A COMMUTATIVITY CONDITION FOR SUBSETS IN QUANDLES — A GENERALIZATION OF ANTIPODAL SUBSETS

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Dedicated to the memory of Professor Tadashi Nagano

Abstract. In this paper, we introduce some commutativity condition for subsets in quandles, which we call the s-commutativity. Note that quandles can be regarded as a generalization of symmetric spaces, and the notion of s-commutative subsets is a generalization of antipodal subsets. We study maximal s-commutative subsets in quandles, and show that they have some nice properties. As one example, any maximal s-commutative subsets in quandles are subquandles. We also determine maximal s-commutative subsets in spheres, projective spaces, and dihedral quandles. In these quandles, maximal s-commutative subsets turn out to be unique up to automorphisms.

1. Introduction

A quandle is a set equipped with a binary operator, whose axioms are corresponding to the Reidemeister moves of knots. Although the notion of quandles is originated in knot theory ([5]), it now plays important roles in many branches of mathematics. Quandles have been studied actively from various perspectives and viewpoints. As one aspect, quandles can be regarded as a generalization of symmetric spaces. The definition of quandles can be formulated as follows.

Definition 1.1. Let Q be a set, and s be a map from Q into $\mathrm{Map}(Q, Q)$, that is, a map $s_x : Q \to Q$ is equipped for each $x \in Q$. Then a pair (Q, s) is called a *quandle* if

($Q1$) for any $x \in Q$, $s_x(x) = x$,
($Q2$) for any $x \in Q$, s_x is bijective,
($Q3$) for any $x, y \in Q$, $s_x \circ s_y = s_{s_x(y)} \circ s_x$.

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As has already been mentioned in [5], symmetric spaces are quandles by taking s_x as the point symmetry at x (for symmetric spaces, see Subsection 2.4). It is then natural to study quandles by referring to the theory of symmetric spaces. There have been such studies, for which we refer to [3, 4, 8] and references therein.

This paper also studies quandles from the viewpoint of symmetric spaces. In the theory of Riemannian symmetric spaces, important notions include not only Riemannian geometric properties (such as curvatures), but also ones derived from the point symmetries. A typical and most important example is the notion of antipodal subsets, introduced by Chen and Nagano ([2]). The definition can easily be transferred to quandles as follows.

Definition 1.2. A subset X in a quandle (Q, s) is said to be *antipodal* if $s_x(y) = y$ holds for any $x, y \in X$.

This notion generalizes the antipodal points in the unit sphere S^n. For each $x \in S^n$, the point $-x$ is an antipodal point of x. The subset $\{\pm x\}$ is an antipodal subset in S^n, which is maximal with respect to inclusion relation. Furthermore, every maximal antipodal subset in S^n is of this form (see Section 3 for details and more examples). For other spaces, it would probably be surprising that the classifications of maximal antipodal subsets are involved in general, and it is still open for some symmetric spaces (see Remark 3.12).

As one of important applications of antipodal subsets, the notion of 2-numbers has been introduced ([2]). The 2-*number* of a compact Riemannian symmetric space is defined by the supremum of the cardinalities of antipodal subsets, which is in fact finite. The 2-number is related to topological information of the symmetric space, such as the Euler characteristic ([2]) and the $\mathbb{Z}_2$-Betti number ([7]). For further results and applications, we refer to a survey article [1] and references therein.

In a possible structure theory of quandles, it would be reasonable to expect to have a nice class of subsets, which reflect some properties of the ambient quandles. In this paper, we introduce the notion of s-commutative subsets in quandles, which would form a nice class of subsets. The definition is given as follows (see Section 4 for details).

Definition 1.3. A subset X in a quandle (Q, s) is said to be *s-commutative* if $s_x \circ s_y = s_y \circ s_x$ holds for any $x, y \in X$.

Note that the s-commutativity condition itself can also been seen in [2, Proposition 3.4], but the notion of s-commutative subsets would be new. One of the purposes of this paper is to study some fundamental properties of s-commutative subsets. Recall that antipodal subsets play important roles in the study of symmetric spaces. The notion of s-commutative subsets is a generalization of antipodal subsets.

Proposition 1.4. *Every antipodal subset in a quandle (Q, s) is s-commutative.*

For s-commutative subsets in quandles, it is natural to consider maximal ones with respect to inclusion relation. Recall that a subset X in (Q, s) is called a subquandle if X is normalized by $s_x^{\pm 1}$ for every $x \in X$.

Theorem 1.5. *Every maximal s-commutative subset in a quandle (Q, s) is a subquandle.*

There are notions of the direct products and the interaction-free unions of quandles. We also study, in Section 5, the behaviour of antipodal subsets and s-commutative subsets under these operations.

In Section 6, we determine maximal s-commutative subsets in some symmetric spaces and quandles. The first example is the n-dimensional unit sphere S^n with $n \in \mathbb{Z}_{>0}$. Denote by $\{e_1, \ldots, e_{n+1}\}$ the standard basis of $\mathbb{R}^{n+1}$, and by $\mathrm{SO}(n+1)$ the special orthogonal group of degree $n+1$.

Proposition 1.6. *The subset $\{\pm e_1, \ldots, \pm e_{n+1}\}$ is maximal s-commutative in the unit sphere S^n, which is unique up to the congruence by $\mathrm{SO}(n+1)$.*

Since the above subset is not antipodal, this example clarifies a difference between antipodal subsets and s-commutative subsets. The second example is the n-dimensional real projective space $P(\mathbb{R}^{n+1})$. We denote by $\mathbb{R}x \in P(\mathbb{R}^{n+1})$ the line spanned by $x \in \mathbb{R}^{n+1} \setminus \{0\}$.

Proposition 1.7. *The subset $\{\mathbb{R}e_1, \ldots, \mathbb{R}e_{n+1}\}$ is maximal s-commutative in $P(\mathbb{R}^{n+1})$ if $n > 1$. If $n = 1$, then $\{\mathbb{R}e_1, \mathbb{R}e_2, \mathbb{R}(e_1 + e_2), \mathbb{R}(e_1 - e_2)\}$ is a maximal s-commutative subset in $P(\mathbb{R}^2)$. In both cases, a maximal s-commutative subset in $P(\mathbb{R}^{n+1})$ is unique up to the congruence by $\mathrm{SO}(n+1)$.*

It is natural but interesting that the shape of a maximal s-commutative subset in $P(\mathbb{R}^{n+1})$ depends on n. Note that $\{\mathbb{R}e_1, \mathbb{R}e_2, \mathbb{R}(e_1+e_2), \mathbb{R}(e_1-e_2)\}$ is s-commutative in $P(\mathbb{R}^2)$, but not in $P(\mathbb{R}^3)$. This yields that the s-commutativity is an extrinsic property. The third example is the dihedral quandle R_n. Recall that $R_n := (\mathbb{Z}/n\mathbb{Z}, s)$ with $s_x(y) := 2x - y$ (see Example 2.5 for details).

Proposition 1.8. *For the dihedral quandle R_n, the following subsets are maximal antipodal and maximal s-commutative, respectively, and all of them are unique up to the automorphisms*:

condition	*maximal antipodal*	*maximal s-commutative*
n is odd	$\{0\}$	$\{0\}$
$n = 4l - 2$ $(l \in \mathbb{Z}_{>0})$	$\{0, 2l - 1\}$	$\{0, 2l - 1\}$
$n = 4l$ $(l \in \mathbb{Z}_{>0})$	$\{0, 2l\}$	$\{0, l, 2l, 3l\}$

By the results of the above three propositions, one can see that antipodal subsets and s-commutative subsets are sometimes the same, but sometimes different, both in the finite and infinite cardinality cases. In Remark 6.6, we also mention a difference between poles and antipodal subsets in finite quandles.

Note that the study on s-commutative subsets is new and undeveloped. There are many open problems related to symmetric spaces and quandles. In particular, it would be interesting to determine maximal s-commutative subsets in symmetric spaces and quandles, and study the uniqueness properties. Note that $\mathrm{SO}(n+1)$ (and also the orthogonal group $\mathrm{O}(n+1)$) acts on S^n and $P(\mathbb{R}^{n+1})$ as automorphisms of quandles. Therefore, for the above cases S^n, $P(\mathbb{R}^{n+1})$, and R_n, maximal s-commutative subsets are unique up to automorphisms. However, this would be not true in general.

Our studies on quandles from the viewpoint of symmetric spaces have been influenced strongly by Professor Tadashi Nagano. In particular, the fourth author obtained his PhD degree under the supervision of Professor Tadashi Nagano, and learned many things about symmetric spaces. Without his guidance, our studies would not have been possible. The authors are also grateful to Hiroyuki Tasaki and Makiko Sumi Tanaka for their kind supports and helpful discussions.

2. Preliminaries

In this section, we recall some basic notions on quandles. In particular, the direct product and the interaction-free union of quandles are studied. We also recall some examples of quandles, including symmetric spaces, whose subsets will be studied in the following sections. Throughout this paper, quandles are denoted by $Q = (Q, s)$ as in Definition 1.1, and s is called the quandle structure on Q.

2.1. Quandles.

In this subsection, we recall some basic notions on quandles. First of all, we recall the notion of subquandles.

Definition 2.1. Let (Q, s) be a quandle. A subset X of Q is called a *subquandle* if for any $x \in X$ it satisfies $s_x^{\pm 1}(X) \subset X$.

Note that X is a subquandle if and only if $s_x(X) = X$ for any $x \in X$. It is easy to see that every subquandle is naturally a quandle. We also note that the condition $s_x(X) \subset X$ is not sufficient to define subquandles, since s_x is not necessary involutive.

Definition 2.2. Let (M, s^M) and (N, s^N) be quandles. Then a map $f : M \to N$ is called a *homomorphism* if it satisfies

$$s^N_{f(x)} \circ f = f \circ s^M_x \quad (\forall x \in M).$$

Note that the inverse map of a bijective homomorphism is also a homomorphism. Hence, a bijective homomorphism is called an *isomorphism*. The notions of automorphisms and the automorphism groups are defined naturally.

Definition 2.3. For a quandle $Q = (Q, s)$, a bijective homomorphism from (Q, s) to (Q, s) itself is called an *automorphism*. The group consisting of all automorphisms is called the *automorphism group* of (Q, s), and denoted by $\mathrm{Aut}(Q)$ or $\mathrm{Aut}(Q, s)$.

A quandle Q is said to be *homogeneous* if $\mathrm{Aut}(Q)$ acts transitively on Q. Note that $s_x \in \mathrm{Aut}(Q)$ holds for any $x \in Q$.

Example 2.4. Let M be a set. Then one obtains a quandle structure s on M by putting $s_x := \mathrm{id}_M$ for every $x \in M$. This (M, s) is called a *trivial quandle*. A trivial quandle (M, s) is homogeneous, since any bijection from M to M is an automorphism.

The next example is the dihedral quandle, which would be the simplest non-trivial quandles. We denote by $\mathbb{Z}/n\mathbb{Z}$ the cyclic group of order n.

Example 2.5. Let us fix $n \in \mathbb{Z}_{>0}$, and put $R_n := \mathbb{Z}/n\mathbb{Z}$. Then we have a quandle structure s on R_n by putting

$$s_x : R_n \to R_n : y \mapsto 2x - y,$$

for each $x \in R_n$. This (R_n, s) is called the dihedral quandle of order n. The dihedral quandle (R_n, s) is homogeneous, since the following f is an automorphism:

$$f : R_n \to R_n : x \mapsto x + 1.$$

We note that, in terms of the above automorphism f, the cyclic group $\mathbb{Z}/n\mathbb{Z}$ acts on (R_n, s) as automorphisms.

2.2. **Direct products of quandles.** In this subsection, we consider a family of quandles $\{Q_\lambda = (Q_\lambda, s^\lambda)\}_{\lambda\in\Lambda}$, and study the direct product quandle $\prod_{\lambda\in\Lambda} Q_\lambda$. As a set, it is the direct product set of $\{Q_\lambda\}_{\lambda\in\Lambda}$, whose elements will be denoted as $(x_\lambda)_\lambda$. The next proposition has been known, and can be proved directly.

Proposition 2.6. *Let $Q := \prod_{\lambda\in\Lambda} Q_\lambda$ be the direct product set. Then the map $s^Q : Q \to \mathrm{Map}(Q, Q)$ defined by the following is a quandle structure on Q:*

$$s^Q_{(x_\lambda)_\lambda} : Q \to Q : (y_\lambda)_\lambda \mapsto (s^\lambda_{x_\lambda}(y_\lambda))_\lambda.$$

The obtained quandle is called the *direct product quandle* of a family of quandles $\{Q_\lambda\}_{\lambda\in\Lambda}$, and denoted by $\prod_{\lambda\in\Lambda} Q_\lambda$. In the remaining of this subsection, we show some properties of the direct product quandles. We start from the following.

Proposition 2.7. *For any $\eta \in \Lambda$, the natural projection $p_\eta : \prod_{\lambda\in\Lambda} Q_\lambda \to Q_\eta$ is a quandle homomorphism.*

Proof. We denote by s^Q the quandle structure on the direct product quandle $Q := \prod_{\lambda\in\Lambda} Q_\lambda$. Take any $(x_\lambda)_\lambda, (y_\lambda)_\lambda \in Q$, and any $\eta \in \Lambda$. Then we have

$$(p_\eta \circ s^Q_{(x_\lambda)_\lambda})((y_\lambda)_\lambda) = p_\eta((s^\lambda_{x_\lambda}(y_\lambda))_\lambda) = s^\eta_{x_\eta}(y_\eta) = (s^\eta_{p_\eta((x_\lambda)_\lambda)} \circ p_\eta)((y_\lambda)_\lambda),$$

which proves that p_η is a quandle homomorphism. $\square$

The next proposition gives a key property of direct product quandles. Namely, a family of quandle homomorphisms into Q_λ gives rise to a quandle homomorphism into $\prod_{\lambda \in \Lambda} Q_\lambda$.

Proposition 2.8. *Let Z be a quandle, and $\pi_\lambda : Z \to Q_\lambda$ be a quandle homomorphism for each $\lambda \in \Lambda$. Then the map*

$$h : Z \to \prod_{\lambda \in \Lambda} Q_\lambda : x \mapsto (\pi_\lambda(x))_\lambda$$

is the unique quandle homomorphism satisfying $p_\lambda \circ h = \pi_\lambda$ for any $\lambda \in \Lambda$.

Proof. Denote by s^Z and s^Q the quandle structures on Z and $Q := \prod_{\lambda \in \Lambda} Q_\lambda$, respectively. Then, since each π_λ is a quandle homomorphism, we have

$$\begin{aligned}(h \circ s^Z_x)(y) = h(s^Z_x(y)) = (\pi_\lambda(s^Z_x(y)))_\lambda = ((s^\lambda_{\pi_\lambda(x)} \circ \pi_\lambda)(y))_\lambda \\ = s^Q_{(\pi_\lambda(x))_\lambda}(\pi_\lambda(y))_\lambda = (s^Q_{h(x)} \circ h)(y)\end{aligned}$$

for any $x, y \in Z$. This shows that h is a quandle homomorphism. It is clear that h satisfies $p_\lambda \circ h = \pi_\lambda$ for any $\lambda \in \Lambda$. The uniqueness follows easily. □

By applying Proposition 2.8, we obtain a property of quandle automorphisms on $\prod_{\lambda \in \Lambda} Q_\lambda$. Namely, the direct product group $\prod_{\lambda \in \Lambda} \mathrm{Aut}(Q_\lambda)$ of the automorphism groups of Q_λ can be understood as a subgroup of $\mathrm{Aut}(\prod_{\lambda \in \Lambda} Q_\lambda)$.

Proposition 2.9. *For each $(f_\lambda)_\lambda \in \prod_{\lambda \in \Lambda} \mathrm{Aut}(Q_\lambda)$, the following map $\prod_{\lambda \in \Lambda} f_\lambda$ is an automorphism on the quandle $\prod_{\lambda \in \Lambda} Q_\lambda$:*

$$\prod_{\lambda \in \Lambda} f_\lambda : \prod_{\lambda \in \Lambda} Q_\lambda \to \prod_{\lambda \in \Lambda} Q_\lambda : (x_\lambda)_\lambda \mapsto (f_\lambda(x_\lambda))_\lambda.$$

Proof. Put $Z := \prod_{\lambda \in \Lambda} Q_\lambda$, and $\pi_\lambda := f_\lambda \circ p_\lambda$ for each $\lambda \in \Lambda$. Since $\pi_\lambda : Z \to Q_\lambda$ is a quandle homomorphism, it follows from Proposition 2.8 that

$$h : Z \to \prod_{\lambda \in \Lambda} Q_\lambda : x \mapsto (\pi_\lambda(x))_\lambda$$

is a quandle homomorphism. Then so is $\prod_{\lambda \in \Lambda} f_\lambda$, since

$$h((x_\lambda)_\lambda) = ((f_\lambda \circ p_\lambda)((x_\lambda)_\lambda))_\lambda = (f_\lambda(x_\lambda))_\lambda = (\prod_{\lambda \in \Lambda} f_\lambda)((x_\lambda)_\lambda).$$

Furthermore, this is bijective, since $\prod_{\lambda \in \Lambda}(f_\lambda^{-1})$ is the inverse map. □

As another application of Proposition 2.8, one can see that the quandle structure on $\prod_{\lambda \in \Lambda} Q_\lambda$ is canonical, in the following sense.

Proposition 2.10. *Let s' be a quandle structure on the direct product set $\prod_{\lambda \in \Lambda} Q_\lambda$, and suppose that the projection p_λ is a quandle homomorphism onto Q_λ for any $\lambda \in \Lambda$. Then $(\prod_{\lambda \in \Lambda} Q_\lambda, s')$ coincides with the direct product quandle of $\{Q_\lambda\}_{\lambda \in \Lambda}$.*

Proof. We apply Proposition 2.8 for the quandle $Z := (\prod_{\lambda \in \Lambda} Q_\lambda, s')$ and the quandle homomorphisms $p_\lambda : Z \to Q_\lambda$. Then the obtained quandle homomorphism $h : Z \to \prod_{\lambda \in \Lambda} Q_\lambda$ is the identify map, since

$$h((x_\lambda)_\lambda) = (p_\lambda((x_\lambda)_\lambda))_\lambda = (x_\lambda)_\lambda$$

for any $(x_\lambda)_\lambda \in Z$. Since $h = \mathrm{id}$ is a quandle homomorphism, we conclude that s' coincides with the quandle structure s^Q on the direct product quandle $\prod_{\lambda\in\Lambda} Q_\lambda$. This completes the proof. □

It should be noted that, by Proposition 2.8, the direct product quandle $\prod_{\lambda\in\Lambda} Q_\lambda$ is the product of the family $\{Q_\lambda = (M_\lambda, s^\lambda)\}_{\lambda\in\Lambda}$ in the category of quandles and quandle homomorphisms. This means that it has the following universal property.

Universal property of direct products: Let us fix any quandle Z and any quandle homomorphism $\pi_\lambda : Z \to Q_\lambda$ for each $\lambda \in \Lambda$. Then there exists a unique quandle homomorphism $h : Z \to \prod_{\lambda\in\Lambda} Q_\lambda$ such that $p_\lambda \circ h = \pi_\lambda$ for any $\lambda \in \Lambda$.

2.3. **Interaction-free unions of quandles.** Same as in the previous subsection, we consider a family of quandles $\{Q_\lambda = (Q_\lambda, s^\lambda)\}_{\lambda\in\Lambda}$. In this subsection, we study the interaction-free union quandle $\bigsqcup^{\mathrm{free}}_{\lambda\in\Lambda} Q_\lambda$, defined by the following.

Proposition 2.11. *Let $Q := \bigsqcup_{\lambda\in\Lambda} Q_\lambda$ be the disjoint union as a set. Then the map $s^{\mathrm{free}} : Q \to \mathrm{Map}(Q, Q)$ defined by the following is a quandle structure on Q: for each $x \in Q_\eta$,*

$$s^{\mathrm{free}}_x(y) := \begin{cases} s^\eta_x(y) & (\textit{if } y \in Q_\eta), \\ y & (\textit{otherwise}). \end{cases}$$

Proof. Let $x \in M_\eta$. Then the quandle structure s^{free} can be written as

$$s^{\mathrm{free}}_x|_{M_\eta} = s^\eta_x, \qquad s^{\mathrm{free}}_x|_{M_{\eta'}} = \mathrm{id} \ (\text{for } \eta' \neq \eta).$$

Then one can check Conditions (Q1), (Q2), and (Q3) of quandles easily. □

We denote by $\bigsqcup^{\mathrm{free}}_{\lambda\in\Lambda} Q_\lambda := (\bigsqcup_{\lambda\in\Lambda} Q_\lambda, s^{\mathrm{free}})$, which will be called the *interaction-free union* of the family of quandles $\{Q_\lambda\}_{\lambda\in\Lambda}$ in this paper. Similar to the case of direct product quandles, it satisfies the following.

Proposition 2.12. *The natural inclusion map $\iota_\eta : Q_\eta \to \bigsqcup^{\mathrm{free}}_{\lambda\in\Lambda} Q_\lambda$ is a quandle homomorphism for any $\eta \in \Lambda$.*

Proof. Take any $x, y \in M_\eta$. Then we have

$$(\iota_\eta \circ s^\eta_x)(y) = s^\eta_x(y) = s^{\mathrm{free}}_x(y) = (s^{\mathrm{free}}_{\iota_\eta(x)} \circ \iota_\eta)(y),$$

which completes the proof. □

We here give a property on quandle homomorphisms. Namely, if a family of quandle homomorphisms from Q_λ satisfies some condition, then it gives rise to a quandle homomorphism from $\bigsqcup^{\mathrm{free}}_{\lambda\in\Lambda} Q_\lambda$.

Proposition 2.13. *Let (Z, s^Z) be a quandle, and $\iota'_\lambda : Q_\lambda \to Z$ be a quandle homomorphism for each $\lambda \in \Lambda$. Assume that it satisfies*

$$(s^Z_{\iota'_\eta(x)} \circ \iota'_\xi)(y) = \iota'_\xi(y) \quad (\forall x \in Q_\eta,\ \forall y \in Q_\xi \textit{ with } \eta \neq \xi).$$

Then the map $h : \bigsqcup^{\mathrm{free}}_{\lambda\in\Lambda} Q_\lambda \to Z$ *defined by*

$$h(x) := \iota'_\eta(x) \quad (\textit{if } x \in Q_\eta)$$

is the unique quandle homomorphism satisfying $h \circ \iota_\eta = \iota'_\eta$ *for any* $\eta \in \Lambda$.

Proof. It is easy to see that $h \circ \iota_\eta = \iota'_\eta$ for any $\eta \in \Lambda$, and that the map h with this property is unique. Hence we have only to show that h is a quandle homomorphism. Take any $\eta, \xi \in \Lambda$, $x \in Q_\eta$, and $y \in Q_\xi$. We claim that

$$(s^Z_{h(x)} \circ h)(y) = (h \circ s^{\mathrm{free}}_x)(y).$$

If $\eta = \xi$, then the claim follows from the property of $\iota'_\xi : Q_\xi \to Z$, which is a quandle homomorphism. In fact, we have

$$(s^Z_{h(x)} \circ h)(y) = (s^Z_{\iota'_\eta(x)} \circ \iota'_\eta)(y) = (\iota'_\eta \circ s^\eta_x)(y) = (h \circ s^{\mathrm{free}}_x)(y).$$

If $\eta \neq \xi$, then the claim follows from the definition of s^{free} and the assumption. One can see that

$$(s^Z_{h(x)} \circ h)(y) = (s^Z_{\iota'_\eta(x)} \circ \iota'_\xi)(y) = \iota'_\xi(y) = h(y) = (h \circ s^{\mathrm{free}}_x)(y).$$

This proves the claim, which completes the proof of the proposition. □

The above proposition gives us a relationship between quandle automorphism groups. Let us consider the automorphism group $\mathrm{Aut}(\bigsqcup^{\mathrm{free}}_{\lambda\in\Lambda} Q_\lambda)$ of the interaction-free union quandle, and the direct product group $\prod_{\lambda\in\Lambda} \mathrm{Aut}(Q_\lambda)$ of the automorphism groups of Q_λ. The next proposition means that the latter can be understood as a subgroup of the former.

Proposition 2.14. *Let* $(f_\lambda)_\lambda \in \prod_{\lambda\in\Lambda} \mathrm{Aut}(Q_\lambda)$. *Then the map* $\bigsqcup_{\lambda\in\Lambda} f_\lambda$ *defined by the following is an automorphism on the quandle* $\bigsqcup^{\mathrm{free}}_{\lambda\in\Lambda} Q_\lambda$:

$$\bigsqcup_{\lambda\in\Lambda} f_\lambda : \bigsqcup^{\mathrm{free}}_{\lambda\in\Lambda} Q_\lambda \to \bigsqcup^{\mathrm{free}}_{\lambda\in\Lambda} Q_\lambda : x_\eta \mapsto f_\eta(x_\eta) \quad (\textit{for } x_\eta \in Q_\eta).$$

Proof. We consider $Z := \bigsqcup^{\mathrm{free}}_{\lambda\in\Lambda} Q_\lambda$ and $\iota'_\lambda := \iota_\lambda \circ f_\lambda$. Then they satisfy the assumption of Proposition 2.13. We thus have a quandle homomorphism h, which satisfies for any $x_\eta \in Q_\eta$ that

$$h(x_\eta) = \iota'_\eta(x_\eta) = (\iota_\eta \circ f_\eta)(x_\eta) = f_\eta(x_\eta) = (\bigsqcup_{\lambda\in\Lambda} f_\lambda)(x_\eta).$$

This yields that $\bigsqcup_{\lambda\in\Lambda} f_\lambda$ is a homomorphism. One also knows that it is bijective, since $\bigsqcup_{\lambda\in\Lambda}(f_\lambda^{-1})$ is the inverse map. □

We saw some properties of interaction-free union quandles, which are similar to the case of direct product quandles. However, we cannot say that s^{free} is a canonical quandle structure on $\bigsqcup_{\lambda\in\Lambda} Q_\lambda$, in the following sense.

Remark 2.15. A quandle structure s' on $Q := \bigsqcup_{\lambda\in\Lambda} Q_\lambda$, satisfying that each inclusion map $\iota_\lambda : Q_\lambda \to (Q, s')$ is a quandle homomorphism, is not unique in general. A simple example is given by the dihedral quandle R_4. Let us consider

$$R_4 = \{0,1,2,3\}, \quad Q_1 := \{0,2\}, \quad Q_2 := \{1,3\}.$$

Then one knows $R_4 = Q_1 \bigsqcup Q_2$ as a set, both inclusion maps are quandle homomorphisms, but the quandle structure on R_4 is different from s^{free}.

We also note that, in the category of quandles and quandle homomorphisms, the interaction-free union $\bigsqcup^{\mathrm{free}}_{\lambda\in\Lambda} Q_\lambda$ is not the coproduct of $\{Q_\lambda\}_{\lambda\in\Lambda}$ in general.

2.4. **Symmetric spaces.** Symmetric spaces form an important subclass of quandles. As is well-known, there are several equivalent definitions of symmetric spaces. For the following definition, we refer to the book by Loos ([6]).

Definition 2.16. A smooth manifold M equipped with a family of diffeomorphisms $\{s_x : M \to M\}_{x\in M}$ indexed by M is called a *symmetric space* if

(1) The map $M \times M \to M : (x,y) \mapsto s_x(y)$ is smooth.
(2) For each $x \in M$, the point x is an isolated fixed point of $s_x : M \to M$.
(3) $s_x^2 = \mathrm{id}_M$ for each $x \in M$.
(4) $s_x \circ s_y = s_y \circ s_{s_y(x)}$ for each $x, y \in M$.

The diffeomorphism $s_x : M \to M$ is called the *point symmetry* at $x \in M$. Note that Loos ([6]) uses a binary operator $x \cdot y$, but the above definition is a paraphrase in terms of the correspondence $s_x(y) = x \cdot y$.

Remark 2.17. According to this definition, one can easily see that every symmetric space is a quandle. We identify a family of diffeomorphisms $\{s_x : M \to M\}_{x\in M}$ with a quandle structure $s : M \to \mathrm{Map}(M,M)$ naturally.

In the following we give fundamental examples of symmetric spaces, such as spheres and real projective spaces. For both examples, we use the standard inner-product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{n+1}$, and the reflection $r_V : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with respect to a subspace V. Recall that r_V is defined by

$$r_V(v+w) = v - w \quad (\text{for any } v \in V \text{ and } w \in V^\perp),$$

where $V^\perp$ is the orthogonal compliment of V in $\mathbb{R}^{n+1}$ with respect to $\langle,\rangle$.

Example 2.18. Let $n \in \mathbb{Z}_{>0}$, and we realize the n-dimensional sphere as

$$S^n := \{x \in \mathbb{R}^{n+1} \mid \langle x,x\rangle = 1\}.$$

For each $x \in S^n$, we define $s_x := r_{\mathbb{R}x}$, the reflection with respect to the line $\mathbb{R}x$. One also can write s_x as

$$s_x : S^n \to S^n : y \mapsto -y + 2\langle x,y\rangle x.$$

Then S^n is a compact symmetric space with point symmetries $s := \{s_x\}_{x\in S^n}$. One can easily see that the orthogonal group $\mathrm{O}(n+1)$ can be considered as a subgroup of $\mathrm{Aut}(S^n, s)$. Therefore spheres are homogeneous quandles.

Note that $\mathrm{Aut}(S^n, s)$ is the automorphism group of the quandle (S^n, s) in the sense of Definition 2.3. There is another notion of the automorphism groups of symmetric spaces, but we do not use it in this paper.

Example 2.19. Let $n \in \mathbb{Z}_{>0}$, and we realize the n-dimensional real projective space by

$$P(\mathbb{R}^{n+1}) := \{\text{one-dimensional subspaces in } \mathbb{R}^{n+1}\}.$$

For each $\ell \in P(\mathbb{R}^{n+1})$, in terms of the reflection r_ℓ, let us define

$$s_\ell : P(\mathbb{R}^{n+1}) \to P(\mathbb{R}^{n+1}) : \ell' \mapsto r_\ell(\ell').$$

Then $P(\mathbb{R}^{n+1})$ is a compact symmetric space with respect to the point symmetries $s := \{s_\ell\}_{\ell \in P(\mathbb{R}^{n+1})}$. One can easily see that the orthogonal group $\mathrm{O}(n+1)$ can be considered as a subgroup of $\mathrm{Aut}(P(\mathbb{R}^{n+1}), s)$. Therefore, real projective spaces are homogeneous quandles.

3. Poles and antipodal subsets

There are notions of poles and antipodal subsets in the theory of (Riemannian) symmetric spaces. In this section, we directly transfer these notions to quandles. Throughout this section, $Q = (Q, s)$ denotes a quandle, and $\mathrm{Aut}(Q)$ denotes the automorphism group of Q.

3.1. **Poles.**

In this subsection, we introduce the notion of poles in quandles, and describe some examples. We start from the notion of pole pairs in a quandle (Q, s), which gives a binary relation on Q.

Definition 3.1. Let $x, y \in Q$. Then a pair (x, y) is called a *pole pair* in Q if it satisfies $s_x = s_y$.

By definition, (x, x) is always a pole pair, which is said to be *trivial*. We are interested in studying non-trivial pole pairs in given quandles.

Remark 3.2. The notion of poles has been defined for symmetric spaces by Chen and Nagano ([2]). In fact, the condition of the original definition is different, but $s_x = s_y$ is one of equivalent conditions for (x, y) to be a pole pair ([2, Proposition 2.9]).

In terms of the notion of pole pairs, one can define the notion of pole subsets of quandles.

Definition 3.3. A subset X in Q is called a *pole subset* if (x, y) is a pole pair for any $x, y \in X$.

It is clear that any subset of a pole subset is also a pole subset. Therefore it is natural to consider the maximal ones with respect to inclusion relation. We denote the collection of all maximal pole subsets in Q by

$$\mathcal{MS}(Q; \mathrm{pole}).$$

Proposition 3.4. *Let $f \in \mathrm{Aut}(Q)$. If X is a pole or maximal pole subset in Q, then $f(X)$ is also a pole or maximal pole subset in Q, respectively. In particular, the automorphism group $\mathrm{Aut}(Q)$ acts on $\mathcal{MS}(Q; \mathrm{pole})$.*

Proof. By the definition of quandle automorphisms, we have

$$f \circ s_x \circ f^{-1} = s_{f(x)} \quad (\forall x \in Q).$$

Therefore, (x, y) is a pole pair if and only if so is $(f(x), f(y))$. Then one can easily show the first assertion, that is, if X is a pole subset then so is $f(X)$.

In order to show the assertion on maximality, assume that X is a maximal pole subset in Q. Then $f(X)$ is a pole subset. Take a pole subset B with $f(X) \subset B$. Since $f^{-1}(B)$ is a pole subset containing X, one has $X = f^{-1}(B)$, that is $f(X) = B$. This completes the proof of the maximality of $f(X)$ as a pole subset. □

By this proposition, it is natural to consider the classification of maximal pole subsets, up to automorphisms. This problem is essentially equivalent to determine the orbit space of the action of $\mathrm{Aut}(Q)$,

$$\mathrm{Aut}(Q) \backslash \mathcal{MS}(Q; \mathrm{pole}).$$

We here note that a maximal pole subset is determined by one element in the following sense. For each $x \in Q$, we denote by

$$P(Q; x) := \{y \in Q \mid (x, y) \text{ is a pole pair in } Q\}.$$

Then $P(Q; x)$ is a maximal pole subset in Q, since the concept of pole pairs defines an equivalent relation on Q. Conversely, a maximal pole subset in Q containing x coincides with $P(Q; x)$. By this argument, for a homogeneous quandle Q, the above orbit space is always a singleton, that is, a maximal pole subset in Q is unique up to automorphisms.

Although we know the uniqueness of pole subsets in homogeneous quandles, it is a different problem to give an explicit expression. In the remaining of this subsection, we describe maximal pole subsets in the quandles given in the previous section. Note that the maximal pole subsets in the spheres S^n and the real projective spaces $P(\mathbb{R}^{n+1})$ have been well-known. We here recall them for the reader's convenience.

Example 3.5. Let us consider the n-dimensional sphere S^n as in Example 2.18. Then a pair of points (x, y) on S^n is a pole pair if and only if $x = \pm y$. Therefore every maximal pole subset in S^n is of the form $\{\pm x\}$.

Proof. Take $x, y \in S^n$. Recall that s_x and s_y are the reflections with respect to the lines $\mathbb{R}x$ and $\mathbb{R}y$, respectively. Therefore, (x, y) is a pole pair if and only if $\mathbb{R}x = \mathbb{R}y$, that is, $x = \pm y$. □

We next describe pole subsets in the real projective spaces $P(\mathbb{R}^{n+1})$. Recall that the point symmetry s_ℓ at $\ell \in P(\mathbb{R}^{n+1})$ is defined by the reflection $r_\ell : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with respect to ℓ.

Example 3.6. Let us consider the n-dimensional real projective space $P(\mathbb{R}^{n+1})$ as in Example 2.19. If $n > 1$, then $P(\mathbb{R}^{n+1})$ does not admit non-trivial pole pairs. If $n = 1$, then every maximal pole subset in $P(\mathbb{R}^2)$ is of the form $\{\ell_1, \ell_2\}$, where ℓ_1 is perpendicular to ℓ_2.

Proof. Take $\ell_1, \ell_2 \in P(\mathbb{R}^{n+1})$. First of all, we claim that (ℓ_1, ℓ_2) is a pole pair if and only if the reflections satisfy $r_{\ell_1} = \pm r_{\ell_2}$. The "if"-part is easy to check. In order to show the converse, assume that (ℓ_1, ℓ_2) is a pole pair. By definition, one has $r_{\ell_1}(\ell) = r_{\ell_2}(\ell)$ for every $\ell \in P(\mathbb{R}^{n+1})$. We consider

$$f := r_{\ell_2}^{-1} \circ r_{\ell_1} \in \mathrm{O}(n+1).$$

Since it is orthogonal and normalizes $\mathbb{R}e_i$ for each $i \in \{1, \ldots, n+1\}$, there exists $\varepsilon_i \in \{\pm 1\}$ such that $f(e_i) = \varepsilon_i e_i$. Furthermore, since f normalizes $\mathbb{R}(e_i + e_j)$, one can see that $\varepsilon_i = \varepsilon_j$. This shows $f = \pm \mathrm{id}$, which completes the proof of the claim.

Note that each r_ℓ has eigenvalues 1 and -1 with multiplicities 1 and n, respectively. Therefore, if $n > 1$, then one has $r_{\ell_1} \neq -r_{\ell_2}$. This yields that, in this case, (ℓ_1, ℓ_2) is a pole pair if and only if $r_{\ell_1} = r_{\ell_2}$, that is, $\ell_1 = \ell_2$. This completes the proof of the case of $n > 1$. For the case of $n = 1$, one can see that $r_{\ell_1} = r_{\ell_2}$ is equivalent to $\ell_1 = \ell_2$, and $r_{\ell_1} = -r_{\ell_2}$ is equivalent to $\ell_1 \perp \ell_2$. □

Finally in this subsection, we describe pole subsets of the dihedral quandles R_n. It would be interesting that it depends on the parity of n.

Example 3.7. Let us consider the dihedral quandle R_n of order n as in Example 2.5. Then we have the following:

(1) If n is odd, then R_n does not admit non-trivial pole pairs.
(2) If n is even, say $n = 2l$ with $l \in \mathbb{Z}_{>0}$, then (x, y) is a pole pair if and only if $x \in \{y, y + l\}$. Therefore every maximal pole subset in R_n is of the form $\{x, x + l\}$.

Proof. Recall that $s_x(z) = 2x - z$ for $R_n = \mathbb{Z}/n\mathbb{Z}$. Hence, (x, y) is a pole pair in R_n if and only if

$$2x = 2y \quad (\text{as in } \mathbb{Z}/n\mathbb{Z}).$$

If n is odd, then $2(x - y) = 0$ is equivalent to $x - y = 0$, which completes the proof of (1). If $n = 2l$ with $l \in \mathbb{Z}_{>0}$, then $2(x - y) = 0$ is equivalent to $x - y \in \{0, l\}$. Therefore, in this case, a maximal pole subset in R_n containing x is of the form $\{x, x + l\}$. □

3.2. **Antipodal subsets.** In this subsection, we introduce and study antipodal subsets in quandles Q. Similarly to the studies on poles in the previous subsection, we start from a binary relation on Q.

Definition 3.8. Let $x, y \in Q$. Then (x, y) is called an *antipodal pair* on Q if it satisfies $s_x(y) = y$ and $s_y(x) = x$.

We then define the notion of antipodal subsets. The definition would be simple and natural, but this notion turns out to be quite interesting for many cases.

Definition 3.9. A subset X in Q is called an *antipodal subset* if (x, y) is an antipodal pair in Q for any $x, y \in X$.

It should be noted that any antipodal subset is a subquandle, which is intrinsically a trivial quandle. Namely, one can rephrase antipodal subsets as subquandles which are trivial quandles.

Remark 3.10. As mentioned in the introduction, the notion of antipodal subsets has also been defined for symmetric spaces by Chen and Nagano ([2]). Note that, for connected Riemannian symmetric spaces, $s_x(y) = y$ and $s_y(x) = x$ are equivalent. In general they are not equivalent for quandles.

As in the case of pole subsets, it is easy to see that any subset of an antipodal subset is also antipodal. Therefore it is natural to consider the maximal ones with respect to inclusion relation. We denote the collection of all maximal antipodal subsets in Q by

$$\mathcal{MS}(Q; \mathrm{antipodal}).$$

Proposition 3.11. *Let $f \in \mathrm{Aut}(Q)$. If X is an antipodal or maximal antipodal subset in Q, then $f(X)$ is also an antipodal or maximal antipodal subset in Q, respectively. In particular, $\mathrm{Aut}(Q)$ acts on $\mathcal{MS}(Q; \mathrm{antipodal})$.*

Proof. Recall that $f \circ s_x \circ f^{-1} = s_{f(x)}$ holds for every $x \in Q$. Therefore, a pair (x, y) is antipodal if and only if so is $(f(x), f(y))$. Then the assertions of this proposition can be proved by the same argument as for Proposition 3.4. □

Similarly to the case of poles, it is natural to consider the classification of maximal antipodal subsets up to automorphisms. This problem is essentially equivalent to determine the orbit space

$$\mathrm{Aut}(Q) \backslash \mathcal{MS}(Q; \mathrm{antipodal}).$$

However, in contrast to maximal pole subsets, maximal antipodal subsets are not necessarily unique even in homogeneous quandles and symmetric spaces.

Remark 3.12. The study on maximal antipodal subsets in symmetric spaces has been initiated in [2]. Among others, it has been known that the uniqueness holds for several symmetric spaces, such as compact Hermitian symmetric spaces ([2]) and symmetric R-spaces ([9]). However the uniqueness does not hold in general, in which cases the classifications of all maximal antipodal subsets are sometimes involved. For example, the classification has not been completed for the oriented real Grassmannians $G_k(\mathbb{R}^n)^{\sim}$ of oriented k-planes in $\mathbb{R}^n$ with higher k. For more details and recent results, we refer to [10, 11, 12] and references therein.

We here recall the classification of maximal antipodal subsets in the spheres S^n and the real projective spaces $P(\mathbb{R}^{n+1})$. In both cases, a maximal antipodal subset is unique up to automorphisms. In fact we show a priori stronger statements, that is, it is unique up to the congruence by $\mathrm{SO}(n+1)$.

Example 3.13. Let S^n be the n-dimensional sphere as in Example 2.18. Then every antipodal pair in S^n is a pole pair. Therefore, every antipodal subset in S^n is a pole subset, and a maximal antipodal subset in S^n is unique up to the congruence by $\mathrm{SO}(n+1)$.

Proof. Let (x, y) be an antipodal pair in S^n. Since s_x is the reflection with respect to the line $\mathbb{R}x$, it follows from $s_x(y) = y$ that $y \in \mathbb{R}x$. This yields that (x, y) is a pole pair in S^n. Therefore, every maximal antipodal subset in S^n is of the form $\{\pm x\}$, which is unique up to the congruence by $\mathrm{SO}(n+1)$. □

We next study the real projective space $P(\mathbb{R}^{n+1})$. In this case, antipodal subsets are not necessarily pole subsets.

Example 3.14. Let us consider the n-dimensional real projective space $P(\mathbb{R}^{n+1})$ as in Example 2.19. A subset X in $P(\mathbb{R}^{n+1})$ is maximal antipodal if and only if there exists an orthonormal basis $\{x_1, \ldots, x_{n+1}\}$ of $\mathbb{R}^{n+1}$ such that

$$X = \{\mathbb{R}x_1, \ldots, \mathbb{R}x_{n+1}\}.$$

In particular, a maximal antipodal subset in $P(\mathbb{R}^{n+1})$ is unique up to the congruence by $\mathrm{SO}(n+1)$.

Proof. For each $\ell \in P(\mathbb{R}^{n+1})$, recall that s_ℓ is given as the reflection with respect to the line ℓ. Therefore, for $\ell_1, \ell_2 \in P(\mathbb{R}^{n+1})$, they satisfy $s_{\ell_1}(\ell_2) = \ell_2$ if and only if

$$\ell_2 = \ell_1 \quad \text{or} \quad \ell_2 \perp \ell_1.$$

By applying this condition, one can directly prove the assertion. In fact, the "if"-part of the assertion is easy. In order to show the converse implication, let X be a maximal antipodal subset in $P(\mathbb{R}^{n+1})$. Take $\ell_1 \in X$. By definition of antipodal subsets, one has

$$X \subset \mathrm{Fix}(s_{\ell_1}, P(\mathbb{R}^{n+1})),$$

where the right-hand side denotes the fixed point set of s_{ℓ_1}. Then, according to the above condition, we have

$$\mathrm{Fix}(s_{\ell_1}, P(\mathbb{R}^{n+1})) = \{\ell_1\} \cup \{\ell \in P(\mathbb{R}^{n+1}) \mid \ell \perp \ell_1\}.$$

The second component of the right-hand side is $P(\ell_1^\perp) \cong P(\mathbb{R}^n)$, and $X - \{\ell_1\}$ is a maximal antipodal subset in it. Therefore, by an induction on the dimension, we obtain $X = \{\ell_1, \ldots, \ell_{n+1}\}$, consisting of $n+1$ orthogonal lines. The uniqueness follows from the fact that all orthonormal bases of $\mathbb{R}^{n+1}$ are congruent by $\mathrm{SO}(n+1)$ up to sign. □

The above proof was essentially given by Chen and Nagano (see Proposition 2.10 and Example 2.11 of [2]). Note that they proved a proposition in more general settings, and our proof describes some details for the case of $P(\mathbb{R}^{n+1})$.

Example 3.15. Let R_n be the dihedral quandle R_n of order n as in Example 2.5. Then every antipodal pair in R_n is a pole pair. Therefore, every antipodal subset in R_n is a pole subset, and a maximal antipodal subset in R_n is unique up to automorphisms.

Proof. Let (x, y) be an antipodal pair in R_n. Then it satisfies

$$y = s_x(y) = 2x - y \quad (\text{in } R_n).$$

Therefore, it follows from an argument in the proof of Example 3.7 that (x, y) is a pole pair. The uniqueness follows from the fact that the cyclic group $\mathbb{Z}/n\mathbb{Z}$ acts on R_n as automorphisms. □

4. s-commutative subsets

Let $Q = (Q, s)$ be a quandle. In this section, we introduce the concept of s-commutative subsets in Q, which is a generalization of antipodal subsets. We also prove that any maximal s-commutative subset in Q is a subquandle.

4.1. s-commutative subsets.

In this subsection, we introduce the concept of s-commutative subsets, and study some basic properties. Similarly to the cases of poles and antipodal subsets, we start from s-commutative pairs of points.

Definition 4.1. Let $x, y \in Q$. Then (x, y) is called an *s-commutative pair* in Q if it satisfies $s_x \circ s_y = s_y \circ s_x$.

The following gives a characterization of s-commutative pairs in terms of pole pairs, which is simple but useful.

Lemma 4.2. *Let $x, y \in Q$. Then the following three conditions are mutually equivalent*:

(1) *(x, y) is an s-commutative pair in Q.*
(2) *$(x, s_y(x))$ is a pole pair in Q.*
(3) *$(s_x(y), y)$ is a pole pair in Q.*

Proof. By the axiom of quandles, we have

$$s_x \circ s_y \circ s_x^{-1} = s_{s_x(y)}. \tag{4.1}$$

Therefore, s_x and s_y are commutative if and only if $s_y = s_{s_x(y)}$ holds. This proves the equivalence of (1) and (3). By changing the roles of x and y, one can easily see that (1) and (2) are equivalent. □

Note that every pole pair is an antipodal pair. The next proposition gives a relationship between antipodal pairs and s-commutative pairs. Recall that a pole pair (x, y) is said to be trivial if $x = y$ holds.

Proposition 4.3. *Every antipodal pair in Q is an s-commutative pair in Q. The converse also holds if Q does not admit non-trivial pole pairs.*

Proof. Let (x, y) be an antipodal pair in Q. Then we have $s_y(x) = x$. In particular, $(x, s_y(x)) = (x, x)$ is a pole pair. It then follows from Lemma 4.2 that (x, y) is an s-commutative pair in Q, which proves the first assertion.

We show the second assertion. Let (x, y) be an s-commutative pair in Q. Then, by Lemma 4.2, both of $(x, s_y(x))$ and $(s_x(y), y)$ are pole pairs in Q. Since Q does not admit non-trivial pole pairs by assumption, we have $x = s_y(x)$ and $s_x(y) = y$, which yields that (x, y) is antipodal. □

In terms of the notion of s-commutative pairs, one can naturally define s-commutative subsets in quandles.

Definition 4.4. A subset X in Q is said to be *s-commutative* if (x, y) is an s-commutative pair in Q for any $x, y \in X$.

According to Proposition 4.3, one can easily obtain the following relationship between antipodal subsets and s-commutative subsets.

Proposition 4.5. *Every antipodal subset in Q is an s-commutative subset in Q. The converse also holds if Q does not admit non-trivial pole pairs.*

One can apply this proposition to the real projective space $P(\mathbb{R}^{n+1})$ with $n \geq 2$. In such case, a subset is s-commutative if and only if it is antipodal. However, as we mentioned in the introduction, an s-commutative subset in a quandle Q is not necessarily antipodal in general.

4.2. **Maximal s-commutative subsets.** As for the pole subsets and antipodal subsets, any subset of an s-commutative subset is also s-commutative. Therefore it is natural to consider the maximal ones with respect to inclusion relation. In this subsection, we study maximal s-commutative subsets in quandles. We denote the collection of all maximal s-commutative subsets in Q by

$$\mathcal{MS}(Q; s\text{-commutative}).$$

Proposition 4.6. *Let $f \in \mathrm{Aut}(Q)$. If X is an s-commutative or maximal s-commutative subset in Q, then $f(X)$ is also s-commutative or maximal s-commutative in Q, respectively. In particular, the automorphism group $\mathrm{Aut}(Q)$ acts on $\mathcal{MS}(Q; s\text{-commutative})$.*

Proof. One can prove the proposition similarly to the proofs of Propositions 3.4 and 3.11. □

As for the cases of maximal pole subsets and maximal antipodal subsets, the classification of maximal s-commutative subsets in Q up to automorphisms is essentially equivalent to determine the orbit space

$$\mathrm{Aut}(Q) \backslash \mathcal{MS}(Q; s\text{-commutative}).$$

For example, this orbit space is a singleton if and only if a maximal s-commutative subset in Q is unique up to automorphisms. In fact, such phenomena do occur for many symmetric spaces and quandles.

The following is the main theorem of this section. Recall that a subset X in Q is a subquandle if and only if it satisfies $s_x(X) = X$ for any $x \in X$.

Theorem 4.7. *Let X be a maximal s-commutative subset in Q. Then X is a subquandle in Q.*

Proof. Take any $x \in X$. First of all, we claim that $X \cup s_x(X)$ is an s-commutative subset in Q. Note that X is s-commutative by assumption, and so is $s_x(X)$ by Proposition 4.6. Take any $y, z \in X$, and show that $(y, s_x(z))$ is an s-commutative pair. Since (x, z) is an s-commutative pair, $(z, s_x(z))$ is a pole pair by Lemma 4.2. Then we have

$$s_y \circ s_{s_x(z)} = s_y \circ s_z = s_z \circ s_y = s_{s_x(z)} \circ s_y,$$

which completes the proof of the claim.

One knows $X \subset X \cup s_x(X)$. Since X is maximal s-commutative and $X \cup s_x(X)$ is s-commutative, we have $X = X \cup s_x(X)$. In particular, it satisfies

$$s_x(X) \subset X.$$

Since $s_x(X)$ is maximal s-commutative, one can prove $s_x(X) = X$ by the same argument. □

5. Subsets in direct products and interaction-free unions of quandles

In this section, we consider the direct product quandle $\prod_{\lambda\in\Lambda} Q_\lambda$ and interaction-free union quandle $\bigsqcup^{\text{free}}_{\lambda\in\Lambda} Q_\lambda$ of a family of quandles $\{Q_\lambda\}_{\lambda\in\Lambda}$. We study pole, antipodal, and s-commutative subsets in these quandles. The results of this section yield that, reduction arguments to each component Q_λ work for many cases, but not always.

5.1. **Subsets in direct products of quandles.** In this subsection, we consider the direct product quandle $\prod_{\lambda\in\Lambda} Q_\lambda$ of a family of quandles $\{Q_\lambda\}_{\lambda\in\Lambda}$, defined in Subsection 2.2. For simplicity of the notation we denote it just by $\prod Q_\lambda$. First of all, we study properties of pairs of points.

Proposition 5.1. *Let $(x_\lambda)_\lambda, (y_\lambda)_\lambda \in \prod Q_\lambda$. Then, $((x_\lambda)_\lambda, (y_\lambda)_\lambda)$ is a pole, antipodal, or s-commutative pair in $\prod Q_\lambda$ if and only if (x_η, y_η) is a pole, antipodal, or s-commutative pair in Q_η for every $\eta \in \Lambda$, respectively.*

Proof. Denote the quandle structures on Q_λ and $\prod Q_\lambda$ by s^λ and s, respectively. Then, by definition, one knows

$$s_{(x_\lambda)_\lambda}((z_\lambda)_\lambda) = (s^\lambda_{x_\lambda}(z_\lambda))_\lambda.$$

Therefore, $s_{(x_\lambda)_\lambda} = s_{(y_\lambda)_\lambda}$ holds if and only if it satisfies $s^\eta_{x_\eta} = s^\eta_{y_\eta}$ for every $\eta \in \Lambda$. This proves the assertion on pole pairs. Similarly one can show the other assertions on antipodal pairs and s-commutative pairs. □

By applying this properties of pairs, one can obtain properties of subsets. Recall that $p_\eta : \prod Q_\lambda \to Q_\eta$ denotes the projection for $\eta \in \Lambda$. For a family of subsets $\{A_\lambda \subset Q_\lambda\}_{\lambda\in\Lambda}$, one can naturally define the subset $\prod A_\lambda := \prod_{\lambda\in\Lambda} A_\lambda$ in $\prod Q_\lambda$.

Proposition 5.2. *Let A be a subset in $\prod Q_\lambda$. Then, A is a pole, antipodal, or s-commutative subset in $\prod Q_\lambda$ if and only if $p_\eta(A)$ is a pole, antipodal, or s-commutative subset in Q_η for every $\eta \in \Lambda$, respectively. In particular, if A_λ is pole, antipodal, or s-commutative in Q_λ for every $\lambda \in \Lambda$, then $\prod A_\lambda$ is pole, antipodal, or s-commutative in $\prod Q_\lambda$, respectively.*

Proof. The first assertion is a direct consequence of Proposition 5.1. The second one follows easily from $p_\eta(\prod A_\lambda) = A_\eta$ and the first assertion. □

Note that it is not true in general that every pole, antipodal, or s-commutative subset in $\prod Q_\lambda$ is of the form $\prod A_\lambda$. However, this is in fact true for the maximal cases.

Proposition 5.3. *Let A be a subset in $\prod Q_\lambda$. Then, A is a maximal pole, maximal antipodal, or maximal s-commutative subset in $\prod Q_\lambda$ if and only if there exists a maximal pole, maximal antipodal, or maximal s-commutative subset A_λ in Q_λ for every $\lambda \in \Lambda$, respectively, such that $A = \prod A_\lambda$.*

Proof. The proof follows from Proposition 5.2. We only prove the case of s-commutative subsets, since the other cases are completely the same.

First of all, assume that A_λ is maximal s-commutative in Q_λ for each $\lambda \in \Lambda$, and prove that $\prod A_\lambda$ is maximal s-commutative in $\prod Q_\lambda$. One knows $\prod A_\lambda$ is s-commutative. Let us take an s-commutative subset A' in $\prod Q_\lambda$ with $\prod A_\lambda \subset A'$. Then one has $A_\eta \subset p_\eta(A')$ for each $\eta \in \Lambda$. Since A_η is maximal s-commutative and $p_\eta(A')$ is s-commutative, we have $A_\eta = p_\eta(A')$. This yields $\prod A_\lambda = A'$, which shows the maximality of $\prod A_\lambda$.

In order to show the converse, let A be a maximal s-commutative subset in $\prod Q_\lambda$. Put $A_\lambda := p_\lambda(A)$. Then it satisfies $A \subset \prod A_\lambda$ by definition. Note that each A_λ is s-commutative, and hence so is $\prod A_\lambda$. Therefore the maximality of A as an s-commutative subset yields that $A = \prod A_\lambda$. One can also show that each A_λ is maximal s-commutative in Q_λ, which follows from the maximality of A and a similar argument. This completes the proof. □

This proposition can be rephrased in terms of bijective correspondences as follows. Recall that, for a quandle Q, we denote by

$$\mathcal{MS}(Q; \text{pole}), \quad \mathcal{MS}(Q; \text{antipodal}), \quad \mathcal{MS}(Q; s\text{-commutative}),$$

the set of all maximal pole, maximal antipodal, and maximal s-commutative subsets in Q, respectively.

Theorem 5.4. *Let A_λ be a subset in Q_λ for each $\lambda \in \Lambda$, and consider the correspondence to a subset $\prod A_\lambda$ in $\prod Q_\lambda$. Then, this defines the following bijective maps, which are equivariant with respect to the natural actions by $\prod \mathrm{Aut}(Q_\lambda)$:*

(1) $\prod \mathcal{MS}(Q_\lambda; \mathrm{pole}) \to \mathcal{MS}(\prod Q_\lambda; \mathrm{pole})$.
(2) $\prod \mathcal{MS}(Q_\lambda; \mathrm{antipodal}) \to \mathcal{MS}(\prod Q_\lambda; \mathrm{antipodal})$.
(3) $\prod \mathcal{MS}(Q_\lambda; s\text{-commutative}) \to \mathcal{MS}(\prod Q_\lambda; s\text{-commutative})$.

Note that, by Proposition 2.9, the direct product group $\prod \mathrm{Aut}(Q_\lambda)$ acts naturally on the direct product quandle $\prod Q_\lambda$ as automorphisms.

5.2. **Subsets in interaction-free unions of quandles.** Let us consider the interaction-free union $\bigsqcup^{\mathrm{free}}_{\lambda\in\Lambda} Q_\lambda$ of a family of quandles $\{Q_\lambda\}_{\lambda\in\Lambda}$, defined in Subsection 2.3. As in the previous subsection, we denote it by $\bigsqcup^{\mathrm{free}} Q_\lambda$ for simplicity, and start with studying properties of pairs of points.

Proposition 5.5. *Let $x \in Q_\lambda$ and $y \in Q_\eta$ with $\lambda, \eta \in \Lambda$.*

(1) *Assume $\lambda = \eta$. Then, (x, y) is a pole, antipodal, or s-commutative pair in $\bigsqcup^{\mathrm{free}} Q_\lambda$ if and only if (x, y) is a pole, antipodal, or s-commutative pair in Q_λ, respectively.*
(2) *Assume $\lambda \neq \eta$. Then, (x, y) is a pole pair in $\bigsqcup^{\mathrm{free}} Q_\lambda$ if and only if it satisfies $s^\lambda_x = \mathrm{id}_{Q_\lambda}$ and $s^\eta_y = \mathrm{id}_{Q_\eta}$. On the other hand, (x, y) is always an antipodal and s-commutative pair in $\bigsqcup^{\mathrm{free}} Q_\lambda$.*

Proof. Recall that $s^{\mathrm{free}}_x = s^\eta_x$ on Q_η if $x \in Q_\eta$, and s^{free}_x is the identity map on Q_η if $x \notin Q_\eta$. Then all the assertions follow from this definition of s^{free}. For example, assume that (x, y) is a pole pair in $\bigsqcup^{\mathrm{free}} Q_\lambda$ with $x \in Q_\lambda$, $y \in Q_\eta$, and $\lambda \neq \eta$. Then one can show that

$$s^\lambda_x = s^{\mathrm{free}}_x |_{Q_\lambda} = s^{\mathrm{free}}_y |_{Q_\lambda} = \mathrm{id}_{Q_\lambda}.$$

Other assertions can be proved easily. □

One can obtain properties of subsets A in $\bigsqcup^{\mathrm{free}} Q_\lambda$, by using the properties of pairs. In the case of direct products, we have used the projections p_η for $\eta \in \Lambda$. For the case of interaction-free unions, the intersections $A \cap Q_\eta$ play similar roles.

Proposition 5.6. *Let A be a subset in $\bigsqcup^{\mathrm{free}} Q_\lambda$. If A is a pole, antipodal, or s-commutative subset in $\bigsqcup^{\mathrm{free}} Q_\lambda$, then $A \cap Q_\eta$ is a pole, antipodal, or s-commutative subset in Q_η for every $\eta \in \Lambda$, respectively. The converse statements also hold for antipodal and s-commutative subsets.*

Proof. The first and second assertions follow from (1) and (2) of Proposition 5.5, respectively. □

Remark 5.7. One can easily show from Proposition 5.6 that, if A_λ is an antipodal or s-commutative subset in Q_λ for every $\lambda \in \Lambda$, then $\bigsqcup A_\lambda$ is an antipodal or s-commutative subset in $\bigsqcup^{\mathrm{free}} Q_\lambda$, respectively. However, the same statement for pole subsets is not true in general. Namely, the union $\bigsqcup A_\lambda$ is not necessarily a pole subset even if all A_λ are pole. A simple example can be given by $R_3 \bigsqcup^{\mathrm{free}} R_3$, the interaction-free union of two copies of R_3. Let us denote its element as $(a)_i$ with $a \in R_3$ and $i \in \{1, 2\}$. Then $\{(0)_1, (0)_2\}$ is the union of pole subsets, but not pole in $R_3 \bigsqcup^{\mathrm{free}} R_3$.

We then study maximal subsets. Note that there are differences between properties of maximal pole subsets and those of maximal antipodal or s-commutative subsets. The following describes maximal pole subsets.

Proposition 5.8. *Let A be a nonempty subset in $\bigsqcup^{\mathrm{free}} Q_\lambda$. Then, A is a maximal pole subset in $\bigsqcup^{\mathrm{free}} Q_\lambda$ if and only if one of the following holds*:

(i) *there exists $\eta \in \Lambda$ such that A is a maximal pole subset in Q_η, and there exists $x \in A$ satisfying $s^\eta_x \neq \mathrm{id}_{Q_\eta}$, or*
(ii) $A = \bigsqcup_{\lambda \in \Lambda} \{x \in Q_\lambda \mid s^\lambda_x = \mathrm{id}_{Q_\lambda}\}$.

Proof. First of all, assume that A satisfies (i). Since A is a pole subset in Q_η, it is also pole in $\bigsqcup^{\mathrm{free}} Q_\lambda$ by the definition of s^{free}. It also satisfies the maximality, since there exists $x \in A$ such that $s^\eta_x \neq \mathrm{id}_{Q_\eta}$. In fact, if (x, y) is a pole pair, then $y \in Q_\eta$ by Proposition 5.5, and hence $y \in A$ by the maximality of A in Q_η.

We next consider the case that A is of the form (ii). Since $A \neq \emptyset$ by assumption, there exists $x \in A$. Note that it satisfies $s^{\mathrm{free}}_x = \mathrm{id}$. Therefore, (x, y) is a pole pair if and only if $s^{\mathrm{free}}_y = \mathrm{id}$, which is equivalent to $y \in A$. This proves that A is a maximal pole subset in $\bigsqcup^{\mathrm{free}} Q_\lambda$.

One can prove the converse implication by a similar argument. Let A be a maximal pole subset in $\bigsqcup^{\mathrm{free}} Q_\lambda$. Since $A \neq \emptyset$, there exists $x \in A \cap Q_\eta$ for some $\eta \in \Lambda$. If it satisfies $s^\eta_x \neq \mathrm{id}_{Q_\eta}$, then one can show that A is of the form of (i). If it satisfies $s^\eta_x = \mathrm{id}_{Q_\eta}$, then A must be of the form of (ii). □

The properties of maximal antipodal and s-commutative subsets in $\bigsqcup^{\mathrm{free}} Q_\lambda$ are described in the next proposition. Namely, a reduction argument to each component Q_λ works for these subsets.

Proposition 5.9. *Let A be a subset in $\bigsqcup^{\mathrm{free}} Q_\lambda$. Then, A is a maximal antipodal or s-commutative subset in $\bigsqcup^{\mathrm{free}} Q_\lambda$ if and only if there exists a maximal antipodal or maximal s-commutative subset A_λ in Q_λ for every $\lambda \in \Lambda$, respectively, such that $A = \bigsqcup A_\lambda$.*

Proof. The proof follows from Proposition 5.6. We only give a sketch of the proof for the s-commutative case. Let A be a maximal s-commutative subset in $\bigsqcup^{\mathrm{free}} Q_\lambda$, and put $A_\lambda := A \cap Q_\lambda$ for each $\lambda \in \Lambda$. Then A_λ is s-commutative. If A'_λ is an s-commutative subset in Q_λ with $A_\lambda \subset A'_\lambda$, then one can see that

$$A = \bigsqcup A_\lambda \subset \bigsqcup A'_\lambda.$$

Note that $\bigsqcup A'_\lambda$ is s-commutative, and A has the maximality. Hence it satisfies $A_\lambda = A'_\lambda$, which yields that each A_λ is maximal s-commutative. The converse implication can be proved in a similar way. □

This proposition can be rephrased in terms of bijective correspondences as follows. Recall that a family of maximal antipodal subsets $\{A_\lambda \subset Q_\lambda\}_{\lambda \in \Lambda}$ is regarded as an element in $\prod \mathcal{MS}(Q_\lambda; \text{antipodal})$.

Theorem 5.10. *Let A_λ be a subset in Q_λ for each $\lambda \in \Lambda$, and consider the correspondence to a subset $\bigsqcup A_\lambda$ in $\bigsqcup^{\text{free}} Q_\lambda$. Then, this defines the following bijective maps, which are equivariant with respect to the natural actions by $\prod \mathrm{Aut}(Q_\lambda)$:*

(1) $\prod \mathcal{MS}(Q_\lambda; \text{antipodal}) \to \mathcal{MS}(\bigsqcup^{\text{free}} Q_\lambda; \text{antipodal})$.
(2) $\prod \mathcal{MS}(Q_\lambda; s\text{-commutative}) \to \mathcal{MS}(\bigsqcup^{\text{free}} Q_\lambda; s\text{-commutative})$.

We refer to Proposition 2.14 for the action of $\prod \mathrm{Aut}(Q_\lambda)$ on $\bigsqcup^{\text{free}} Q_\lambda$. Note that a similar statement on maximal pole subsets cannot be true.

6. Examples

In this section, we determine maximal s-commutative subsets in the spheres S^n, the real projective spaces $P(\mathbb{R}^{n+1})$, and the dihedral quandles R_n. In these quandles, a maximal s-commutative subset is turned out to be unique up to the automorphism.

First of all, we study maximal s-commutative subsets in the spheres S^n, and prove Proposition 1.6. For this purpose, it is enough to show the next proposition.

Proposition 6.1. *Let S^n be the n-dimensional sphere as in Example 2.18. Then we have the following*:

(1) *Let $x, y \in S^n$. Then a pair (x, y) is an s-commutative pair if and only if $x = \pm y$ or $\langle x, y \rangle = 0$.*
(2) *A subset $X \subset S^n$ is maximal s-commutative if and only if there exists an orthonormal basis $\{x_1, \ldots, x_{n+1}\}$ of $\mathbb{R}^{n+1}$ such that*
$$X = \{\pm x_1, \ldots, \pm x_{n+1}\}.$$
(3) *A maximal s-commutative subset in S^n is unique up to the congruence by $\mathrm{SO}(n+1)$.*

Proof. Recall that (x, y) is an s-commutative pair if and only if $(x, s_y(x))$ is a pole pair by Lemma 4.2. Since every pole subset in the sphere S^n is of the form $\{\pm x\}$, the above condition is equivalent to
$$s_y(x) = \pm x.$$
In the case of $s_y(x) = x$, the pair (x, y) is antipodal, which is equivalent to $y = \pm x$. In the case of $s_y(x) = -x$, one can easily obtain $\langle x, y \rangle = 0$, which completes the proof of (1).

We show (2). The "if"-part follows from (1) easily. In order to show the "only if"-part, let X be a maximal s-commutative subset in S^n. Take $x_1 \in X$. Since any element in X is s-commutative with x_1, it follows from (1) that
$$X \subset \{\pm x_1\} \cup \{y \in S^n \mid \langle y, x_1 \rangle = 0\}.$$

Note that $X \cup \{-x_1\}$ is s-commutative, since $s_{x_1} = s_{-x_1}$. Hence the maximality of X yields that $\{\pm x_1\} \subset X$. Furthermore, since X is maximal s-commutative in S^n, we can show that $X - \{\pm x_1\}$ is maximal s-commutative in

$$\{y \in S^n \mid \langle y, x_1 \rangle = 0\} \cong S^{n-1}.$$

By an inductive argument, one can complete the proof of (2).

The assertion (3) is a direct consequence of (2). □

Remark 6.2. Recall that every maximal s-commutative subset is a subquandle by Theorem 4.7. Therefore, $X := \{\pm x_1, \ldots, \pm x_{n+1}\}$ is a subquandle of S^n, where $\{x_1, \ldots, x_{n+1}\}$ is an orthonormal basis of $\mathbb{R}^{n+1}$. The quandle structure is given by

$$s_{x_i} = s_{-x_i}, \quad s_{x_i}(\pm x_j) = \begin{cases} \pm x_j & (\text{if } i = j), \\ \mp x_j & (\text{if } i \neq j). \end{cases}$$

If $n = 1$, then the quandle $\{\pm x_1, \pm x_2\}$ is isomorphic to the dihedral quandle R_4 of order 4. One can also see that, for every $n \in \mathbb{N}$, the above quandle is homogeneous and disconnected. For more details of this quandle, we refer to [3].

We next study maximal s-commutative subsets in the real projective spaces $P(\mathbb{R}^{n+1})$. Proposition 1.7 follows from the next proposition. Recall that $\mathbb{R}x \in P(\mathbb{R}^{n+1})$ denotes the line spanned by $x \in \mathbb{R}^{n+1}$.

Proposition 6.3. *Let $P(\mathbb{R}^{n+1})$ be the real projective space as in Example 2.19. Then we have the following*:

(1) *Let $n > 1$. Then a subset $X \subset P(\mathbb{R}^{n+1})$ is maximal s-commutative if and only if it is maximal antipodal, that is, there exists an orthonormal basis $\{x_1, \ldots, x_{n+1}\}$ of $\mathbb{R}^{n+1}$ such that*

$$X = \{\mathbb{R}x_1, \ldots, \mathbb{R}x_{n+1}\}.$$

(2) *Let $n = 1$. Then a subset $X \subset P(\mathbb{R}^2)$ is maximal s-commutative if and only if there exists an orthonormal basis $\{x_1, x_2\}$ of $\mathbb{R}^2$ such that*

$$X = \{\mathbb{R}x_1,\ \mathbb{R}x_2,\ \mathbb{R}(x_1 + x_2),\ \mathbb{R}(x_1 - x_2)\}.$$

(3) *For every n, a maximal s-commutative subset in $P(\mathbb{R}^{n+1})$ is unique up to the congruence by $\mathrm{SO}(n + 1)$.*

Proof. The case of $n > 1$ is easy. In this case, recall that $P(\mathbb{R}^{n+1})$ does not admit non-trivial pole pairs by Example 3.6. Therefore, Proposition 4.5 yields that $X \subset P(\mathbb{R}^{n+1})$ is maximal s-commutative if and only if it is maximal antipodal. Hence one can complete the proof of (1) by Example 3.14.

We then consider the case of $n = 1$. For simplicity of the notation, we denote by $\angle(\ell_1, \ell_2) \in [0, \pi/2]$ the angle between $\ell_1, \ell_2 \in P(\mathbb{R}^2)$. We claim that (ℓ_1, ℓ_2) is an s-commutative pair in $P(\mathbb{R}^2)$ if and only if

$$\angle(\ell_1, \ell_2) \in \{0, \pi/4, \pi/2\}.$$

It follows from Lemma 4.2 that (ℓ_1, ℓ_2) is an s-commutative pair if and only if $(\ell_1, s_{\ell_2}(\ell_1))$ is a pole pair. By Example 3.6, the latter condition is equivalent to

$$\angle(\ell_1, s_{\ell_2}(\ell_1)) \in \{0, \pi/2\}.$$

One can see that $\angle(\ell_1, s_{\ell_2}(\ell_1)) = 0$ is equivalent to $\angle(\ell_1, \ell_2) \in \{0, \pi/2\}$. Furthermore, $\angle(\ell_1, s_{\ell_2}(\ell_1)) = \pi/2$ is equivalent to $\angle(\ell_1, \ell_2) = \pi/4$. This completes the proof of the claim. By using this claim, the assertion (2) can be proved directly.

One can then show (3) by the shapes of maximal s-commutative subsets described in (1) and (2). □

Finally in this section, we study maximal s-commutative subsets in the dihedral quandles R_n, and prove Proposition 1.8. This follows from the next proposition. Recall Example 2.5, where the dihedral quandle is defined by $R_n := \mathbb{Z}/n\mathbb{Z}$ with quandle structure $s_x(y) = 2x - y$.

Proposition 6.4. *Let R_n be the dihedral quandle of order n. Then we have the following*:

1. *Let $x, y \in R_n$. Then a pair (x, y) is s-commutative if and only if it satisfies $4(x - y) = 0$ as in $\mathbb{Z}/n\mathbb{Z}$.*
2. *If n is odd, then every maximal s-commutative subset is of the form $\{x\}$, which is a maximal pole subset.*
3. *If $n = 4l - 2$ with $l \in \mathbb{Z}_{>0}$, then every maximal s-commutative subset is of the form $\{x, x + 2l - 1\}$, which is a maximal pole subset.*
4. *If $n = 4l$ with $l \in \mathbb{Z}_{>0}$, then every maximal s-commutative subset is of the form $\{x, x + l, x + 2l, x + 3l\}$, which is not antipodal.*
5. *For any $n \in \mathbb{Z}_{>0}$, a maximal s-commutative subset in R_n is unique up to the congruence by $\mathrm{Aut}(R_n, s)$.*

Proof. Let $x, y \in R_n = \mathbb{Z}/n\mathbb{Z}$. In the following, we calculate everything as elements of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. By the definition of the quandle structure of R_n, we have

$$s_x \circ s_y(z) = s_x(2y - z) = 2x - (2y - z) = 2x - 2y + z.$$

Therefore, a pair (x, y) is s-commutative if and only if

$$2x - 2y = 2y - 2x.$$

This completes the proof of (1). Then, by using this condition, one can prove (2), (3), and (4) as follows.

If n is odd, then $4(x-y) = 0$ is equivalent to $x - y = 0$. Therefore, in this case, every maximal s-commutative subset consists of a single point. This is also a maximal pole subset by Example 3.7, which completes the proof of (2).

If $n = 4l - 2$, then $4(x - y) = 0$ is equivalent to $x - y \in \{0, 2l - 1\}$. In this case, every maximal s-commutative subset is of the form $\{x, x+2l-1\}$. This is a maximal pole subset by Example 3.7, which completes the proof of (3).

If $n = 4l$, then $4(x - y) = 0$ is equivalent to $x - y \in \{0, l, 2l, 3l\}$. In this case, every maximal s-commutative subset consists of four points as desired. This is not antipodal by Example 3.15, which proves (4).

The uniqueness claimed in (5) is also easy to see, since the cyclic $\mathbb{Z}/n\mathbb{Z}$-action on R_n is an automorphism. □

Remark 6.5. Recall that maximal s-commutative subsets are subquandles. In the case of R_{4l-2} with $l \in \mathbb{Z}_{>0}$, the maximal s-commutative subset is isomorphic to R_2, which is a trivial quandle. In the case of R_{4l} with $l \in \mathbb{Z}_{>0}$, the maximal s-commutative subset is isomorphic to R_4. One can see the pictures for the cases of $n \in \{6, 7, 8\}$ in Figure 1.

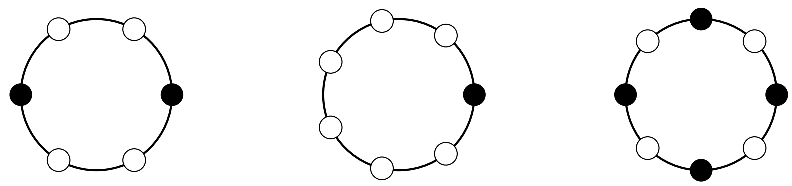

FIGURE 1. Maximal s-commutative subsets in R_n with $n \in \{6, 7, 8\}$

Remark 6.6. Recall that every pole subset is antipodal, and every antipodal subset is s-commutative. These three notions are different for symmetric spaces. Recall that a maximal antipodal subset in the real projective space $P(\mathbb{R}^{n+1})$ is not pole, and a maximal s-commutative subset in the sphere S^n is not antipodal. We here note that these three notions are also different for finite quandles. By the above proposition, a maximal s-commutative subset in R_{4l} with $l \in \mathbb{Z}_{>0}$ is not antipodal. One can see the difference between pole subsets and antipodal subsets as follows. Consider the interaction-free union $X = \{a, 0, 1, 2\}$ of the trivial quandle $\{a\}$ and the dihedral quandle $R_3 = \{0, 1, 2\}$. Then the subset $\{a, 0\}$ is (maximal) antipodal, but not a pole subset, since $s_a^{\mathrm{free}} = \mathrm{id}_X \neq s_0^{\mathrm{free}}$.

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