A LIE THEORETIC INTERPRETATION OF REALIZATIONS OF SOME CONTACT METRIC MANIFOLDS

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Abstract. We study (κ, μ)-spaces whose Boeckx invariants satisfy $I \leq -1$, from the viewpoint of submanifold geometry. We give a Lie theoretic proof that these spaces can be realized as homogeneous hypersurfaces in noncompact real two-plane Grassmannians.

1. Introduction

In [8], Blair, Koufogiorgos and Papantoniou introduced the following class of contact metric manifolds:

Definition 1.1. Let $(\kappa, \mu) \in \mathbb{R}^2$. A contact metric manifold $(M, \eta, \xi, \varphi, g)$ is called a (κ, μ)-space if the Riemannian curvature tensor R satisfies

$$R(X, Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y)$$

for any vector fields X and Y on M, where I denotes the identity transformation and $h := (1/2)\mathcal{L}_\xi \varphi$ is the Lie derivative of φ along ξ.

We remark that (κ, μ)-spaces satisfy the inequality $\kappa \leq 1$, and if $\kappa = 1$ then they are Sasakian. This class of contact metric manifolds contains not only Sasakian manifolds, but also many non-Sasakian manifolds including standard examples of contact metric manifolds, such as the unit tangent sphere bundles of Riemannian manifolds with constant sectional curvature $c \neq 1$. Moreover (κ, μ)-spaces have fruitful geometric properties. Among others, (κ, μ)-spaces are stable under D-homothetic transformations, and have a strongly pseudoconvex CR-structure. For more details, we refer to [8].

In [9, 10], Boeckx has studied (κ, μ)-spaces deeply. He proved that every non-Sasakian (κ, μ)-space is locally homogeneous, and its local geometry is completely determined by the dimension and the numbers (κ, μ). Furthermore he introduced an invariant I_M defined by $I_M = (1 - \mu/2)/\sqrt{1 - \kappa}$ for a non-Sasakian (κ, μ)-space M. This invariant completely determines a (κ, μ)-space locally up to equivalence and D-homothetic transformations. Also local models of non-Sasakian (κ, μ)-spaces have been obtained. The

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unit tangent sphere bundles of Riemannian manifolds with constant sectional curvature $c \neq 1$ provide examples of (κ, μ)-spaces with Boeckx invariant $I > -1$. For the case of $I \leq -1$, Boeckx gave examples of (κ, μ)-spaces with any odd dimension and value $I \leq -1$ by a two-parameter family of Lie groups endowed with certain left-invariant contact metric structure. Later, another geometric construction of (κ, μ)-spaces with $I \leq -1$ has been obtained by Loiudice and Lotta ([17]). Namely, these spaces can be constructed as the tangent hyperquadric bundles of Lorentzian manifolds with constant sectional curvature c ($c \leq 0$, $c \neq -1$).

On the other hand, from the view points of CR geometry and submanifold geometry, it would be a natural question whether a given (κ, μ)-space can be realized as a real hypersurface in a Kähler manifold. In [1], Adachi, Kameda and Maeda proved that a Sasakian space form with constant ϕ-sectional curvature $c + 1$ ($c \neq 0$) can be realized as a real hypersurface in a nonflat complex space form $\widetilde{M}(c)$. The realization problem has also been studied for non-Sasakian cases. Cho and Inoguchi ([13]) proved that for any $I > 0$, there exists a (κ, μ)-space with Boeckx invariant I, which can be realized as a homogeneous hypersurface in a non-flat complex space form. Cho and the authors ([12]) showed that the $(0, 4)$-space (whose Boeckx invariant is $I = -1$) can be realized as a homogeneous hypersurface in the noncompact real two-plane Grassmannian $G_2^*(\mathbb{R}^{n+3}) = \mathrm{SO}^0(2, n+1)/(\mathrm{SO}(2) \times \mathrm{SO}(n+1))$. Recently, Cho ([11]) proved that for any $I \neq 1$, there exists a (κ, μ)-space with Boeckx invariant I, which can be realized as a homogeneous hypersurface in the real two-plane Grassmannian $G_2(\mathbb{R}^{n+3})$ or its noncompact dual $G_2^*(\mathbb{R}^{n+3})$. We will summarize the details in Subsection 2.2.

In the present paper, we study (κ, μ)-spaces whose Boeckx invariants satisfy $I \leq -1$, from the viewpoint of submanifold geometry. As a result, we give a Lie theoretic proof that (κ, μ)-spaces with $I < -1$ can be realized as homogeneous hypersurfaces in $G_2^*(\mathbb{R}^{n+3})$. In fact, we describe realizations of $(0, \mu)$-spaces for all $\mu > 4$. As mentioned above, a realization of the $(0, 4)$-space (thus $I = -1$) has been obtained. Our argument gives an explicit description of a deformation from (κ, μ)-spaces with $I < -1$ to the one with $I = -1$.

2. Notes on (κ, μ)-spaces

2.1. Preliminaries for contact metric manifolds. In this subsection, we recall necessary notions for contact metric manifolds, especially for (κ, μ)-spaces. We refer to [7]. Let M be a $(2n + 1)$-dimensional manifold, and denote by $\mathfrak{X}(M)$ the set of all smooth vector fields on M.

Definition 2.1. We call M an *almost contact manifold* if it is equipped with a 1-form η, a vector field $\xi \in \mathfrak{X}(M)$, and a $(1, 1)$-tensor field φ such that

$$\eta(\xi) = 1, \quad \varphi^2 X = -X + \eta(X)\xi \quad (\forall X \in \mathfrak{X}(M)). \tag{2.1}$$

An almost contact manifold is denoted by a quadruplet (M, η, ξ, φ). The vector field ξ is called the *characteristic vector field.* Note that it follows that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{2.2}$$

Definition 2.2. Let (M, η, ξ, φ) be an almost contact manifold. Then, a Riemannian metric g is called an *associated metric* if it satisfies

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (\forall X, Y \in \mathfrak{X}(M)). \tag{2.3}$$

We call such $(M, \eta, \xi, \varphi, g)$ an *almost contact metric manifold*, or an *almost contact Riemannian manifold.* Note that, for an almost contact manifold (M, η, ξ, φ), there always exists an associated metric (see [19]). It follows from (2.1), (2.2), and (2.3) that

$$\eta(X) = g(X, \xi) \quad (\forall X \in \mathfrak{X}(M)).$$

For an almost contact metric manifold $(M, \eta, \xi, \varphi, g)$, the fundamental 2-form Φ on M is defined by

$$\Phi(X, Y) = g(X, \varphi Y) \quad (X, Y \in \mathfrak{X}(M)).$$

Definition 2.3. An almost contact metric manifold $(M, \eta, \xi, \varphi, g)$ is called a *contact metric manifold*, or a *contact Riemannian manifold* if $\Phi = d\eta$ holds.

A contact metric manifold is denoted by $(M, \eta, \xi, \varphi, g)$, and (η, ξ, φ, g) is called a *contact metric structure* on M. Note that $\Phi = d\eta$ implies

$$\eta \wedge (d\eta)^n \neq 0.$$

The notion of (κ, μ)-spaces was introduced by Blair, Koufogiorgos, and Papantoniou in [8] (see Definition 1.1). We here recall some known facts on (κ, μ)-spaces according to [7]. Let $(M, \eta, \xi, \varphi, g)$ be a (κ, μ)-space. Recall that $\kappa \leq 1$. Moreover, if $\kappa = 1$, then $\mu = 0$ and hence $(M, \eta, \xi, \varphi, g)$ is a Sasakian manifold. If $\kappa < 1$, then $(M, \eta, \xi, \varphi, g)$ is not Sasakian, and its Riemannian curvature tensor is completely determined by the (κ, μ)-condition. For a contact metric manifold $(M, \eta, \xi, \varphi, g)$, a D-homothetic deformation means the change of the structure tensors by

$$\bar{\eta} := a\eta, \quad \bar{\xi} := (1/a)\xi, \quad \bar{\varphi} := \varphi, \quad \bar{g} := ag + a(a-1)\eta \otimes \eta,$$

where a is a positive constant. Then, $(M, \bar{\eta}, \bar{\xi}, \bar{\varphi}, \bar{g})$ is again a contact metric manifold. The class of all (κ, μ)-spaces are preserved by D-homothetic deformations, namely, a D-homothetic deformation maps a (κ, μ)-space to a $(\bar{\kappa}, \bar{\mu})$-space, where

$$\bar{\kappa} = (\kappa + a^2 - 1)/a^2, \quad \bar{\mu} = (\mu + 2a - 2)/a.$$

Boeckx ([9]) proved that every non-Sasakian (κ, μ)-space is locally homogeneous. Moreover, he introduced an invariant

$$I_M = (1 - \mu/2)/\sqrt{1 - \kappa}$$

for non-Sasakian (κ, μ)-space M, which is called the *Boeckx invariant*. He also proved in [10] that a non-Sasakian (κ, μ)-space is locally isometric (as contact metric manifolds) to a (κ', μ')-space up to D-homothetic deformation if and only if they have the same Boeckx invariant.

Local models of non-Sasakian (κ, μ)-spaces have been obtained. The models for the case of $I > -1$ can be given as follows:

Theorem 2.4 ([8]). *Every (κ, μ)-space with Boeckx invariant $I > -1$ is locally isometric, up to a D-homothetic deformation, to the unit tangent sphere bundle of a Riemannian manifold with constant sectional curvature $c \neq 1$, endowed with the standard contact metric structure.*

The models for the case of $I \leq -1$ are more involved, but can be constructed explicitly as follows:

Definition 2.5 ([10]). Let $\alpha, \beta \in \mathbb{R}$. Then, we define a real $(2n+1)$-dimensional Lie algebra $\mathfrak{g}_{\alpha,\beta}$ with basis $\{\xi, X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n\}$ as follows:

(1) the bracket product $[\xi, X_i]$ is given by

$$\begin{aligned} [\xi, X_1] &= -(1/2)\alpha\beta X_2 - (1/2)\alpha^2 Y_1, \\ [\xi, X_2] &= (1/2)\alpha\beta X_1 - (1/2)\alpha^2 Y_2, \\ [\xi, X_i] &= -(1/2)\alpha^2 Y_i, \qquad (i \neq 1, 2) \end{aligned}$$

(2) the bracket product $[\xi, Y_i]$ is given by

$$\begin{aligned} [\xi, Y_1] &= (1/2)\beta^2 X_1 - (1/2)\alpha\beta Y_2, \\ [\xi, Y_2] &= (1/2)\beta^2 X_2 + (1/2)\alpha\beta Y_1, \\ [\xi, Y_i] &= (1/2)\beta^2 X_i, \qquad (i \neq 1, 2) \end{aligned}$$

(3) the bracket product $[X_i, X_j]$ is given by

$$\begin{aligned} [X_1, X_i] &= \alpha X_i, \qquad (i \neq 1) \\ [X_i, X_j] &= 0, \qquad (i, j \neq 1) \end{aligned}$$

(4) the bracket product $[Y_i, Y_j]$ is given by

$$\begin{aligned} [Y_2, Y_i] &= \beta Y_i, \qquad (i \neq 2) \\ [Y_i, Y_j] &= 0, \qquad (i, j \neq 2) \end{aligned}$$

(5) the bracket product $[X_i, Y_j]$ is given by

$$\begin{aligned}
&[X_1, Y_1] = -\beta X_2 + 2\xi, \\
&[X_1, Y_i] = 0, && (i \neq 1) \\
&[X_2, Y_1] = \beta X_1 - \alpha Y_2, \\
&[X_2, Y_2] = \alpha Y_1 + 2\xi, \\
&[X_2, Y_i] = \beta X_i, && (i \neq 1, 2) \\
&[X_i, Y_1] = -\alpha Y_i, && (i \neq 1, 2) \\
&[X_i, Y_2] = 0, && (i \neq 1, 2) \\
&[X_i, Y_j] = \delta_{ij}(-\beta X_2 + \alpha Y_1 + 2\xi). && (i, j \neq 1, 2)
\end{aligned}$$

It follows from long but direct calculations that $\mathfrak{g}_{\alpha,\beta}$ is a Lie algebra, that is, the above defined bracket product satisfies the Jacobi identity. We denote by $G_{\alpha,\beta}$ the simply-connected Lie group with Lie algebra $\mathfrak{g}_{\alpha,\beta}$. We now define some left-invariant structures on $G_{\alpha,\beta}$ as follows:

- the left-invariant metric g is defined so that the above basis is orthonormal,
- the characteristic vector field is given by ξ,
- the 1-form η is the metric dual of ξ, that is, $\eta(X) = g(X, \xi)$,
- the $(1,1)$-tensor field φ is defined by

$$\varphi(\xi) = 0, \quad \varphi(X_i) = Y_i, \quad \varphi(Y_i) = -X_i.$$

Then, Boeckx showed the following.

Proposition 2.6 ([10])**.** *Assume that $\beta > \alpha \geq 0$. Then, $(G_{\alpha,\beta}, \eta, \xi, \varphi, g)$ is a (κ, μ)-space with Boeckx invariant $I_{G_{\alpha,\beta}} = -\frac{\beta^2+\alpha^2}{\beta^2-\alpha^2} \leq -1$, where*

$$\kappa = 1 - \frac{(\beta^2 - \alpha^2)^2}{16}, \qquad \mu = 2 + \frac{\alpha^2 + \beta^2}{2}.$$

Loiudice and Lotta gave another construction of (κ, μ)-spaces with Boeckx invariant $I \leq -1$.

Theorem 2.7 ([17])**.** *Every non-Sasakian (κ, μ)-space with Boeckx invariant $I \leq -1$ is locally isometric, up to a D-homothetic deformation, to the tangent hyperquadric bundle of a Lorentzian manifold with constant sectional curvature $c \leq 0$ with $c \neq -1$, endowed with the standard contact metric structure.*

We also note that, in [18], Loiudice and Lotta gave other homogeneous models of non-Sasakian (κ, μ)-spaces, which provide geometric interpretations of the Boeckx invariants.

2.2. **Realizations of (κ, μ)-spaces as real hypersurfaces.** In this subsection, we mention some known facts for realization problems of (κ, μ)-spaces as homogeneous real hypersurfaces in a Kähler manifold.

First of all, we recall that any hypersurfaces in a Kähler manifold are almost contact metric manifolds with respect to the following structures:

Proposition 2.8 ([3, 7])**.** *Let $(\overline{M}, J, g)$ be a Kähler manifold, and M be a connected oriented real hypersurface in $\overline{M}$. Denote by N a unit normal vector field of M. We define the structures (η, ξ, φ, g) on M as follows:*

- *the Riemannian metric g is induced from the Riemannian metric on $\overline{M}$,*
- *the characteristic vector field ξ is defined by $\xi := -JN$,*
- *the 1-form η is the metric dual of ξ, that is, $\eta(X) = g(X, \xi)$,*
- *the $(1,1)$-tensor field φ is defined by $\varphi X = JX - \eta(X)N$.*

Then, $(M, \eta, \xi, \varphi, g)$ is an almost contact metric manifold.

From now on, we always consider the above induced almost contact metric structure on a hypersurface in a Kähler manifold. For realizations of non-Sasakian (κ, μ)-spaces, it is well-known that $\mathbb{R}^n \times S^{n-1}(4)$ in $\mathbb{C}^n$ is a $(0,0)$-space, whose Boeckx invariant is $I = 0$. If we take, as an ambient space, the complex projective space $\mathbb{C}\mathrm{P}^n(c)$ or the complex hyperbolic space $\mathbb{C}\mathrm{H}^n(c)$ with constant holomorphic sectional curvature c, the following facts have been known:

Theorem 2.9 ([13])**.** (1) *For any $c > 0$, the tube of radius*

$$r = \frac{2}{\sqrt{c}} \tan^{-1}\left(\sqrt{c+4} - \frac{\sqrt{c}}{2}\right) < \frac{\pi}{2\sqrt{c}}$$

around the complex quadric Q^{n-1} in $\mathbb{C}\mathrm{P}^n(c)$ is a $(-c/4, -\sqrt{c}/2)$-space, whose Boeckx invariant is $I = \sqrt{1 + c/4} > 1$.

(2) *For any $c \in (-4, 0)$, the tube of radius*

$$r = \frac{1}{\sqrt{-c}} \tanh^{-1}\left(\frac{\sqrt{-c}}{2}\right)$$

around the totally geodesic real hyperbolic space $\mathbb{R}\mathrm{H}^n(c/4)$ in $\mathbb{C}\mathrm{H}^n(c)$ is a $(3c/4, -c/2)$-space, whose Boeckx invariant is

$$0 < I = \frac{c+4}{2\sqrt{4-3c}} < 1.$$

On the realization problem in terms of the Boeckx invariant I, we have mentioned the cases for $0 \leq I < 1$ and $1 < I$. For the remaining cases, it was shown that they can be realized as homogeneous real hypersurfaces in real two-plane Grassmannians and its noncompact duals. Let $n \geq 2$, and $G_2(\mathbb{R}^{n+3})$ be the real two-plane Grassmannian. Note that this can be identified with the complex quadric Q^{n+1} as Kähler manifolds. We also denote by $G_2^*(\mathbb{R}^{n+3})$ the noncompact real two-plane Grassmannian, which is the noncompact dual of $G_2(\mathbb{R}^{n+3})$. Recall that $G_2(\mathbb{R}^{n+3})$ and $G_2^*(\mathbb{R}^{n+3})$ have nonnegative and nonpositive sectional curvatures, respectively. The normalizations of the Riemannian metrics on these Grassmannians will be expressed in terms of the maximal or minimal sectional curvatures.

Theorem 2.10 ([11, 12])**.** *A horosphere in $G_2^*(\mathbb{R}^{n+3})$ whose center at infinity is determined by an A-principal geodesic in $G_2^*(\mathbb{R}^{n+3})$ with minimal sectional curvature -8 is a $(0,4)$-space, whose Boeckx invariant is $I=-1$.*

For the A-principal geodesics in $G_2^*(\mathbb{R}^{n+3})$, we refer to [3].

Theorem 2.11 ([11])**.**

(1) *For any $c>0$, the tube of radius*

$$r=\sqrt{\frac{2}{c}}\tan^{-1}\frac{2\sqrt{2}}{\sqrt{c}}<\frac{\pi}{\sqrt{2c}}$$

around the real form S^{n+1} of $G_2(\mathbb{R}^{n+3})$ with maximal sectional curvature c is a $(0,-c/2)$-space, whose Boeckx invariant is $I=1+c/4>1$.

(2) *For any $c\in(-8,0)$, the tube of radius*

$$r=\sqrt{\frac{2}{|c|}}\coth^{-1}\frac{2\sqrt{2}}{\sqrt{|c|}}<\infty$$

around the totally real submanifold $\mathbb{R}\mathrm{H}^{n+1}$ in $G_2^*(\mathbb{R}^{n+3})$ with minimal sectional curvature c is a $(0,-c/2)$-space, whose Boeckx invariant is $-1<I=1+c/4<1$.

(3) *For any $c<-8$, the tube of radius*

$$r=\sqrt{\frac{2}{|c|}}\tanh^{-1}\frac{2\sqrt{2}}{\sqrt{|c|}}<\infty$$

around the totally geodesic $G_2^*(\mathbb{R}^{n+2})$ in $G_2^*(\mathbb{R}^{n+3})$ with minimal sectional curvature c is a $(0,-c/2)$-space, whose Boeckx invariant is $I=1+c/4<-1$.

For details of geometry of the above contact real hypersurfaces in $G_2^*(\mathbb{R}^{n+3})$, we refer to [3].

3. Noncompact real two-plane Grassmannians

From now on, we will denote by M_r a tube around the totally geodesic $G_2^*(\mathbb{R}^{n+2})$ with radius $r>0$ in the noncompact real two-plane Grassmannian $G_2^*(\mathbb{R}^{n+3})$. In this section, we recall a expression of M_r as a homogeneous space.

3.1. Review for $G_2^*(\mathbb{R}^{n+3})$.

In this subsection, we recall some basic facts on Riemannian symmetric spaces, and particularly, the noncompact real two-plane Grassmannians. We refer to [14].

Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{k}$ be a compact subalgebra of $\mathfrak{g}$, and θ be an involutive automorphism of $\mathfrak{g}$. Then the triplet $(\mathfrak{g},\mathfrak{k},\theta)$ is called an *orthogonal symmetric Lie algebra* if it satisfies

$$\mathfrak{k}=\mathfrak{g}^{\theta}:=\{X\in\mathfrak{g}\mid\theta(X)=X\}.$$

It is well-known that orthogonal symmetric Lie algebras correspond to Riemannian symmetric spaces. In fact, for each $(\mathfrak{g}, \mathfrak{k}, \theta)$, one obtains a Riemannian symmetric space G/K, by putting G a connected Lie group with Lie algebra $\mathfrak{g}$ and K a Lie subgroup of G with Lie algebra $\mathfrak{k}$. Recall that the point $[e] = eK \in G/K$ is called the origin of G/K, and usually denoted by o.

We here introduce the noncompact real two-plane Grassmannians. For simplicity of notation, we denote by I_m the identity matrix, and also by

$$I_{p,q} := \begin{bmatrix} I_p & \\ & -I_q \end{bmatrix} \in M(p+q; \mathbb{R}).$$

We denote by X^T the transposed matrix of X. The proof of the following proposition is an easy exercise of linear algebra.

Proposition 3.1. *The triplet* $(\mathfrak{g}, \mathfrak{k}, \theta)$ *defined as follows is an orthogonal symmetric Lie algebra*:

(1) $\mathfrak{g} := \mathfrak{so}(2, n+1) := \{X \in \mathfrak{gl}(n+3; \mathbb{R}) \mid XI_{2,n+1} + I_{2,n+1}X^T = 0\}$,
(2) $\theta : \mathfrak{g} \to \mathfrak{g} : X \mapsto I_{2,n+1}XI_{2,n+1}$,
(3) $\mathfrak{k} := \mathfrak{so}(2) \oplus \mathfrak{so}(n+1)$.

Let us consider the corresponding Riemannian symmetric space. We need the indefinite special orthogonal group

$$\mathrm{SO}(2, n+1) := \{g \in \mathrm{SL}(n+3; \mathbb{R}) \mid gI_{2,n+1}g^T = I_{2,n+1}\}.$$

Let $\mathrm{SO}^0(2, n+1)$ be the connected component of $\mathrm{SO}(2, n+1)$ containing the identity. Then the corresponding Riemannian symmetric space can be expressed as

$$G_2^*(\mathbb{R}^{n+3}) = \mathrm{SO}^0(2, n+1)/(\mathrm{SO}(2) \times \mathrm{SO}(n+1)),$$

which we call the noncompact real two-plane Grassmannian. This is an irreducible Hermitian symmetric space of noncompact type if $n \geq 2$. Note that this symmetric space is of dimension $2(n+1)$, and of rank 2.

3.2. **Some homogeneous hypersurfaces in $G_2^*(\mathbb{R}^{n+3})$.** For the Lie algebra $\mathfrak{g} = \mathfrak{so}(2, n+1)$, one has an automorphism

$$\rho : \mathfrak{g} \to \mathfrak{g} : X \mapsto I_{n+2,1}XI_{n+2,1}. \tag{3.1}$$

Then one can see that $\mathfrak{g}^\rho = \mathfrak{so}(2, n)$, and the corresponding connected Lie subgroup is $\mathrm{SO}^0(2, n)$. In this subsection, we study orbits of this group.

Recall that a reflective submanifold of a Riemannian manifold is a connected component of the fixed point set with respect to an involutive isometry. It is easy to see that every reflective submanifold is totally geodesic, that is, the second fundamental form vanishes identically. For reflective submanifolds, we refer to [15].

Proposition 3.2. *For the action of* $\mathrm{SO}^0(2, n)$ *on* $G_2^*(\mathbb{R}^{n+3})$*, the orbit through the origin coincides with* $G_2^*(\mathbb{R}^{n+2})$*, which is reflective (and hence totally geodesic).*

Proof. In general, let G/K be a Riemannian symmetric space, and $(\mathfrak{g}, \mathfrak{k}, \theta)$ be its orthogonal symmetric Lie algebra. Then $\rho : \mathfrak{g} \to \mathfrak{g}$ is called an involutive automorphism of $(\mathfrak{g}, \mathfrak{k}, \theta)$ if ρ is a Lie algebra automorphism of $\mathfrak{g}$, $\rho(\mathfrak{k}) = \mathfrak{k}$, and ρ commutes with θ. One knows that there is a one-to-one correspondence between reflective submanifolds in G/K and involutive automorphisms of $(\mathfrak{g}, \mathfrak{k}, \theta)$. In fact, for an involutive automorphism ρ of $(\mathfrak{g}, \mathfrak{k}, \theta)$, the fixed point set $\mathfrak{g}^{\rho}$ is a subalgebra, and the orbit of the corresponding connected Lie subgroup through the origin is a reflective submanifold (which is a connected component of the involutive isometry of G/K induced from ρ).

We apply this general theory to the orthogonal symmetric Lie algebra $(\mathfrak{g}, \mathfrak{k}, \theta)$ given in Proposition 3.1. It is easy to see that ρ defined in (3.1) is an involutive automorphism of $(\mathfrak{g}, \mathfrak{k}, \theta)$. Therefore, the orbit of $\mathrm{SO}^0(2, n)$ through the origin is reflective.

It remains to show that the orbit through the origin is $G_2^*(\mathbb{R}^{n+2})$. For simplicity of notation, we put $H := \mathrm{SO}^0(2, n)$. The orbit through the origin $o \in G_2^*(\mathbb{R}^{n+3})$ is represented as $H.o \cong H/H_o$, where H_o denotes the isotropy subgroup of H at o. One can easily see that

$$H_o = H \cap (\mathrm{SO}(2) \times \mathrm{SO}(n+1)) = \mathrm{SO}(2) \times \mathrm{SO}(n),$$

which shows that $H.o$ coincides with $G_2^*(\mathbb{R}^{n+2})$. This completes the proof. □

For an isometric action on a Riemannian manifold, maximal dimensional orbits are said to be *regular*, and other orbits *singular*. The *cohomogeneity* of an isometric action is the codimension of a regular orbit. Therefore, regular orbits of cohomogeneity one actions are homogeneous hypersurfaces.

Cohomogeneity one actions on Riemannian symmetric spaces of noncompact type have been studied in [4, 5, 6]. In this case, it has been known that each action admits at most one singular orbit. Furthermore, if there exists a singular orbit, then all regular orbits are tubes around it. The classification of cohomogeneity one actions up to orbit equivalence has been obtained just for some spaces. However, the case of noncompact real two-plane Grassmannians has been completed by Berndt and Domínguez-Vázquez ([2]). In their list one can find the following.

Proposition 3.3 ([2])**.** *The action of* $\mathrm{SO}^0(2, n)$ *on* $G_2^*(\mathbb{R}^{n+3})$ *is a cohomogeneity one action. Therefore, every tube* M_r *around* $G_2^*(\mathbb{R}^{n+2})$ *with radius* $r > 0$ *is a homogeneous hypersurface.*

4. Proof of the main theorem

In this section, we prove that every tube M_r around $G_2^*(\mathbb{R}^{n+2})$ with radius $r > 0$ is obtained as an orbit of certain smaller subgroup $Q \subsetneq \mathrm{SO}^0(2, n)$ (Subsection 4.2), and for some particular radius it is a (κ, μ)-space (Subsection 4.3). Our proof is based on Lie theoretic arguments, in which the solvable model of the noncompact real two-plane Grassmannian plays a key role.

4.1. **The solvable model of $G_2^*(\mathbb{R}^{n+3})$.** In this subsection, we review some general facts on Iwasawa decompositions and the solvable model of the noncompact real two-plane Grassmannian $G_2^*(\mathbb{R}^{n+3})$. Refer to [14] and [12].

Let G/K be a Riemannian symmetric space of noncompact type, and $(\mathfrak{g}, \mathfrak{k}, \theta)$ be the corresponding orthogonal symmetric Lie algebra. Denote the Killing form of $\mathfrak{g}$ by B, and the orthogonal complement of $\mathfrak{k}$ with respect to B by $\mathfrak{p}$. One then obtains the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let us fix $\mathfrak{a}$ as a maximal abelian subspace of $\mathfrak{p}$, and denote the dual space of $\mathfrak{a}$ by $\mathfrak{a}^*$. Then, for each $\lambda \in \mathfrak{a}^*$, we define

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid \ \mathrm{ad}(H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a} \ \},$$

and call $\lambda \in \mathfrak{a}$ a (restricted) root if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$. Denote by Σ the set of roots. Then, one obtains the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda.$$

Now we review the Iwasawa decompositions. Let us fix Λ as a set of simple roots, and then denote the set of positive roots associated with Λ by Σ^+. Then, one can consider

$$\mathfrak{n} := \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda, \quad \text{and} \quad \mathfrak{s} := \mathfrak{a} \oplus \mathfrak{n},$$

which are a nilpotent and solvable Lie subalgebra of $\mathfrak{g}$, respectively. We then obtain $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{k} \oplus \mathfrak{s}$ (as a vector space), which is called the Iwasawa decomposition of $\mathfrak{g}$, and we call $\mathfrak{s}$ the solvable part of the Iwasawa decomposition.

Recall that G/K denotes a Riemannian symmetric space of noncompact type. As usual, we identify the tangent space $T_o(G/K)$ with $\mathfrak{p}$, where $o := eK$. Let us denote by S the connected subgroup of G corresponding to $\mathfrak{s}$. One knows that S acts on G/K simply-transitively, and hence, $\mathfrak{s}$ can be identified with $T_o(G/K) \cong \mathfrak{p}$. Therefore, the geometrical structures (e.g., the Riemannian metric) on G/K derive ones on $\mathfrak{s}$. We call a collection of the bracket relation on $\mathfrak{s}$ and its related structures *the solvable model* of G/K.

From now on, we consider the noncompact real two-plane Grassmannians $G_2^*(\mathbb{R}^{n+3})$ with $n \geq 2$. Note that it is a Hermitian symmetric space of noncompact type, and we hereafter assume that it has the minimal sectional curvature $-c^2$ with $c > 0$.

Let us describe the solvable model of $G_2^*(\mathbb{R}^{n+3})$. We keep to use the notations mentioned above. One knows that the root system Σ of $\mathfrak{g} = \mathfrak{so}(2, n+1)$ is of B_2-type, and therefore, we can put

$$\Sigma^+ := \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\},$$

where α_1 and α_2 stand for simple roots satisfying $|\alpha_1| > |\alpha_2|$. Then, the solvable part of the Iwasawa decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}) \subset \mathfrak{so}(2, n+1).$$

Let $\langle , \rangle_{\mathfrak{s}}$ and J be the induced metric and complex structure on $\mathfrak{s}$ from $G_2^*(\mathbb{R}^{n+3})$, respectively. According to [12, Theorem 4.2] and its proof, one obtains the solvable model of $G_2^*(\mathbb{R}^{n+3})$ with minimal sectional curvature $-c^2$ as follows:

Theorem 4.1 ([12])**.** *There exists a basis*

$$\mathfrak{s} = \mathrm{span}\{A_1, A_2, X_0, Y_1, \dots, Y_{n-1}, Z_1, \dots, Z_{n-1}, W_0\}$$

such that

(1) $A_i \in \mathfrak{a}$, $X_0 \in \mathfrak{g}_{\alpha_1}$, $Y_i \in \mathfrak{g}_{\alpha_2}$, $Z_i \in \mathfrak{g}_{\alpha_1+\alpha_2}$, $W_0 \in \mathfrak{g}_{\alpha_1+2\alpha_2}$, *and they have the following bracket relations:*
- $[A_1, X_0] = cX_0$, $[A_1, Y_i] = -(c/2)Y_i$, $[A_1, Z_i] = (c/2)Z_i$,
- $[A_2, Y_i] = (c/2)Y_i$, $[A_2, Z_i] = (c/2)Z_i$, $[A_2, W_0] = cW_0$,
- $[X_0, Y_i] = cZ_i$, $[Y_i, Z_i] = cW_0$,
- *and other relations vanish;*

(2) *they are orthonormal with respect to* $\langle , \rangle_{\mathfrak{s}}$*;*

(3) J *satisfies that* $JA_1 = -X_0$, $JA_2 = W_0$, $JY_i = Z_i$.

4.2. **The Lie group Q.** We have studied the totally geodesic submanifold $G_2^*(\mathbb{R}^{n+2})$ of $G_2^*(\mathbb{R}^{n+3})$ in the previous section. In this subsection, we show that $G_2^*(\mathbb{R}^{n+2})$ and its tube M_r with radius $r > 0$ are obtained as orbits of a smaller subgroup $Q \subsetneq \mathrm{SO}^0(2, n)$.

Definition 4.2. We define $\mathfrak{q} := (\mathfrak{s} \ominus \mathrm{span}\{Y_1, Z_1\}) \oplus \mathfrak{g}_{-\alpha_1}$.

One can easily see that $\mathfrak{q}$ is a Lie subalgebra of $\mathfrak{so}(2, n+1)$. In particular, we have

$$\mathfrak{q} = \begin{cases} \mathrm{span}\{A_1, A_2, X_0, W_0, \theta X_0\} & \text{if } n = 2, \\ \mathrm{span}\{A_1, A_2, X_0, Y_2, \dots Y_{n-1}, Z_2, \dots Z_{n-1}, W_0, \theta X_0\} & \text{if } n > 2. \end{cases}$$

Especially, the bracket relations on $\mathfrak{q}$ can be calculated from Theorem 4.1 and the following Lemma.

Lemma 4.3. *One has*

(1) $[\theta X_0, A_1] = c\theta X_0$, $[\theta X_0, A_2] = 0$,

(2) $[\theta X_0, X_0] = 2cA_1$, $[\theta X_0, Y_i] = 0$, $[\theta X_0, Z_i] = -cY_i$, $[\theta X_0, W_0] = 0$.

Proof. First, it follows from Theorem 4.1 that

$$[\theta X_0, A_i] = \theta[X_0, -A_i] = \delta_{1i} \cdot c\theta X_0,$$

which shows the assertion (1).

Next, we show (2). Let us take a constant $k > 0$ such that $kB(X, Y)$ coincides with the metric on $T_o(G/K) = \mathfrak{p}$, and define a metric $\langle , \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ by $\langle X, Y \rangle_{\mathfrak{g}} = -kB(X, \theta X)$. Then one knows $\langle [\theta X, Y], Z \rangle_{\mathfrak{g}} = \langle X, [Y, Z] \rangle_{\mathfrak{g}}$ for any $X, Y, Z \in \mathfrak{g}$, and $\langle X, Y\rangle_{\mathfrak{s}} = (1/2)\langle X, Y\rangle_{\mathfrak{g}}$ for any $X, Y \in \mathfrak{n}$. See [12, Section 4] for more details.

The property of root space decompositions shows that

$$[\theta X_0, X_0] \in \mathfrak{a}, \quad [\theta X_0, Z_i] \in \mathfrak{g}_{\alpha_2}, \quad [\theta X_0, Y_i] = [\theta X_0, W_0] = 0.$$

If we put $[\theta X_0, X_0] = \sum a_i A_i \in \mathfrak{a}$, then it satisfies

$$a_i = \langle [\theta X_0, X_0], A_i\rangle_{\mathfrak{g}} = \delta_{1i} \cdot c\langle X_0, X_0\rangle_{\mathfrak{g}} = \delta_{1i} \cdot 2c.$$

This shows $[\theta X_0, X_0] = 2cA_1$. Similarly, putting $[\theta X_0, Z_i] = \sum b_j Y_j \in \mathfrak{g}_{\alpha_2}$, we have

$$b_j = (1/2)\langle [\theta X_0, Z_i], Y_j\rangle_{\mathfrak{g}} = (1/2)(-c)\langle Z_i, Z_j\rangle_{\mathfrak{g}}, = \delta_{ij} \cdot (-c)$$

which shows $[\theta X_0, Z_i] = -cY_i$. This completes the proof. □

Throughout this paper, let us denote by Q the connected subgroup of G corresponding to $\mathfrak{q}$.

Proposition 4.4. *The action of Q on $G_2^*(\mathbb{R}^{n+3})$ is orbit equivalent to the action of $\mathrm{SO}^0(2, n)$. Therefore, it is of cohomogeneity one, and the orbits of Q are the totally geodesic $G_2^*(\mathbb{R}^{n+2})$ and its tubes M_r with radius $r > 0$.*

Proof. According to the general theory, a cohomogeneity one action is determined by one orbit up to orbit equivalence (since the other orbits are tubes around it or equidistant hypersurfaces). Therefore, it is enough to show that the orbit of Q through o is the totally geodesic $G_2^*(\mathbb{R}^{n+2})$, and the action of Q is of cohomogeneity one.

First of all, we show the former claim, that is,

$$Q.o = G_2^*(\mathbb{R}^{n+2}). \tag{4.1}$$

Recall that $\mathfrak{so}(2, n+1) = \mathfrak{g}^\rho$, where the involutive automorphism ρ is defined in (3.1). By the construction of the solvable model in [12], one can show that $\mathfrak{q} \subset \mathfrak{g}^\rho$. In fact, it satisfies

$$\rho(A_i) = A_i, \quad \rho(X_0) = X_0, \quad \rho(W_0) = W_0,$$
$$\rho(Y_i) = Y_i, \quad \rho(Z_i) = Z_i \quad (i \in \{1, \ldots, n-2\}),$$
$$\rho(Y_{n-1}) = -Y_{n-1}, \quad \rho(Z_{n-1}) = -Z_{n-1}.$$

This yields that

$$Q.o \subset \mathrm{SO}^0(2, n).o = G_2^*(\mathbb{R}^{n+2}).$$

In order to show the converse inclusion, recall $Q.o = Q/Q_o$. For the Lie algebra $\mathfrak{q}_o$ of the isotropy subgroup Q_o, one has

$$\mathfrak{q}_o = \mathfrak{q} \cap (\mathfrak{so}(2) \oplus \mathfrak{so}(n+1)) = \mathfrak{q}^\theta = \mathrm{span}\{(1+\theta)X_0\},$$

which is one-dimensional. Therefore, one can calculate that $Q.o$ and $G_2^*(\mathbb{R}^{n+2})$ have the same dimension. Since both are connected and complete, this completes the proof of (4.1).

It remains to show the latter claim, that is, the action of Q is of cohomogeneity one. This follows from the slice theorem. In general, the cohomogeneity of an isometric action coincides with the cohomogeneity of the slice representation at some point. Note that the slice representation of Q at the origin o is the action of Q_o on the normal space $\nu_o(Q.o)$. Since the tangent space of $Q.o$ at o coincides with the image of the orthogonal projection of $\mathfrak{q}$ onto $\mathfrak{p} = \mathfrak{g}^{-\theta}$, the normal space is given by

$$\nu_o(Q.o) = \mathrm{span}\{(1-\theta)Y_{n-1},\ (1-\theta)Z_{n-1}\}.$$

By the bracket relations described in the definition of the solvable model and Lemma 4.3, $\mathfrak{q}_o = \mathrm{span}\{(1+\theta)X_0\}$ acts nontrivially on $\nu_o(Q.o)$. Therefore, the slice representation is isomorphic to the standard action of $\mathrm{SO}(2)$ on $\mathbb{R}^2$, which is of cohomogeneity one. This completes the proof. □

4.3. **The Lie group $Q(s)$ and the main theorem.** In this subsection, we prove that the tube M_r is a (κ, μ)-space for some particular radius $r > 0$. First of all, we give a parameter change of the radius function, which makes the latter calculations simpler.

Lemma 4.5. *For $t > 0$, let us put $p := \exp(-tZ_1).o$, and denote by $r(t)$ the distance between $Q.p$ and $Q.o = G_2^*(\mathbb{R}^{n+2})$. Then the function $r = r(t)$ is monotonic increasing and can take any positive values.*

Proof. We consider $\gamma(u) = \exp(-u(1-\theta)Z_1).o$, which is the unit speed geodesic starting at o. Then γ is perpendicular to $Q.o$ at o. Since the action of Q is of cohomogeneity one and $Q.o$ is a singular orbit, the geodesic ray $\gamma([0,\infty))$ can be identified with the orbit space. Hence $\gamma([0,\infty))$ intersects with $Q.p$ at exactly one point, which is $\gamma(r(t))$.

One can see this picture in some $\mathbb{R}\mathrm{H}^2$. Let H be the connected Lie subgroup with Lie algebra $\mathrm{span}\{[\theta Z_1, Z_1], Z_1, \theta Z_1\} \cong \mathfrak{sl}(2,\mathbb{R})$. Then the orbit $H.o$ is a totally geodesic submanifold and isometric to $\mathbb{R}\mathrm{H}^2$. By definition, the point p and the geodesic γ are contained in this $\mathbb{R}\mathrm{H}^2$. The two points p and $\gamma(r(t))$ are related as follows. Define $A' := \exp(\mathbb{R}[\theta Z_1, Z_1]) \subset H$. Then $H.o$ is a geodesic, and $H.p$ is an equidistant curve of $H.o$ in $\mathbb{R}\mathrm{H}^2$. By a general property of $\mathbb{R}\mathrm{H}^2$, the curve $H.p$ intersects with the geodesic γ at exactly one point. Note that $A' \subset Q$, and hence $A'.p \subset Q.p$. Therefore, this intersecting point is $\gamma(r(t))$. This yields that the function $r(t)$ can be defined by using only $\mathbb{R}\mathrm{H}^2$, and the assertion follows from properties of curves in $\mathbb{R}\mathrm{H}^2$. □

Then we again replace the parameter from $t \in (0,\infty)$ to $s \in (0,\pi/2)$ by $t(s) := (\sqrt{2}/c)\tan(s)$, and define the subgroup $Q(s)$ as follows:

Definition 4.6. For $s \in (0,\pi/2)$, we define $Q(s) := gQg^{-1}$, where $g := \exp(t(s)Z_1)$.

The orbit $Q(s).o$ is isometrically congruent to M_r. Indeed,

$$g^{-1}.(Q(s).o) = g^{-1}.(gQg^{-1}.o) = Q.p = M_r,$$

where $p := g^{-1}.o = \exp(-tZ_1).o$ and $r := d(p, Q.o)$. Lemma 4.5 states that the converse holds.

Proposition 4.7. *Every tube M_r around $G_2^*(\mathbb{R}^{n+2})$ with radius $r > 0$ is isometrically congruent to the orbit $Q(s).o$ through the origin for some unique $s \in (0, \pi/2)$.*

Proof. From Lemma 4.5, for $r > 0$, there uniquely exists $t > 0$ such that $d(p, Q.o) = r$, where $p := \exp(-tZ_1).o$. We can also choose $s \in (0, \pi/2)$ satisfying $t = (\sqrt{2}/c)\tan(s)$ uniquely. Then, the congruency of M_r and $Q(s).o$ follows from the above argument, which completes the proof. □

Let $s \in (0, \pi/2)$. Since $Q(s).o = Q(s)/\{e\}$, we hereafter identify $Q(s).o$ with the Lie group $Q(s)$, and study the geometry at the Lie algebra level. We denote by $\mathfrak{q}(s)$ the Lie algebra of $Q(s)$. Since $\mathfrak{q}(s) = \mathrm{Ad}(\exp(tZ_1))\mathfrak{q}$, we have

$$\begin{aligned}\mathfrak{q}(s) = \mathrm{span}\{&A_1 - (1/\sqrt{2})\tan(s)Z_1, A_2 - (1/\sqrt{2})\tan(s)Z_1,\\ &X_0, W_0, \theta X_0 + \sqrt{2}\tan(s)Y_1\}\end{aligned}$$

when $n = 2$, and

$$\begin{aligned}\mathfrak{q}(s) = \mathrm{span}\{&A_1 - (1/\sqrt{2})\tan(s)Z_1, A_2 - (1/\sqrt{2})\tan(s)Z_1,\\ &X_0, Y_2, \dots, Y_{n-1}, Z_2, \dots, Z_{n-1}, W_0, \theta X_0 + \sqrt{2}\tan(s)Y_1\}\end{aligned}$$

when $n > 2$.

For later convenience, we define

$$\begin{aligned}\xi &:= \sin(s)(-X_0 + W_0)/\sqrt{2} - \cos(s)Y_1\\ &\qquad - \cos(s)\cot(s)(X_0 + \theta X_0)/\sqrt{2},\\ \xi^\perp &:= (-X_0 - W_0)/\sqrt{2},\\ T &:= (-A_1 + A_2)/\sqrt{2},\\ \widetilde{Y}_1 &:= \cos(s)(-X_0 + W_0)/\sqrt{2} + \sin(s)Y_1\\ &\qquad + \cos(s)(X_0 + \theta X_0)/\sqrt{2},\\ \widetilde{Z}_1 &:= \cos(s)(-A_1 - A_2)/\sqrt{2} + \sin(s)Z_1.\end{aligned}$$

Then, one can see that

$$\mathfrak{q}(s) = \begin{cases}\mathrm{span}\{\xi, \xi^\perp, T, \widetilde{Y}_1, \widetilde{Z}_1\} & \text{if } n = 2,\\ \mathrm{span}\{\xi, \xi^\perp, T, \widetilde{Y}_1, Y_2, \dots, Y_{n-1}, \widetilde{Z}_1, Z_2, \dots, Z_{n-1}\} & \text{if } n > 2.\end{cases} \tag{4.2}$$

Now we describe the almost contact metric structure of $Q(s).o = Q(s)$. Let us identify the tangent space $T_o(G_2^*(\mathbb{R}^{n+3}))$ with $\mathfrak{s}$, and denote by $\varpi : \mathfrak{g} \cong \mathfrak{k} \oplus \mathfrak{s} \to \mathfrak{s}$ the orthogonal projection. One can see that $T_o(Q(s).o) \cong \varpi(\mathfrak{q}(s)) \subset \mathfrak{s}$ is of codimension one, whose unit normal vector is given by

$$N := \sin(s)(-A_1 - A_2)/\sqrt{2} - \cos(s)Z_1 \in \mathfrak{s}.$$

Then, according to Proposition 2.8, we equip $Q(s).o$ (and hence $Q(s)$) with the almost contact metric structure with respect to N.

Proposition 4.8. *For the almost contact metric structure on $Q(s)$, one has*

(1) *the metric $\langle , \rangle$ makes the basis of (4.2) orthonormal;*
(2) *ξ defined above is the characteristic vector field;*
(3) *the $(1,1)$-tensor φ satisfies that $\varphi(\xi) = 0$, $\varphi(\xi^{\perp}) = T$, $\varphi(\widetilde{Y}_1) = \widetilde{Z}_1$, and $\varphi(Y_i) = Z_i$ for $i \in \{2, \ldots, n-1\}$.*

Proof. First, we note that for the elements in the basis (4.2),

$$\varpi(\xi) = \sin(s)(-X_0 + W_0)/\sqrt{2} - \cos(s)Y_1,$$
$$\varpi(\widetilde{Y}_1) = \cos(s)(-X_0 + W_0)/\sqrt{2} + \sin(s)Y_1,$$

and otherwise $\varpi(X) = X$ holds. Since the metric on $\mathfrak{q}(s)$ is given by $\langle X, Y\rangle := \langle \varpi(X), \varpi(Y)\rangle_{\mathfrak{s}}$, the assertion (1) follows from Theorem 4.1 (2). The assertion (2) follows from Theorem 4.1 (3), namely,

$$-JN = \sin(s)(-X_0 + W_0)/\sqrt{2} - \cos(s)Y_1 = \varpi(\xi).$$

The assertion (3) can be shown similarly. □

Now, we are in position to prove the main theorem.

Theorem 4.9. *For $s \in (0, \pi/2)$, the hypersurface $Q(s).o$ in $G_2^*(\mathbb{R}^{n+3})$ with minimal sectional curvature $-8\csc(s)^2$ is a $(0, 4\csc(s)^2)$-space, whose Boeckx invariant is $I = 1 - 2\csc(s)^2 < -1$.*

Proof. Take any $s \in (0, \pi/2)$, and normalize the metric on $G_2^*(\mathbb{R}^{n+3})$ so that the minimal sectional curvature is $-8\csc(s)^2$. One can prove the theorem by giving an explicit isomorphism between $\mathfrak{q}(s)$ and $\mathfrak{g}_{\alpha,\beta}$ as (almost) contact metric Lie algebras. Let us consider the following correspondence from $\mathfrak{q}(s)$ to $\mathfrak{g}_{\alpha,\beta}$:

$$\xi \mapsto \hat{\xi}, \quad -\widetilde{Z}_i \mapsto \hat{X}_1, \quad \xi^{\perp} \mapsto \hat{X}_2, \quad \widetilde{Y}_i \mapsto \hat{Y}_1, \quad T \mapsto \hat{Y}_2,$$

in addition, if $n > 2$,

$$-Z_i \mapsto \hat{X}_{i+1}, \quad Y_i \mapsto \hat{Y}_{i+1},$$

where $\{\hat{\xi}, \hat{X}_1, \ldots, \hat{X}_n, \hat{Y}_1, \ldots, \hat{Y}_n\}$ is the basis of $\mathfrak{g}_{\alpha,\beta}$ mentioned in Definition 2.5. By calculating the bracket relations on $\mathfrak{q}(s)$ and comparing them to ones on $\mathfrak{g}_{\alpha,\beta}$, one can show that the above correspondence is an isomorphism between $\mathfrak{q}(s)$ and $\mathfrak{g}_{\alpha,\beta}$ as Lie algebras. Similarly, by comparing their contact structures, one can show that they are isomorphic as contact metric algebras. □

This theorem gives realizations of (κ, μ)-spaces with Boeckx invariant $I < -1$, in fact $(0, \mu)$-spaces for $\mu > 4$, as homogeneous hypersurfaces in $G_2^*(\mathbb{R}^{n+3})$.

Remark 4.10. (1) In Theorem 4.9, let us consider the limit $s \to \pi/2$. Then, the hypersurface $Q(s).o$ is deformed to $S_N.o$, where S_N is the connected subgroup of S corresponding to $\mathfrak{s} \ominus \mathrm{span}\{(-A_1 - A_2)/\sqrt{2}\}$. It has been known in [12] that the hypersurface $S_N.o$ is a $(0,4)$-space. Therefore, our theorem describes such deformation explicitly.

(2) Since two Lie algebras $\mathfrak{q}(s)$ and $\mathfrak{q}$ are conjugate, $\mathfrak{g}_{\alpha,\beta}$ is also isomorphic to $\mathfrak{q}$ as Lie algebras. This fact gives us another expression of $\mathfrak{g}_{\alpha,\beta}$ as a Lie algebra. In particular, when $n = 2$, one see that $\mathfrak{g}_{\alpha,\beta} \cong \mathfrak{sl}_3(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, which has been mentioned in [16].

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