Global existence of solutions to a parabolic attraction-repulsion chemotaxis system in $\mathbb{R}^2$: the attractive dominant case

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Abstract

We discuss the Cauchy problem for the following parabolic attraction-repulsion chemotaxis system:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u\nabla(\beta_1 v_1 - \beta_2 v_2)), & t > 0,\ x \in \mathbb{R}^2, \\ \partial_t v_j = \Delta v_j - \lambda_j v_j + u, & t > 0,\ x \in \mathbb{R}^2 \quad (j = 1, 2), \\ u(t, 0) = u_0(x),\ v_{j0}(t, 0) = v_{j0}(x), & x \in \mathbb{R}^2 \quad (j = 1, 2) \end{cases}$$

with constants $\beta_j,\ \lambda_j > 0$ $(j = 1, 2)$. In this paper we prove that the nonnegative solutions exist globally in time under the assumption $(\beta_1 - \beta_2)\int_{\mathbb{R}^2} u_0\, dx < 8\pi$ in the attractive dominant case $\beta_1 > \beta_2$.

Key words: Global existence; A priori estimate; Modified entropy
2020 Mathematics subject classification: 35A01; 35B45; 35K45; 35Q92

1 Introduction

In this paper we consider the Cauchy problem for a parabolic attraction-repulsion chemotaxis system:

$$\text{(CP)} \qquad \begin{cases} \partial_t u = \Delta u - \nabla \cdot (\beta_1 u \nabla v_1) + \nabla \cdot (\beta_2 u \nabla v_2), & t > 0,\ x \in \mathbb{R}^2, \\ \tau_j \partial_t v_j = \Delta v_j - \lambda_j v_j + u, & t > 0,\ x \in \mathbb{R}^2 \quad (j = 1, 2), \\ u(0, x) = u_0(x),\ \tau_j v_j(0, x) = \tau_j v_{j0}(x), & x \in \mathbb{R}^2 \quad (j = 1, 2), \end{cases}$$

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where β_j, λ_j $(j = 1, 2)$ are positive constants, $\tau_1, \tau_2 \in \{0, 1\}$, and u_0, v_{10}, and v_{20} are nonnegative functions. For initial data, we impose the following regularity conditions:

$$u_0 \geq 0,\ u_0 \not\equiv 0,\ u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2), \tag{1.1}$$

$$v_{j0} \geq 0,\ v_{j0},\ \nabla v_{j0} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \quad (j = 1, 2). \tag{1.2}$$

This system was proposed in [11] to describe the aggregation process of *Microglia*. In the system, the functions $u(t, x)$, $v_1(t, x)$, and $v_2(t, x)$ on $[0, \infty) \times \mathbb{R}^2$ represent the density of *Microglia*, the chemical concentration of attractive, and repulsive signals, respectively.

Various types of Chemotaxis model have been widely and extensively studied in the past decades. In particular, the parabolic-elliptic-elliptic counterpart:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (\beta_1 u \nabla v_1) + \nabla \cdot (\beta_2 u \nabla v_2), & t > 0,\ x \in \mathbb{R}^2, \\ 0 = \Delta v_j - \lambda_j v_j + u, & t > 0,\ x \in \mathbb{R}^2 \quad (j = 1, 2), \\ u(0, x) = u_0(x), & x \in \mathbb{R}^2 \end{cases} \tag{1.3}$$

attracts lots of attention by several researchers, for instance, Shi–Wang [22] and Nagai–Yamada [18, 20]. The main question posed in these works is to ask if the system (1.3) has global-in-time (classical) solutions depending on the relation between β_1, β_2 and on the size of initial mass $\|u_0\|_{L^1}$. We just recall some known results. For the repulsion-dominant case, i.e., $\beta_1 < \beta_2$, Shi–Wang [22] proved that, without any restriction on the size of initial mass $\|u_0\|_{L^1(\mathbb{R}^2)}$, every local-in-time solution of the system (1.3) may be extended for all time and remains bounded in $\mathbb{R}^2$ uniformly with respect to t. Nagai–Yamada [18] proved that this result continues to hold for the balanced case, i.e., $\beta_1 = \beta_2$. In view of the relation between β_1 and β_2, this last result is optimal in the sense that the result does not necessarily hold for the attraction-dominant case, i.e., $\beta_1 > \beta_2$. In this case, Nagai–Yamada [18,20] showed that all solutions exist globally in time if $\|u_0\|_{L^1(\mathbb{R}^2)} \leq 8\pi/(\beta_1 - \beta_2)$. Moreover, the boundedness of global in time solutions was discussed in Nagai–Yamada [21]. On the other hand, Shi–Wang [22] proved that finite-time blow up does occur for some initial data satisfying $\|u_0\|_{L^1(\mathbb{R}^2)} > 8\pi/(\beta_1 - \beta_2)$. More precisely, it was proved that there exists a small number $r_0 > 0$ such that if the size of initial mass $\|u_0\|_{L^1(\mathbb{R}^2)}$ is larger than $8\pi/(\beta_1 - \beta_2)$ and

$$\int_{\mathbb{R}^2} |x - x_0|^2\, u_0(x) dx < r_0$$

with some point $x_0 \in \mathbb{R}^2$, then the solution blows up in finite time. Here, by *finite-time blow-up*, we mean

$$\limsup_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} = +\infty \tag{1.4}$$

for some $T_{\max} < \infty$. Such a value $T_{\max}$ is called *the maximal existence time.* In this sense, we understand that the number $8\pi/(\beta_1 - \beta_2)$ is the threshold value for initial mass, below which solutions are global in time and above which some solutions blow up in finite time. This critical mass phenomenon is well-known for the classical Keller–Segel model, which corresponds to the case $\beta_2 = 0$ in (1.3). See, for instance, [3, 14–17, 19].

We shall turn our attention to the fully parabolic system (CP). The Cauchy–Neumann problem on bounded domains Ω in $\mathbb{R}^2$ have been treated by many researchers (see [2, 4, 7–10, 24]). Fujie–Suzuki [4] especially showed that the global existence of solutions holds if $\|u_0\|_{L^1(\Omega)} < 4\pi/(\beta_1 - \beta_2)$, or the radially symmetric function u_0 satisfies $\|u_0\|_{L^1(\Omega)} < 8\pi/(\beta_1 - \beta_2)$. Concerning the Cauchy problem for (CP) in $\mathbb{R}^2$, the result obtained for (1.3) with $\beta_1 = \beta_2$ was extended therein to the system (CP) by Jin–Liu [6]. For the repulsion-dominant case, i.e., $\beta_1 < \beta_2$, the third author [26] has recently proved that every global solution is bounded uniformly in time. For the attraction-dominant case, i.e., $\beta_1 > \beta_2$, Shi–You [23] recently asserted that nonnegative solutions to (CP) with $\tau_1 = 1$ and $\tau_2 = 0$ exist globally in time under the condition $\|u_0\|_{L^1(\mathbb{R}^2)} < 8\pi/(\beta_1 - \beta_2)$. However, to the best of our knowledge, the case $\beta_1 > \beta_2$ and $\tau_1 = \tau_2 = 1$ has been left open. Our aim in this article is to fill this gap. We are now in a position to state our main result.

Theorem 1.1. *Let u_0 and v_{j0} $(j = 1, 2)$ satisfy* (1.1) *and* (1.2)*, respectively. Assume that $\beta_1 > \beta_2$ holds. If the initial mass is subcritical in the sense that*

$$\int_{\mathbb{R}^2} u_0\, dx < \frac{8\pi}{\beta_1 - \beta_2} \tag{1.5}$$

is true, then the nonnegative solution of (CP) *with $\tau_1 = \tau_2 = 1$ exists globally in time.*

Our strategy for proving Theorem 1.1 is to use the characterization of maximal existence time in terms of the L^∞-norm of $u(t)$ (cf. (iv) of Proposition 2.1). In order to obtain a priori estimates on $\|u(t)\|_{L^\infty(\mathbb{R}^2)}$, we rely on Moser iteration scheme, which has been used in a number of PDE problems. It is essential to show its first step, i.e., obtaining an *a priori* estimate on $\|u(t)\|_{L^2(\mathbb{R}^2)}$. To this end, we introduce a functional $\mathcal{F}(u, v, w)(t)$(cf. (3.3) below), what we call **modified free energy functional**, for the particular system (CP) with $\tau_1 = \tau_2 = 1$. This nontrivial definition of $\mathcal{F}(u, v, w)(t)$ captures a feature of the fully parabolic system and is different from the one introduced in [23] for partially elliptic simplified systems. In fact, we first introduce a change of unknown functions and then define the functional $\mathcal{F}(u, v, w)(t)$ for the new unknown functions. In particular, it involves three absorption terms, which make our analysis successful in deriving desired estimates. We then combine a useful modified free energy identity on $\mathcal{F}(u, v, w)(t)$ with the Trudinger–Moser inequality, using the idea of [12] that makes the inequality useful even in unbounded domains. Consequently, we obtain an estimate of the form:

$$\delta_0 \int_{\mathbb{R}^2} (1 + u(t)) \log(1 + u(t))\, dx + \frac{1}{\beta_1 - \beta_2} \int_0^t \int_{\mathbb{R}^2} (\beta_1 \partial_t v_1(s) - \beta_2 \partial_t v_2(s))^2\, dx d\tau \le C, \tag{1.6}$$

where $\delta_0 \in (0, 1]$ is some constant. Once this is shown, we can argue as in [23], to conclude the proof of Theorem 1.1.

The rest of this article is structured as follows: In Section 2 we collect some tools used in Theorem 1.1. Section 3 is devoted to the proof of the modified free energy identity (3.4) below. In Section 4 we derive a priori estimates (1.6) by applying the modified free energy identity and the Trudinger–Moser inequality. We prove Theorem 1.1 in Section 5. For the convenience of readers, we demonstrate the Moser iteration technique in the Appendix. As a result, we show that (1.6) implies an L^∞ bound for $u(t)$ for any time-interval.

Notation. For $1 \le p \le \infty$ and $T > 0$, let L^p be the standard Lebesgue space on $\mathbb{R}^2$ with the norm $\|\cdot\|_p$ and let $L^p(0,T;X)$ be the set of all p-integrable functions over interval $(0,T)$ with values in a Banach space X, whose norm is denoted as $\|\cdot\|_{L^p(0,T;X)}$. For $k \in \mathbb{N}$ and $1 \le p \le \infty$, $W^{k,p}$ stands for the standard Sobolev space on $\mathbb{R}^2$ with the norm $\|\cdot\|_{W^{k,p}}$ and $W^{k,2} =: H^k$. Symbol $\mathbb{Z}_+$ is the set of all nonnegative integers. We set $|\alpha| = \alpha_1 + \alpha_2$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$. Partial derivatives of order m with respect to t and x_j are denoted by ∂_t^m and ∂_j^m, respectively, and set $\nabla = {}^t(\partial_1, \partial_2)$ and $\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$. Symbol C is a positive constant which may vary line to line. In particular, $C(*, \dots, *)$ denotes a positive constant depending on the quantities in parentheses.

2 Preliminaries

First of all, we state that the existence of local in time solutions to (CP) and some properties of the solutions are established by virtue of the method in [22, §2] (see also [23]).

Proposition 2.1. *Let u_0 and v_{j0} $(j = 1, 2)$ satisfy* (1.1) *and* (1.2)*, respectively. Then there exists a positive constant T_0 such that the system* (CP) *has a unique smooth solution (u, v_1, v_2) on $[0, T_0] \times \mathbb{R}^2$. Furthermore, the following assertions hold:*

(i) $u, v_1, v_2 \in C([0, T_0]; L^p)$ $(1 \le p < \infty)$, $\sup_{0<t<T_0} \|(u, v_1, v_2)\|_\infty < \infty$.

(ii) $\partial_t^k \partial_x^\alpha u, \partial_t^k \partial_x^\alpha v_1, \partial_t^k \partial_x^\alpha v_2 \in C((0, T_0]; L^p)$ $(1 < p \le \infty,\ k \in \mathbb{Z}_+,\ \alpha \in \mathbb{Z}_+^2,\ k + |\alpha| \ge 1)$.

(iii) $u(t,x) > 0,\ v_1(t,x) > 0,\ v_2(t,x) > 0$ $(0 < t < T_0,\ x \in \mathbb{R}^2)$.

(iv) *If the maximal existence time T_{max} is finite, then*

$$\limsup_{t \uparrow T_{max}} \|u(\cdot, t)\|_\infty = +\infty. \tag{2.1}$$

Proposition 2.2. *For every $0 < t < T$, we have*

$$\|u(t)\|_1 = \|u_0\|_1, \tag{2.2}$$

$$\|v_j(t)\|_1 = e^{-\lambda_j t}\|v_{j0}\|_1 + \lambda_j^{-1}(1 - e^{-\lambda_j t})\|u_0\|_1 \quad (j = 1, 2). \tag{2.3}$$

Given a function $f \in L^q$ $(1 \le q \le \infty)$, we define, as usual, the heat semigroup $e^{t\Delta}f$ as

$$(e^{t\Delta}f)(x) := \int_{\mathbb{R}^2} G(t, x-y) f(y)\, dy, \quad t > 0,$$
$$\text{where } G(t,x) = \frac{1}{4\pi t}\exp\left(-\frac{|x|^2}{4t}\right).$$

We just recall some basic estimates concerning the heat semigroup as well as the Trudinger–Moser inequality:

Proposition 2.3 (L^p-L^q estimates [5]). *Let $1 \le q \le p \le \infty$ and $\alpha \in \mathbb{Z}_+^2$ hold. Then there exists a positive constant $C(p,q,\alpha)$ depending only on p, q, and α such that*

$$\|\partial_x^\alpha e^{t\Delta} f\|_p \le C(p,q,\alpha) t^{-1/q+1/p-|\alpha|/2} \|f\|_q, \quad t > 0. \tag{2.4}$$

In particular, $C(p,q,\alpha) = 1$ if $|\alpha| = 0$ and $p = q$.

Proposition 2.4. *Let $\lambda > 0$, $0 < T \le \infty$, and $f \in L^\infty(0,T;L^q)$ $(1 \le q \le \infty)$ be given. Then the functions $F_\lambda(t) \in W^{1,p}$, $0 \le t < T$, defined as*

$$F_\lambda(t) := \int_0^t e^{-\lambda(t-s)} e^{(t-s)\Delta} f(s)\, ds, \quad 0 < t < T \tag{2.5}$$

enjoy the following estimates:

(i) *If $1 < q \le p \le \infty$ or $1 = q \le p < \infty$, then:*

$$\|F_\lambda(t)\|_p \le C(p,q) \lambda^{-(1-1/q+1/p)} \|f\|_{L^\infty(0,T;L^q)}, \quad 0 < t < T.$$

(ii) *If $1 \le q \le p < 2q/(2-q)$, $2 < q \le p \le \infty$ or $2 = q \le p < \infty$, then:*

$$\|\nabla F_\lambda(t)\|_p \le C(p,q) \lambda^{-(1/2-1/q+1/p)} \|f\|_{L^\infty(0,T;L^q)}, \quad 0 < t < T.$$

Proof. The proof is the same as in [20, Lemma 2.3], so we omit it. □

Proposition 2.5 (Trudinger–Moser inequality [13, 25]). *Let Ω be a two-dimensional domain with finite Lebesgue measure. Then there exists a positive constant C_{TM}, independent of Ω, such that inequality*

$$\frac{1}{|\Omega|} \int_\Omega e^{|g|}\, dx \le C_{TM} \exp\left(\frac{1}{16\pi} \|\nabla g\|_{L^2(\Omega)} \right) \tag{2.6}$$

holds for every $g \in H_0^1(\Omega)$, where $|\Omega|$ denotes the Lebesgue measure of Ω.

Remark 2.6. (i) The Trudinger–Moser inequality holds for open sets Ω with $|\Omega| < \infty$ by the proof of Moser [13] using rearrangement techniques.

(ii) We have $C_{TM} \ge 1$ by taking $g \equiv 0$ in (2.6).

3 Modified free energy identity

In what follows, we denote by (u, v_1, v_2) the nonnegative solution of (CP) defined on $[0,T]$ for some $0 < T < \infty$. Let us set

$$v = \beta_1 v_1 - \beta_2 v_2, \quad w = v_1 - v_2, \quad \beta = \beta_1 - \beta_2. \tag{3.1}$$

Under the assumption $\beta_1 \neq \beta_2$, system (CP) is reduced to

$$\partial_t u = \Delta u - \nabla \cdot (u \nabla v), \tag{3.2a}$$

$$\partial_t v = \Delta v - a_1 v + a_2 w + \beta u, \tag{3.2b}$$

$$\partial_t w = \Delta w - b_1 w - b_2 v, \tag{3.2c}$$

where

$$a_1 = \frac{\lambda_1\beta_1 - \lambda_2\beta_2}{\beta}, \quad a_2 = \frac{\beta_1\beta_2(\lambda_1 - \lambda_2)}{\beta}, \quad b_1 = \frac{\lambda_2\beta_1 - \lambda_1\beta_2}{\beta}, \quad b_2 = \frac{\lambda_1 - \lambda_2}{\beta}.$$

Putting $a = b(\lambda_1 - \lambda_2)/\beta$ and $b = \beta_1\beta_2/\beta$, we now define a functional $\mathcal{F}(u, v, w)(t)$ as

$$\begin{aligned} \mathcal{F}(u, v, w)(t) = &\int_{\mathbb{R}^2} (1 + u(t)) \log(1 + u(t))\, dx - \int_{\mathbb{R}^2} u(t)v(t)\, dx \\ &+ \frac{1}{2\beta} \int_{\mathbb{R}^2} (|\nabla v(t)|^2 + a_1 v^2(t))\, dx - a \int_{\mathbb{R}^2} v(t)w(t)\, dx \\ &- \frac{b}{2} \int_{\mathbb{R}^2} (|\nabla w(t)|^2 + b_1 w^2(t))\, dx \end{aligned} \tag{3.3}$$

and call it **modified free energy functional** for the system (3.2).

Due to the regularity properties of solutions and the elementary estimates

$$(1 + s) \log(1 + s) = \begin{cases} O(s) & \text{as } s \to 0 \\ O(s^{1+\alpha}) & \text{as } s \to \infty \end{cases}$$

for every $\alpha > 0$, it turns out that the functional $\mathcal{F}(u, v, w)(t)$ $(0 \leq t < T_{\max})$ is well-defined.

Remark 3.1. In the case $\lambda_1 = \lambda_2$, we have $a_1 = b_1 = \lambda_1$ and $a_2 = b_2 = 0$, so system (3.2) is reduced to a classical parabolic Keller–Segel system. For this system, the global existence has been already discussed in [12]. Although, the second component, which corresponds to v above is assumed to be nonnegative in [12], the proof there works without any change even if it is sign-changing. We therefore assume $\lambda_1 \neq \lambda_2$ throughout this article.

We now state a modified free energy identity.

Lemma 3.2 (Modified free energy identity)**.** *For every $0 < t < T$, one has*

$$\mathcal{F}(u, v, w)(t) + \mathcal{D}(t) = \mathcal{F}(u, v, w)(0) + \int_0^t \int_{\mathbb{R}^2} \left(\frac{1}{4} |\nabla v|^2 + b(\partial_t w)^2 \right) dxds, \tag{3.4}$$

where

$$\begin{aligned} \mathcal{D}(t) = &\frac{1}{\beta} \int_0^t \int_{\mathbb{R}^2} (\partial_t v)^2\, dxds + \int_0^t \int_{\mathbb{R}^2} u |\nabla(\log(1 + u) - v)|^2\, dxds \\ &+ \int_0^t \int_{\mathbb{R}^2} \left| \nabla \left(\log(1 + u) - \frac{v}{2} \right) \right|^2 dxds. \end{aligned}$$

Proof. Noting $\int_{\mathbb{R}^2} \partial_t u \, dx = 0$ due to (2.2), we have

$$\frac{d}{dt}\int_{\mathbb{R}^2} \{(1+u)\log(1+u) - uv\}\, dx = \int_{\mathbb{R}^2} \partial_t u(\log(1+u) - v)\, dx - \int_{\mathbb{R}^2} u\partial_t v\, dx.$$

Using $\partial_t u = \nabla\cdot(u\nabla(\log(1+u)-v)) + \Delta\log(1+u)$ and integration by parts, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \partial_t u(\log(1+u) - v)\, dx \\
&= \int_{\mathbb{R}^2} \nabla\cdot(u\nabla(\log(1+u)-v))(\log(1+u)-v)\, dx + \int_{\mathbb{R}^2} \Delta\log(1+u)(\log(1+u)-v)\, dx \\
&= -\int_{\mathbb{R}^2} u|\nabla(\log(1+u)-v)|^2\, dx - \int_{\mathbb{R}^2} \nabla\log(1+u)\cdot\nabla(\log(1+u)-v)\, dx \\
&= -\int_{\mathbb{R}^2} u|\nabla(\log(1+u)-v)|^2\, dx - \int_{\mathbb{R}^2} |\nabla\log(1+u)|^2\, dx + \int_{\mathbb{R}^2} \nabla\log(1+u)\cdot\nabla v\, dx \\
&= -\int_{\mathbb{R}^2} u|\nabla(\log(1+u)-v)|^2\, dx - \int_{\mathbb{R}^2} \left|\nabla\left(\log(1+u) - \frac{v}{2}\right)\right|^2 dx + \frac{1}{4}\int_{\mathbb{R}^2} |\nabla v|^2\, dx,
\end{aligned}$$

whence:

$$\begin{aligned}
(3.5)\quad &\frac{d}{dt}\int_{\mathbb{R}^2} \{(1+u)\log(1+u) - uv\}\, dx + \int_{\mathbb{R}^2} u|\nabla(\log(1+u)-v)|^2\, dx \\
&\quad + \int_{\mathbb{R}^2} \left|\nabla\left(\log(1+u) - \frac{v}{2}\right)\right|^2 dx + \int_{\mathbb{R}^2} u\partial_t v\, dx \;=\; \frac{1}{4}\int_{\mathbb{R}^2} |\nabla v|^2\, dx.
\end{aligned}$$

By use of (3.2b), we obtain

(3.6)

$$\begin{aligned}
\int_{\mathbb{R}^2} u\partial_t v\, dx =& \frac{1}{\beta}\int_{\mathbb{R}^2} (\partial_t v - \Delta v + a_1 v - a_2 w)\partial_t v\, dx \\
=& \frac{1}{\beta}\int_{\mathbb{R}^2} (\partial_t v)^2\, dx + \frac{1}{\beta}\int_{\mathbb{R}^2} \nabla v\cdot\nabla\partial_t v\, dx + \frac{a_1}{\beta}\int_{\mathbb{R}^2} v\partial_t v\, dx - \frac{a_2}{\beta}\int_{\mathbb{R}^2} w\partial_t v\, dx \\
=& \frac{1}{\beta}\int_{\mathbb{R}^2} (\partial_t v)^2\, dx + \frac{1}{2\beta}\frac{d}{dt}\int_{\mathbb{R}^2} (|\nabla v|^2 + a_1 v^2)\, dx - \frac{a_2}{\beta}\frac{d}{dt}\int_{\mathbb{R}^2} vw\, dx \\
&+ \frac{a_2}{\beta}\int_{\mathbb{R}^2} v\partial_t w\, dx.
\end{aligned}$$

Also, we see from $-\partial_t w + \Delta w - b_1 w = b_2 v$ that

$$\begin{aligned}
\int_{\mathbb{R}^2} v\partial_t w\, dx =& \frac{1}{b_2}\int_{\mathbb{R}^2} (-\partial_t w + \Delta w - b_1 w)\partial_t w\, dx \\
=& -\frac{1}{b_2}\int_{\mathbb{R}^2} (\partial_t w)^2\, dx - \frac{1}{2b_2}\frac{d}{dt}\int_{\mathbb{R}^2} (|\nabla w|^2 + b_1 w^2)\, dx,
\end{aligned}$$

which together with $a_2/(\beta b_2) = \beta_1\beta_2/\beta = b$ implies that

$$(3.7)\qquad \frac{a_2}{\beta}\int_{\mathbb{R}^2} v\partial_t w\, dx = -b\int_{\mathbb{R}^2} (\partial_t w)^2\, dx - \frac{b}{2}\frac{d}{dt}\int_{\mathbb{R}^2} (|\nabla w|^2 + b_1 w^2)\, dx.$$

Substituting (3.7) into (3.6) and then making use of $a_2/\beta = bb_2 = a$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} u\partial_t v\, dx =& \frac{1}{\beta}\int_{\mathbb{R}^2}(\partial_t v)^2\, dx + \frac{1}{2\beta}\frac{d}{dt}\int_{\mathbb{R}^2}(|\nabla v|^2 + a_1 v^2)\, dx - a\frac{d}{dt}\int_{\mathbb{R}^2} vw\, dx \\
& - \frac{b}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|\nabla w|^2 + b_1 w^2)\, dx - b\int_{\mathbb{R}^2}(\partial_t w)^2\, dx.
\end{aligned} \tag{3.8}
$$

Combining (3.5) with (3.8) gives that

(3.9)

$$
\begin{aligned}
& \frac{d}{dt}\int_{\mathbb{R}^2}\{(1+u)\log(1+u) - uv\}\, dx + \frac{1}{2\beta}\frac{d}{dt}\int_{\mathbb{R}^2}(|\nabla v|^2 + a_1 v^2)\, dx \\
& - a\frac{d}{dt}\int_{\mathbb{R}^2} vw\, dx - \frac{b}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|\nabla w|^2 + b_1 w^2)\, dx \\
& + \int_{\mathbb{R}^2} u|\nabla(\log(1+u) - v)|^2\, dx + \int_{\mathbb{R}^2}\left|\nabla\left(\log(1+u) - \frac{v}{2}\right)\right|^2 dx + \frac{1}{\beta}\int_{\mathbb{R}^2}(\partial_t v)^2\, dx \\
=& \frac{1}{4}\int_{\mathbb{R}^2}|\nabla v|^2\, dx + b\int_{\mathbb{R}^2}(\partial_t w)^2\, dx.
\end{aligned}
$$

The integration of the last identity over $[0, T]$ completes the proof. □

4 *A priori* estimates for the system (CP)

In this section let v, w and β be the same symbols as in (3.1). We begin with showing some auxiliary estimates.

Lemma 4.1. *The following estimates are true:*

(i) *For every $1 \le p < \infty$ and each $j = 1, 2$, there exists a constant $C_1 = C_1(p, \lambda_j)$ such that*

$$\|v_j(t)\|_p \le e^{-\lambda_j t}\|v_{j0}\|_p + C_1\|u_0\|_1 \quad (0 < t < T).$$

(ii) *For every $1 \le p < \infty$ and $0 < t < T$, there exists a constant $C_2 = C_2(p, \lambda_1, \lambda_2) > 0$ such that*

$$\|w(t)\|_p \le e^{-\lambda_1 t}\|w(0)\|_p + C_2(e^{-\lambda_2 t}\|v_{20}\|_1 + \|u_0\|_1) \quad (0 < t < T).$$

(iii) *For every $2 \le p < \infty$ and $0 < t < T$, there exists a constant $C_3 = C_3(p, \lambda_1, \lambda_2) > 0$ such that*

$$\|\nabla w(t)\|_p \le e^{-\lambda_1 t}\|\nabla w(0)\|_p + C_3(e^{-\lambda_2 t}\|v_{20}\|_2 + \|u_0\|_1) \quad (0 < t < T).$$

(iv) *There exists a constant $C_4 = C_4(\lambda_1, \lambda_2) > 0$ such that*

$$\int_0^T \|\partial_t w(t)\|_2^2\, dt \le C_4(\|(v_{10}, v_{20})\|_{H^1}^2 + T\|u_0\|_1^2).$$

Proof. The first claim (i) is an immediate consequence of (2.4) and Proposition 2.4(i).

Notice that w satisfies equation $\partial_t w = \Delta w - \lambda_1 w + (\lambda_2 - \lambda_1)v_2$. By means of the heat semigroup, this can be recast as the integral equation

$$w(t) = e^{-\lambda_1 t}e^{t\Delta}w(0) + (\lambda_2 - \lambda_1)\int_0^t e^{-\lambda_1(t-s)}e^{(t-s)\Delta}v_2(s)\,ds.$$

For $1 \le p \le \infty$, the L^p-L^p estimate (2.4) yields

$$\|e^{t\Delta}w(0)\|_p \le \|w(0)\|_p, \quad 0 < t < T.$$

Taking advantage of Proposition 2.4(i) and Lemma 4.1(i), we may obtain

$$\left\|(\lambda_2 - \lambda_1)\int_0^t e^{-\lambda_1(t-s)}e^{(t-s)\Delta}v_2(s)\,ds\right\|_p \le C(p, \lambda_1, \lambda_2)(e^{-\lambda_2 t}\|v_{20}\|_1 + \|u_0\|_1)$$

for $0 < t < T$. Due to these estimates, we deduce the second claim (ii).

Assume that $2 \le p < \infty$. Since $\nabla e^{t\Delta}w(0) = e^{t\Delta}\nabla w(0)$, it follows from (2.4) that

$$\|\nabla e^{t\Delta}w(0)\|_p \le \|\nabla w(0)\|_p, \quad 0 < t < T.$$

Applying Proposition 2.4(ii) and Lemma 4.1(i), we obtain

$$\begin{aligned}&\left\|(\lambda_2 - \lambda_1)\int_0^t e^{-\lambda_1(t-s)}\nabla e^{(t-s)\Delta}v_2(s)\,ds\right\|_p\\ &\le C(p, \lambda_1, \lambda_2)\|v_2\|_{L^\infty(0,T;L^2)} \le C(p, \lambda_1, \lambda_2)(e^{-\lambda_2 t}\|v_{20}\|_2 + \|u_0\|_1)\end{aligned}$$

for $0 < t < T$. The third claim (iii) then follows.

We finally show the fourth claim (iv). Multiplying the equation $\partial_t w = \Delta w - \lambda_1 w + (\lambda_2 - \lambda_1)v_2$ by $\partial_t w$ and integrating the identity over $\mathbb{R}^2$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}(\partial_t w)^2\,dx &= \int_{\mathbb{R}^2}\Delta w\partial_t w\,dx - \lambda_1\int_{\mathbb{R}^2}w\partial_t w\,dx + (\lambda_2 - \lambda_1)\int_{\mathbb{R}^2}v_2\partial_t w\,dx\\ &= -\int_{\mathbb{R}^2}\nabla w\cdot\partial_t\nabla w\,dx - \frac{d}{dt}\left(\frac{\lambda_1}{2}\int_{\mathbb{R}^2}w^2\,dx\right) + (\lambda_2 - \lambda_1)\int_{\mathbb{R}^2}v_2\partial_t w\,dx\\ &\le -\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^2}|\nabla w|^2\,dx + \frac{\lambda_1}{2}\int_{\mathbb{R}^2}w^2\,dx\right) + \frac{(\lambda_2 - \lambda_1)^2}{2}\int_{\mathbb{R}^2}v_2^2\,dx\\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2}(\partial_t w)^2\,dx,\end{aligned}$$

whence:

$$\|\partial_t w(t)\|_2^2 + \frac{d}{dt}(\|\nabla w(t)\|_2^2 + \lambda_1\|w(t)\|_2^2) \le (\lambda_2 - \lambda_1)^2\|v_2(t)\|_2^2.$$

An integration of the last inequality in time leads to

$$\begin{aligned}&\int_0^T\|\partial_t w(t)\|_2^2\,dt + \|\nabla w(T)\|_2^2 + \lambda_1\|w(T)\|_2^2\\ &\qquad \le \|\nabla w(0)\|_2^2 + \lambda_1\|w(0)\|_2^2 + (\lambda_2 - \lambda_1)^2\int_0^T\|v_2(t)\|_2^2\,dt.\end{aligned}$$

Notice that $\|\nabla w(0)\|_2^2 + \lambda_1 \|w(0)\|_2^2 \le C(\lambda_1)(\|v_{10}\|_{H^1}^2 + \|v_{20}\|_{H^1}^2)$. Due to Lemma 4.1(i), we have

$$\int_0^T \|v_2(t)\|_2^2 \, dt \le C(\lambda_2)(\|v_{20}\|_{H^1}^2 + T\|u_0\|_1^2).$$

Therefore the fourth claim (iv) follows. The proof is now complete. □

Lemma 4.2. *For $0 < t \le T$, $s > 0$, let us set*

$$M(t) = \int_{D(t,s)} u(t) \, dx, \quad D(t,s) = \{x \in \mathbb{R}^2 \mid v(t,x) > s\}.$$

Then for every $\delta \in [0,1)$, the inequality

$$\begin{aligned}(4.1) \qquad \int_{\mathbb{R}^2} u(t)v(t)\,dx \le{}& (1-\delta)\int_{D(t,s)} (1+u(t))\log(1+u(t))\,dx \\ &+ \frac{1}{16\pi(1-\delta)} \left\{ \|u_0\|_1 + \frac{\beta_1}{s}\left(\|v_{10}\|_1 + \frac{1}{\lambda_1}\|u_0\|_1 \right) \right\} \|\nabla v(t)\|_2^2 + C(\delta,s)\end{aligned}$$

holds, where

$$(4.2) \quad C(\delta,s) = \begin{cases} s\|u_0\|_1, & D(t,s) = \emptyset, \\ (1-\delta)\left\{ \|u_0\|_1 + \frac{\beta_1}{s}\left(\|v_{10}\|_1 + \frac{1}{\lambda_1}\|u_0\|_1 \right) \right\} \log C_{TM} + s\|u_0\|_1, & D(t,s) \ne \emptyset, \end{cases}$$

and $C_{TM} (\ge 1)$ is the constant in the Trudinger–Moser inequality (2.6)*.*

Proof. Consider the case $D(t,s) = \emptyset$. Due to $v(t,x) \le s$ ($x \in \mathbb{R}^2$), we have

$$\int_{\mathbb{R}^2} u(t)v(t)\,dx \le s \int_{\mathbb{R}^2} u(t)\,dx = s\|u_0\|_1.$$

Hence (4.1) holds in this case.

Consider next the case $D(t,s) \ne \emptyset$. Since $v(t) \in C(\mathbb{R}^2)$, the set $D(t,s)$ is open in $\mathbb{R}^2$. Due to the fact $v(t) \in L^1(\mathbb{R}^2)$, we have $|D(t,s)| < \infty$. It follows that

$$\begin{aligned}\int_{\mathbb{R}^2} u(t)v(t)\,dx &= \int_{D(t,s)} u(t)\{v(t)-s\}\,dx + s\int_{D(t,s)} u(t)\,dx + \int_{\mathbb{R}^2\setminus D(t,s)} u(t)v(t)\,dx \\ &\le \int_{D(t,s)} u(t)(v(t)-s)^+\,dx + s\int_{D(t,s)} u(t)\,dx + s\int_{\mathbb{R}^2\setminus D(t,s)} u(t)\,dx \\ (4.3) \qquad &\le \int_{D(t,s)} (1+u(t))(v(t)-s)^+\,dx + s\|u_0\|_1.\end{aligned}$$

Let us write

$$g(t) := (1-\delta)(1+u(t)), \qquad h(t) := \frac{(v(t)-s)^+}{1-\delta},$$

so that

$$(4.4)\qquad \int_{D(t,s)} (1+u(t))(v(t)-s)^+\,dx = \int_{D(t,s)} g(t)h(t)\,dx,$$

$$(4.5)\qquad \int_{D(t,s)} g(t)\,dx = (1-\delta)(|D(t,s)| + M(t)) =: \widetilde{M}(t,s).$$

Applying Jensen's inequality for a convex function $-\log\cdot$, we obtain

$$(4.6)\quad \frac{1}{\widetilde{M}(t,s)}\left(\int_{D(t,s)} g(t)h(t)\,dx - \int_{D(t,s)} g(t)\log g(t)\,dx\right) = \int_{D(t,s)} \frac{g(t)}{\widetilde{M}(t,s)}\log\frac{e^{h(t)}}{g(t)}\,dx$$
$$\leq \log\left(\int_{D(t,s)} \frac{e^{h(t)}}{\widetilde{M}(t,s)}\,dx\right).$$

It then follows from (4.3)–(4.6) that

(4.7)

$$\begin{aligned}
&\int_{\mathbb{R}^2} u(t)v(t)\,dx\\
&\leq \int_{D(t,s)} g(t)\log g(t)\,dx + \widetilde{M}(t,s)\log\left(\int_{D(t,s)} \frac{e^{h(t)}}{\widetilde{M}(t,s)}\,dx\right) + s\|u_0\|_1\\
&\leq \int_{D(t,s)} g(t)\log g(t)\,dx + \widetilde{M}(t,s)\log\left(\int_{D(t,s)} e^{h(t)}\,dx\right) - \widetilde{M}(t,s)\log\widetilde{M}(t,s) + s\|u_0\|_1.
\end{aligned}$$

A straightforward calculation shows that

$$\int_{D(t,s)} g(t)\log g(t)\,dx = \widetilde{M}(t,s)\log(1-\delta) + (1-\delta)\int_{D(t,s)} (1+u(t))\log(1+u(t))\,dx,$$

where $\widetilde{M}(t,s)$ is as in (4.5). Due to this and (4.7), we have

$$\begin{aligned}
(4.8)\qquad \int_{\mathbb{R}^2} u(t)v(t)\,dx &\leq (1-\delta)\int_{D(t,s)} (1+u(t))\log(1+u(t))\,dx\\
&+\widetilde{M}(t,s)\log\left(\int_{D(t,s)} e^{h(t)}\,dx\right) + \widetilde{M}(t,s)(\log(1-\delta) - \log\widetilde{M}(t,s)) + s\|u_0\|_1.
\end{aligned}$$

We shall now pay attention to the second term of the right-hand side of (4.8). Since $v(t) \in H^1(\mathbb{R}^2)\cap C(\mathbb{R}^2)$, we have

$$(v(t)-s)^+ = 0 \text{ on } \partial D(t,s), \quad \nabla(v(t)-s)^+ = \begin{cases} \nabla v(t), & \text{in } D(t,s),\\ 0, & \text{in } \mathbb{R}^2\setminus D(t,s),\end{cases}$$

whence $(v(t)-s)^+ \in H_0^1(D(t,s))$. Applying the Trudinger–Moser inequality (2.6) and

$$|D(t,s)| + M(t) = \frac{\widetilde{M}(t,s)}{1-\delta}$$

for the second term of the right-hand side of (4.8), we then obtain

(4.9)

$$
\begin{aligned}
&\widetilde{M}(t,s)\log\left(\int_{D(t,s)} e^{h(t)}\,dx\right)\\
&\le \widetilde{M}(t,s)\left[\frac{1}{16\pi}\|\nabla h(t)\|^2_{L^2(D(t,s))}+\log(C_{TM}|D(t,s)|)\right]\\
&\le \frac{\widetilde{M}(t,s)}{16\pi(1-\delta)^2}\|\nabla v(t)\|_2^2+\widetilde{M}(t,s)\log C_{TM}+\widetilde{M}(t,s)\log\frac{\widetilde{M}(t,s)}{1-\delta}\\
&\le \frac{|D(t,s)|+M(t)}{16\pi(1-\delta)}\|\nabla v(t)\|_2^2+\widetilde{M}(t,s)\log C_{TM}+\widetilde{M}(t,s)(\log\widetilde{M}(t,s)-\log(1-\delta)).
\end{aligned}
$$

Using (4.9) in (4.8), we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} u(t)v(t)\,dx \le (1-\delta)\int_{D(t,s)} &(1+u(t))\log(1+u(t))\,dx\\
&+\frac{|D(t,s)|+M(t)}{16\pi(1-\delta)}\|\nabla v(t)\|_2^2+\widetilde{M}(t,s)\log C_{TM}+s\|u_0\|_1.
\end{aligned}
\tag{4.10}
$$

In order to estimate the measure $|D(t,s)|$, we recall $v=\beta_1 v_1-\beta_2 v_2$ and (2.3). Then:

$$
\begin{aligned}
\int_{D(t,s)} s\,dx &\le \int_{D(t,s)} v(t,x)\,dx \le \beta_1\int_{\mathbb{R}^2} v_1(t,x)\,dx\\
&\le \beta_1 e^{-\lambda_1 t}\int_{\mathbb{R}^2} v_{10}\,dx+\frac{\beta_1}{\lambda_1}(1-e^{-\lambda_1 t})\int_{\mathbb{R}^2} u_0\,dx \le \beta_1\left(\|v_{10}\|_1+\frac{1}{\lambda_1}\|u_0\|_1\right),
\end{aligned}
$$

whence:

$$
|D(t,s)| \le \frac{\beta_1}{s}\left(\|v_{10}\|_1+\frac{1}{\lambda_1}\|u_0\|_1\right). \tag{4.11}
$$

Combining (4.10) with (4.11) as well as an obvious estimate $M(t)\le\|u_0\|_1$, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^2} u(t)v(t)\,dx \le &(1-\delta)\int_{D(t,s)}(1+u(t))\log(1+u(t))\,dx\\
&+\frac{1}{16\pi(1-\delta)}\left\{\|u_0\|_1+\frac{\beta_1}{s}\left(\|v_{10}\|_1+\frac{1}{\lambda_1}\|u_0\|_1\right)\right\}\|\nabla v(t)\|_2^2\\
&+(1-\delta)\left\{\|u_0\|_1+\frac{\beta_1}{s}\left(\|v_{10}\|_1+\frac{1}{\lambda_1}\|u_0\|_1\right)\right\}\log C_{TM}+s\|u_0\|_1
\end{aligned}
$$

for every $\delta\in[0,1)$ and $t\in(0,T]$. Here we have used the fact that $C_{TM}\ge 1$ as well (See Remark 2.6). The claim (4.1) then follows and the proof is complete. □

Lemma 4.3. *Assume that the initial mass is subcritical, i.e.,* $\|u_0\|_1 < 8\pi/\beta$. *Then there exist constants* $\delta_0 \in (0,1)$ *and* $s_0 > 0$ *such that*

$$
\begin{aligned}
(4.12)\qquad \mathcal{F}(u,v,w)(t) \geq& \delta_0 \int_{\mathbb{R}^2} (1+u(t))\log(1+u(t))\,dx - C(\delta_0,s_0) + G(t)\\
\text{with}\ \ G(t) =& \frac{a_1}{2\beta}\|v(t)\|_2^2 - a\int_{\mathbb{R}^2} v(t)w(t)\,dx - \frac{b}{2}(\|\nabla w(t)\|_2^2 + b_1\|w(t)\|_2^2),
\end{aligned}
$$

where $\mathcal{F}(u,v,w)(t)$ *and* $C(\delta_0,s_0)$ *are given in* (3.3) *and* (4.2), *respectively.*

Proof. Fix $\delta \in [0,1)$. Due to (3.3) and (4.1), we have

$$
\begin{aligned}
\mathcal{F}(u,v,w)(t) \geq& \delta \int_{\mathbb{R}^2} (1+u(t))\log(1+u(t))\,dx\\
&+ (1-\delta)\int_{D(t,s)} (1+u(t))\log(1+u(t))\,dx - \int_{\mathbb{R}^2} u(t)v(t)\,dx + \frac{1}{2\beta}\|\nabla v(t)\|_2^2\\
&+ \frac{a_1}{2\beta}\|v(t)\|_2^2 - a\int_{\mathbb{R}^2} v(t)w(t)\,dx - \frac{b}{2}(\|\nabla w(t)\|_2^2 + b_1\|w(t)\|_2^2)\\
\geq& \delta \int_{\mathbb{R}^2} (1+u(t))\log(1+u(t))\,dx\\
&+ \left[\frac{1}{2\beta} - \frac{1}{16\pi(1-\delta)}\left\{\|u_0\|_1 + \frac{\beta_1}{s}\left(\|v_{10}\|_1 + \frac{1}{\lambda_1}\|u_0\|_1\right)\right\}\right]\|\nabla v(t)\|_2^2\\
&- C(\delta,s) + G(t).
\end{aligned}
$$

We now take a number $\delta_0 \in (0,1)$ such that

$$
0 < \delta_0 < \frac{8\pi - \beta\|u_0\|_1}{8\pi}
$$

(Note that we can certainly take such a δ_0 since $\|u_0\|_1 < 8\pi/\beta$ by assumption). Set

$$
\frac{1}{2\beta} - \frac{\|u_0\|_1}{16\pi(1-\delta_0)} =: A(\beta,\delta_0,\|u_0\|_1) > 0.
$$

For such a δ_0, we choose a number $s_0 > 0$ sufficiently large so that

$$
s_0 > \frac{\beta_1(\lambda_1\|v_{10}\|_1 + \|u_0\|_1)}{16\pi\lambda_1 A(\beta,\delta_0,\|u_0\|_1)(1-\delta_0)}
$$

or equivalently,

$$
A(\delta_0,\beta,\|u_0\|_1) - \frac{\beta_1}{16\pi(1-\delta_0)s_0}\left(\|v_{10}\|_1 + \frac{1}{\lambda_1}\|u_0\|_1\right) > 0.
$$

It then follows that

$$
\frac{1}{2\beta} - \frac{1}{16\pi(1-\delta_0)}\left\{\|u_0\|_1 + \frac{\beta_1}{s_0}\left(\|v_{10}\|_1 + \frac{1}{\lambda_1}\|u_0\|_1\right)\right\} > 0,
$$

whence the claim holds. The proof is now complete. □

Lemma 4.4. *Under the assumption of* $\|u_0\|_1 < 8\pi/\beta$*, there holds*

$$\delta_0 \int_{\mathbb{R}^2} (1+u(t))\log(1+u(t))\,dx + \frac{1}{\beta}\int_0^t \int_{\mathbb{R}^2} (\partial_t v)^2\,dxds \le C(T) \tag{4.13}$$

for $0<t<T$*, where* $\delta_0 \in (0,1)$ *is the constant defined in Lemma 4.3.*

Proof. To estimate $\int_{\mathbb{R}^2} u(t)v(t)dx$, we shall use the inequality (4.1) from Lemma 4.2. Due to (4.1) with $\delta = 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^2} u(t)v(t)\,dx \le &\int_{D(t,s)} (1+u(t))\log(1+u(t))\,dx \\ &+ \frac{1}{16\pi}\left\{\|u_0\|_1 + \frac{\beta_1}{s}\left(\|v_{10}\|_1 + \frac{1}{\lambda_1}\|u_0\|_1\right)\right\}\|\nabla v(t)\|_2^2 + C(0,s).\end{aligned} \tag{4.14}$$

Recalling the definition of the modified free energy functional, we deduce from (4.14),

$$\begin{aligned}&\frac{1}{2\beta}\|\nabla v(t)\|_2^2 \\ =&\mathcal{F}(u,v,w)(t) - \int_{\mathbb{R}^2}(1+u(t))\log(1+u(t))\,dx + \int_{\mathbb{R}^2} u(t)v(t)\,dx - G(t) \\ \le&\mathcal{F}(u,v,w)(t) + \left\{\int_{\mathbb{R}^2} u(t)v(t)\,dx - \int_{D(t,s)}(1+u(t))\log(1+u(t))\,dx\right\} - G(t) \\ \le&\mathcal{F}(u,v,w)(t) + \frac{1}{16\pi}\left\{\|u_0\|_1 + \frac{\beta_1}{s}\left(\|v_{10}\|_1 + \frac{1}{\lambda_1}\|u_0\|_1\right)\right\}\|\nabla v(t)\|_2^2 + C(0,s) + |G(t)|,\end{aligned}$$

which implies that for $0<t<T$,

$$\begin{aligned}k(s)\|\nabla v(t)\|_2^2 \le&\mathcal{F}(u,v,w)(t) + C(0,s) + |G(t)| \\ \text{with}\quad k(s) :=& \frac{1}{2\beta} - \frac{1}{16\pi}\left\{\|u_0\|_1 + \frac{\beta_1}{s}\left(\|v_{10}\|_1 + \frac{1}{\lambda_1}\|u_0\|_1\right)\right\}.\end{aligned}$$

Since $\|u_0\|_1 < 8\pi/\beta$ by assumption, there exists $s_1 > 0$ such that $k(s_1) > 0$ holds, whence:

$$\|\nabla v(t)\|_2^2 \le \frac{1}{k(s_1)}\{\mathcal{F}(u,v,w)(t) + C(0,s_1) + |G(t)|\} \quad (0<t<T). \tag{4.15}$$

Summarizing (3.4) and (4.15), we obtain

$$\begin{aligned}&\mathcal{F}(u,v,w)(t) + \mathcal{D}(t) \\ &\le \mathcal{F}(u,v,w)(0) + \frac{1}{4k(s_1)}\int_0^t \mathcal{F}(u,v,w)(s)\,ds + \frac{1}{4k(s_1)}\int_0^t \{C(0,s_1) + |G(s)|\}\,ds \\ &\qquad + b\int_0^t \|\partial_t w(s)\|_2^2\,ds.\end{aligned} \tag{4.16}$$

Here Lemma 4.1(i)–(iii) give

$$
\begin{aligned}
(4.17)\qquad |G(t)| \le & \frac{|a_1|}{2\beta}\|v(t)\|_2^2 + |a|\|v(t)\|_2\|w(t)\|_2 + \frac{b}{2}(\|\nabla w(t)\|_2^2 + |b_1|\|w(t)\|_2^2) \\
\le & C(\|(u_0, v_{10}, v_{20})\|_1, \|(v_{10}, v_{20})\|_{H^1}),
\end{aligned}
$$

and also Lemma 4.1(iv) yields

$$
\int_0^T \|\partial_t w(s)\|_2^2\, ds \le C_4(\|(v_{10}, v_{20})\|_{H^1}^2 + T\|u_0\|_1^2).
$$

Hence,

$$
\mathcal{F}(u,v,w)(0) + \frac{1}{4k(s_1)}\int_0^t \{C(0,s_1) + |G(s)|\}\, ds + b\int_0^t \|\partial_t w(s)\|_2^2\, ds \le \widetilde{C}(T),
$$

where $\widetilde{C}(T) := \mathcal{F}(u,v,w)(0) + C(\|(u_0, v_{10}, v_{20})\|_1, \|(v_{10}, v_{20})\|_{H^1}, T) > 0$. This implies

$$
(4.18)\qquad \mathcal{F}(u,v,w)(t) + \mathcal{D}(t) \le \widetilde{C}(T) + \frac{1}{4k(s_1)}\int_0^t \mathcal{F}(u,v,w)(s)\, ds \quad (0 < t < T).
$$

By noticing the positivity of $\mathcal{D}(t)$, the application of the Gronwall inequality to (4.18) then shows that the inequality

$$
\mathcal{F}(u,v,w)(t) + \mathcal{D}(t) \le \widetilde{C}(T) + \frac{\widetilde{C}(T)}{4k(s_1)} e^{\frac{T}{4k(s_1)}} =: \widehat{C}(T)
$$

holds for $0 < t < T$. Due to this, (4.12) and (4.17), we have

$$
\begin{aligned}
\delta_0 \int_{\mathbb{R}^2} (1 + u(t)) \log(1 + u(t))\, dx + \frac{1}{\beta}\int_0^t \int_{\mathbb{R}^2} (\partial_t v)^2\, dxds \\
\le \mathcal{F}(u,v,w)(t) + \mathcal{D}(t) + C(\delta_0, s_0) - G(t) \le C(T)
\end{aligned}
$$

for $0 < t < T$, where $C(T) := \widehat{C}(T) + C(\delta_0, s_0) + C(\|(u_0, v_{10}, v_{20})\|_1, \|(v_{10}, v_{20})\|_{H^1}, T)$. The proof is now complete. □

5 Proof of Theorem 1.1

The following proposition is key one to show Theorem 1.1.

Proposition 5.1. *Let $0 < T < \infty$. Assume that the nonnegative solution (u, v_1, v_2) to* (CP) *on $[0,T] \times \mathbb{R}^2$ satisfies*

$$
(5.1)\qquad \delta_0 \int_{\mathbb{R}^2} (1 + u(t)) \log(1 + u(t))\, dx + \frac{1}{\beta_1 - \beta_2}\int_0^t \int_{\mathbb{R}^2} (\partial_t v(s))^2\, dxd\tau \le C,
$$

where $v = \beta_1 v_1 - \beta_2 v_2$ and $\delta_0 \in (0,1)$ is some constant. Then:

$$
(5.2)\qquad \sup_{0<t<T} \|u(t)\|_{L^\infty} \le C(T).
$$

The proof of Proposition 5.1 is the same as in Shi–You [23, §5] (see also [12, 16]), but for the reader's convenience, we give its proof in the Appendix.

We now begin the proof of Theorem 1.1. Assume $T_{\max} < \infty$. Since $\|u_0\|_1 < 8\pi/(\beta_1 - \beta_2)$ by assumption, Lemma 4.4 guarantees that the a priori estimate (4.13) holds for $T = T_{\max}$. Proposition 5.1 then guarantees $\sup_{0<t<T_{\max}} \|u(t)\|_\infty \le C(T_{\max})$, which contradicts (2.1). The proof is complete.

A Proof of Proposition 5.1

The following Gagliardo–Nirenberg inequality in $\mathbb{R}^2$ is used in the course of the proof of Proposition 5.1 (for the inequality, see, e.g., [5]): Let $1 \le q \le p < \infty$ and $\sigma = 1 - q/p$. Then there is a positive constant C depending only on p and q such that for all $f \in L^q$ with $|\nabla f| \in L^2$,

$$\|f\|_p \le C\|\nabla f\|_2^\sigma \|f\|_q^{1-\sigma}. \tag{A.1}$$

Let v, w, and β be the ones defined as in (3.1). Following Shi–You [23, §5], we show Proposition 5.1. We prepare some lemmas.

Lemma A.1. *There is a postive constant $C(T)$ depending on T such that*

$$\sup_{0<t<T} \|u(t)\|_2 \le C(T). \tag{A.2}$$

Proof. Multiplying equation (3.2a) by u and then integrating by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_2^2 &= -\frac{1}{2}\int_{\mathbb{R}^2} u^2 \Delta v\, dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^2} u^2 \partial_t v\, dx - \frac{1}{2}\int_{\mathbb{R}^2} u^2\left(a_1 v - a_2 w\right) dx + \frac{\beta}{2}\|u\|_3^3.\end{aligned}$$

Here we have used $\Delta v = \partial_t v + (a_1 v - a_2 w) - \beta u$ by (3.2b). The Hölder inequality and the Gagliardo–Nirenberg inequality (A.1) with $p = 4, q = 2$ imply that

$$\begin{aligned}-\frac{1}{2}\int_{\mathbb{R}^2} u^2 \partial_t v\, dx &\le \frac{1}{2}\|u\|_4^2 \|\partial_t v\|_2 \le C\Big(\|\nabla u\|_2^{1/2}\|u\|_2^{1/2}\Big)^2 \|\partial_t v\|_2 \\ &= C\|\nabla u\|_2 \|u\|_2 \|\partial_t v\|_2 \le \frac{1}{4}\|\nabla u\|_2^2 + C\|\partial_t v\|_2^2 \|u\|_2^2.\end{aligned}$$

Using the Hölder inequality and Young's inequality, we get

$$\begin{aligned}-\frac{1}{2}\int_{\mathbb{R}^2} u^2(a_1 v - a_2 w)\, dx \le& \frac{1}{2}\left(\int_{\mathbb{R}^2} u^3\, dx\right)^{\frac{2}{3}}\left(\int_{\mathbb{R}^2} |a_1 v - a_2 w|^3\, dx\right)^{\frac{1}{3}} \\ \le& \frac{\beta}{2}\int_{\mathbb{R}^2} u^3\, dx + C\int_{\mathbb{R}^2} |a_1 v - a_2 w|^3\, dx \\ \le& \frac{\beta}{2}\|u\|_3^3 + C\left(\|v\|_3^3 + \|w\|_3^3\right),\end{aligned}$$

which yields that

$$-\frac{1}{2}\int_{\mathbb{R}^2} u^2\,(a_1 v - a_2 w)\;dx + \frac{\beta}{2}\|u\|_3^3 \le \beta\|u\|_3^3 + C(\|v\|_3^3 + \|w\|_3^3).$$

It here follows from [16, Lemma 2.1, (2.3)] that for any $\varepsilon > 0$,

$$\beta\|u\|_3^3 \le C\beta\varepsilon\|(1+u)\log(1+u)\|_1\|\nabla u\|_2^2 + C(\varepsilon)\beta\|u\|_1^2.$$

Notice that a bound

$$\|(1+u(t))\log(1+u(t))\|_1 \le C(T), \qquad 0 < t < T,$$

holds due to (5.1). Taking $\varepsilon > 0$ such that $C\beta\varepsilon C(T) \le 1/4$, we observe that

$$-\frac{1}{2}\int_{\mathbb{R}^2} u^2\,(a_1 v - a_2 w)\;dx + \frac{\beta}{2}\|u\|_3^3 \le \frac{1}{4}\|\nabla u\|_2^2 + C(\|v\|_3^3 + \|w\|_3^3 + \|u_0\|_1^2).$$

Hence

$$\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_2^2 \le C\|\partial_t v\|_2^2\|u\|_2^2 + C(\|v\|_3^3 + \|w\|_3^3 + \|u_0\|_1^2).$$

Applying the Gronwall inequality to the differential inequality above yields that

$$\begin{aligned} \|u(t)\|_2^2 \le{}& \|u_0\|_2^2 \exp\left(C\int_0^t \|\partial_t v\|_2^2\,d\tau\right) \\ &+ C\int_0^t (\|v\|_3^3 + \|w\|_3^3 + \|u_0\|_1^2)\exp\left(C\int_s^t \|\partial_t v\|_2^2\,d\tau\right) ds. \end{aligned} \tag{A.3}$$

As a consequence, we obtain (A.2) because the right hand side of (A.3) is bounded in $(0,T)$ due to (5.1) and Lemma 4.1 (i)–(ii). □

Lemma A.2. *The following estimate holds:*

$$\sup_{0<t<T}\|\nabla v(t)\|_4 \le C(T). \tag{A.4}$$

Proof. By the heat semigroup $e^{t\Delta}$, we have

$$\nabla v_j(t) = e^{-\lambda_j t}e^{t\Delta}\nabla v_{j0} + \int_0^t e^{-\lambda_j(t-s)}\nabla e^{(t-s)\Delta}u(s)\,ds, \quad j = 1,2. \tag{A.5}$$

Due to L^p-L^q estimate (2.4) for $e^{t\Delta}$ and Lemma 2.4(ii), we see that

$$\|\nabla v_j(t)\|_4 \le e^{-\lambda_j t}\|\nabla v_0\|_4 + C\lambda_j^{-3/4}\sup_{0<t<T}\|u(t)\|_2.$$

Hence this together with (A.2) and $v = \beta_1 v_1 - \beta_2 v_2$ implies the desired estimate (A.4). □

Lemma A.3. *There exists a positive constant $C(T)$ such that*

$$\sup_{0<t<T}\|u(t)\|_3 \le C(T). \tag{A.6}$$

Proof. Multiplying equation (3.2a) by u^2 and integrating by parts, we obtain

$$\frac{1}{3}\frac{d}{dt}\|u\|_3^3 + \frac{8}{9}\|\nabla u^{3/2}\|_2^2 = -\frac{4}{3}\int_{\mathbb{R}^2} u^{3/2}\nabla u^{3/2}\cdot\nabla v\,dx.$$

Using the Hölder inequality and applying the Gagliardo-Nirenberg inequality (A.?) as $p = 4, q = 4/3$ and $f = u^{3/2}$ yield that

$$\begin{aligned}
&-\frac{4}{3}\int_{\mathbb{R}^2} u^{3/2}\nabla u^{3/2}\cdot\nabla v\,dx \le \frac{4}{3}\|u^{3/2}\|_4\|\nabla u^{3/2}\|_2\|\nabla v\|_4\\
&\le C\big(\|\nabla u^{3/2}\|_2^{2/3}\|u^{3/2}\|_{4/3}^{1/3}\big)\|\nabla u^{3/2}\|_2\|\nabla v\|_4 = C\|\nabla u^{3/2}\|_2^{5/3}\|u\|_2^{1/2}\|\nabla v\|_4\\
&\le \frac{5}{9}\|\nabla u^{3/2}\|_2^2 + C\|u\|_2^3\|\nabla v\|_4^6.
\end{aligned}$$

Hence:

$$\frac{d}{dt}\|u\|_3^3 + \|\nabla u^{3/2}\|_2^2 \le C\|u\|_2^3\|\nabla v\|_4^6. \tag{A.7}$$

By the application of the Gagliardo–Nirenberg inequality (A.1) with $p = 2, q = 4/3$ and $f = u^{3/2}$ again, we observe that

$$\|\nabla u^{3/2}\|_2^2 \ge \|u\|_3^3 - C\|u\|_2^3.$$

Combining this estimate with (A.7), we get

$$\frac{d}{dt}\|u\|_3^3 + \|u\|_3^3 \le C(1 + \|\nabla v\|_4^6)\|u\|_2^3.$$

Therefore (A.6) is derived by applying the Gronwall inequality and using Lemmas A.1 and A.2. □

Lemma A.4. *There exists a positive constant $C(T)$ such that*

$$\sup_{0<t<T}\|\nabla v(t)\|_\infty \le C(T). \tag{A.8}$$

Proof. By taking the L^∞-norm in (A.5), the L^p-L^q estimate (2.4) for $e^{t\Delta}$ and Proposition 2.4(ii) yield that

$$\|\nabla v_j(t)\|_\infty \le e^{-\lambda_j t}\|\nabla v_0\|_\infty + C\lambda_j^{-1/6}\sup_{0<t<T}\|u(t)\|_3.$$

Consequently, by Lemma A.3 and $v = \beta_1 v_1 - \beta_2 v_2$ we observe the desired estimate (A.8). □

Proof of Proposition 5.1. The proof is based on the Moser's iteration technique (see [1] for example), which is often used in the study of PDEs. For the reader's convenience, we are going to give the detailed proof.

Multiplying equation (3.2a) by u^{p-1} ($p \geq 2$) and then integrating by parts, we have

$$\frac{d}{dt}\|u\|_p^p + \frac{4(p-1)}{p}\|\nabla u^{p/2}\|_2^2 = 2(p-1)\int_{\mathbb{R}^2} u^{p/2}\nabla u^{p/2}\cdot\nabla v\,dx.$$

By use of the Hölder inequality and the Gagliardo–Nirenberg inequality, we see that

$$\begin{aligned}\|u^{p/2}\nabla u^{p/2}\cdot\nabla v\|_1 &\leq \|\nabla v\|_\infty\|u^{p/2}\|_2\|\nabla u^{p/2}\|_2\\ &\leq \|\nabla v\|_\infty\|\nabla u^{p/2}\|_2^{3/2}\|u^{p/2}\|_1^{1/2}\\ &\leq p^{-1}\|\nabla u^{p/2}\|_2^2 + Cp^3\|\nabla v\|_\infty^4\|u^{p/2}\|_1^2,\end{aligned}$$

which gives

$$\frac{d}{dt}\|u\|_p^p + \frac{2(p-1)}{p}\|\nabla u^{p/2}\|_2^2 \leq Cp^3(p-1)\|\nabla v\|_\infty^4\|u^{p/2}\|_1^2. \tag{A.9}$$

By the application of the Gagliardo–Nirenberg inequality, we obtain

$$\frac{2(p-1)}{p}\|\nabla u^{p/2}\|_2^2 \geq p(p-1)\|u\|_p^p - Cp^3(p-1)\|u^{p/2}\|_1^2.$$

This together with (A.9) yields that

$$\frac{d}{dt}\|u\|_p^p + p(p-1)\|u\|_p^p \leq Cp^3(p-1)(1+\|\nabla v\|_\infty^4)\|u^{p/2}\|_1^2.$$

Therefore, applying the Gronwall ineqality and using Lemma A.4, we have

$$\begin{aligned}\|u(t)\|_p^p \leq& e^{-p(p-1)t}\|u_0\|_p^p + Cp^3(p-1)\int_0^t e^{-p(p-1)(t-s)}(1+\|\nabla v\|_\infty^4)\|u\|_{p/2}^p\,ds\\ \leq& e^{-p(p-1)t}\|u_0\|_p^p + Cp^3(p-1)\left(\sup_{0<t<T}\|u(t)\|_{p/2}^{p/2}\right)^2\int_0^t e^{-p(p-1)(t-s)}\,ds\\ \leq& e^{-p(p-1)t}\|u_0\|_p^p + Cp^2\left(\sup_{0<t<T}\|u(t)\|_{p/2}^{p/2}\right)^2.\end{aligned}$$

Set $p = 2^k$ ($k = 1, 2, \ldots$) and

$$\Phi_k := \sup_{0<t<T}\|u(t)\|_{2^k}^{2^k}.$$

Then for each $k = 1, 2, \ldots$, we have

$$\begin{aligned}\Phi_k \leq& e^{-2^k(2^k-1)t}\|u_0\|_1\|u_0\|_\infty^{2^k-1} + C2^{2k}\Phi_{k-1}^2\\ \leq& e^{-2^k(2^k-1)t}d^{2^k} + C2^{2k}\Phi_{k-1}^2\\ \leq& C2^{2k}\max\{d^{2^k}, \Phi_{k-1}^2\},\end{aligned} \tag{A.10}$$

where $d := \max\{\|u_0\|_1, \|u_0\|_\infty\}$. Because $\Phi_{k-1}^2 \leq C^2(2^{2(k-1)})^2 \max\{d^{2^k}, \Phi_{k-2}^{2^2}\}$ due to (A.10), we find that

$$\Phi_k \leq C^{1+2} 2^{2k+2^2(k-1)} \max\{d^{2^k}, \Phi_{k-2}^{2^2}\}.$$

Repeating this procedure, we obtain

$$\Phi_k \leq C^{\sum_{j=1}^k 2^{j-1}} 2^{\sum_{j=1}^k 2^j(k+1-j)} \max\{d^{2^k}, \Phi_0^{2^k}\},$$

which yields that

$$\begin{aligned}\|u(t)\|_{2^k} &\leq C^{\sum_{j=1}^k 2^{-(k-j)-1}} 2^{\sum_{j=1}^k 2^{-(k-j)}(k-j+1)} \max\{d, \Phi_0\} \\ &= C^{\sum_{j=1}^k 2^{-j}} 2^{\sum_{j=1}^k j2^{-(j-1)}} \max\{d, \Phi_0\}.\end{aligned}$$

Passing to the limit $k \to \infty$, we obtain the desired estimate (5.2). The proof is complete. □

Acknowledgement. The second author was partly supported by Grant-in-Aid for scientific research (18K03373). This work is partially supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849).

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