.$$

At each point $\mathbf{x} \in S^{2n+1}(1)$, the restriction of the differential of π to the horizontal subspace $(d\pi)_{\mathbf{x}} : \mathcal{H}_{\mathbf{x}}S^{2n+1}(1) \to T_{\pi(\mathbf{x})}\mathbb{C}P^n$ is a linear isometry and we have $(d\pi)_{\mathbf{x}}(\tilde{X}) = X$ for each vector $X \in T_{\pi(\mathbf{x})}\mathbb{C}P^n$. Each horizontal subspace $\mathcal{H}_{\mathbf{x}}S^{2n+1}(1)$ is invariant under the scalar multiplication by $\sqrt{-1}$ on $\mathbb{C}^{n+1}$. The complex structure tensor J of $\mathbb{C}P^n$ is induced by this multiplication $\sqrt{-1}\times$ and it can be described as

$$JX = J(d\pi)_{\mathbf{x}}(\tilde{X}) = (d\pi)_{\mathbf{x}}(\sqrt{-1}\tilde{X}) \quad \text{or} \quad \widetilde{JX} = \sqrt{-1}\tilde{X} \in \mathcal{H}_{\mathbf{x}} S^{2n+1}(1).$$

Under the identification $T_{\pi(\mathbf{x})}\mathbb{C}P^n \cong T^{1,0}_{\pi(\mathbf{x})}\mathbb{C}P^n = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\mathbf{x}, (\mathbb{C}\mathbf{x})^{\perp})$, for each $X \in T_{\pi(\mathbf{x})}\mathbb{C}P^n$ we have $\tilde{X} = X(\mathbf{x}) \in (\mathbb{C}\mathbf{x})^{\perp} = \mathcal{H}_{\mathbf{x}} S^{2n+1}(1) \in \mathbb{C}^{n+1}$ and

$$(JX)(\mathbf{x}) = X(\sqrt{-1}\mathbf{x}) = \sqrt{-1}X(\mathbf{x}),$$

where $(\mathbb{C}\mathbf{x})^{\perp}$ denotes an n-dimensional complex vector subspace of $\mathbb{C}^{n+1}$ defined by

$$(\mathbb{C}\mathbf{x})^{\perp} := \{v \in \mathbb{C}^{n+1} \mid \langle\langle v, w\rangle\rangle = 0 \ (\forall w \in \mathbb{C}\mathbf{x})\}.$$

In this case the O'Neill tensors $\mathcal{T}$ and $\mathcal{A}$ for the Riemannian submersion ([**14**]) are given by $\mathcal{T} = 0$ and

$$\mathcal{A}_{\tilde{X}}\tilde{Y} = -\langle\sqrt{-1}\tilde{X}, \tilde{Y}\rangle\sqrt{-1}\mathbf{x} \tag{3.1}$$

for each horizontal vectors $\tilde{X}$,$\tilde{Y}$ at $\mathbf{x} \in S^{2n+1}(1)$.

Suppose that M^m is a complex m-dimensional complex submanifold immersed in $\mathbb{C}P^n$. The inverse image of the submanifold M under the Hopf fibration $\pi : S^{2n+1}(1) \longrightarrow \mathbb{C}P^n$ is defined as

$$\begin{aligned} \hat{M} =& \pi^{-1}(M) \\ =& \{(p, \mathbf{x}) \in M \times S^{2n+1}(1) \mid p \in M, \mathbf{x} \in \pi^{-1}(\varphi(p)) \subset S^{2n+1}(1)\}. \end{aligned}$$

$$\begin{array}{ccc} & & \mathbb{C}^{n+1} \\ & & \cup \\ \hat{M} = \pi^{-1}(M) & \xrightarrow{\ \hat{\varphi}\ } & S^{2n+1}(1) \\ \pi \Big\downarrow S^1 & & \pi \Big\downarrow S^1 \\ M & \xrightarrow{\ \varphi\ } & \mathbf{C}P^n \end{array}$$

Then $\hat{M}$ is a real $2m+1$-dimensional submanifold immersed in $S^{2n+1}(1) \subset \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ and the projection $\pi : \hat{M} \to M$ is also a Riemannian submersion, so that we have an orthogonal direct sum decomposition of the tangent vector bundle of $\hat{M}$ into vertical and horizontal subbundles:

$$T\hat{M} = \mathcal{V}\hat{M} \oplus \mathcal{H}\hat{M}.$$

Here note that the vertical subspace at each point $\mathbf{x} \in \hat{M}$ is given by

$$\mathcal{V}_{\mathbf{x}}\hat{M} = \mathcal{V}_{\mathbf{x}} S^{2n+1}(1) = \mathbb{R}\sqrt{-1}\mathbf{x}.$$

We shall construct explicitly a homogeneous structure tensor field D in the sense of Olmos and Sánchez on the inverse image $\hat{M} = \pi^{-1}(M)$.

Now we define the tensor field D of type $(1,2)$ on $\hat{M} = \pi^{-1}(M) \subset S^{2n+1}(1)$ by

$$\begin{cases} D_{\tilde{X}}(\tilde{Y}) := -\langle\sqrt{-1}\tilde{X}, \tilde{Y}\rangle\sqrt{-1}\mathbf{x} \in \mathcal{V}_{\mathbf{x}}\hat{M}, \\ D_{\tilde{X}}(V) := \sqrt{-1}\tilde{X} = \widetilde{JX} \in \mathcal{H}_{\mathbf{x}}\hat{M}, \\ D_V(\tilde{X}) := \dfrac{1}{2}\sqrt{-1}\tilde{X} = \dfrac{1}{2}\widetilde{JX} \in \mathcal{H}_{\mathbf{x}}\hat{M}, \\ D_V(V) := 0 \end{cases} \tag{3.2}$$

for each horizontal vectors $\tilde{X}, \tilde{Y}$ and the vertical vector $V = \sqrt{-1}\mathbf{x}$ on $\hat{M}$.

Remark 3.1. We can also define the tensor field $\mathbf{D}$ of type $(1,2)$ on $S^{2n+1}(1)$ by

$$\begin{cases} \mathbf{D}_{\tilde{X}}(\tilde{Y}) := -\langle\sqrt{-1}\tilde{X}, \tilde{Y}\rangle\sqrt{-1}\mathbf{x} \in \mathcal{V}_{\mathbf{x}}S^{2n+1}(1), \\ \mathbf{D}_{\tilde{X}}(V) := \sqrt{-1}\tilde{X} = \widetilde{JX} \in \mathcal{H}_{\mathbf{x}}S^{2n+1}(1), \\ \mathbf{D}_V(\tilde{X}) := \frac{1}{2}\sqrt{-1}\tilde{X} = \frac{1}{2}\widetilde{JX} \in \mathcal{H}_{\mathbf{x}}S^{2n+1}(1), \\ \mathbf{D}_V(V) := 0 \end{cases} \tag{3.3}$$

for each horizontal vectors $\tilde{X}, \tilde{Y}$ on $S^{2n+1}(1)$ and the vertical vector $V = \sqrt{-1}\mathbf{x}$ on $S^{2n+1}(1)$. Notice that if M is a complex manifold of $\mathbb{C}P^n$, then the restriction of $\mathbf{D}$ to its inverse image $\hat{M} = \pi^{-1}(M)$ coincides with the tensor field D of type $(1,2)$ on $\hat{M}$ defined by (3.2). Also note that $\mathbf{D}_{\tilde{X}}(\tilde{Y}) = \mathcal{A}_{\tilde{X}}(\tilde{Y})$.

Then the main result of this article is described as follows:

Theorem 3.2. *Suppose that M is a complex submanifold immersed in $\mathbb{C}P^n$. Let $\nabla^{\hat{M}}$ be the Levi-Civita connection of its inverse image $\hat{M} = \pi^{-1}(M) \subset S^{2n+1}(1)$ and D be a tensor field of type $(1,2)$ defined by (3.2) on $\hat{M} = \pi^{-1}(M)$. Then*

(1) *The affine connection $\nabla^c := \nabla^{\hat{M}} - D$ of $\hat{M}$ is a (non-trivial) canonical connection on $\hat{M}$.*
(2) *M has parallel second fundamental form if and only if $\hat{M}$ satisfies*

$$\nabla^c \alpha^{\hat{M}} = 0. \tag{3.4}$$

We prove this theorem by showing the following Lemmas 3.3, 3.4 and **??**.

Lemma 3.3. *The affine connection ∇^c is a metric connection on $\hat{M}$, that is.*

$$g_{\hat{M}}(D_u v, w) + g_{\hat{M}}(v, D_u w) = 0 \tag{3.5}$$

for all vectors $u, v, w \in T\hat{M}$.

Proof. Following the definition of D, for each horizontal vectors $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and a vertical vector $V = \sqrt{-1}\mathbf{x}$ we compute

$$\begin{aligned} g_{\hat{M}}(D_{\tilde{X}}\tilde{Y}, \tilde{Z}) + g_{\hat{M}}(\tilde{Y}, D_{\tilde{X}}\tilde{Z}) =& 0 + 0 = 0, \\ g_{\hat{M}}(D_{\tilde{X}}V, \tilde{Z}) + g_{\hat{M}}(V, D_{\tilde{X}}\tilde{Z}) =& \langle\sqrt{-1}\tilde{X}, \tilde{Z}\rangle - \langle\sqrt{-1}\tilde{X}, \tilde{Z}\rangle = 0 + 0 = 0, \\ g_{\hat{M}}(D_{\tilde{X}}V, V) + g_{\hat{M}}(V, D_{\tilde{X}}V) =& 0 + 0 = 0, \\ g_{\hat{M}}(D_V\tilde{Y}, \tilde{Z}) + g_{\hat{M}}(\tilde{Y}, D_V\tilde{Z}) =& \frac{1}{2}\langle\sqrt{-1}\tilde{Y}, \tilde{Z}\rangle + \frac{1}{2}\langle\tilde{Y}, \sqrt{-1}\tilde{Z}\rangle \\ =& \frac{1}{2}\langle\sqrt{-1}\tilde{Y}, \tilde{Z}\rangle - \frac{1}{2}\langle\sqrt{-1}\tilde{Y}, \tilde{Z}\rangle \\ =& 0, \\ g_{\hat{M}}(D_V V, \tilde{Z}) + g_{\hat{M}}(V, D_V\tilde{Z}) =& 0 + 0 = 0, \\ g_{\hat{M}}(D_V V, V) + g_{\hat{M}}(V, D_V V) =& 0. \end{aligned}$$

□

Lemma 3.4. *The tensor field D on $\hat{M}$ satisfies the equation*

$$\nabla^{\hat{M}} D = D \cdot D. \tag{3.6}$$

Proof. Let $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ be any basic horizontal vector fields on $\hat{M}$ and $V = \sqrt{-1}\mathbf{x}$ be a vertical vector field on $\hat{M}$. First we show the equation

$$(\nabla^{\hat{M}}_{\tilde{Z}} D)_{\tilde{X}}(\tilde{Y}) = (D_{\tilde{Z}} \cdot D)_{\tilde{X}}(\tilde{Y}). \tag{3.7}$$

By the definition of D we compute

$$\begin{aligned}
&(\nabla^{\hat{M}}_{\tilde{Z}} D)_{\tilde{X}}(\tilde{Y}) \\
=&\nabla^{\hat{M}}_{\tilde{Z}}(D_{\tilde{X}}\tilde{Y}) - D_{\nabla^{\hat{M}}_{\tilde{Z}}\tilde{X}}\tilde{Y} - D_{\tilde{X}}(\nabla^{\hat{M}}_{\tilde{Z}}\tilde{Y}) \\
=&\nabla^{\hat{M}}_{\tilde{Z}}(D_{\tilde{X}}\tilde{Y}) - D_{-\langle\sqrt{-1}\tilde{Z},\tilde{X}\rangle V}\tilde{Y} - D_{\tilde{X}}(-\langle\sqrt{-1}\tilde{Z},\tilde{Y}\rangle V) \\
=&-\langle\sqrt{-1}\tilde{X},\tilde{Y}\rangle\sqrt{-1}\tilde{Z} + \langle\sqrt{-1}\tilde{Z},\tilde{X}\rangle\frac{\sqrt{-1}}{2}\tilde{Y} + \langle\sqrt{-1}\tilde{Z},\tilde{Y}\rangle\sqrt{-1}\tilde{X}.
\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}
&(D_{\tilde{Z}} \cdot D)_{\tilde{X}}(\tilde{Y}) \\
=&D_{\tilde{Z}}(D_{\tilde{X}}\tilde{Y}) - D_{D_{\tilde{Z}}\tilde{X}}\tilde{Y} - D_{\tilde{X}}(D_{\tilde{Z}}\tilde{Y}) \\
=&-\langle\sqrt{-1}\tilde{X},\tilde{Y}\rangle\sqrt{-1}\tilde{Z} + \langle\sqrt{-1}\tilde{Z},\tilde{X}\rangle\frac{\sqrt{-1}}{2}\tilde{Y} + \langle\sqrt{-1}\tilde{Z},\tilde{Y}\rangle\sqrt{-1}\tilde{X}
\end{aligned}$$

Hence we obtain the equation (3.7). Following the definition of D, we can check all of the following other equations:

$$(\nabla^{\hat{M}}_{V} D)_{\tilde{X}}(\tilde{Y}) = (D_{V} \cdot D)_{\tilde{X}}(\tilde{Y}) \tag{3.8}$$

$$(\nabla^{\hat{M}}_{\tilde{Z}} D)_{\tilde{X}}(V) = (D_{\tilde{Z}} \cdot D)_{\tilde{X}}(V) \tag{3.9}$$

$$(\nabla^{\hat{M}}_{\tilde{Z}} D)_{V}(\tilde{Y}) = (D_{\tilde{Z}} \cdot D)_{V}(\tilde{Y}) \tag{3.10}$$

$$(\nabla^{\hat{M}}_{V} D)_{\tilde{X}}(V) = (D_{V} \cdot D)_{\tilde{X}}(V) \tag{3.11}$$

$$(\nabla^{\hat{M}}_{\tilde{Z}} D)_{V}(V) = (D_{\tilde{Z}} \cdot D)_{V}(V) \tag{3.12}$$

$$(\nabla^{\hat{M}}_{V} D)_{V}(\tilde{Y}) = (D_{V} \cdot D)_{V}(\tilde{Y}) \tag{3.13}$$

$$(\nabla^{\hat{M}}_{V} D)_{V}(V) = (D_{V} \cdot D)_{V}(V) \tag{3.14}$$

However they are also quite elementary computations. □

Since (3.6) is equivalent to (2.4), it follows from Lemmas 3.5 and 3.6 that ∇^c is a canonical connection on $\hat{M}$.

Lemma 3.5. *$\nabla^*\alpha^M = 0$ if and only if $\nabla^c\alpha^{\hat{M}} = 0$, that is,*

$$(\nabla^*_u\alpha^{\hat{M}})(v,w) + \alpha^{\hat{M}}(D_u v, w) + \alpha^{\hat{M}}(v, D_u w) = 0 \tag{3.15}$$

for all vectors $u, v, w \in T\hat{M}$.

Proof. We use some results from the fundamental equations of Riemannian submersions (cf. [**14**]). The second fundamental forms of $\hat{M}$ and M are related as

$$\alpha^{\hat{M}}(\tilde{X},\tilde{Y}) = (\alpha^{M}(X,Y))^{\sim} \tag{3.16}$$

for any tangent vectors X, Y on M. By tanking the covariant derivative of the equation (3.16) in the horizontal direction $\tilde{Z}$ we have

$$(\nabla^*_{\tilde{Z}}\alpha^{\hat{M}})(\tilde{X},\tilde{Y}) + \alpha^{\hat{M}}(\mathcal{A}_{\tilde{Z}}\tilde{X},\tilde{Y}) + \alpha^{\hat{M}}(\tilde{X},\mathcal{A}_{\tilde{Z}}\tilde{Y}) = ((\nabla^*_{Z}\alpha^{M})(X,Y))^{\sim} \tag{3.17}$$

for any tangent vectors X, Y, Z on M. Since the fibers of $\pi : S^{2n+1}(1) \to \mathbb{C}P^n$ are totally geodesic, we have

$$\alpha^{\hat{M}}(V,W) = 0 \tag{3.18}$$

for any vertical vectors V, W on $\hat{M}$. It follows from (3.18) that

$$(\nabla^*_{U}\alpha^{\hat{M}})(V,W) = 0 \tag{3.19}$$

for any vertical vectors U, V, W on $\hat{M}$. Moreover, since M is a complex submanifold of $\mathbb{C}P^n$, for each normal vector field ξ to M we compute

$$\begin{aligned} g_{S^{2n+1}}(\alpha^{\hat{M}}(V,\tilde{Y}),\tilde{\xi}) =& g_{S^{2n+1}}(\nabla^{S^{2n+1}}_{\tilde{Y}}V,\tilde{\xi}) \\ =& - g_{S^{2n+1}}(V,\nabla^{S^{2n+1}}_{\tilde{Y}}\tilde{\xi}) \\ =& - g_{S^{2n+1}}(V,\mathcal{A}_{\tilde{Y}}\tilde{\xi}) \\ =& - g_{S^{2n+1}}(V,-\langle\sqrt{-1}\tilde{Y},\tilde{\xi}\rangle\sqrt{-1}\mathbf{x}) \\ =& g_{S^{2n+1}}(V,\langle\widetilde{JY},\tilde{\xi}\rangle\sqrt{-1}\mathbf{x}) \\ =& \langle\widetilde{JY},\tilde{\xi}\rangle g_{S^{2n+1}}(V,\sqrt{-1}\mathbf{x}) \\ =& g_{\mathbb{C}P^n}(JY,\xi) g_{S^{2n+1}}(V,\sqrt{-1}\mathbf{x}) = 0. \end{aligned}$$

Hence we have

$$\alpha^{\hat{M}}(V,\tilde{Y}) = 0 \tag{3.20}$$

for any vertical vector V and any horizontal vector $\tilde{Y}$ on M. It follows from (3.20) that

$$(\nabla^*_{U}\alpha^{\hat{M}})(V,\tilde{Y}) = 0 \tag{3.21}$$

for any vertical vectors U, V and any horizontal vector $\tilde{Y}$ on $\hat{M}$. It follows from (3.18) and (3.20) that

$$(\nabla^*_{\tilde{X}}\alpha^{\hat{M}})(V,W) = 0 \tag{3.22}$$

for any vertical vectors V, W and any horizontal vector $\tilde{X}$ on $\hat{M}$.

By (3.17), (3.18), (3.20) and the definition of D, we have

$$\begin{aligned} &(\nabla^*_{\tilde{Z}}\alpha^{\hat{M}})(\tilde{X},\tilde{Y}) + \alpha^{\hat{M}}(D_{\tilde{Z}}\tilde{X},\tilde{Y}) + \alpha^{\hat{M}}(\tilde{X},D_{\tilde{Z}}\tilde{Y}) \\ =&(\nabla^*_{\tilde{Z}}\alpha^{\hat{M}})(\tilde{X},\tilde{Y}) = ((\nabla^*_{Z}\alpha^{M})(X,Y))^{\sim}. \end{aligned} \tag{3.23}$$

By differentiating (3.16) in the vertical direction $V = \sqrt{-1}\mathbf{x}$, we have

$$\begin{aligned} &g_{S^{2n+1}}((\nabla^*_{V}\alpha^{\hat{M}})(\tilde{X},\tilde{Y}),\tilde{\xi}) \\ &+g_{S^{2n+1}}(\alpha^{\hat{M}}(\widetilde{JX},\tilde{Y}),\tilde{\xi}) + g_{S^{2n+1}}(\alpha^{\hat{M}}(\tilde{X},\widetilde{JY}),\tilde{\xi}) + g_{S^{2n+1}}(\alpha^{\hat{M}}(\tilde{X},\tilde{Y}),\widetilde{J\xi}) = 0 \end{aligned}$$

for any tangent vectors X, Y on M and any normal vector ξ to M. By (3.16), the identity for a Kähler submanifold M and the definition of D we compute

$$\begin{aligned}
&g_{S^{2n+1}}(\alpha^{\hat{M}}(\widetilde{JX},\tilde{Y}),\tilde{\xi})+g_{S^{2n+1}}(\alpha^{\hat{M}}(\tilde{X},\widetilde{JY}),\tilde{\xi})+g_{S^{2n+1}}(\alpha^{\hat{M}}(\tilde{X},\tilde{Y}),\widetilde{J\xi})\\
=&g_{\mathbb{C}P^n}(\alpha^{M}(JX,Y),\xi)\circ\pi+g_{\mathbb{C}P^n}(\alpha^{M}(X,JY),\xi)\circ\pi+g_{\mathbb{C}P^n}(\alpha^{M}(X,Y),J\xi)\circ\pi\\
=&g_{\mathbb{C}P^n}(\alpha^{M}(JX,Y),\xi)\circ\pi\\
=&\frac{1}{2}g_{\mathbb{C}P^n}(\alpha^{M}(JX,Y),\xi)\circ\pi+\frac{1}{2}g_{\mathbb{C}P^n}(\alpha^{M}(X,JY),\xi)\circ\pi\\
=&\frac{1}{2}g_{S^{2n+1}}(\alpha^{\hat{M}}(\widetilde{JX},\tilde{Y}),\tilde{\xi})+\frac{1}{2}g_{S^{2n+1}}(\alpha^{\hat{M}}(\tilde{X},\widetilde{JY}),\tilde{\xi})\\
=&g_{S^{2n+1}}(\alpha^{\hat{M}}(\frac{1}{2}\sqrt{-1}\tilde{X},\tilde{Y}),\tilde{\xi})+g_{S^{2n+1}}(\alpha^{\hat{M}}(\tilde{X},\frac{1}{2}\tilde{Y}),\tilde{\xi})\\
=&g_{S^{2n+1}}(\alpha^{\hat{M}}(D_V\tilde{X},\tilde{Y}),\tilde{\xi})+g_{S^{2n+1}}(\alpha^{\hat{M}}(\tilde{X},D_V\tilde{Y}),\tilde{\xi}).
\end{aligned}$$

Hence we have

$$(\nabla^*_V\alpha^{\hat{M}})(\tilde{X},\tilde{Y})+\alpha^{\hat{M}}(D_V\tilde{X},\tilde{Y})+\alpha^{\hat{M}}(\tilde{X},D_V\tilde{Y})=0. \tag{3.24}$$

It follows from (3.19) and the definition of D that

$$\begin{aligned}(\nabla^*_U\alpha^{\hat{M}})(V,W)+\alpha^{\hat{M}}(D_UV,W)+\alpha^{\hat{M}}(V,D_UW)=&(\nabla^*_U\alpha^{\hat{M}})(V,W)\\=&0.\end{aligned} \tag{3.25}$$

By using (3.18), (3.20) and the definition of D, we compute

$$\begin{aligned}(\nabla^*_{\tilde{X}}\alpha^{\hat{M}})(V,W)+\alpha^{\hat{M}}(D_{\tilde{X}}V,W)+\alpha^{\hat{M}}(V,D_{\tilde{X}}W)=&(\nabla^*_{\tilde{X}}\alpha^{\hat{M}})(V,W)\\=&0.\end{aligned} \tag{3.26}$$

By differentiating (3.20) in the horizontal direction $\tilde{X}$, we have

$$(\nabla^*_{\tilde{X}}\alpha^{\hat{M}})(V,\tilde{Y})+\alpha^{\hat{M}}((\mathcal{A}_{\tilde{X}}V)^{T\hat{M}},\tilde{Y})=0.$$

By (3.18) and the definition of D it becomes

$$\begin{aligned}&(\nabla^*_{\tilde{X}}\alpha^{\hat{N}})(V,\tilde{Y})+\alpha^{\hat{N}}(D_{\tilde{X}}V,\tilde{Y})+\alpha^{\hat{N}}(V,D_{\tilde{X}}\tilde{Y})\\=&(\nabla^*_{\tilde{X}}\alpha^{\hat{N}})(V,\tilde{Y})+\alpha^{\hat{N}}(D_{\tilde{X}}V,\tilde{Y})\\=&(\nabla^*_{\tilde{X}}\alpha^{\hat{N}})(V,\tilde{Y})+\alpha^{\hat{N}}((\mathcal{A}_{\tilde{X}}V)^{T\hat{N}},\tilde{Y})=0.\end{aligned} \tag{3.27}$$

By using (3.22), (3.18), (3.20) we compute

$$(\nabla^*_W\alpha^{\hat{M}})(V,\tilde{Y})+\alpha^{\hat{M}}(D_WV,\tilde{Y})+\alpha^{\hat{M}}(V,D_W\tilde{Y})=(\nabla^*_W\alpha^{\hat{M}})(V,\tilde{Y})=0. \tag{3.28}$$

By those six equations (3.23), (3.24), (3.25), (3.26), (3.27) and (3.28), we obtain Lemma 3.5. □

Since (3.15) is equivalent to (3.4), we obtain Theorem 3.2. Therefore by Olmos and Sánchez's theorem 2.1 $\hat{M}$ is obtained as a standardly embedded R-space, that is, an orbit of the isotropy representation of a Riemannian symmetric pair (G,K).

In order to construct extrinsic symmetries of $\hat{M}\subset S^{2n+1}(1)\subset\mathbb{C}^{n+1}$, we recall the argument of [**13**]. For any $p,q\in\hat{M}$, let $\tau=\tau(t)$ $(t\in[a,b])$ be a piecewise smooth curve on $\hat{M}$ joining from $p=\tau(a)$ to $q=\tau(b)$. Let

$$\tau_a^b: T_p\hat{M}\longrightarrow T_q\hat{M}$$

denote the parallel displacement along τ with respect to the canonical connection ∇^c. Since the curvature tensor field R^c and the torsion tensor field T^c of ∇^c satisfy $\nabla^c R^c = 0$ and $\nabla^c T^c = 0$, there is a local isometry $s : U_p \to U_q$ of $\hat{M}$ such that

$$\begin{cases} s(p) = \tilde{s}(p) = q, \\ s(p') = \tilde{s}(p') \quad (\forall p' \in U_p), \\ (ds)_p = \tau_a^b. \end{cases}$$

Let

$$(\tau^\perp)_a^b : T_p^\perp \hat{M} \longrightarrow T_q^\perp \hat{M}$$

denote the parallel displacement along τ with respect to the normal connection $\nabla^\perp$. By using the orthogonal direct sum decompositions as real vector subspaces

$$\begin{aligned} \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2} =& \mathbb{R}\mathbf{x}(p) \oplus T_p\hat{M} \oplus T_p^\perp \hat{M} \\ =& \mathbb{R}\mathbf{x}(q) \oplus T_q\hat{M} \oplus T_q^\perp \hat{M}, \end{aligned}$$

we define an isometry $\tilde{s}$ of $S^{2n+1}(1)$, that is, $\tilde{s} \in SO(2n+2)$ by

$$\begin{cases} \tilde{s}(\mathbf{x}(p)) := \mathbf{x}(q), \\ \tilde{s}|_{T_p\hat{M}} := (ds)_p = \tau_a^b, \\ \tilde{s}|_{T_p^\perp \hat{M}} := (\tau^\perp)_a^b. \end{cases}$$

Then it follows from the condition $\nabla^c \alpha^{\hat{M}} = 0$ and the linearity of $\tilde{s}$ that

$$\begin{cases} s(p) = \tilde{s}(p) = q, \\ s(p') = \tilde{s}(p') \quad (\forall p' \in U_p), \\ (ds)_p = (d\tilde{s})_p|_{T_pM} = \tilde{s}|_{T_pM} = \tau_a^b, \\ (d\tilde{s})_p|_{T_p^\perp M} = \tilde{s}|_{T_p^\perp M} = (\tau^\perp)_a^b. \end{cases}$$

Moreover we can show

Lemma 3.6.

$$\tilde{s}(\sqrt{-1}\mathbf{x}) = \sqrt{-1}\tilde{s}(\mathbf{x}) \quad (\forall \mathbf{x} \in \mathbb{C}^{n+1})$$

and hence $\tilde{s} \in U(n+1)$.

Proof. We consider the trivial vector bundle

$$\underline{\mathbb{C}}^{n+1} := \hat{M} \times \mathbb{C}^{n+1}$$

over $\hat{N}$ with fiber $\mathbb{C}^{n+1}$ as a *real* vector bundle. For each point $\mathbf{x} \in \hat{M}$ we have orthogonal direct sum decompositions

$$\begin{aligned} \mathbb{C}^{n+1} =& \mathbb{R}\mathbf{x} \oplus T_{\mathbf{x}}\hat{M} \oplus T_{\mathbf{x}}^\perp \hat{M} \\ =& \mathbb{R}\mathbf{x} \oplus (\mathcal{V}_{\mathbf{x}}\hat{M} \oplus \mathcal{H}_{\mathbf{x}}\hat{M}) \oplus T_{\mathbf{x}}^\perp \hat{M}. \end{aligned}$$

Let

$$\mathcal{R}\hat{M} := \coprod_{\mathbf{x} \in \hat{M}} \mathbb{R}\mathbf{x}$$

be the real trivial line bundle over $\hat{M}$ equipped with the subbundle connection $\partial^{\mathcal{R}\hat{M}}$ on $\mathcal{R}\hat{M} \subset \underline{\mathbb{C}}^{n+1}$ induced from the trivial connection ∂, so that the position vector $\mathbf{x}$ of points of $\hat{M}$ gives a parallel global section of $\mathcal{R}\hat{M}$ with respect to $\partial^{\mathcal{R}\hat{M}}$. Then we have the following orthogonal direct sum decompositions of $\underline{\mathbb{C}}^{n+1}$ as real vector subbundles:

$$\begin{aligned}\underline{\mathbb{C}}^{n+1} =& \mathcal{R}\hat{M} \oplus T\hat{M} \oplus T\hat{M} \\ =& \mathcal{R}\hat{M} \oplus (\mathcal{V}\hat{M} \oplus \mathcal{H}\hat{M}) \oplus T^{\perp}\hat{M}.\end{aligned}$$

Along the first orthogonal decomposition we endow the vector bundle $\underline{\mathbb{C}}^{n+1}$ with the direct sum connection

$$\partial^{\mathcal{R}\hat{M}} \oplus \nabla^c \oplus \nabla^{\perp}.$$

Let $\mathrm{End}_{\mathbb{R}}(\underline{\mathbb{C}}^{n+1})$ be the vector bundles of $\mathbb{R}$-linear endomorphisms at each fiber of $\underline{\mathbb{C}}^{n+1}$. The multiplication $\sqrt{-1}\times : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ by $\sqrt{-1}$ on $\mathbb{C}^{n+1}$ can be regarded as $\mathbb{R}$-linear endomorphisms at each fiber of $\underline{\mathbb{C}}^{n+1}$, and thus it defines a smooth section $\sqrt{-1}\times$ of the real vector bundle $\mathrm{End}_{\mathbb{R}}(\underline{\mathbb{C}}^{n+1})$. Then we have only to show that *this section* $\sqrt{-1}\times$ *of* $\mathrm{End}_{\mathbb{R}}(\underline{\mathbb{C}}^{n+1})$ *is parallel with respect to the connection* $\partial^{\mathcal{R}\hat{M}} \oplus \nabla^c \oplus \nabla^{\perp}$, .

Since M is a complex submanifold of $\mathbb{C}P^n$, at each point $\mathbf{x} \in \hat{M}$ the real vector subspaces $\mathbb{R}\mathbf{x} \oplus \mathbb{R}\sqrt{-1}\mathbf{x}$, $\mathcal{H}_{\mathbf{x}}\hat{M}$ and $T^{\perp}_{\mathbf{x}}\hat{M}$ are invariant under the multiplication by $\sqrt{-1}$, respectively. Thus the real vector subbundles $\mathcal{R}\hat{N} \oplus (\mathcal{V}\hat{M},\, \mathcal{H}\hat{M})$ and $T^{\perp}\hat{M}$ are invariant under the action of the section $\sqrt{-1}\times$ of $\mathrm{End}_{\mathbb{R}}(\underline{\mathbb{C}}^{n+1})$, respectively.

Set $(V)_{\mathbf{x}} := \sqrt{-1}\mathbf{x}\quad (\forall \mathbf{x} \in \hat{M})$, which is a vertical vector field on $\hat{N}$. First we observe that $V = \sqrt{-1}\mathbf{x}$ is a parallel vector field on $\hat{M}$ with respect to the canonical connection ∇^c. Indeed, Since $\nabla^{\hat{M}}_V V = 0$ and $D^c_V V = 0$, we have $\nabla^c_V V = \nabla^{\hat{M}}_V V - D^c_V V = 0$. Since $\nabla^{\hat{M}}_{\tilde{X}} V = (\partial_{\tilde{X}} V)^{T\hat{M}} = \sqrt{-1}\tilde{X}$ and $D^c_{\tilde{X}} V = \sqrt{-1}\tilde{X}$ we have $\nabla^c_{\tilde{X}} V = \nabla^{\hat{M}}_{\tilde{X}} V - D^c_{\tilde{X}} V = \sqrt{-1}\tilde{X} - \sqrt{-1}\tilde{X} = 0$. It implies that

$$\nabla^c(\sqrt{-1}\mathbf{x}) = 0 = \sqrt{-1}\partial^{\mathcal{R}\hat{M}}\mathbf{x}. \tag{3.29}$$

Particularly the vertical subbundle $\mathcal{V}\hat{M}$ and horizontal subbundle $\mathcal{H}\hat{M}$ are invariant under parallel translations relative to ∇^c respectively. Moreover, by $\nabla^c D_V = 0$ and elementary computations we have

$$\nabla^c(\sqrt{-1}\tilde{X}) = \nabla^c(2D_V\tilde{X}) = 2D_V(\nabla^c\tilde{X}) = \sqrt{-1}\nabla^c\tilde{X}, \tag{3.30}$$

$$\nabla^{\perp}(\sqrt{-1}\tilde{\xi}) = \sqrt{-1}\nabla^{\perp}\tilde{\xi} \tag{3.31}$$

for any tangent vector field X and normal vector field ξ on M. Those equations (3.29), (3.30) and (3.31) mean the parallelism of $\sqrt{-1}\times \in \mathrm{End}_{\mathbb{R}}(\underline{\mathbb{C}}^{n+1})$ with respect to $\partial^{\mathcal{R}\hat{M}} \oplus \nabla^c \oplus \nabla^{\perp}$, Since a linear isomorphism $\tilde{s} : (\underline{\mathbb{C}}^{n+1})_{\mathbf{x}(p)} \to (\underline{\mathbb{C}}^{n+1})_{\mathbf{x}(q)}$ is a parallel displacement along τ with respect to $\partial^{\mathcal{R}\hat{M}} \oplus \nabla^c \oplus \nabla^{\perp}$, it preserves the parallel section $\sqrt{-1}\times$, namely, we obtain

$$\tilde{s}(\sqrt{-1}\mathbf{x}) = \sqrt{-1}\tilde{s}(\mathbf{x}) \quad (\forall \mathbf{x} \in \mathbb{C}^{n+1}).$$

□

Hence (G, K) must be an Hermitian symmetric pair. Moreover by the following lemma we see that (G, K) is irreducible.

LEMMA 3.7. *Let N be a complex submanifold of $\mathbb{C}P^n$ and $\hat{N} = \pi^{-1}(N) \subset S^{2n+1}(1)$ be the inverse image of N under the Hopf fibration $\pi : S^{2n+1}(1) \to \mathbb{C}P^n$. Suppose that*

$$\mathbb{C}^{n+1} = \mathbb{C}^{\ell_1+1} \oplus \mathbb{C}^{\ell_2+1}$$

and

$$\hat{N} = \pi^{-1}(N) = \hat{N}_1 \times \hat{N}_2 \subset S^{2\ell_1+1}(r_1) \times S^{2\ell_2+1}(r_2) \subset \mathbb{C}^{\ell_1+1} \oplus \mathbb{C}^{\ell_2+1},$$

where $\hat{N}_1 \times \hat{N}_2$ *is a Riemannian direct product of submanifolds* $\hat{N}_i \subset S^{2\ell_i+1}(r_1) \subset \mathbb{C}^{\ell_i+1}$ *with* $r_i \geq 0$ *for* $i = 1, 2$ *and* $(r_1)^2 + (r_2)^2 = 1$. *Then* $r_1 = 0$ *or* $r_2 = 0$.

Proof. Let $\mathbf{x} \in \hat{N}$. By the Riemannian product $\hat{N} = \hat{N}_1 \times \hat{N}_2$, the position vector $\mathbf{x} \in \hat{N}$ can be orthogonally decomposed as

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2, \quad \mathbf{x}_i \in \hat{N}_i \subset S^{2\ell_i+1}(r_1) \subset \mathbb{C}^{\ell_i+1} \ (i = 1, 2)$$

and the tangent space of $\hat{N}$ at $\mathbf{x}$ can be orthogonally decomposed as

$$T_{\mathbf{x}}\hat{N} = T_{\mathbf{x}_1}\hat{N}_1 \oplus T_{\mathbf{x}_2}\hat{N}_2.$$

By using the orthogonal decomposition of the tangent vector space into vertical and horizontal subspaces

$$T_{\mathbf{x}}\hat{N} = \mathcal{V}_{\mathbf{x}}\hat{N} \oplus \mathcal{H}_{\mathbf{x}}\hat{N},$$

we have

$$\sqrt{-1}\mathbf{x}_1 + \sqrt{-1}\mathbf{x}_2 = \sqrt{-1}\mathbf{x} \in \mathcal{V}_{\mathbf{x}}\hat{N} \subset T_{\mathbf{x}}\hat{N} = T_{\mathbf{x}_1}\hat{N}_1 \oplus T_{\mathbf{x}_2}\hat{N}_2 \subset \mathbb{C}^{\ell_1+1} \oplus \mathbb{C}^{\ell_2+1}.$$

From this equation we see that

$$\sqrt{-1}\mathbf{x}_1 \in T_{\mathbf{x}_1}\hat{N}_1, \ \sqrt{-1}\mathbf{x}_2 \in T_{\mathbf{x}_2}\hat{N}_2.$$

If we set

$$\mathcal{H}_{\mathbf{x}_i}\hat{N}_i := \{v \in T_{\mathbf{x}_i}\hat{N}_i \mid \langle v, \sqrt{-1}\mathbf{x}_i\rangle = 0\} \quad (i = 1, 2),$$

then each tangent vector spaces can be orthogonally decomposed as

$$T_{\mathbf{x}_1}\hat{N}_1 = \mathbb{R}\sqrt{-1}\mathbf{x}_1 \oplus \mathcal{H}_{\mathbf{x}_1}\hat{N}_1, \quad T_{\mathbf{x}_2}\hat{N}_2 = \mathbb{R}\sqrt{-1}\mathbf{x}_2 \oplus \mathcal{H}_{\mathbf{x}_2}\hat{N}_2,$$

On the other hand, since $\hat{M}$ is the inverse image of M we have

$$T_{\mathbf{x}}S^{2n+1}(1) = \mathcal{V}_{\mathbf{x}}S^{2n+1}(1) \oplus \mathcal{H}_{\mathbf{x}}S^{2n+1}(1),$$

and

$$\mathcal{V}_{\mathbf{x}}\hat{N} = \mathcal{V}_{\mathbf{x}}S^{2n+1}(1) = \mathbb{R}\sqrt{-1}\mathbf{x} = \mathbb{R}\sqrt{-1}(\mathbf{x}_1 + \mathbf{x}_2).$$

Assume that $r_1 > 0$ and $r_2 > 0$. Then we can take a non-zero vector

$$\sqrt{-1}\left(\frac{1}{r_1^2}\mathbf{x}_1 - \frac{1}{r_2^2}\mathbf{x}_2\right) \in T_{\mathbf{x}_1}\hat{N}_1 \oplus T_{\mathbf{x}_1}\hat{N}_2 = T_{\mathbf{x}}\hat{N},$$

so that we have

$$\sqrt{-1}\mathbf{x} = \sqrt{-1}(\mathbf{x}_1 + \mathbf{x}_2) \perp \sqrt{-1}\left(\frac{1}{r_1^2}\mathbf{x}_1 - \frac{1}{r_2^2}\mathbf{x}_2\right).$$

Because

$$\begin{aligned} &\left\langle \sqrt{-1}\mathbf{x}, \sqrt{-1}\left(\frac{1}{r_1^2}\mathbf{x}_1 - \frac{1}{r_2^2}\mathbf{x}_2\right)\right\rangle \\ =&\left\langle \sqrt{-1}(\mathbf{x}_1 + \mathbf{x}_2), \sqrt{-1}\left(\frac{1}{r_1^2}\mathbf{x}_1 - \frac{1}{r_2^2}\mathbf{x}_2\right)\right\rangle \\ =&\frac{1}{r_1^2}\langle \mathbf{x}_1, \mathbf{x}_1\rangle - \frac{1}{r_2^2}\langle \mathbf{x}_2, \mathbf{x}_2\rangle = \frac{1}{r_1^2}r_1^2 - \frac{1}{r_2^2}r_2^2 = 1 - 1 = 0. \end{aligned}$$

Hence we obtain

$$\sqrt{-1}\left(\frac{1}{r_1^2}\mathbf{x}_1 - \frac{1}{r_2^2}\mathbf{x}_2\right) \in \mathcal{H}_{\mathbf{x}}\hat{N}.$$

However, since M is a complex submanifold of $\mathbb{C}P^n$, we have $\sqrt{-1}\mathcal{H}_{\mathbf{x}}\hat{N} = \mathcal{H}_{\mathbf{x}}\hat{N}$ and thus

$$\begin{aligned} -\frac{1}{r_1^2}\mathbf{x}_1 + \frac{1}{r_2^2}\mathbf{x}_2 =& \sqrt{-1}\sqrt{-1}\left(\frac{1}{r_1^2}\mathbf{x}_1 - \frac{1}{r_2^2}\mathbf{x}_2\right) \\ \in& \sqrt{-1}\mathcal{H}_{\mathbf{x}}\hat{N} = \mathcal{H}_{\mathbf{x}}\hat{N} \subset T_{\mathbf{x}}\hat{N} = T_{\mathbf{x}_1}\hat{N}_1 \oplus T_{\mathbf{x}_2}\hat{N}_2 \subset \mathbb{C}^{\ell_1+1} \oplus \mathbb{C}^{\ell_2+1}. \end{aligned}$$

Hence $\mathbf{x}_1 \in T_{\mathbf{x}_1}\hat{N}_1$ and $\mathbf{x}_2 \in T_{\mathbf{x}_2}\hat{N}_2$, a contradiction. Therefore we obtain that $r_1 = 0$ or $r_2 = 0$. □

Therefore we obtain a result of Takeuchi (Theorem 1.2) as a corollary:

Corollary 3.8. *Assume that M is a parallel Kähler submanifold of $\mathbb{C}P^n$. Then its inverse image $\hat{M} = \pi^{-1}(M) \subset S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ is a standardly embedded R-space which is obtained as an orbit of the isotropy representation of an irreducible Hermitian symmetric pair (G, K).*

By using results of [**18**] we can also determine explicitly *seven Kähler submanifols* of complex projective spaces, which give all parallel Kähler submanifolds of $\mathbb{C}P^n$.

M^m	m	n	(G, K)
$\mathbb{C}P^m(4)$	m	m	$(SU(m+2), S(U(m+1) \times U(1)))$
$\mathbb{C}P^m(2)$	m	$m + \frac{m(m+1)}{2}$	$(Sp(m+1), U(m+1))$
$\mathbb{C}P^{m-s}(1) \times \mathbb{C}P^s(1)$	m	$m + s(m-s)$	$(SU(m+2), S(U(m-s+1) \times U(s+1)))$
$Q_m(\mathbb{C})$	m	$m+1$	$(SO(m+4), SO(m+2) \times SO(2))$
$SU(s+2)/S(U(2) \times U(s))$	$2s$	$2s + \frac{s(s+1)}{2}$	$(SO(2(s+2)), U(s+2))$
$SO(10)/U(5)$	10	15	$(E_6, (Spin(10) \times U(1))/\mathbb{Z}_4)$
$E_6/((Spin(10) \times U(1))/\mathbb{Z}_4)$	16	26	$(E_7, (E_6 \times U(1))/\mathbb{Z}_3)$

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