CALABI-YAU STRUCTURE AND BARGMANN TYPE TRANSFORMATION ON THE CAYLEY PROJECTIVE PLANE

KURANDO BABA AND KENRO FURUTANI

Abstract. Our purpose is to show the existence of a Calabi-Yau structure on the punctured cotangent bundle $T_0^*(P^2\mathbb{O})$ of the Cayley projective plane $P^2\mathbb{O}$ and to construct a Bargmann type transformation from a space of holomorphic functions on $T_0^*(P^2\mathbb{O})$ to L_2-space on $P^2\mathbb{O}$. The space of holomorphic functions corresponds to the Fock space in the case of the original Bargmann transformation. A Kähler structure on $T_0^*(P^2\mathbb{O})$ was shown by identifying it with a quadrics in the complex space $\mathbb{C}^{27}\backslash\{0\}$ and the natural symplectic form of the cotangent bundle $T_0^*(P^2\mathbb{O})$ is expressed as a Kähler form. Our method to construct the transformation is the pairing of polarizations, one is the natural Lagrangian foliation given by the projection map $\mathbf{q} : T_0^*(P^2\mathbb{O}) \longrightarrow P^2\mathbb{O}$ and the polarization given by the Kähler structure.

The transformation gives a quantization of the geodesic flow in terms of one parameter group of elliptic Fourier integral operators whose canonical relations are defined by the graph of the geodesic flow action at each time. It turn out that for the Cayley projective plane the results are not same with other cases of the original Bargmann transformation for Euclidean space, spheres and other projective spaces.

1. Introduction

The fundamental and historical problem in the quantization theory will be how to assign a function on a phase space to an operator acting on the space of quantum states and the assignment satisfies some algebraic condition, like a Lie algebra homomorphism. The phase space appearing in the theory has a structure, a symplectic structure. There are many theory relating with this problem. One method is the theory of deformation quantization. Also there is the opposite theory, an assignment of operators to functions, from an operator to a function. In the (pseudo)differential operator theory and Fourier integral operator theory, the basic assignment of operators to their principal symbol (and sub-principal symbol) is a fundamental isomorphism between the spaces of operators and functions on the phase space modulo lower order classes.

The famous transformation, called Bargmann transformation was introduced in [Ba] and gives one aspect of the quantization of the unitary representation. The method to construct such a transformation is given by the pairing of two polarizations, real polarization and complex polarization, on $\mathbb{C}^n$ interpreted as $\mathbb{C}^n \cong T^*(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}^n$, complex space and fiber space by Lagrangian fibers $\pi : \mathbb{C}^n \to \mathbb{R}^n$. Under precise treatments of this method it was given a similar operator for the case of the sphere in [Ra2], in [FY] for the complex projective space and for the quaternion projective spaces in [Fu1].

Among the projective spaces the Cayley projective plane $P^2\mathbb{O}$ is the exceptional one and our purpose in this paper is to show that we can also construct such an operator for this manifold in the same method. This case will be one of the non-trivial examples to which we can apply this method, "*pairing of two polarizations*" ([Ra2], [Ii1], [Ii2], [Fu1], [FY]).

In the paper [Fu2] a Kähler structure on its punctured cotangent bundle $T_0^*(P^2\mathbb{O})$ was constructed by embedding it into the complex space $\mathbb{C}^{27}\backslash\{0\}$ as an intersection of null sets of several quadric polynomials, which gives the realization of the natural symplectic form as a Kähler form. Here we show the holomorphic triviality of the canonical line bundle of this complex manifold by giving a nowhere vanishing global holomorphic 16-form explicitly.

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There are several study of the existence of Kähler structure on the (punctured) cotangent bundle of a certain class of manifolds, like [Ra1], [Sz1], [Sz2], [Koi], [Li], [FT], also see [Be], [So] in relation with a special property of the geodesic flow, SC_ℓ-manifolds.

The classical Bargmann transformation gives a correspondence between monomials on $\mathbb{C}^n$ and Hermite functions on $\mathbb{R}^n$, which are the eigenfunctions of the harmonic oscillator and this facts were applied to various problems, especially to Töplitz operator theory (there are so many, but here I just cite one book [BS]). Also there are many precise treatments and modifications of this transformation (for examples, [Ii1], and recent works in [Ch1], [Ch2]).

For our case we show the restrictions of monomials defined on $\mathbb{C}^{27}\backslash\{0\}$ to the embedded punctured cotangent bundle $T_0^*(P^2\mathbb{O})$ are mapped to eigenfunctions of the Laplacian on $P^2\mathbb{O}$.

This paper is organized as follows. In §2, we explain a realization of quaternion and octanion number fields, $\mathbb{H}$ and $\mathbb{O}$, in a complex matrix algebra. Multiplication law in the octanion is interpreted in the two 2×2-complex matrix algebra $\mathbb{C}(2) \times \mathbb{C}(2)$.

In §3, we introduce the Jordan algebra $\mathcal{J}(3)$ of 3×3 octanion matrices. Cayley projective plane $P^2\mathbb{O}$ is realized in this Jordan algebra. Following an earlier result in [Fu2] we explain the embedding of the punctured cotangent bundle $T_0^*(P^2\mathbb{O})$ of the Cayley projective plane into the complexified Jordan algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{J}(3) =: \mathcal{J}(3)^{\mathbb{C}}$ of 3×3 complexified octanion matrices:

$$\tau_{\mathbb{O}} : T^*(P^2\mathbb{O}) \longrightarrow \mathcal{J}(3)^{\mathbb{C}}.$$

We denote the image $\tau_{\mathbb{O}}(T^*(P^2\mathbb{O})) = \mathbb{X}_{\mathbb{O}}$. Also we state that the natural symplectic form $\omega^{P^2\mathbb{O}}$ is a Kähler form.

In §4, using the defining equations of the punctured cotangent bundle of the Cayley projective plane embedded in the complex Jordan algebra $\mathcal{J}(3)^{\mathbb{C}}$, we give an open covering by complex coordinates neighborhoods and show by an elementary way that the canonical line bundle of the complex structure is holomorphically trivial by explicitly constructing a nowhere vanishing holomorphic global section (we put it $\Omega_{\mathbb{O}}$), that is, a 16-degree holomorphic differential form which coincides with the restriction of a smooth 16-degree differential form on the whole complexified Jordan algebra $\mathcal{J}(3)^{\mathbb{C}}$.

In §5, we resume a basic fact on symplectic manifolds with integral symplectic form and a method of the geometric quantization. Here we consider two types of typical polarizations (real and positive complex). Then we apply the method to our case ($= T_0^*(P^2\mathbb{O})$) and give a Bargmann type transformation in the form of a fiber integration on the punctured cotangent bundle $T_0^*(P^2\mathbb{O})$ to the base space $P^2\mathbb{O}$.

In §6, first we show the nowhere vanishing holomorphic global section $\Omega_{\mathbb{O}}$ constructed in §4 is F_4-invariant. Incidentally, we determine the product $\Omega_{\mathbb{O}}\wedge\overline{\Omega_{\mathbb{O}}}$ in terms of the Liouville volume form $dV_{T^*(P^2\mathbb{O})} := \dfrac{1}{16\,!}\left(\omega^{P^2\mathbb{O}}\right)^{16}$ of the cotangent bundle $T^*(P^2\mathbb{O})$.

Also we introduce a class of subspaces consisting of holomorphic functions on $\mathbb{X}_{\mathbb{O}}$ satisfying some L_2 conditions. These will correspond to the Fock space in the Euclidean case.

In §7, we determine the exterior product of the Riemann volume form pull-backed to the cotangent bundle $T_0^*(P^2\mathbb{O})$ and the nowhere vanishing global holomorphic section $\Omega_{\mathbb{O}}$ in terms of the Liouville volume form. For this purpose we fix a local coordinates at a point in $P^2\mathbb{O}$ which is also used in the section §9.

In §8, we discuss invariant polynomials and a similar feature to harmonic polynomials with respect to the natural representation of the group F_4 to the Jordan algebra $\mathcal{J}(3)$ and its extension to the polynomial algebra. Then, based on a general theorem in [He] (also [HL] and [Ko]) we state the eigenfunction decomposition of L_2 space of $P^2\mathbb{O}$.

In §9, based on the data obtained until §8 we discuss our Bargmann type transformation is a bounded operator, or isomorphism or unbounded according to the Hilbert space structures in the Fock-like space. Some cases says there are quantum states in $L_2(P^2\mathbb{O})$ which are approximated by classical phenomena, but can not be observed directly by classical mechanical way.

Finally in §10 we mention that our Fock-like spaces have the reproducing kernel and a relation with the geodesic flow action.

2. Representations of quaternion and octanion algebras by complex matrix algebras

First, we fix a representation of quaternion numbers $h = h_0\mathbf{1} + h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k}$ ($\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are standard basis of the quaternion number field $\mathbb{H}$ and $h_i \in \mathbb{R}$) as a 2×2 complex matrix in the following way:

$$\rho_{\mathbb{H}} : \mathbb{H} \ni h \longmapsto \begin{pmatrix} h_0 + h_1\sqrt{-1} & h_2 + h_3\sqrt{-1} \\ -h_2 + h_3\sqrt{-1} & h_0 - h_1\sqrt{-1} \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ -\overline{\mu} & \overline{\lambda} \end{pmatrix} \in \mathbb{C}(2), \tag{2.1}$$

where we understand that quaternions $h_0\mathbf{1} + h_1\mathbf{i}$ and $h_2 + h_3\mathbf{i} \in \mathbb{H}$ are complex numbers $\lambda = h_0 + \sqrt{-1}h_1$ and $\mu = h_2 + h_3\sqrt{-1} \in \mathbb{C}$ respectively. Hence by this representation the complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ is isomorphic to the "algebra" of the whole 2×2 complex matrix algebra $\mathbb{C}(2)$ (we put $z_i = x_i + \sqrt{-1}y_i \in \mathbb{C}$):

$$\mathbb{C} \otimes_{\mathbb{R}}\mathbb{H} \ni h = z_0\mathbf{1} + z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k} \longmapsto \begin{pmatrix} z_0 + \sqrt{-1}z_1 & z_2 + \sqrt{-1}z_3 \\ \\ -z_2 + \sqrt{-1}z_3 & z_0 - \sqrt{-1}z_1 \end{pmatrix} \in \mathbb{C}(2). \tag{2.2}$$

We denote this map also by $\rho_{\mathbb{H}}$ and the inverse map is

$${\rho_{\mathbb{H}}}^{-1} : \mathbb{C}(2) \ni A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \longmapsto {\rho_{\mathbb{H}}}^{-1}(A) = \frac{z_1 + z_4}{2}\mathbf{1} + \frac{z_1 - z_4}{2\sqrt{-1}}\mathbf{i} + \frac{z_2 - z_3}{2}\mathbf{j} + \frac{z_2 + z_3}{2\sqrt{-1}}\mathbf{k}. \tag{2.3}$$

For $h = h_0\mathbf{1} + h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k} \in \mathbb{H}$ (or $\in \mathbb{C} \otimes_{\mathbb{R}}\mathbb{H}$), we denote its conjugation by $\theta(h) = h_0\mathbf{1} - h_1\mathbf{i} - h_2\mathbf{j} - h_3\mathbf{k}$, then for $\rho_{\mathbb{H}}(h) = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$, $\rho_{\mathbb{H}}(\theta(h)) = \begin{pmatrix} w_4 & -w_2 \\ -w_3 & w_1 \end{pmatrix}$ and the product $\rho_{\mathbb{H}}(\theta(h))\rho_{\mathbb{H}}(h) = \rho_{\mathbb{H}}(h)\rho_{\mathbb{H}}(\theta(h)) = (w_1w_4 - w_2w_3) \cdot \mathrm{Id} = \det\, \rho_{\mathbb{H}}(h) \cdot \mathrm{Id}$, where Id is 2×2 identity matrix.

Let $\{\mathbf{e}_i\}_{i=0}^{7}$ be the standard basis of the octanion number field $\mathbb{O}$ such that $\mathbf{e}_0$ is the basis of the center. We identify $\mathbf{e}_0 = \mathbf{1}$, $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$ and $\mathbf{e}_3 = \mathbf{k}$ with the basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ of the quaternion number field. By the multiplication law $\mathbf{e}_i\mathbf{e}_4 = \mathbf{e}_{i+4}$ $(i = 0, 1, 2, 3)$ we express an (complexified) octanion number $x = \sum x_i\mathbf{e}_i$ as the sum of two quaternion numbers:

$$x = \sum_{i=0}^{3} x_i\mathbf{e}_i + \sum_{i=0}^{3}\, x_{i+4}\mathbf{e}_i \cdot \mathbf{e}_4 = a + b \cdot \mathbf{e}_4 \in \mathbb{H} \oplus \mathbb{H}\mathbf{e}_4 \text{ or } \in \mathbb{C} \otimes_{\mathbb{R}}\mathbb{H} \oplus \mathbb{C} \otimes_{\mathbb{R}}\mathbb{H}\mathbf{e}_4.$$

The complexification $\mathbb{C} \otimes_{\mathbb{R}}\mathbb{O}$ is identified as

$$\mathbb{C} \otimes_{\mathbb{R}}\mathbb{O} \cong \mathbb{C}(2) \oplus \mathbb{C}(2)\mathbf{e}_4$$

through the map $\rho_{\mathbb{H}} \oplus \rho_{\mathbb{H}} =: \rho_{\mathbb{O}}$.

We define the conjugation operation in $\mathbb{O}$ (and also in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$) with the same notation θ for the quaternion case as

$$\theta : h = \sum h_i\mathbf{e}_i \longmapsto \theta(h) = h_0\mathbf{1} - \sum_{i=1}^{7} h_i\mathbf{e}_i.$$

The conjugation θ is interpreted in the matrix representation through the representation $\rho_{\mathbb{O}}$ as

$$\begin{aligned} \theta : \mathbb{C}(2) \oplus \mathbb{C}(2)\mathbf{e}_4 \ni Z + W\mathbf{e}_4 &= \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}\mathbf{e}_4 \\ \longmapsto \theta(Z + W\mathbf{e}_4) = \theta(Z) - W\mathbf{e}_4 &= \begin{pmatrix} z_4 & -z_2 \\ -z_3 & z_1 \end{pmatrix} - \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}\mathbf{e}_4. \end{aligned} \tag{2.4}$$

Remark 1. *The multiplication law of the octanions in the matrix form is given in* (4.4).

Remark 2. *We use the conjugation $\overline{z} = x - \sqrt{-1}y$ only for the complex number $z = x + \sqrt{-1}y$ and do not use the operation θ for the conjugate of complex numbers to avoid confusion. So, for a complex octanion number $z = \sum \{z\}_i\mathbf{e}_i$, $\{z\}_i \in \mathbb{C}$, we mean $\overline{z} = \sum \overline{\{z\}_i}\mathbf{e}_i$ and it holds $\theta(\overline{z}) = \overline{\theta(z)}$. Also for an octanion matrix $A = \Big(z_{ij} \Big)$ we mean $\overline{A} := \Big(\overline{z_{ij}} \Big)$ and $\theta(A) := \Big(\theta(z_{ij}) \Big)$.*

3. Cayley projective plane and its punctured cotangent bundle

In this section, we refer [SV], [M] and [Yo] for all the necessary facts on the exceptional group F_4 and the Cayley projective plane.

Let $\mathcal{J}(3)$ be a subspace of the 3×3 octanion matrices:

$$\mathcal{J}(3) = \left\{ \begin{pmatrix} t_1 & z & \theta(y) \\ \theta(z) & t_2 & x \\ y & \theta(x) & t_3 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{O}, t_i \in \mathbb{R} \right\}.$$

We introduce a product in $\mathcal{J}(3)$, called a "Jordan product", by

$$\mathcal{J}(3) \times \mathcal{J}(3) \ni (A, B) \longmapsto A \circ B := \frac{AB + BA}{2} \in \mathcal{J}(3).$$

It is called an exceptional Jordan algebra and of 27-dimensional over $\mathbb{R}$. Then the group of $\mathbb{R}$-linear algebra automorphisms is the exceptional Lie group F_4:

$$F_4 := \{\, g \in GL(\mathcal{J}(3)) \cong GL(27, \mathbb{R}) \mid g(A \circ B) = g(A) \circ g(B),\ g(Id) = Id,\ A, B \in \mathcal{J}(3) \}\,. \tag{3.1}$$

There are various characterizations for the group F_4 (see for examples, [Yo], [SV]).

The complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{J}(3) =: \mathcal{J}(3)^{\mathbb{C}}$ consists of 3×3 matrices with components of the complexified octanions of the form:

$$\mathcal{J}(3)^{\mathbb{C}} = \left\{ \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \;\middle|\; x, y, z \ \in\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{O},\ \xi_i \in \mathbb{C} \right\}$$

and is an exceptional Jordan algebra over $\mathbb{C}$ of the complex dimension 27. The complex linear automorphisms $\alpha : \mathcal{J}(3)^{\mathbb{C}} \longrightarrow \mathcal{J}(3)^{\mathbb{C}}$ satisfying the conditions

$$\alpha(A \circ B) = \alpha(A) \circ \alpha(B),\ \alpha(Id) = Id$$

is the complex Lie group ${F_4}^{\mathbb{C}}$. We may regard $F_4 \subset {F_4}^{\mathbb{C}}$ in a natural way.

Definition 3.1. *The Cayley projective plane $P^2\mathbb{O}$ is defined as*

$$P^2\mathbb{O} = \left\{\ X \in \mathcal{J}(3) \mid X^2 = X,\ \mathrm{tr}\,(X) = \xi_1 + \xi_2 + \xi_3 = 1\ \right\}.$$

It is known that the group F_4 acts on $P^2\mathbb{O}$ in two point homogeneous way.

Let $X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in P^2\mathbb{O}$, then it is known that the stationary subgroup of the point X_1 in F_4 is isomorphic to $Spin(9)$ and $F_4 \ni g \longmapsto g \cdot X_1$ gives an isomorphism:

$$F_4/Spin(9) \cong P^2\mathbb{O}. \tag{3.2}$$

For $X = \begin{pmatrix} \xi_1 & x_3 & \theta(x_2) \\ \theta(x_3) & \xi_2 & x_1 \\ x_2 & \theta(x_1) & \xi_3 \end{pmatrix}$, $Y = \begin{pmatrix} \eta_1 & y_3 & \theta(y_2) \\ \theta(y_3) & \eta_2 & y_1 \\ y_2 & \theta(y_1) & \eta_3 \end{pmatrix} \in \mathcal{J}(3)$, we define their inner product by

$$< X, Y >^{\mathcal{J}(3)} := \mathrm{tr}(X \circ Y) = \sum_{i=1}^{3} \xi_i \eta_i + 2 < x_i, y_i >^{\mathbb{R}^8}, \tag{3.3}$$

where $< \cdot, \cdot >^{\mathbb{R}^8}$ denotes the standard Euclidean inner product of x_i and $y_i \in \mathbb{O} \cong \mathbb{R}^8$.

This inner product has a property

$$< X \circ Y,\, Z >^{\mathcal{J}(3)} = < X, Y \circ Z >^{\mathcal{J}(3)},\ X, Y, Z \in \mathcal{J}(3). \tag{3.4}$$

In particular, since the trace function $\mathcal{J}(3) \ni A \longmapsto \mathrm{tr}\,(A)$ is invariant under the F_4 action, that is

$$\mathrm{tr}\,(g \cdot A) = \mathrm{tr}\,(A),\ g \in F_4,\ A \in \mathcal{J}(3), \tag{3.5}$$

this inner product is invariant under the action by F_4 (hence F_4 can be seen as $F_4 \subset SO(27)$):

$$<\ g \cdot A\,,\, g \cdot B >^{\mathcal{J}(3)} = \mathrm{tr}(\, g \cdot A \circ g \cdot B\,) = \mathrm{tr}\,\big(g \cdot (A \circ B)\big) = \mathrm{tr}\,(A \circ B) = < A, B >^{\mathcal{J}(3)}\,. \tag{3.6}$$

The tangent bundle $T(P^2\mathbb{O})$ is identified with a subspace in $\mathcal{J}(3)\times\mathcal{J}(3)$ such that

$$T(P^2\mathbb{O}) = \left\{ (X,Y)\in\mathcal{J}(3)\times\mathcal{J}(3) \;\middle|\; X\in P^2\mathbb{O},\ X\circ Y = \frac{1}{2}Y \right\}.$$

We consider the Riemannian metric $g^{P^2\mathbb{O}}$ on the manifold $P^2\mathbb{O}$ being induced from the inner product in $\mathcal{J}(3)$: $g_X^{P^2\mathbb{O}}(Y_1,Y_2) :=< Y_1, Y_2 >^{\mathcal{J}(3)}$, $Y_1,\ Y_2\in T_X(P^2\mathbb{O})$.

Using this metric, hereafter we identify the tangent bundle $T(P^2\mathbb{O})$ and the cotangent bundle $T^*(P^2\mathbb{O})$.

Let $Y_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_{X_1}(P^2\mathbb{O})$. The stationary subgroup at the point $(X_1,Y_1)\in T(P^2\mathbb{O})$ is known as being isomorphic to $Spin(7)$ and the two point homogeneity of the action by F_4 gives us the isomorphism $F_4/Spin(7)\cong S(P^2\mathbb{O})$, the unit (co)tangent sphere bundle of $P^2\mathbb{O}$.

The inner product on $\mathcal{J}(3)$, $<\cdot,\cdot>^{\mathcal{J}(3)}$, is extended to the complexification $\mathcal{J}(3)^{\mathbb{C}}$ as a *complex bi-linear form* in a natural way, which we denote by $<\cdot,\cdot>^{\mathcal{J}(3)^{\mathbb{C}}}$. Then the extension as the *Hermitian inner product* on the complexification $\mathcal{J}(3)^{\mathbb{C}}$ is given by $< A, \overline{B} >^{\mathcal{J}(3)^{\mathbb{C}}}$, $A,\ B\in\mathcal{J}(3)^{\mathbb{C}}$ (see Remark 2 for the matrix $\overline{B}$).

We will denote the norm of $a\in\mathbb{O}$ by $|a| = \sqrt{< a, a >^{\mathbb{R}^8}}$ and by $||X|| = \sqrt{< X, X >^{\mathcal{J}(3)}}$ the norm of $X\in\mathcal{J}(3)$, respectively. Also with the same way for elements $a\in\mathbb{C}\otimes_{\mathbb{R}}\mathbb{O}$ and $A\in\mathcal{J}(3)^{\mathbb{C}}$, we denote their norms.

The punctured cotangent bundle $T^*(P^2\mathbb{O})\backslash\{0\} =: T_0^*(P^2\mathbb{O})$ is realized as a subspace in $\mathcal{J}(3)^{\mathbb{C}}$ with the following form:

Theorem 3.2. ([Fu2]) *Let $\mathbb{X}_{\mathbb{O}}$ be a subspace in $\mathcal{J}(3)^{\mathbb{C}}$:*

$$\mathbb{X}_{\mathbb{O}} = \left\{ A = \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \;\middle|\; x,y,z\in\mathbb{C}\otimes_{\mathbb{R}}\mathbb{O},\ \xi_i\in\mathbb{C},\ A^2=0,\ A\neq 0 \right\}. \tag{3.7}$$

Then the correspondence between $T_0^*(P^2\mathbb{O})$ and $\mathbb{X}_{\mathbb{O}}$ is given by

$$\tau_{\mathbb{O}} : T_0^*(P^2\mathbb{O}) \ni (X,Y) \longmapsto \tau_{\mathbb{O}}(X,Y) = 1\otimes\left(||Y||^2X - Y^2\right) + \sqrt{-1}\otimes\frac{||Y||Y}{\sqrt{2}}. \tag{3.8}$$

Then

Theorem 3.3. ([Fu2])

$$\tau_{\mathbb{O}}{}^*\left(\sqrt{-2}\,\overline{\partial}\partial\,||A||^{1/2}\right) = \omega^{P^2\mathbb{O}}, \tag{3.9}$$

where we denote by $\omega^{P^2\mathbb{O}}$ the natural symplectic form on the cotangent bundle $T^*(P^2\mathbb{O})$.

The inverse $\tau_{\mathbb{O}}{}^{-1}$ is given by

$$\tau_{\mathbb{O}}{}^{-1} : \mathbb{X}_{\mathbb{O}} \ni A \longmapsto (X,Y) = (X(A),Y(A)) \in \mathcal{J}(3)\times\mathcal{J}(3),$$

$$\begin{cases} X(A) = \frac{1}{2||A||}\cdot\left(A+\overline{A}\right) + \frac{A\circ\overline{A}}{||A||^2}, \\ \\ Y(A) = -\frac{\sqrt{-1}}{\sqrt{2}}\cdot||A||^{-1/2}\left(A-\overline{A}\right). \end{cases} \tag{3.10}$$

4. Complex coordinate neighborhoods and Calabi-Yau structure

We denote the holomorphic part of the complexified cotangent bundle $T^*(\mathbb{X}_{\mathbb{O}})\otimes\mathbb{C}$ by $T^{*'}(\mathbb{X}_{\mathbb{O}})^{\mathbb{C}}$ (and likewise $T^{*''}(\mathbb{X}_{\mathbb{O}})^{\mathbb{C}}$ is the anti-holomorphic subbundle).

In this section we show that the canonical line bundle $\bigwedge^{16} T^{*'}(\mathbb{X}_{\mathbb{O}})^{\mathbb{C}}$ is holomorphically trivial by explicitly constructing a nowhere vanishing global holomorphic section (Theorem 4.7).

For this purpose we consider an open covering by explicit coordinates neighborhoods and show that the Jacobians of the coordinates transformations is a coboundary form of $\mathbb{C}^*$-valued zero form.

The condition in (3.7) is expressed in the following six equations in terms of octanions:

$$(\xi_3+\xi_2)x+\theta(yz)=0,\ (\xi_1+\xi_3)y+\theta(zx)=0,\ (\xi_2+\xi_1)z+\theta(xy)=0, \tag{4.1}$$

$$\xi_1{}^2 + z\theta(z) + \theta(y)y = 0,\ \xi_2{}^2 + \theta(z)z + x\theta(x) = 0,\ \xi_3{}^2 + \theta(x)x + y\theta(y) = 0. \tag{4.2}$$

The condition $0 \neq A \in \mathbb{X}_{\mathbb{O}}$ is equivalent to one of the components x, y, or z being non zero. Then this implies

Proposition 4.1.

$$\mathbb{X}_{\mathbb{O}} \ni A,\ \textit{then}\ \mathrm{tr}\,(A) = \xi_1 + \xi_2 + \xi_3 = 0.$$

This property does not appear in an explicit form in (4.1) and (4.2) but plays an important role in §8. Although it is proved in [Fu2], we give an elementary proof based on the permitted regulations in the octanion.

Proof. Since the associativity

$$a \cdot \theta(a)b = a\theta(a) \cdot b$$

holds, by multiplying z from the left to the equality $(\xi_3 + \xi_2)x + \theta(yz) = 0$ it holds the equality:

$$z \cdot (\xi_3 + \xi_2)x + z \cdot \theta(z)\theta(y) = (\xi_3 + \xi_2)zx + z\theta(z) \cdot \theta(y) = -(\xi_3 + \xi_2)(\xi_1 + \xi_3)\theta(y) + z\theta(z) \cdot \theta(y) = 0.$$

Hence if we assume $y \neq 0$

$$(\xi_3 + \xi_2)(\xi_1 + \xi_3) = z\theta(z)$$

and by the same way

$$(\xi_2 + \xi_1)(\xi_3 + \xi_1) = \theta(x)x.$$

These imply that

$$(\xi_3 + \xi_2)(\xi_1 + \xi_3) + (\xi_2 + \xi_1)(\xi_3 + \xi_1) + \xi_2{}^2 = (\xi_1 + \xi_2 + \xi_3)^2 = 0.$$

and we have

$$\xi_1 + \xi_2 + \xi_3 = 0.$$

From the arguments above the same holds for other cases of $x \neq 0$ or $z \neq 0$. □

Remark 3. *The property above can be seen easily, if we use the transitivity of the action of the group F_4 on the* (co)*tangent sphere bundle.*

Also from the definition of the map $\tau_{\mathbb{O}}$, $\mathrm{tr}(A) = \mathrm{tr}(\tau_{\mathbb{O}}(X, Y)) = 0$ *is equivalent to* $\mathrm{tr}\,(Y) = 0$.

Here we mention the following fact, which is a special case described in Proposition 8.20.

Lemma 4.2. *Assume that a linear function $f : \mathcal{J}(3)^{\mathbb{C}} \to \mathbb{C}$*

$$f(A) = 2\sum_{i=0}^{7} (a_i\{w\}_i + b_i\{v\}_i + c_i\{u\}_i) + \sum_{i=1}^{3} \alpha_i\xi_i$$

vanishes on $\mathbb{X}_{\mathbb{O}}$. Then f is a constant multiple of the trace function $A \mapsto \mathrm{tr}\,(A),\ A \in \mathcal{J}(3)^{\mathbb{C}}$.

Proof. Put $a = \sum a_i\mathbf{e}_i$, $b = \sum b_i\mathbf{e}_i$, $c = \sum c_i\mathbf{e}_i \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$ and $B = \begin{pmatrix} \alpha_1 & a & \theta(b) \\ \theta(a) & \alpha_2 & c \\ b & \theta(c) & \alpha_3 \end{pmatrix} \in \mathcal{J}(3)^{\mathbb{C}}$.

Then

$$f(A) = \mathrm{tr}\,(A \circ B) := f_B(A).$$

Let $Y = \begin{pmatrix} 0 & z & \theta(y) \\ \theta(z) & 0 & 0 \\ y & 0 & 0 \end{pmatrix} \in T_{X_1}(P^2\mathbb{O})$, where $z = \sum z_i\mathbf{e}_i, y = \sum y_i\mathbf{e}_i \in \mathbb{O}$, then

$$\tau_{\mathbb{O}}(X_1, Y) = \begin{pmatrix} |z|^2 + |y|^2 & 0 & 0 \\ 0 & -|z|^2 & -\theta(yz) \\ 0 & -yz & -|y|^2 \end{pmatrix} + \sqrt{|z|^2 + |y|^2} \begin{pmatrix} 0 & \sqrt{-1}z & \sqrt{-1}\theta(y) \\ \sqrt{-1}\theta(z) & 0 & 0 \\ \sqrt{-1}y & 0 & 0 \end{pmatrix} \in \mathbb{X}_{\mathbb{O}}.$$

Here we put $y = 0$, and we may assume for such $A = \tau_{\mathbb{O}}(X_1, Y)$

$$f_B(A) = \mathrm{tr}\,(B \circ A) = (|z|^2)(\alpha_1 - \alpha_2) + 2\sum \sqrt{-1}|z|z_ia_i = 0,$$

for any $\pm z_i \in \mathbb{R}$. Then $\alpha_1 = \alpha_2$ and also $a_i = 0$ for $i = 0, \cdots, 7$. Likewise we have $\alpha_1 = \alpha_3$ and $b_i = 0$ for $i = 0, \cdots, 7$.

Then we may assume

$$f_B(\tau_{\mathbb{O}}(X_1, Y)) =< \sum c_i \mathbf{e}_i, \theta(yz) >^{\mathbb{R}^8} = 0 \text{ for any } y, z \in \mathbb{O}.$$

Hence $c_i = 0$ for $i = 0, \cdots, 7$, which shows our assertion, that is $a = b = c = 0$, $\alpha := \alpha_1 = \alpha_2 = \alpha_3$ and

$$f_B(A) = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3 = \alpha \cdot \operatorname{tr}(A).$$

□

Corollary 4.3. *The space spanned by* $\mathbb{X}_{\mathbb{O}}$ $(:= [\,\mathbb{X}_{\mathbb{O}}\,])$ *is a* 26*-dimensional subspace in* $\mathcal{J}(3)^{\mathbb{C}}$.

Let $z, y, x \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$ and put

$$\begin{cases} \rho_{\mathbb{O}}(z) = Z + W\mathbf{e}_4 = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} \mathbf{e}_4, \\ \rho_{\mathbb{O}}(y) = Y + V\mathbf{e}_4 = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} + \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \mathbf{e}_4, \\ \rho_{\mathbb{O}}(x) = X + U\mathbf{e}_4 = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} + \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \mathbf{e}_4, \quad \text{where } z_i, w_i, y_i, v_i, x_i, u_i \in \mathbb{C}. \end{cases} \tag{4.3}$$

Then the conditions (4.1) and (4.2) are rewritten in terms of the matrices Z, W, Y, V, X, U as

$$\begin{cases} \xi_1(\theta(X) - U\mathbf{e}_4) = (Y + V\mathbf{e}_4)(Z + W\mathbf{e}_4) = YZ - \theta(W)V + (WY + V\theta(Z))\mathbf{e}_4, \\ \xi_2(\theta(Y) - V\mathbf{e}_4) = (Z + W\mathbf{e}_4)(X + U\mathbf{e}_4) = ZX - \theta(U)W + (UZ + W\theta(X))\mathbf{e}_4, \\ \xi_3(\theta(Z) - W\mathbf{e}_4) = (X + U\mathbf{e}_4)(Y + V\mathbf{e}_4) = XY - \theta(V)U + (VX + U\theta(Y))\mathbf{e}_4, \end{cases} \tag{4.4}$$

$$\begin{cases} {\xi_1}^2 + \det Z + \det W + \det Y + \det V = 0, \\ {\xi_2}^2 + \det Z + \det W + \det X + \det U = 0, \\ {\xi_3}^3 + \det Y + \det V + \det X + \det U = 0. \end{cases} \tag{4.5}$$

Hereafter (until §7), we denote the matrix $A = \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \in \mathcal{J}(3)^{\mathbb{C}}$ *in the form of a vector* $\in \mathbb{C}^{27}$*:*

$$\begin{aligned} A \longleftrightarrow &(\xi_1, \xi_2, \xi_3, z_1, \dots, z_4, w_1, \dots, w_4, y_1, \dots, y_4, v_1, \dots, v_4, x_1, \dots, x_4, u_1, \dots, u_4) \\ &= (a_1, a_2, a_3, a_4, \dots, \dots, a_{27}) \in \mathbb{C}^{27}, \end{aligned} \tag{4.6}$$

using the components given in (4.3) *by the map* $\rho_{\mathbb{O}}$.

The conditions for matrices in $\mathbb{X}_{\mathbb{O}}$ require that at least one of the off-diagonal components in the matrix A is non-zero. Hence, for example, we assume that there is at least one component in the matrix $\rho_{\mathbb{O}}(z) = Z + W\mathbf{e}_4 = \begin{pmatrix} z_1, & z_2 \\ z_3 & z_4 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} \mathbf{e}_4$, say $z_1 \neq 0$ and put $O_{z_1} = \{A \in \mathbb{X}_{\mathbb{O}} \mid z_1 \neq 0\}$. Also we define other open subsets $\{O_{z_i}, O_{w_i}, O_{y_i}, O_{v_i}, O_{x_i}, O_{u_i}\}_{i=1}^4$ in a same way like O_{z_1}. Then we have

Proposition 4.4. *The* 24 *subsets*

$$\{O_{z_i}, O_{w_i}, O_{y_i}, O_{v_i}, O_{x_i}, O_{u_i}\}_{i=1}^4 =: \mathcal{U}_0 \tag{4.7}$$

are all open coordinate neighborhoods and totally is an open covering of $\mathbb{X}_{\mathbb{O}}$.

Proof. We show a coordinates for the case O_{z_1}. Other cases will be shown by the same way.

From the equations in (4.4) we select 5 equations expressed in 2×2 complex matrices including the complex variable z_1 and from the equation (4.5) we select one equation also including the complex variable z_1:

$$\begin{cases} \xi_1\begin{pmatrix} x_4 & -x_2\\ -x_3 & x_1\end{pmatrix} = \begin{pmatrix} y_1 & y_2\\ y_3 & y_4\end{pmatrix}\begin{pmatrix} z_1 & z_2\\ z_3 & z_4\end{pmatrix} - \begin{pmatrix} w_4 & -w_2\\ -w_3 & w_1\end{pmatrix}\begin{pmatrix} v_1 & v_2\\ v_3 & v_4\end{pmatrix},\\ -\xi_1\begin{pmatrix} u_1 & u_2\\ u_3 & u_4\end{pmatrix} = \begin{pmatrix} w_1 & w_2\\ w_3 & w_4\end{pmatrix}\begin{pmatrix} y_1 & y_2\\ y_3 & y_4\end{pmatrix} + \begin{pmatrix} v_1 & v_2\\ v_3 & v_4\end{pmatrix}\begin{pmatrix} z_4 & -z_2\\ -z_3 & z_1\end{pmatrix},\\ \xi_2\begin{pmatrix} y_4 & -y_2\\ -y_3 & y_1\end{pmatrix} = \begin{pmatrix} z_1 & z_2\\ z_3 & z_4\end{pmatrix}\begin{pmatrix} x_1 & x_2\\ x_3 & x_4\end{pmatrix} - \begin{pmatrix} u_4 & -u_2\\ -u_3 & u_1\end{pmatrix}\begin{pmatrix} w_1 & w_2\\ w_3 & w_4\end{pmatrix},\\ -\xi_2\begin{pmatrix} v_1 & v_2\\ v_3 & v_4\end{pmatrix} = \begin{pmatrix} u_1 & u_2\\ u_3 & u_4\end{pmatrix}\begin{pmatrix} z_1 & z_2\\ z_3 & z_4\end{pmatrix} + \begin{pmatrix} w_1 & w_2\\ w_3 & w_4\end{pmatrix}\begin{pmatrix} x_4 & -x_2\\ -x_3 & x_1\end{pmatrix},\\ \xi_3\begin{pmatrix} z_4 & -z_2\\ -z_3 & z_1\end{pmatrix} = \begin{pmatrix} x_1 & x_2\\ x_3 & x_4\end{pmatrix}\begin{pmatrix} y_1 & y_2\\ y_3 & y_4\end{pmatrix} - \begin{pmatrix} v_4 & -v_2\\ -v_3 & v_1\end{pmatrix}\begin{pmatrix} u_1 & u_2\\ u_3 & u_4\end{pmatrix},\\ {\xi_2}^2 + z_1z_4 - z_2z_3 + w_1w_4 - w_2w_3 + x_1x_4 - x_2x_3 + u_1u_4 - u_2u_3 = 0. \end{cases} \tag{4.8}$$

From these we can select 10 equations including the variable z_1:

$$\begin{cases} f_1 = -\xi_2y_4 + z_1x_1 + z_2x_3 - (u_4w_1 - u_2w_3) = 0,\\ f_2 = \xi_2y_2 + z_1x_2 + z_2x_4 - (u_4w_2 - u_2w_4) = 0,\\ f_3 = \xi_2v_1 + u_1z_1 + u_2z_3 + (w_1x_4 - w_2x_3) = 0,\\ f_4 = \xi_2v_3 + u_3z_1 + u_4z_3 + (w_3x_4 - w_4x_3) = 0,\\ f_5 = -\xi_1x_4 + y_1z_1 + y_2z_3 - (w_4v_1 - w_2v_3) = 0,\\ f_6 = \xi_1x_3 + y_3z_1 + y_4z_3 - (-w_3v_1 + w_1v_3) = 0,\\ f_7 = \xi_1u_2 - v_1z_2 + v_2z_1 + w_1y_2 + w_2y_4 = 0,\\ f_8 = \xi_1u_4 - v_3z_2 + v_4z_1 + w_3y_2 + w_4y_4 = 0,\\ f_9 = -\xi_3z_1 + x_3y_2 + x_4y_4 - (-v_3u_2 + v_1u_4) = 0,\\ f_{10} = {\xi_2}^2 + z_1z_4 - z_2z_3 + w_1w_4 - w_2w_3 + x_1x_4 - x_2x_3 + u_1u_4 - u_2u_3 = 0. \end{cases} \tag{4.9}$$

The 10 variables

$$x_1, x_2, u_1, u_3, y_1, y_3, v_2, v_4, \xi_3, z_4 \tag{4.10}$$

are coefficients of the variable z_1, and can be solved easily.

In fact, with one more additional equation

$$f_{11} = \xi_1 + \xi_2 + \xi_3 = 0, \tag{4.11}$$

we can solve the 11 variables

$$\{x_1, x_2, u_1, u_3, y_1, y_3, v_2, v_4,\ z_4,\ \xi_3,\ \xi_1\} \tag{4.12}$$

in terms of the remaining 16 variables

$$\{x_3, x_4, u_2, u_4, y_2, y_4, v_1, v_3,\ z_1, z_2, z_3, w_1, w_2, w_3, w_4\ ,\ \xi_2\}, \tag{4.13}$$

in which, except $z_1 \neq 0$ other variables can take any values in $\mathbb{C}$.

Here, if we choose the equation

$$f_{10} = {\xi_1}^2 + z_1z_4 - z_2z_3 + w_1w_4 - w_2w_3 + y_1y_4 - y_2y_3 + v_1v_4 - v_2v_3 = 0,$$

instead of the tenth equation f_{10} in (4.9)(= first equation in (4.5)), then the variable ξ_1 should be chosen as an independent variable.

In any choice(in O_{z_1} case, ξ_1 or ξ_2) once we fix them (here we choose as above), and denote by P_{z_1} the projection map

$$\begin{aligned} P_{z_1} : O_{z_1} \ni &(\xi_1, \xi_2, \xi_3, z_1, \dots, z_4, w_1, \dots, w_4, y_1, \dots, y_4, v_1, \dots, v_4, x_1, \dots, x_4, u_1, \dots, u_4)\\ &= (a_1, \dots, a_{27}) \longmapsto (\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, w_4, y_2, y_4, v_1, v_3, x_3, x_4, u_2, u_4)\\ &= (a_2, a_4, a_5, a_6, a_8, a_9, a_{10}, a_{11}, a_{13}, a_{15}, a_{16}, a_{18}, a_{22}, a_{23}, a_{25}, a_{27}) \in \mathbb{C}\times\mathbb{C}^*\times\mathbb{C}^{14}. \end{aligned}$$

Then, the pair (O_{z_1}, P_{z_1}) is a local coordinates neighborhood (note that $\dim_{\mathbb{C}} \mathbb{X}_{\mathbb{O}} = 16$). □

In any case in $\mathcal{U}_0$*, once we fix independent variables, then we denote the dependent variables as* $x_1(*), x_2(*)\cdots$ *, (or* $a_1(*), a_2(*), \dots,$*) etc., where* $*$ *means the independent variables.*

Corollary 4.5. *Each coordinate neighborhood O_ℓ in $\mathcal{U}_0$ is dense in $\mathbb{X}_{\mathbb{O}}$. Hence any number of intersections of open sets in $\mathcal{U}_0$ is also open dense.*

Proof. It will be enough to show the case O_{z_1}. So, let $A \in \mathbb{X}_{\mathbb{O}} \backslash O_{z_1}$. Assume, say $A \in O_{x_1}$, then the subset $z_1 = 0$ is defined by an rational equation: $z_1 = \frac{\xi_2 y_4 - z_2 x_3 + u_4 w_1 - u_2 w_3}{x_1} = 0$. Hence the subset $z_1 = 0$ must be at most codimension 1 in $\mathbb{X}_{\mathbb{O}}$. □

Proposition 4.6. *Let O_{a_i} and O_{a_j} be any of two open coordinate neighborhoods in $\mathcal{U}_0 = \{O_{a_i}\}_{i=1}^{24}$. Then the Jacobian $J_{a_j,a_i} = \det d\big(P_{a_j} \circ P_{a_i}^{-1}\big)$ of the coordinate transformation $P_{a_j} \circ P_{a_i}^{-1}$ is given by*

$$J_{a_j,a_i} = \left(\frac{a_j}{a_i}\right)^5 \text{ on } P_{a_i}(O_{a_j} \cap O_{a_i}). \tag{4.14}$$

Proof. Let $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and define a map

$$\tilde{\sigma} : \mathbb{C}(2) \ni S \longmapsto \tilde{\sigma}(S) := \sigma \cdot S \cdot \sigma \in \mathbb{C}(2), \tag{4.15}$$

then $\theta(\tilde{\sigma}(S)) = \tilde{\sigma}(\theta(S))$. This property of $\tilde{\sigma}$ naturally induces an automorphism of $\mathcal{J}(3)^{\mathbb{C}}$, which we denote by the same notation $\tilde{\sigma} : \mathcal{J}(3)^{\mathbb{C}} \to \mathcal{J}(3)^{\mathbb{C}}$.

By the Lemma 4.5, it will be enough to determine the Jacobian J_{z_1,a_i} for the cases of

$$O_{z_1} \cap O_{a_i} = O_{a_4} \cap O_{a_i}, \text{ for } i \geq 5.$$

Also by the symmetry of the components y and x it is enough for the cases of

$$O_{z_1} \cap O_{a_i} = O_{a_4} \cap O_{a_i}, \text{ for } i = 17 \sim 24.$$

Finally, by the automorphism $\tilde{\sigma}$ explained above and the symmetry between z and x we see that it is enough to determine them for the 5 cases

$$O_{z_1} \cap O_{z_2}, O_{z_1} \cap O_{z_4}, O_{z_1} \cap O_{w_1}, O_{z_1} \cap O_{w_2}, O_{z_1} \cap O_{x_1}.$$

All the determinations can be done by the basic way of the calculation of the determinants. So we show two cases $O_{z_1} \cap O_{z_2}$ and $O_{z_1} \cap O_{w_1}$ how they look like.

[I] $O_{z_1} \cap O_{z_2} = O_{a_4} \cap O_{a_5}$ case: For this case we consider the coordinate transformation $P_{z_2} \circ P_{z_1}^{-1}$, which is given by the correspondence:

$$\begin{aligned} &(\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, w_4, y_2, y_4, v_1, v_3, x_3, x_4, u_2, u_4) \\ &\longmapsto (\xi_2, z_1, z_2, z_4, w_1, w_2, w_3, w_4, y_2, y_4, v_2, v_4, x_1, x_2, u_2, u_4) \end{aligned}$$

where the coordinates $(\xi_2, z_1, z_2, z_4, w_1, w_2, w_3, w_4, y_2, y_4, v_2, v_4, x_1, x_2, u_2, u_4)$ are given by the rational functions:

$$\begin{cases} x_1 = \frac{\xi_2 y_4 - z_2 x_3 + (u_4 w_1 - u_2 w_3)}{z_1}, \ x_2 = \frac{-\xi_2 y_2 - z_2 x_4 + (u_4 w_2 - u_2 w_4)}{z_1}, \quad u_2 = u_2, \ u_4 = u_4, \\ y_2 = y_2, \ , y_4 = y_4, \quad v_2 = \frac{-\xi_1 u_2 + v_1 z_2 - w_1 y_2 - w_2 y_4}{z_1}, \ v_4 = \frac{-\xi_1 u_4 + v_3 z_2 - w_3 y_2 - w_4 y_4}{z_1}, \\ z_1 = z_1, \ z_2 = z_2, \ z_4 = \frac{-\xi_2{}^2 + z_2 z_3 - w_1 w_4 + w_2 w_3 - x_1 x_4 + x_2 x_3 - u_1 u_4 + u_2 u_3}{z_1}, \\ w_1 = w_1, \quad w_2 = w_2, \quad w_3 = w_3, \quad w_4 = w_4, \quad \xi_2 = \xi_2. \end{cases} \tag{4.16}$$

We change the orderings of the coordinates in $P_{z_1}(O_{z_1})$ with "*even*" permutations as

$$(\xi_2, z_1, z_2, w_1, w_2, w_3, w_4, y_2, y_4, u_2, u_4, z_3, v_1, v_3, x_3, x_4)$$

and $P_{z_2}(O_{z_2})$ as

$$(\xi_2, z_1, z_2, w_1, w_2, w_3, w_4, y_2, y_4, u_2, u_4, z_4, v_2, v_4, x_1, x_2).$$

Then the Jacobi matrix is of the form that

$$\begin{pmatrix} Id_{11} & C \\ 0_{5,11} & D \end{pmatrix}, \tag{4.17}$$

where Id_{11} is 11×11 identity matrix, $0_{5,11}$ is 5×11 zero matrix and D is given by

$$D = \begin{pmatrix} \frac{z_2}{z_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{z_2}{z_1} & 0 & 0 & 0 \\ 0 & * & \frac{z_2}{z_1} & 0 & 0 \\ x_2 & * & * & -\frac{z_2}{z_1} & 0 \\ -x_1 & * & * & * & -\frac{z_2}{z_1} \end{pmatrix} \tag{4.18}$$

(the 11×5 matrix C and components $*$ are given by some functions). Hence the Jacobian J_{z_2,z_1} is

$$J_{z_2,z_1} = \det D = \left(\frac{z_2}{z_1}\right)^5.$$

[II] $O_{w_1} \cap O_{z_1} = O_{a_8} \cap O_{a_4}$ case: For this case we consider the coordinate transformation $P_{w_1} \circ P_{z_1}^{-1}$, which is given by the correspondence:

$$\begin{aligned} &(\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, w_4, y_2, y_4, v_1, v_3, x_3, x_4, u_2, u_4) \\ &\longmapsto (\xi_2, z_1, z_2, z_3, z_4, w_1, w_2, w_3, y_3, y_4, v_1, v_2, x_1, x_3, u_1, u_2), \end{aligned}$$

where the coordinates $(x_1, x_3, u_1, u_2, y_3, y_4, v_1, v_2, z_1, z_2, z_3, z_4, w_1, w_2, w_3, \xi_2)$ are given by the rational functions:

$$\begin{cases} x_1 = \frac{\xi_2 y_4 - z_2 x_3 + (u_4 w_1 - u_2 w_3}{z_1}, \quad x_3 = x_3, \quad u_1 = \frac{-\xi_2 v_1 - u_2 z_3 - (w_1 x_4 - w_2 x_3}{z_1}, \quad u_2 = u_2, \\ y_3 = \frac{-\xi_1 x_3 - y_4 z_3 + (-w_3 v_1 + w_1 v_3}{z_1}, \quad y_4 = y_4, \quad v_1 = v_1, \quad v_2 = \frac{-\xi_1 u_2 + v_1 z_2 - w_1 y_2 - w_2 y_4}{z_1}, \\ z_1 = z_1, \quad z_2 = z_2, \quad z_3 = z_3, \quad z_4 = \frac{-{\xi_2}^2 + z_2 z_3 - w_1 w_4 + w_2 w_3 - x_1 x_4 + x_2 x_3 - u_1 u_4 + u_2 u_3}{z_1}, \\ w_1 = w_1, \quad w_2 = w_2, \quad w_3 = w_3, \quad \xi_2 = \xi_2. \end{cases} \tag{4.19}$$

We change the orderings of the coordinates in $P_{z_1}(O_{z_1})$ by the "*odd*" permutation as

$$\begin{aligned} &(\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, w_4, y_2, y_4, v_1, v_3, x_3, x_4, u_2, u_4) \\ &\longmapsto (\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, y_4, v_1, x_3, u_2, w_4, y_2, v_3, x_4, u_4) \end{aligned}$$

and $P_{w_1}(O_{w_1})$ by the *even* permutation as

$$\begin{aligned} &(\xi_2, z_1, z_2, z_3, z_4, w_1, w_2, w_3, y_3, y_4, v_1, v_2, x_1, x_3, u_1, u_2) \\ &\longmapsto (\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, y_4, v_1, x_3, u_2, z_4, y_3, v_2, x_1, u_1) \end{aligned}$$

Then the Jacobi matrix is of the form that

$$\begin{pmatrix} Id_{11} & C' \\ 0_{5,11} & D' \end{pmatrix}, \tag{4.20}$$

where the matrix D' is given by

$$D' = \begin{pmatrix} -\frac{w_1}{z_1} & 0 & 0 & 0 & 0 \\ * & 0 & -\frac{w_1}{z_1} & 0 & 0 \\ * & \frac{w_1}{z_1} & 0 & 0 & 0 \\ * & 0 & 0 & 0 & -\frac{w_1}{z_1} \\ * & 0 & 0 & \frac{w_w}{z_1} & 0 \end{pmatrix}. \tag{4.21}$$

(the matrix C' and components $*$ are given by some functions) Hence the Jacobian J_{w_1,z_1} is

$$J_{w_1,z_1} = \det D' = -\left(\frac{w_1}{z_1}\right)^5.$$

□

From the above Proposition 4.6 we can see that the $\mathbb{C}^*$-valued 1-cocycle defined by $\{J_{a_j,a_i}\}_{a_i,a_j \in \{z_1,\cdots,\cdots,u_4\}}$ is the coboundary of $\mathbb{C}^*$-valued 0-cochain $\{h_i = \frac{1}{a_i^5}\}$, we have

Theorem 4.7. *The set of holomorphic sections*

$$\left\{h_{z_i}=\frac{1}{{z_i}^5},h_{w_i}=\frac{1}{{w_i}^5},h_{y_i}=\frac{1}{{y_i}^5},h_{v_i}=\frac{1}{{v_i}^5},h_{x_i}=\frac{1}{{x_i}^5},h_{u_i}=\frac{1}{{u_i}^5}\right\},$$

each function is defined on the open coordinate neighborhood O_{z_i}, O_{w_i} and so on, together define a nowhere vanishing global holomorphic section $\Omega_{\mathbb{O}}$ of the canonical line bundle $\bigwedge^{16} T^{*'}(\mathbb{X}_{\mathbb{O}})^{\mathbb{C}}$.

Here the above local sections, for example h_{z_1} defined on O_{z_1}, should be understood as the coefficient of a 16-degree (= highest degree) holomorphic differential form:

$$\begin{aligned}&h_{z_1}(\xi_2,z_1,z_2,z_3,w_1,w_2,w_3,w_4,y_2,y_4,v_1,v_3,x_3,x_4,u_2,u_4)\\&=\frac{1}{{z_1}^5}\,d\xi_2\wedge dz_1\wedge dz_2\wedge dz_3\wedge dw_1\wedge dw_2\wedge dw_3\wedge dw_4\wedge dy_2\wedge dy_4\wedge dv_1\wedge dv_3\wedge dx_3\wedge dx_4\wedge du_2\wedge du_4\\&=\frac{1}{{a_4}^5}da_2\wedge da_4\wedge da_5\wedge da_6\wedge da_8\wedge da_9\wedge da_{10}\wedge da_{13}\wedge da_{15}\wedge da_{16}\wedge da_{18}\wedge da_{22}\wedge da_{23}\wedge da_{25}\wedge da_{27}.\end{aligned}$$

Remark 4. *As in the case for the sphere, the nowhere vanishing global holomorphic 16-form $\Omega_{\mathbb{O}}$ coincides with the restriction of a smooth 16-form $\tilde{\Omega}_{\mathbb{O}}$ defined on the whole space $\mathcal{J}(3)^{\mathbb{C}}$ and there is a smooth 11-form $\mathfrak{n}$ on $\mathcal{J}(3)^{\mathbb{C}}$ with the property that*

$$\tilde{\Omega}_{\mathbb{O}}\wedge\mathfrak{n}=da_1\wedge a_2\wedge a_3\cdots\cdot\wedge a_{27}.$$

For the description of these smooth forms we need a troublesome preparation for the coordinates choices and we do not use the forms later so that we omit the construction.

We mention that since the transition function of the canonical line bundle on $\mathbb{X}_{\mathbb{O}}$ is invariant under the multiplication by non-zero complex numbers, it is a pull-back of a complex line bundle on the quotient space $\overline{\mathbb{X}}_{\mathbb{O}}:=\mathbb{C}^*\backslash\mathbb{X}_{\mathbb{O}}$. More precisely

Proposition 4.8. (1) *Interpreting the calculations above in terms of the homogeneous coordinates we see that the canonical line bundle $\mathcal{K}^{\overline{\mathbb{X}}_{\mathbb{O}}}=\bigwedge^{15} T^{*'}(\overline{\mathbb{X}}_{\mathbb{O}})^{\mathbb{C}}$ of the quotient space $\overline{\mathbb{X}}_{\mathbb{O}}$ is isomorphic to $\overset{5}{\otimes}\mathcal{L}^*{}_{|\overline{\mathbb{X}}_{\mathbb{O}}}$, where $\mathcal{L}$ is the tautological line bundle on the projective space $P^{26}\mathbb{C}$, $\mathcal{L}\subset P^{26}\mathbb{C}\times\mathbb{C}^{27}$.*

(2) *Let $\mathcal{V}$ be the kernel of the projection map $\pi:\mathbb{X}_{\mathbb{O}}\longrightarrow\overline{\mathbb{X}}_{\mathbb{O}}$,*

$$\mathcal{V}:=\ker\,d\pi\subset T(\mathbb{X}_{\mathbb{O}}),$$

which can be seen naturally as a complex line bundle trivialized by the holomorphic vector field corresponding to the dilation action

$$\mathbb{X}_{\mathbb{O}}\ni A\longmapsto t\cdot A\in\mathbb{X}_{\mathbb{O}}.$$

In this sense we denote it by $\mathcal{V}_{\mathbb{C}}$. Then by the exact sequence

$$\{0\}\longrightarrow\pi^*(T^{*'}(\overline{\mathbb{X}}_{\mathbb{O}})^{\mathbb{C}})\longrightarrow T^{*'}(\mathbb{X}_{\mathbb{O}})^{\mathbb{C}}\longrightarrow{\mathcal{V}_{\mathbb{C}}}^*\longrightarrow\{0\}$$

we know that the canonical line bundle $\mathcal{K}^{\mathbb{X}_{\mathbb{O}}}\cong\pi^*(\mathcal{K}^{\overline{\mathbb{X}}_{\mathbb{O}}})\otimes{\mathcal{V}_{\mathbb{C}}}^*$ is holomorphically trivial, since $\pi^*(\mathcal{L})$ is holomorphically trivial.

5. Symplectic manifolds and polarizations

In this section we review an aspect of a geometric quantization theory in a restricted framework fitting only to our purpose. In the subsections §5.3 and §5.4 and in section §6 we explain how the framework is adapted to our case.

5.1. **Integral symplectic manifold.** Let (M,ω^M) be a symplectic manifold with the symplectic form ω^M. In this paper we assume that

[In1] the map $H^2(M,\mathbb{Z})\to H^2_{dR}(M,\mathbb{R})$ is injective, or the group $H^2(M,\mathbb{Z})$ has no torsion and,

[In2] the de Rham cohomology class $[\omega^M]$ of the symplectic form ω^M is in this image.

Then the complex line bundle $\mathbb{L}_{\omega^M}:=\mathbb{L}\in H^1(M,\mathbb{C}^*)\cong H^2(M,\mathbb{Z})$ corresponding to the cohomology class $[\omega^M]$ is unique(of course, up to isomorphism). The first condition is satisfied, for example if M is simply connected and our case $M=\mathbb{X}_{\mathbb{O}}$ satisfies both of these conditions trivially, since $H^2(\mathbb{X}_{\mathbb{O}},\mathbb{Z})=\{0\}$.

Under these assumptions, the unique complex line bundle $\mathbb{L}$ has the canonically defined connection ∇, which is defined as follows:

Let $\{U_i\}$ be an open covering of M with several "good" properties required in the arguments below (it is always possible for manifolds). Then there are one-forms $\{f_i\}$, each of which is defined on U_i and $df_i = \omega^M$. Then the correction of smooth functions $\{c_{ij}\}$ defined by $dc_{ij} = f_j - f_i$ on $U_i \cap U_j$ satisfy that $c_{jk} - c_{ik} + c_{ij}$ takes integers on $U_i \cap U_j \cap U_k$, and the transition functions $\{g_{ij} = e^{2\pi\sqrt{-1}c_{ij}}\}$ defines the line bundle $\pi : \mathbb{L} \to M$.

The connection ∇ on $\mathbb{L}$ is defined as

$$\nabla_X(s_i) = 2\pi\sqrt{-1} < f_i, X > s_i \text{ on } U_i, \ (X \text{ is a vector field})$$

where s_i is a nowhere vanishing section on U_i identifying $U_i \times \mathbb{C}$ and $\pi^{-1}(U_i) \subset \mathbb{L}$ in such a way that

$$U_i \times \mathbb{C} \ni (x, z) \mapsto z \cdot s_i(x) \in \pi^{-1}(U_i).$$

Here $< f_i\,, X >$ *denotes the pairing of a one-form* f_i *and a tangent vector* X*.*

If we choose all the functions c_{ij} being real valued, we may regard that the line bundle $\mathbb{L}$ is equipped with an Hermitian inner product, which we denote by $< \cdot\,, \cdot >_x^{\mathbb{L}}$ at $x \in M$. *Hereafter we assume that the line bundle* $\mathbb{L}$ *is equipped with such an Hermitian inner product.*

We may regard that the space $C^\infty(M)$ is a Lie algebra by the Poisson bracket $\{f, g\} := \omega^M(H_f, H_g)$, where H_f denotes the Hamilton vector field with the Hamiltonian f defined by $< df, \bullet >= \omega^M(H_f, \bullet)$. The space $\Gamma(\mathbb{L}, M)$ is a central object in the quantization theory. There is a basic fact that the correspondence from $g \in C^\infty(M)$ to the operator T_g, assignment of a function to an operator,

$$T_g : \Gamma(\mathbb{L}, M) \ni s \longmapsto \nabla_{H_g}(s) + 2\pi\sqrt{-1}g \cdot s$$

is a Lie algebra homomorphism, $[T_g\,, T_h] = T_{\{g\,,\,h\}}$, and it is the main theme in the quantization theory how to assign a function on a phase space to an operator on the configuration space.

5.2. **Real and complex polarizations.** Let (M, ω^M) be a symplectic manifold with the symplectic form ω^M ($\dim M = 2n$). The skew-symmetric bi-linear form ω_p^M at each point $p \in M$ is naturally extended to the complexification $T(M) \otimes \mathbb{C} := T(M)^{\mathbb{C}}$ as the skew-symmetric complex bi-linear form which we denote with the same notation.

Let F be a subbundle of the complex fiber dimension n in $T(M)^{\mathbb{C}}$ satisfying the properties that

(1) F is maximal isotropic with respect to the skew-symmetric bi-linear form ω^M,

(2) F is integral, that is $F \cap \overline{F}$ has constant rank and $F,\ F + \overline{F}$ is closed under bracket operation of vector fields taking values in these subbundles.

In this paper we only treat two extreme cases,

$(P1)$ $\quad F = \overline{F}$, and

$(P2)$ $\quad F + \overline{F} = T(M)^{\mathbb{C}}$.

First one is the complexification of a Lagrangian foliation $L \subset T(M)$, $F = L \otimes \mathbb{C}$ and we call it a real polarization. The second case is called a complex polarization.

If there is a polarization satisfying the second condition $F + \overline{F} = T(M)^{\mathbb{C}}$, then M has a almost complex structure J and the subbundle F is identified with $(0, 1)$-vectors in $T(M)^{\mathbb{C}}$ (anti-complex subbundle). The integrability condition implies that M becomes a complex manifold. When we put

$$g(\alpha, \beta) := \omega^M(J(\alpha), \beta), \ \alpha, \beta \text{ vector fields on } M,$$

then g is a non-singular symmetric bi-linear form on $T(M)$ and moreover it defines an Hermitian form on $T(M)^{\mathbb{C}}$. Under the condition that the form g is positive definite, then it is equivalent that M has a Kähler structure. We call such a polarization a *positive polarization*.

Hence it is equivalent that if there is a positive complex polarization on the symplectic manifold M, then M is a Kähler manifold and the symplectic form ω^M is a Kähler form. Also real polarization is always positive.

In this paper we consider two polarizations on the space $\mathbb{X}_{\mathbb{O}}$*, one is the real polarization* $\mathcal{F}$ *naturally defined on the cotangent bundle and a Kähler polarization* (= *positive complex polarization*) $\mathcal{G}$ *described in* (3.7) *and Theorem* (3.3)*.*

5.3. **Hilbert space structure on the spaces of polarized sections.** Now let M be a symplectic manifold satisfying the conditions [In1] and [In2] as in the subsection § 5.1 and fix a line bundle $\mathbb{L}$ corresponding to the cohomology class $[\omega^M]$ with the connection ∇ and the Hermitian inner product explained in the above subsections and assume that there is a polarization F on M.

Let U be an open subset in M. We introduce a space $C_F(U) \subset C^\infty(U)$ by

$$C_F(U) = \{h \in C^\infty(U) \mid X(h) = 0, \ {}^\forall X \in \Gamma(F, U), \text{ vector fields taking values in } F\}$$

and a subspace $\Gamma_F(\mathbb{L}, U)$ of smooth sections in $\Gamma(\mathbb{L}, U)$ by

$$\Gamma_F(\mathbb{L}, U) = \{s \in \Gamma(\mathbb{L}, U) \mid \nabla_X(s) = 0, \ {}^\forall X \in \Gamma(F, U)\}.$$

Let U be an open subset such that there is an one-form θ on U satisfying

$$d\theta = \omega^M, \text{ and } <\theta, X>= 0 \text{ for vectors } X \in F.$$

Although it is not canonical, we may locally identify the spaces $C_F(U)$ and $\Gamma_F(\mathbb{L}, U)$ by fixing a nowhere vanishing section $s : U \to \mathbb{L}$ with the property that $\nabla_X(s) = 0$ for $X \in F$ in such a way that

$$C_F(U) \ni \varphi \mapsto \varphi \cdot s \in \Gamma_F(\mathbb{L}, U).$$

Then under this identification, the connection ∇ is

$$\nabla_X(\varphi \cdot s) = X(\varphi) \cdot s + 2\pi\sqrt{-1}\varphi < \theta, X > \cdot s = X(\varphi) \cdot s,$$

for vector field X taking values in F.

If F is a real polarization, then the function space $C_F(U)$ consists of such functions that are constant along each leaf $\cap\, U$ of the Lagrangian foliation, and if F is a complex polarization, then $C_F(U)$ consists of holomorphic functions on U.

We call these sections $\in \Gamma_F(\mathbb{L}, U)$ "*polarized sections*" (with respect to a polarization F) and are the main objects in the geometric quantization theory. We may regard, according to the polarization, that they express quantum states in the real polarization case and that they express good classical observables in the complex polarization. The above identification indicates the local nature of the polarized sections according to the polarization.

One basic problem is to introduce an inner product on the space $\Gamma_F(\mathbb{L}, M)$ of $\mathbb{L}$-valued polarized sections and a related space (which will be explained later) in a reasonable way (or without additional assumptions) to make it a (pre-)Hilbert space and the most interesting problem is to see a transformation from one space of polarized sections $\Gamma_G(\mathbb{L}, M)$ (by a polarization G) to another space $\Gamma_F(\mathbb{L}, M)$ of polarized sections by another polarization F.

We discuss two cases according to the polarizations (real and positive complex) how we introduce an inner product below in [RP] (real polarization) and in [CP] (complex polarization).

Under our assumptions we work only on density, (partial) half density, or (partial)1/4-density spaces. The meaning of "partial" will be explained in Remark 5.

[RP] Let F be a *real polarization*. In this paper, for avoiding unnecessary generality, we assume more strongly that

(RP1) there is a submersion to an orientable manifold N,

$$\Phi : M \longrightarrow N$$

whose fibers are connected Lagrangian submanifolds.

So, the real polarization F is defined as the kernel $F = \operatorname{Ker} d\Phi$ of a surjective submersion $\Phi : M \longrightarrow N$ and the functions in $C_F(M)$ are naturally descended to the base space N, that is $\Phi^*(C^\infty(N)) = C_F(M)$.

Let $\alpha,\ \beta \in \Gamma_F(\mathbb{L}, M)$, then by the equality

$$0 =< \nabla_X(\alpha), \beta >^{\mathbb{L}} + < \alpha, \nabla_X(\beta) >^{\mathbb{L}}= X(< \alpha, \beta >^{\mathbb{L}}), \text{ for } X \in F,$$

the function $< \alpha\, , \beta >^{\mathbb{L}}$ is constant on each fiber. Hence it can be naturally identified with a function on the base manifold N. For such functions we need not integrate along the leaves and it will be enough to consider the integration to the transversal direction of the leaves. This is realized by the integration on the base space to make the space $\Gamma_F(\mathbb{L}, M)$ into a (pre) Hilbert space. There are many way to choose a measure on N, say a Riemann volume form to integrate it.

Instead of the space $\Gamma_F(\mathbb{L}, M)$, we consider $\mathbb{L}$-*valued polarized* (or exactly to say, we call horizontal and partial) *half-densities* $\varphi \in \Gamma_F\Big(\mathbb{L} \otimes |\overset{max}{\bigwedge} F^0|, M\Big)$ *and/or* horizontal and partial $\frac{1}{4}$-*densities* $\varphi \in \Gamma_F\Big(\mathbb{L} \otimes \sqrt{|\overset{max}{\bigwedge} F^0|}, M\Big)$, where F^0 is the annihilator of F,

$$F^0 = \{\xi \in T^*(M) \mid \ <\xi, X>= 0, \ ^\forall X \in F \ \}.$$

We can introduce a (partial) connection $\nabla\!\!\!\!/_X(\xi) := i_X(d\xi)$ on $\overset{max}{\bigwedge} F^0 = \overset{\max}{\bigwedge} (d\Phi)^*(\Phi^*(T^*(N)))$, where ξ is a differential form $\in \Gamma\Big(\overset{max}{\bigwedge} F^0, M\Big)$, $X \in F$ and i_X denotes the interior product with a tangent vector $X \in F$.

Note that

$$i_X(d\xi) = i_X \circ d\xi \in \Gamma\Big(\overset{max}{\bigwedge} F^0, M\Big), \quad \text{for } X \in F \text{ and } \xi \in \Gamma\Big(\overset{max}{\bigwedge} F^0, M\Big).$$

Since $i_X(\xi) = 0$ for $\xi \in \Gamma\Big(\overset{max}{\bigwedge} F^0, M\Big)$ by $X \in F$

$$\begin{aligned} \nabla\!\!\!\!/_X(f \cdot \xi) &= i_X \circ d(f \cdot \xi) = i_X \circ (df \wedge \xi + f \cdot d\xi) = X(f) \cdot \xi - df \wedge i_X(\xi) + f \cdot i_X(d\xi) \\ &= X(f)\xi + f \cdot \nabla\!\!\!\!/_X(f \cdot \xi), \text{ for } X \in F \text{ and } f \in C^\infty(M), \end{aligned}$$

the vector fields taking values in F work as a differentiation on the space of the differential forms $\Gamma\Big(\overset{max}{\bigwedge} F^0, M\Big)$. Hence we can consider the differentiation $\nabla\!\!\!\!/_X$ along the polarization F for the sections $\in \Gamma\Big(\overset{max}{\bigwedge} F^0, M\Big)$ and also sections $\in \Gamma\Big(|\overset{max}{\bigwedge} F^0|, M\Big)$ too.

Then under our assumption (RP1) and according to the definition of the partial connection, the sections $\xi \in \Gamma_F\Big(\overset{max}{\bigwedge} F^0, M\Big)$ can be descended to the sections $\in \Gamma(\overset{max}{\bigwedge} T^*(N), N)$, hence it holds

$$\Phi^*\left(\Gamma\Big(\overset{max}{\bigwedge} T^*(N), N\Big)\right) \cong \Gamma_F\Big(\overset{max}{\bigwedge} F^0, M\Big). \tag{5.1}$$

We may regard a differential form in $\Gamma_F\Big(\overset{max}{\bigwedge} F^0, M\Big)$ a polarized (or horizontal) "partial" half density (or half degree form) on M.

Remark 5. *By our assumption* (RP1)*, there is an exact sequence*

$$\{0\} \longrightarrow F^0 \longrightarrow T^*(M) \longrightarrow F^* \longrightarrow \{0\}, \tag{5.2}$$

and the injective bundle map on M, $(d\Phi)^* : \Phi^*(T^*(N)) \to T^*(M)$, which is the dual of the differential $d\Phi$. Since the polarization F coincides with the vertical subbundle of the projection map Φ, the image $(d\Phi)^*(\Phi^*(T^*(N))) = F^0$.

By the assumption (RP1) *we regard that* $\overset{max}{\bigwedge} T^*(N) \cong |\overset{max}{\bigwedge} T^*(N)|$ (line bundles of the highest degree differential form and density (volume form) line bundle) *and we consider the square root bundle* $\sqrt{\overset{max}{\bigwedge} F^0}$.

Sections in $\Gamma_F(\overset{max}{\bigwedge} F^0, M)$ *or* $\Gamma_F(\sqrt{\overset{max}{\bigwedge} F^0}, M)$ *are not the half densities or 1/4-densities, since* $\overset{max}{\bigwedge} T^*(M) \cong \overset{max}{\bigwedge} F^* \otimes \overset{max}{\bigwedge} F^0 =$ *trivial bundle given by the Liouville volume form. So we should call the sections in* $\Gamma_F(\overset{max}{\bigwedge} F^0, M)$ *or in* $\Gamma_F(\sqrt{\overset{max}{\bigwedge} F^0}, M)$ *polarized "partial" half density or "partial" 1/4-density.*

Differential forms $\mu \in \Gamma_F\Big(\overset{max}{\bigwedge} F^0, M\Big)$ is descended to densities $\mu_* \in \Gamma(\overset{max}{\bigwedge} T^*(N), N)$ (highest degree differential form) on the base manifold N, that is there is a unique highest degree differential form $\mu_* \in \Gamma(\overset{max}{\bigwedge} T^*(N), N)$ such that $\Phi^*(\mu_*) = \mu$ by the isomorphism (5.1), and then we can integrate μ_* on N. Hence we have a natural linear form

$$\mathrm{I}_N : \Gamma_F\Big(\overset{max}{\bigwedge} F^0, M\Big) \ni \mu \longmapsto \mathrm{I}_N(\mu) := \int_N \mu_* \in \mathbb{C}.$$

If we denote the inverse map of Φ^* of (5.1) by Φ_*, then

$$\int_N \mu_* = \int_N \Phi_*(\mu).$$

In turn, we consider the square root bundle $\sqrt{\overset{max}{\bigwedge} F^0}$, which can be seen as a partial 1/4-density bundle on M. Then we can also introduce a partial connection ${\nabla_X}^{1/2}$ on the line bundle $\sqrt{\overset{max}{\bigwedge} F^0}$ and as well it is defined also on the line bundle $\mathbb{L} \otimes \sqrt{\overset{max}{\bigwedge} F^0}$. Hence we consider *"$\mathbb{L}$-valued polarized (or horizontal) partial $\frac{1}{4}$-densities"* $\alpha \otimes \eta \in \Gamma_F\Big(\mathbb{L} \otimes \sqrt{\overset{max}{\bigwedge} F^0}, M\Big)$ and define their product by making use of the Hermitian inner product on $\mathbb{L}$ with the formula

$$\begin{array}{ccc} \Gamma_F\Big(\mathbb{L} \otimes \sqrt{\overset{max}{\bigwedge} F^0}, M\Big) \times \Gamma_F\Big(\mathbb{L} \otimes \sqrt{\overset{max}{\bigwedge} F^0}, M\Big) & \longrightarrow & \Gamma_F\Big(\overset{max}{\bigwedge} F^0, M\Big) \\ \cup & & \cup \\ (\alpha \otimes \mu, \beta \otimes \nu) & \longmapsto & < \alpha\, , \, \beta >^{\mathbb{L}} \cdot \mu \otimes \nu \in \Gamma_F\Big(\overset{max}{\bigwedge} F^0, M\Big). \end{array} \tag{5.3}$$

The resulting horizontal partial half density $< \alpha\, , \, \beta >^{\mathbb{L}} \cdot \mu \otimes \nu \in \Gamma_F(\overset{max}{\bigwedge} F^0, M)$, is identified with a density on N. Hence we can define a pairing (or an inner product) for the sections in $\Gamma_F\Big(\mathbb{L} \otimes \sqrt{\overset{max}{\bigwedge} F^0}, M\Big)$ by the integration of the corresponding density on N in a natural way,

$$\begin{array}{c} \Gamma_F\Big(\mathbb{L}\otimes\sqrt{\overset{max}{\bigwedge} F^0}, M\Big) \times \Gamma_F\Big(\mathbb{L} \otimes \sqrt{\overset{max}{\bigwedge} F^0}, M\Big) \longrightarrow \mathbb{C}, \\ \cup \\ (a \otimes \mu, b \otimes \nu) \longmapsto \mathrm{I}_N(\Phi_*(< a, b >^{\mathbb{L}} \cdot \mu \otimes \nu)) = \int_N \Phi_*(< a, b >^{\mathbb{L}} \cdot \mu \otimes \nu). \end{array}$$

For the real polarization $\mathcal{F}$ on our space $\mathbb{X}_{\mathbb{O}}$, first we trivialize the line bundle $\mathbb{L}$ by a nowhere vanishing polarized section $\mathbf{s}_0 \in \Gamma_{\mathcal{F}}(\mathbb{L}, \mathbb{X}_{\mathbb{O}})$ with $< \mathbf{s}_0, \mathbf{s}_0 >^{\mathbb{L}} \equiv 1$. We call this trivialization of the line bundle $\mathbb{L}$ a "unitary trivialization".

Next, let $dv_{P^2\mathbb{O}}$ be the Riemann volume form on $P^2\mathbb{O}$. We consider the square root

$$\sqrt{\{\mathbf{q} \circ (\tau_{\mathbb{O}})^{-1}\}^*(dv_{P^2\mathbb{O}})} = \{\mathbf{q} \circ (\tau_{\mathbb{O}})^{-1}\}^*(\sqrt{dv_{P^2\mathbb{O}}}) \in \Gamma_{\mathcal{F}}\Big(\sqrt{\overset{max}{\bigwedge} \mathcal{F}^0}, \mathbb{X}_{\mathbb{O}}\Big),$$

and identify a $\mathbb{L}$-valued polarized partial 1/4-density $\xi \otimes \mu \in \Gamma_{\mathcal{F}}\Big(\mathbb{L}\otimes\sqrt{\overset{max}{\bigwedge} \mathcal{F}^0}, \mathbb{X}_{\mathbb{O}}\Big)$ with $f \cdot \mathbf{s}_0 \otimes \sqrt{\{\mathbf{q} \circ (\tau_{\mathbb{O}})^{-1}\}^*(dv_{P^2\mathbb{O}})}$, where the function f can been as a pull-back of a function $g \in C^\infty(P^2\mathbb{O})$, $f = \mathbf{q}^*(g)$. Then we may identify it with a half density on N of the form $g \cdot \sqrt{dv_{P^2\mathbb{O}}}$. Hence we identify the L_2-space with respect to the Riemann volume form $dv_{P^2\mathbb{O}}$ (we denote it by $L_2(P^2\mathbb{O}, dv_{P^2\mathbb{O}})$) *and the space of $\mathbb{L}$-valued polarized partial 1/4-densities* $\Gamma_{\mathcal{F}}\Big(\mathbb{L} \otimes \sqrt{\overset{max}{\bigwedge} \mathcal{F}^0},\ \mathbb{X}_{\mathbb{O}}\Big)$ (*after taking completion*).

[CP] Let G be a positive complex polarization on M whose symplectic form ω^M is expressed as a Kähler form:

$$\sqrt{-1}\ \overline{\partial}\,\partial\phi = \omega^M.$$

The line bundle $\mathbb{L}$ corresponding to the cohomology class $[\omega^M]$ is equipped with an Hermitian inner product $< \cdot, \cdot >^{\mathbb{L}}$ as was explained in 5.1.

The inner product $< a, b >^{\mathbb{L}}$ of two sections $a, b \in \Gamma_G(\mathbb{L}, M)$ is a function on M and can be integrated with respect to the Liouville volume form $dV_M := \frac{(-1)^{n(n-1)/2}}{n!}\left\{\omega^M\right\}^n$ ($\dim M = 2n$). Hence we can introduce an inner product on the space $\Gamma_G(\mathbb{L}, M)$ intrinsically, since we do not depend on any other additional assumptions.

We can also introduce an inner product on the space of $\mathbb{L}$-valued "polarized" sections of the canonical line bundle K^G for the complex polarization G.

The canonical line bundle $K^G = \overset{max}{\bigwedge} T^{*'}(M)^{\mathbb{C}}$ is the line bundle of the highest degree exterior product of the holomorphic part $T^{*'}(M)^{\mathbb{C}}$ of the complexified cotangent bundle $T^*(M)^{\mathbb{C}}$ ($(1,0)$ type cotangent vectors), which is the annihilator of the complex polarization G ((0,1) tangent vectors), like F^0 for the real polarization F. The sections of the canonical line bundle can be thought as half densities (or complex valued half density) by the isomorphism $K^G \otimes \overline{K^G} = \overset{max}{\bigwedge} T^*(M)^{\mathbb{C}}$. We can introduce a partial connection $\nabla\!\!\!\!/_X{}^G$ $(X \in G)$ along the complex polarization G in the similar way as for the real polarization. Then we consider the space $\Gamma_G(\mathbb{L} \otimes K^G, M)$ of "$\mathbb{L}$*-valued polarized sections of the canonical line bundle*" and using the Hermitian inner product on $\mathbb{L}$ we have a highest degree differential form

$$< a \otimes \mu, b \otimes \nu >=< a, b >^{\mathbb{L}} \cdot \mu \wedge \overline{\nu} \in \Gamma\Big(\overset{max}{\bigwedge} T^*(M)^{\mathbb{C}}, M \Big),$$

where $a, b \in \Gamma_G(\mathbb{L}, M)$ and $\mu, \nu \in \Gamma_G(K^G, M)$. The quantity $\mu \wedge \overline{\nu}$ can be seen as a (complex valued) density on M. Hence we have an intrinsic (pre-)Hilbert space structure on the space $\Gamma_G(\mathbb{L} \otimes K^G, M)$.

For the complex polarization $\mathcal{G}$ on our space $\mathbb{X}_{\mathbb{O}}$, we use a structure so called Calabi-Yau structure on $\mathbb{X}_{\mathbb{O}}$ to identify the space $\Gamma_{\mathcal{G}}(\mathbb{L} \otimes K^{\mathcal{G}}, \mathbb{X}_{\mathbb{O}})$ and the space $C_{\mathcal{G}}(\mathbb{X}_{\mathbb{O}})$ of holomorphic functions on $\mathbb{X}_{\mathbb{O}}$ by the correspondence

$$\gamma : C_{\mathcal{G}}(\mathbb{X}_{\mathbb{O}}) \ni h \longmapsto \gamma(h) = h \cdot \mathbf{t}_0 \otimes \Omega_{\mathbb{O}} \in \Gamma_{\mathcal{G}}(\mathbb{L} \otimes K^{\mathcal{G}}, \mathbb{X}_{\mathbb{O}}). \tag{5.4}$$

The existence of the nowhere vanishing holomorphic 16-form $\Omega_{\mathbb{O}}$ on $\mathbb{X}_{\mathbb{O}}$ was proved in Proposition (4.7) and $\mathbf{t}_0$ is taken for trivializing the line bundle $\mathbb{L}$ satisfying the property $\nabla\!\!\!\!/_X{}^{\mathcal{G}}(\mathbf{t}_0) = 0$.

We call a trivialization of the line bundle $\mathbb{L}$ by the section $\mathbf{t}_0$ a "holomorphic trivialization". We will determine the relation of the sections $\mathbf{s}_0$ and $\mathbf{t}_0$, $\mathbf{t}_0 = g_0\mathbf{s}_0$ in the subsection § 6.1.

5.4. **Pairing of polarizations and a Bargmann type transformation.** First, we recall the fiber integration. Let $\phi : M \to N$ be a differentiable map between two manifolds.

Let $\sigma \in \Gamma\Big(\overset{p}{\bigwedge} T^*(M), M \Big)$ be a differential form with the degree $p \geq \dim M - \dim N := d$. For $g \in \Gamma\Big(\overset{q}{\bigwedge} T^*(N), N \Big)$ with compact support satisfying $q = m - p = \dim M - p \geq 0$ (we denote the space of sections with compact support by $\Gamma_0(* , *)$). We assume

$$\int_M |\sigma \wedge \phi^*(g)| < +\infty$$

for any $g \in \Gamma_0\Big(\overset{q}{\bigwedge} T^*(N) \Big)$ and define a linear functional

$$g \longmapsto \int_M \sigma \wedge \phi^*(g),$$

which is understood as a distribution on the space $\Gamma_0\Big(\overset{q}{\bigwedge} T^*(N) \Big)$. *We denote this distribution by $\phi_*(\sigma)$ and express as*

$$\phi_*(\sigma)(g) := \int_N \phi_*(\sigma) \wedge g = \int_M \sigma \wedge \phi^*(g). \tag{5.5}$$

If ϕ is a submersion, then $\phi_*(\sigma)$ is a smooth differential form of degree $p - d$.

In the last subsection we introduced inner products on the spaces $\Gamma_F\left(\mathbb{L} \otimes \sqrt{\overset{max}{\bigwedge} F^0}, M \right)$ and $\Gamma_G\left(\mathbb{L} \otimes \overset{max}{\bigwedge} T^{*'}(M)^{\mathbb{C}}, M \right) = \Gamma_G\left(\mathbb{L} \otimes K^G, M \right)$ for a real polarization F satisfying the condition (RP1) and a positive complex polarization G on an integral symplectic manifold M. Our main purpose is to construct a transformation

$$\mathfrak{B} : \Gamma_G\left(\mathbb{L} \otimes K^G, M \right) \longrightarrow \Gamma_F\Big(\mathbb{L} \otimes \sqrt{\overset{max}{\bigwedge} F^0}, M \Big) \tag{5.6}$$

or it may be understood as a sesqui-linear form on

$$\Gamma_F\Big(\mathbb{L} \otimes \sqrt{\overset{max}{\bigwedge} F^0}, M \Big) \times \Gamma_G\Big(\mathbb{L} \otimes K^G, M \Big) \overset{Id \times \mathfrak{B}}{\longrightarrow} \tag{5.7}$$

$$\Gamma_F\Big(\mathbb{L}\otimes\sqrt{\bigwedge^{max}F^0}, M\Big)\times\Gamma_F\Big(\mathbb{L}\otimes\sqrt{\bigwedge^{max}F^0}, M\Big)\longrightarrow\mathbb{C}.$$

For the sections $(\alpha\otimes\mu\,,\,\beta\otimes\nu)\in\Gamma_F\Big(\mathbb{L}\otimes\sqrt{\bigwedge^{max}F^0}, M\Big)\times\Gamma_G\Big(\mathbb{L}\otimes K^G,\, M\Big)$ their product

$$<\alpha,\beta>^{\mathbb{L}}\cdot|\mu\otimes\nu|$$

($|\ast|$ means a section of $\big|K^G\otimes\sqrt{\bigwedge^{max}F^0}\big|$) is understood as a partial 3/4-density on M and so we need some modification to integrate it, since there are no manifold of the dimension $3/4\times\dim M$.

Since we identify the half density space $\Gamma\Big(\sqrt{\bigwedge^{max}T^*(N)}, N\Big)$ with a L_2-space by fixing a Riemann volume form dv_N, we define a sequi-linear form

$$\Gamma_F\Big(\mathbb{L}\otimes\sqrt{\bigwedge^{max}F^0}, M\Big)\times\Gamma_G\Big(\mathbb{L}\otimes K^G, M\Big)\longrightarrow\mathbb{C}$$

by

$$\begin{aligned}\Gamma_F\Big(\mathbb{L}\otimes\sqrt{\bigwedge^{max}F^0}, M\Big)\times\Gamma_G\Big(\mathbb{L}\otimes K^G, M\Big)&\ni(\alpha\otimes\mu\,,\,\beta\otimes\nu)\\ \longmapsto\int_M<\alpha,\beta>^{\mathbb{L}}&\cdot\Phi^*(f_\mu dv_N)\wedge\overline{\nu},\end{aligned}\tag{5.8}$$

where we can put $\mu=\Phi^*(f_\mu)\sqrt{\Phi^*(dv_N)}$ with a function $f_\mu\in C^\infty(N)$, that is we multiply the partial 1/4-density $\sqrt{\Phi^*(dv_N)}$ to the partial 1/4-density $\nu=\Phi^*(f_\nu)\sqrt{\Phi^*(dv_N)}$, then $\sqrt{\Phi^*(dv_N)}\otimes\sqrt{\Phi^*(dv_N)}\otimes\mu$ is a (complex valued) highest degree differential form (or can be thought as a density) on M and we can define a sesqui-linear form.

Once we have a sesqui-linear form

$$P:\Gamma_F\Big(\mathbb{L}\otimes\sqrt{\bigwedge^{max}F^0}, M\Big)\times\Gamma_G\Big(\mathbb{L}\otimes K^G, M\Big)\longrightarrow\mathbb{C}$$

it is rewritten as

$$P(\alpha\otimes\mu,\beta\otimes\nu)=\sum\mathrm{I}_N(\Phi_*<\alpha,\alpha_i>^{\mathbb{L}}\cdot\mu\otimes\mu_i),$$

where we put $\mathfrak{B}(\beta\otimes\nu)=\sum\alpha_i\otimes\mu_i$.

In our space $T_0^*(P^2\mathbb{O})\overset{\tau_{\mathbb{O}}}{\cong}\mathbb{X}_{\mathbb{O}}$ *we have two polarizations* $\mathcal{F}$ (real) *and* $\mathcal{G}$ (*Kähler*) *and we identify the spaces* $C_{\mathcal{G}}(\mathbb{X}_{\mathbb{O}})$ *and* $\Gamma_{\mathcal{G}}(\mathbb{L}\otimes K^{\mathcal{G}},\mathbb{X}_{\mathbb{O}})$ *by* (5.4). *The inner product on the space* $C_{\mathcal{G}}(\mathbb{X}_{\mathbb{O}})$ *induced by the map* γ *will be explicitly described in* (6.9) *at the end of* § *6.2 in terms of the Liouville volume form. There we will introduce a parameter family of inner products on the space* $\Gamma_{\mathcal{G}}(\mathbb{L}\otimes K^{\mathcal{G}},\mathbb{X}_{\mathbb{O}})$.

We recall the sections $\mathbf{s}_0$ *and* $\mathbf{t}_0$ *and describe our Bargmann type transformation including the quantity* $<\mathbf{s}_0,\mathbf{t}_0>^{\mathbb{L}}$.

Let $\theta^{P^2\mathbb{O}}$ *be the canonical one-form on the cotangent bundle* $T^*(P^2\mathbb{O})$, *then* $d\theta^{P^2\mathbb{O}}=\omega^{P^2\mathbb{O}}$ *and for any* $X\in\mathcal{F}$, $<\theta^{P^2\mathbb{O}},X>=0$. *So let* $\mathbf{s}_0$ *be a nowhere vanishing polarized* (*with respect to the real polarization* $\mathcal{F}$) *global section of* $\mathbb{L}$ *defining a trivialization* $\mathbb{X}_{\mathbb{O}}\times\mathbb{C}\cong\mathbb{L}$ *by the correspondence*

$$\mathbb{X}_{\mathbb{O}}\times\mathbb{C}\ni(A,z)\longleftrightarrow z\cdot\mathbf{s}_0(A)\in\mathbb{L},\tag{5.9}$$

with $<\mathbf{s}_0\,,\,\mathbf{s}_0>^{\mathbb{L}}\equiv 1$.

Also by the relation

$${\tau_{\mathbb{O}}}^*\left(\sqrt{-2}\;\overline{\partial}\partial\,||A||^{1/2}\right)=\omega^{P^2\mathbb{O}},$$

given in Theorem (3.3), *we take a* (*complex*)*one-form*

$$\theta_{\mathcal{G}}=\sqrt{-2}\,\partial\,||A||^{1/2},$$

then $d{\tau_{\mathbb{O}}}^*(\theta_{\mathcal{G}})=\omega^{P^2\mathbb{O}}$ *and* $\theta_{\mathcal{G}}(X)=0$ *for* $X\in\mathcal{G}$, *since* X *is a* $(0,1)$ *tangent vector.*

Then we can trivialize the line bundle $\mathbb{L}$ *by making use of a nowhere vanishing global section* $\mathbf{t}_0$ *in a similar way to* (5.9).

Using the identifications (5.4) *and the correspondence*

$$C^\infty(P^2\mathbb{O}) \ni g \longmapsto \{\mathbf{q}\circ {\tau_{\mathbb{O}}}^{-1}\}^*(g)\cdot \mathbf{s}_0 \otimes \sqrt{\{\mathbf{q}\circ {\tau_{\mathbb{O}}}^{-1}\}^*(dv_{P^2\mathbb{O}})}$$
$$\longmapsto \{\mathbf{q}\circ {\tau_{\mathbb{O}}}^{-1}\}^*(g)\cdot \mathbf{s}_0 \otimes \{\mathbf{q}\circ {\tau_{\mathbb{O}}}^{-1}\}^*(dv_{P^2\mathbb{O}})$$

the integral (5.8) *is rewritten as*

$$\int_{\mathbb{X}_{\mathbb{O}}} \{\mathbf{q}\circ {\tau_{\mathbb{O}}}^{-1}\}^*(g)\cdot \overline{h}\cdot <\mathbf{s}_0,\mathbf{t}_0>^{\mathbb{L}} \cdot \{\mathbf{q}\circ {\tau_{\mathbb{O}}}^{-1}\}^*(dv_{P^2\mathbb{O}})\wedge \overline{\Omega_{\mathbb{O}}}, \tag{5.10}$$

and it is also expressed in terms of the fiber integration as follows:

$$\int_{\mathbb{X}_{\mathbb{O}}} \{\mathbf{q}\circ {\tau_{\mathbb{O}}}^{-1}\}^*(g)\cdot \overline{h}\cdot <\mathbf{s}_0,\mathbf{t}_0>^{\mathbb{L}} \cdot \{\mathbf{q}\circ {\tau_{\mathbb{O}}}^{-1}\}^*(dv_{P^2\mathbb{O}})\wedge \overline{\Omega_{\mathbb{O}}}$$
$$= \int_{P^2\mathbb{O}} g\cdot \{\mathbf{q}\circ {\tau_{\mathbb{O}}}^{-1}\}_*(<\mathbf{s}_0,\mathbf{t}_0>^{\mathbb{L}} \cdot \overline{h}\cdot \overline{\Omega_{\mathbb{O}}})\, dv_{P^2\mathbb{O}}. \tag{5.11}$$

Then the Bargmann type transformation

$$\mathfrak{B} : C_{\mathcal{G}}(\mathbb{X}_{\mathbb{O}}) \to C^\infty(P^2\mathbb{O}),\ C_{\mathcal{G}} \ni h \longmapsto \mathfrak{B}(h),$$

is defined as

$$\mathfrak{B}(h) = \{\mathbf{q}\circ {\tau_{\mathbb{O}}}^{-1}\}_*(h\cdot <\mathbf{t}_0,\mathbf{s}_0>^{\mathbb{L}} \Omega_{\mathbb{O}}). \tag{5.12}$$

Hence we can express the integral (5.11) *as*

$$\int_{P^2\mathbb{O}} g\cdot \overline{\mathfrak{B}(h)}\, dv_{P^2\mathbb{O}}.$$

Remark 6. *The section* $\mathbf{s}_0$ *is free of* $U(1)$*-multiple and* $\mathbf{t}_0$ *is of free from a constant* $\in \mathbb{C}^*$.

6. Bargmann type transformation

For expressing the Bargmann type transformation explicitly and to determine its L_2 continuity, we need to know the function $<\mathbf{s}_0,\mathbf{t}_0>^{\mathbb{L}}= g_0$, and relations of $\Omega_{\mathbb{O}}\wedge\overline{\Omega_{\mathbb{O}}}$ and $\{\mathbf{q}\circ {\tau_{\mathbb{O}}}^{-1}\}^*(dv_{P^2\mathbb{O}})\wedge\overline{\Omega_{\mathbb{O}}}$ with the Liouville volume form $dV_{T^*(P^2\mathbb{O})}$ explicitly. In this section we determine them.

6.1. **Holomorphic trivialization and unitary trivialization.** The relation of the sections $\mathbf{s}_0$ and $\mathbf{t}_0$ is given by a function $g_0 =<\mathbf{s}_0,\mathbf{t}_0>^{\mathbb{L}}$, that is

$$\mathbf{t}_0 = g_0\cdot \mathbf{s}_0. \tag{6.1}$$

The function g_0 satisfies an equation

$$\nabla_X(\mathbf{t}_0) = 2\pi\sqrt{-1} < \sqrt{-2}\,\partial\,||A||^{1/2}, X > g_0\cdot \mathbf{s}_0$$
$$= \nabla_X(g_0\mathbf{s}_0) = X(g_0)\mathbf{s}_0 + 2\pi\sqrt{-1}g_0\cdot <\theta^{P^2\mathbb{O}}, X> \mathbf{s}_0,$$

and we have an equation for the function g_0:

$$2\pi\sqrt{-1}\cdot\left({\tau_{\mathbb{O}}}^*\left(\sqrt{-1}\sqrt{2}\,\partial||A||^{1/2}\right) - \theta^{P^2\mathbb{O}}\right) g_0 = dg_0. \tag{6.2}$$

Put $g_0 = e^{2\pi\sqrt{-1}\lambda}$, then the equation (6.2) reduces to the equation

$$d\lambda = {\tau_{\mathbb{O}}}^*\left(\sqrt{2}\sqrt{-1}\partial\,||A||^{1/2}\right) - \theta^{P^2\mathbb{O}}. \tag{6.3}$$

To get a solution λ we need to consider the real and imaginary parts in the formula

$$\sqrt{2}\sqrt{-1}\partial\,||A||^{1/2}$$

separately. So, put

$${\tau_{\mathbb{O}}}^*\big(\sqrt{2}\sqrt{-1}\partial\,||A||^{1/2}\big) := a + \sqrt{-1}b$$

with real and imaginary parts of the one-form ${\tau_{\mathbb{O}}}^*\big(\sqrt{2}\sqrt{-1}\partial\,||A||^{1/2}\big)$ on $\mathcal{J}(3)\times\mathcal{J}(3)$. Then

$$d\,(d\lambda) = d\big({\tau_{\mathbb{O}}}^*\big(\sqrt{2}\sqrt{-1}\partial\,||A||^{1/2}\big) - \theta^{P^2\mathbb{O}}\big) = 0$$

implies that there are real valued functions λ_{Re} and λ_{Im} such that

$$a-\theta^{P^2\mathbb{O}}=d\lambda_{Re},\ \text{and}$$
$$d\lambda_{Im}=b.$$

The problem to solve the equation (6.3) reduces to find explicitly the functions λ_{Re} and λ_{Im}.

Let $(X,Y)\in T^*_0(P^2\mathbb{O})\subset\mathcal{J}(3)\times\mathcal{J}(3)$. Here again we remark that we are identifying the cotangent space $T^*_X(P^2\mathbb{O})$ and the tangent space $T_X(P^2\mathbb{O})$ by the Riemannian metric defined by $(Y_1,Y_2)_X^{P^2\mathbb{O}}:=\operatorname{tr}(Y_1\circ Y_2)$ for $Y_i\in T_X(P^2\mathbb{O})\cong\mathcal{J}(3)$, that is for $Y_i\in T_X(P^2\mathbb{O})\subset\mathcal{J}(3),\ i=1,2,$

$$Y_1=\begin{pmatrix}\epsilon_1 & u_3 & \theta(u_2)\\ \theta(u_3) & \epsilon_2 & u_1\\ u_2 & \theta(u_1) & \epsilon_3\end{pmatrix},\quad Y_2=\begin{pmatrix}\eta_1 & v_3 & \theta(v_2)\\ \theta(v_3) & \eta_2 & v_1\\ v_2 & \theta(v_1) & \eta_3\end{pmatrix},\quad \epsilon_i,\eta_i\in\mathbb{R},\ u_i,v_i\in\mathbb{O}\cong\mathbb{R}^8,$$

$$(Y_1,Y_2)_X^{P^2\mathbb{O}}:=\operatorname{tr}(Y_1\circ Y_2)=\sum\epsilon_i\eta_i+2\sum(u_i,v_i)^{\mathbb{R}^8}. \tag{6.4}$$

Based on this expression, by the notation (Y,dX) for

$$X=\begin{pmatrix}\xi_1 & x_3 & \theta(x_2)\\ \theta(x_3) & \xi_2 & x_1\\ x_2 & \theta(x_1) & \xi_3\end{pmatrix}\in\mathcal{J}(3),\ Y=\begin{pmatrix}\epsilon_1 & u_3 & \theta(u_2)\\ \theta(u_3) & \epsilon_2 & u_1\\ u_2 & \theta(u_1) & \epsilon_3\end{pmatrix}\in\mathcal{J}(3),$$

we mean the canonical one-form

$$(Y,dX):=\sum\epsilon_i d\xi_i+2\sum\{u_1\}_i d\{x_1\}_i+\{u_2\}_i d\{x_2\}_i+\{u_3\}_i d\{x_3\}_i$$

on $T^*(\mathcal{J}(3))\cong\mathcal{J}(3)\times(\mathcal{J}(3))^*\cong\mathcal{J}(3)\times(\mathcal{J}(3))$, or its restriction to $T^*(P^2\mathbb{O})$, that is, in the inner product expression (6.4) we understand as η_i and $\{v_k\}_i$ $(k=1,2,3,i=0,\cdots,7)$ are replaced by the differentials $d\xi_i$ and $d\{x_k\}_i$ of the corresponding components in $X\in\mathcal{J}(3)$, respectively.

Also for $A\in\mathcal{J}(3)^{\mathbb{C}}$ and an one form B on $\mathcal{J}(3)^{\mathbb{C}}$ we express the complex one form (A,dB) in the same way.

Let $(X,Y)\in T(P^2\mathbb{O})\cong T^*(P^2\mathbb{O})$ and put $A=\tau_{\mathbb{O}}(X,Y)=1\otimes\big(||Y||^2X-Y^2\big)+\sqrt{-1}\otimes\frac{||Y||Y}{\sqrt{2}}$, then

Proposition 6.1. (*see* [Fu2])

$$\frac{1}{2}||Y||^4=||a||^2=||b||^2,\quad ||A||^2=||a||^2+||b||^2=||Y||^4,\ \textit{and}\quad (da,a)=||Y||^2(Y,dY)=(db,b).$$

In the expression

$$\begin{aligned}\tau^*(dA,\overline{A})&=(\tau^*(dA),\tau^*(\overline{A})=(a-\sqrt{-1}b,da+\sqrt{-1}db)=d||A||^2\\ &=(a,da)+(b,db)+\sqrt{-1}\big((a,db)-(b,da)\big),\quad\text{and}\\ (a,db)-(b,da)&=2(db,a)=\frac{2}{\sqrt{2}}\cdot\big\{||Y||^3(dY,X)-||Y||(dY,Y\circ Y)\big\}\end{aligned}$$

and it is proved in [Fu2] (page 179) that

$$(dY,Y\circ Y)=0.$$

Hence

$$\begin{aligned}\tau^*_{\mathbb{O}}(\sqrt{2}\sqrt{-1}\partial\,||A||^{1/2})-\theta^{P^2\mathbb{O}}&=\sqrt{2}\sqrt{-1}\frac{1}{4||Y||^3}\{\sqrt{-2}\cdot(dY,X)-2||Y||^2(Y,dY)-\theta^{P^2\mathbb{O}}\\ &=-(dY,X)-(Y,dX)+\frac{\sqrt{-1}}{\sqrt{2}||Y||}(Y,dY)=\frac{\sqrt{-1}}{\sqrt{2}}d||Y||,\end{aligned}$$

since $d(X,Y)=(dX,Y)+(Y,dX)=d\operatorname{tr}(X\circ Y)=0$ for $(X,Y)\in T^*(P^2\mathbb{O})$. Hence finally we may choose the solutions λ_{Re} and λ_{Im} with

$$\lambda_{Re}\equiv 0\ \text{and}\ \lambda_{Im}=\frac{1}{\sqrt{2}}||Y||.$$

Hence

Proposition 6.2.

$$g_0=e^{-\sqrt{2}\pi\,||Y||},\ \textit{or it is expressed on}\ \mathbb{X}_{\mathbb{O}}\ \textit{as}\ g_0=e^{-\sqrt{2}\pi\,||A||^{1/2}}.$$

Now we have

$$\mathfrak{B} : C_{\mathcal{G}}(\mathbb{X}_{\mathbb{O}}) \to C^{\infty}(P^2\mathbb{O}),$$

$$\mathfrak{B}(h) = \{\mathbf{q}\circ(\tau_{\mathbb{O}})^{-1}\}_*(h\cdot < \mathbf{t}_0, \mathbf{s}_0 >^{\mathbb{L}} \Omega_{\mathbb{O}}) = \{\mathbf{q}\circ(\tau_{\mathbb{O}})^{-1}\}_*(h\cdot e^{-\sqrt{2}\pi||A||^{1/2}}\Omega_{\mathbb{O}}). \tag{6.5}$$

Remark 7. *The solution λ_{Re} can be an arbitrary real constant. However the absolute value $|g_0|$ does not depend on the chosen constant λ_{Re}.*

6.2. **Fock-like space.** We show

Proposition 6.3. *The nowhere vanishing global holomorphic section $\Omega_{\mathbb{O}}$ of the canonical line bundle $K^{\mathcal{G}}$ is F_4-invariant.*

Proof. Let $\alpha \in F_4$. The action of α on $\mathbb{X}_{\mathbb{O}}$ is naturally defined from the action on $P^2\mathbb{O}$ and the action is holomorphic. We denote it with the same notation $\alpha : \mathbb{X}_{\mathbb{O}} \to \mathbb{X}_{\mathbb{O}}$.

We can put $\alpha^*(\Omega_{\mathbb{O}}) = K_\alpha \cdot \Omega_{\mathbb{O}}$ with a nowhere vanishing holomorphic function $K_\alpha = K_\alpha(A)$.

Then

$$\alpha^*(\Omega_{\mathbb{O}})\bigwedge\overline{\alpha^*(\Omega_{\mathbb{O}})} = \alpha^*\big(\Omega_{\mathbb{O}}\bigwedge\overline{\Omega_{\mathbb{O}}}\big) = |K_\alpha|^2\cdot\Omega_{\mathbb{O}}\bigwedge\overline{\Omega_{\mathbb{O}}}.$$

We can express

$$\Omega_{\mathbb{O}}\bigwedge\overline{\Omega_{\mathbb{O}}} = D\cdot\frac{1}{16!}\{\tau_{\mathbb{O}}{}^{-1}\}^*\Big((\omega^{P^2\mathbb{O}})^{16}\Big)$$

by the Liouville volume form $\dfrac{1}{16!}(\omega^{P^2\mathbb{O}})^{16}$ and a function $D = D(A)$ on $\mathbb{X}_{\mathbb{O}}$. Hence

$$\alpha^*\big(\Omega_{\mathbb{O}}\bigwedge\overline{\Omega_{\mathbb{O}}}\big) = \alpha^*(D)\cdot\frac{1}{16!}\{\tau_{\mathbb{O}}{}^{-1}\}^*\Big((\omega^{P^2\mathbb{O}})^{16}\Big),$$

since the action by α on $\mathbb{X}_{\mathbb{O}}$ is symplectic. Hence

$$\alpha^*(D) = |K_\alpha|^2\cdot D.$$

By comparing the behaviours of $\Omega_{\mathbb{O}}$ and the Liouville volume form $dV_{P^2\mathbb{O}}$ under the dilation action by positive numbers:

$$T_t : \mathbb{X}_{\mathbb{O}} \to \mathbb{X}_{\mathbb{O}},\ A \to t\cdot A,$$

we can see on the coordinate neighborhood O_{z_1}

$$\begin{aligned}
T_t^*\big(\Omega_{\mathbb{O}}\bigwedge\overline{\Omega_{\mathbb{O}}}\big) &= \frac{1}{(tz_1)^5}d(tz_1)\wedge\cdots\wedge d(t\xi_2)\bigwedge\frac{1}{(t\overline{z_1})^5}d(t\overline{z_1})\wedge\cdots\wedge d(t\overline{\xi_2})\\
&= t^{22}\frac{1}{z_1{}^5}dz_1\wedge\cdots\wedge d\xi_2\bigwedge\frac{1}{\overline{z_1}^5}d\overline{z_1}\wedge\cdots\wedge d\overline{\xi_2}\\
&= t^{22}\cdot D(A)\cdot\frac{1}{16!}\{\tau_{\mathbb{O}}{}^{-1}\}^*\Big((\omega^{P^2\mathbb{O}})^{16}\Big) = D(t\cdot A)\cdot t^8\frac{1}{16!}\{\tau_{\mathbb{O}}{}^{-1}\}^*\Big((\omega^{P^2\mathbb{O}})^{16}\Big).
\end{aligned}$$

Hence

$$D(t\cdot A) = t^{14}\cdot D(A).$$

Note that the action T_t on $T_0^*(P^2\mathbb{O})$ defined via the map $\tau_{\mathbb{O}}$ is

$$\tau_{\mathbb{O}}{}^{-1}\circ T_t\circ\tau_{\mathbb{O}} : T_0^*(P^2\mathbb{O}) \ni (X,Y) \longmapsto (X,\sqrt{t}Y) \in T_0^*(P^2\mathbb{O}). \tag{6.6}$$

Then since $||\alpha(A)|| = ||A||$

$$\begin{aligned}
\alpha^*(D)(A) = D\left(\alpha(A)\right) &= D\left(||\alpha(A)||\cdot\frac{\alpha(A)}{||\alpha(A)||}\right)\\
&= ||A||^{14}\cdot D\left(\frac{\alpha(A)}{||\alpha(A)||}\right) = ||A||^{14}\cdot|K_\alpha(A)|^2 D\left(\frac{A}{||A||}\right),
\end{aligned}$$

hence

$$D\left(\frac{\alpha(A)}{||\alpha(A)||}\right) = |K_\alpha(A)|^2\, D\left(\frac{A}{||A||}\right). \tag{6.7}$$

This equality implies that the function K_α is bounded on $\mathbb{X}_{\mathbb{O}}$. Especially, if we consider it on the coordinate open subset $O_{z_1} \cong \mathbb{C}^*\times\mathbb{C}^{15}$ ($z_1 \neq 0$), then it can be extended to a holomorphic function on $\mathbb{C}\times\mathbb{C}^{15} \supset O_{z_1}$ and is bounded there. Hence the function K_α is a constant function on the whole space $\mathbb{X}_{\mathbb{O}}$.

Then by the property

$$K_{\alpha\cdot\beta} = K_\alpha \cdot K_\beta, \alpha, \beta \in F_4,$$

$F_4 \ni \alpha \longmapsto K_\alpha$ is a one-dimensional representation of the compact simply connected group F_4, so that we have not only $|K_\alpha| \equiv 1$ for any $\alpha \in F_4$, but also it must hold always $K_\alpha \equiv 1$. This implies $\Omega_{\mathbb{O}}$ is F_4-invariant. □

Corollary 6.4. *Since the action of F_4 on $S(\mathbb{X}_{\mathbb{O}}) = \{ A \in \mathbb{X} \mid ||A|| = 1 \}$ is transitive, the function is of the form $D(A) = C_1 \times ||A||^{14}$ with the constant $C_1 = 2^{26}$. Especially we have*

$$\tau_{\mathbb{O}}{}^* \left(\Omega_{\mathbb{O}} \wedge \overline{\Omega}_{\mathbb{O}}\right)(X,Y) = 2^{26}||Y||^{28}\frac{1}{16!}\left(\omega^{P^2\mathbb{O}}\right)^{16}. \tag{6.8}$$

Proof. It is enough to determine the constant C_1.

Following the expression (4.6) of the matrix $A \in \mathcal{J}(3)^{\mathbb{C}}$ we denote

$$A = \begin{pmatrix} \xi_1 & c' + c''\mathbf{e}_4 & \theta(b' + b''\mathbf{e}_4) \\ \theta(c' + c''\mathbf{e}_4) & \xi_2 & a' + a''\mathbf{e}_4 \\ b' + b''\mathbf{e}_4 & \theta(a' + a''\mathbf{e}_4) & \xi_3 \end{pmatrix} \in \mathcal{J}(3)^{\mathbb{C}}$$

by

$$(\xi_1, \xi_2, \xi_3, z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4, y_1, y_2, y_3, y_4, x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4),$$

where

$$\rho_{\mathbb{H}}(c') + \rho_{\mathbb{H}}(c'')\mathbf{e}_4 = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}\mathbf{e}_4,$$
$$\rho_{\mathbb{H}}(b') + \rho_{\mathbb{H}}(b'')\mathbf{e}_4 = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} + \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}\mathbf{e}_4,$$
$$\rho_{\mathbb{H}}(a') + \rho_{\mathbb{H}}(a'')\mathbf{e}_4 = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} + \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}\mathbf{e}_4$$

The correspondence between $c = c' + c''\mathbf{e}_4 = \sum\{c\}_i\mathbf{e}_i$ (and also $b = b' + b''\mathbf{e}_4 = \sum\{b\}_i\mathbf{e}_i, a = a' + a''\mathbf{e}_4 = \sum\{a\}_i\mathbf{e}_i$), and the components $\{z_i, w_i\}$ is given in (2.2) and (2.3).

By a simple calculation we have

$$\begin{aligned} ||A||^2 &= \sum |\xi_i|^2 + 2\sum |a'|^2 + |a''|^2 + |b'|^2 + |b''|^2 + |c'|^2 + |c''|^2 \\ &= \sum_{i=1}^{3} |\xi_i|^2 + \sum_{i=1}^{4} |z_i|^2 + |w_i|^2 + |y_i|^2 + |v_i|^2 + |x_i|^2 + |u_i|^2. \end{aligned}$$

Then we rewrite for $A \in O_{z_1}$

$$A \longleftrightarrow (\xi_1, \xi_2, \xi_3, z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4, y_1, y_2, y_3, y_4, x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4))$$

as

$$\begin{aligned} &(x_3, x_4, u_2, u_4, y_2, y_4, v_1, v_3, z_1, z_2, z_3, w_1, w_2, w_3, w_4, \xi_2 \ ; \ x_1, x_2, u_1, u_3, y_1, y_3, v_2, v_4, z_4, \xi_1, \xi_3) \\ &= (s_1, \cdots\cdots, s_{16} \ ; \ s_{17}, \cdots, s_{27}), \end{aligned}$$

that is, the first 16 coordinates give local coordinates on O_{z_1} and the remaining coordinates $(s_{17}, \cdots, s_{27})$ are rational functions of the coordinates $(s_1, \cdots, s_{16})$, that is $s_j = s_j(s_1, \cdots, s_{16})$, $j \geq 17$ (especially $s_9 = z_1$ and the explicit form of each s_j for $j > 16$ is given in (4.16)).

In particular, we see from the explicit form of these functions at the point $A = A(z_1) = A(0, \cdots, z_1, \cdots, 0, ; 0, \cdots, 0)$

$$s_j(A(z_1)) = s_j(0, \cdots, 0, z_1, 0, \cdots, 0) = s_j(0, \cdots, 0, s_9, 0, \cdots, 0) = 0, \ 17 \leq j \leq 27,$$

and for $i \leq 16$, $j \geq 17$

$$\frac{\partial s_j}{\partial s_i}(A(z_1)) = 0.$$

On O_{z_1} it holds

$$\Omega_{\mathbb{O}} \bigwedge \overline{\Omega_{\mathbb{O}}} = \frac{1}{|z_1|^{10}}\, dx_3 \wedge dx_4 \wedge \cdots \wedge dw_4 \wedge d\xi_2 \bigwedge d\overline{x_3} \wedge d\overline{x_4} \wedge \cdots \wedge d\overline{w_4} \wedge d\overline{\xi_2}$$

$$= \frac{1}{|s_9|^{10}} ds_1 \wedge \cdots \wedge ds_{16} \bigwedge d\overline{s_1} \wedge \cdots \wedge d\overline{s_{16}} = D(A) \cdot \frac{1}{16!} \{\tau_{\mathbb{O}}{}^{-1}\}^* \Big((\omega^{P^2\mathbb{O}})^{16} \Big).$$

We calculate the right hand side at the point $A(z_1) = (0, \cdots, 0, z_1, 0, \ldots; 0, \cdots, 0) \in O_{z_1}$ using the expression of $\omega^{P^2\mathbb{O}}$ (see Theorem 3.2):

$$\omega^{P^2\mathbb{O}} = \{\tau_{\mathbb{O}}\}^* \Big(\sqrt{-2}\overline{\partial}\partial ||A||^{1/2} \Big).$$

Then

$$\overline{\partial}\partial ||A||^{1/2} = \frac{1}{4} \cdot \overline{\partial} \left(\left(\sum_{i=1}^{27} |s_i|^2 \right)^{-3/4} \cdot \sum \overline{s_i} ds_i \right)$$

$$= \frac{-3}{16} \left(\sum_{i=1}^{27} |s_i|^2 \right)^{-7/4} \sum_{j=1}^{27} s_j d\overline{s_j} \bigwedge \sum_{i=1}^{27} \overline{s_i} ds_i + \frac{1}{4} \left(\sum_{i=1}^{27} |s_i|^2 \right)^{-3/4} \sum_{i=1}^{27} d\overline{s_i} \wedge ds_i.$$

Here we evaluate it at the point $A(z_1)$, then it is given by

$$\frac{-3}{16} |s_9|^{-7/2} \cdot |s_9|^2 \, d\overline{s_9} \wedge ds_9 + \frac{1}{4} |s_9|^{-3/2} \sum_{i=1}^{16} d\overline{s_i} \wedge ds_i$$

$$= \frac{1}{16} |s_9|^{-3/2} d\overline{s_9} \wedge ds_9 + \frac{1}{4} |s_9|^{-3/2} \sum_{1 \le i \le 8 \text{ and } 10 \le i \le 16} d\overline{s_i} \wedge ds_1.$$

Hence

$$(\omega^{P^2\mathbb{O}})^{16} = \Big(\sqrt{-2}\, \overline{\partial}\partial ||A||^{1/2} \Big)^{16} = 16! \cdot \frac{1}{16} \cdot \frac{1}{4^{15}} (\sqrt{-2})^{16} \cdot \frac{1}{|s_9|^{3/2 \times 16}} d\overline{s_1} \wedge ds_1 \wedge \cdots \cdots \wedge d\overline{s_{16}} \wedge ds_{16}$$

$$= 16! \cdot \frac{1}{2^{26} \cdot |s_9|^{24}} d\overline{s_1} \wedge ds_1 \wedge \cdots\cdots \wedge d\overline{s_{16}} \wedge ds_{16}$$

at the point $A(z_1) = A(0, \cdots, 0, s_9, 0, \cdots, 0; 0, \cdots, 0)$ $(s_9 = z_1)$. Consequently we have

$$\Omega_{\mathbb{O}} \wedge \overline{\Omega_{\mathbb{O}}}\Big|_{A(z_1)} = \overline{\Omega_{\mathbb{O}}} \wedge \Omega_{\mathbb{O}}\Big|_{A(z_1)} = D\left(\frac{s_9}{|s_9|} \right) \cdot |s_9|^{14} \cdot \frac{1}{2^{26}} |s_9|^{-24} d\overline{s_1} \wedge ds_1 \wedge \cdots\cdots \wedge d\overline{s_{16}} \wedge ds_{16}$$

and the constant C_1 is

$$C_1 = 2^{26}, \; D(A) = 2^{26} ||A||^{14}, \; \Omega_{\mathbb{O}} \bigwedge \overline{\Omega_{\mathbb{O}}} = 2^{26} ||A||^{14} \frac{1}{16!} \{\tau_{\mathbb{O}}{}^{-1}\}^* \Big((\omega^{P^2\mathbb{O}})^{16} \Big).$$

□

Let $\mathbb{C}[\mathcal{J}(3)^{\mathbb{C}} = \sum \mathcal{P}_k[\mathcal{J}(3)^{\mathbb{C}}]$ be the algebra of polynomials (and of polynomial functions) on $\mathcal{J}(3)^{\mathbb{C}}$ with the 27 complex variables $(\xi_1, \xi_2, \xi_3, z_i, w_i, y_i, v_i, x_i, u_i)$ $(i = 1, \cdots, 4$ and $\mathcal{P}_k$ is a subspace of degree k homogeneous polynomials) and denote their restrictions to $\mathbb{X}_{\mathbb{O}}$ by $\mathbb{C}[\mathbb{X}_{\mathbb{O}}] = \sum \mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]$.

Recall the correspondence

$$\gamma : \mathbb{C}[\mathbb{X}_{\mathbb{O}}] = \sum \mathcal{P}_k[\mathbb{X}_{\mathbb{O}}] \ni p \longmapsto \gamma(p) = p \cdot \mathbf{t}_0 \otimes \Omega_{\mathbb{O}} \in \Gamma_{\mathcal{G}}(\mathbb{L} \otimes K^{\mathbb{G}}, \mathbb{X}_{\mathbb{O}}).$$

We define a parameter family of inner products $\{(*\,,\,*)_\varepsilon\}_{\varepsilon \in \mathbb{R}}$ on the space $\Gamma_{\mathcal{G}}(\mathbb{L} \otimes K^{\mathcal{G}}, \mathbb{X}_{\mathbb{O}})$ by the following way that

$$\Gamma_{\mathcal{G}}(\mathbb{L} \otimes K^{\mathcal{G}}, \mathbb{X}_{\mathbb{O}}) \times \Gamma_{\mathcal{G}}(\mathbb{L} \otimes K^{\mathcal{G}}, \mathbb{X}_{\mathbb{O}}) \ni (h \cdot \mathbf{t}_0 \otimes \Omega_{\mathbb{O}}\,,\, g \cdot \mathbf{t}_0 \otimes \Omega_{\mathbb{O}})$$

$$\longmapsto \int_{\mathbb{X}_{\mathbb{O}}} h \cdot \overline{g} < \mathbf{t}_0, \mathbf{t}_0 >^{\mathbb{L}} \cdot ||A||^{\varepsilon} \cdot \Omega_{\mathbb{O}} \bigwedge \overline{\Omega_{\mathbb{O}}}$$

$$= 2^{26} \int_{\mathbb{X}_{\mathbb{O}}} h \cdot \overline{g} \cdot e^{-2\sqrt{2}\pi ||A||^{1/2}} \cdot ||A||^{14+\varepsilon} \cdot \{\tau_{\mathbb{O}}{}^{-1}\}^* (dV_{P^2\mathbb{O}}) = (h\,,\, g)_\varepsilon, \tag{6.9}$$

then through the map γ we also consider a parameter family of inner products on the space $\mathbb{C}[\mathbb{X}_{\mathbb{O}}]$.

Remark 8. *According to the value of ε, the integral (6.9) for functions $f, g \in \mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]$ need not be finite. In fact, for $k > -11 - \varepsilon/2$ the integral (6.9) converges. We denote by $\mathfrak{F}_\varepsilon$ the completion of the space $\sum_{k > -11 - \varepsilon/2} \mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]$ with respect to the integral (6.9) and the the remaining finite dimensional space $\sum_{k \le -11 - \varepsilon/2} \mathcal{P}[\mathbb{X}_{\mathbb{O}}]$ with a suitable inner product.*

7. Pairing with the Riemann volume form

Let $dv_{P^2\mathbb{O}}$ be the Riemann volume form on $P^2\mathbb{O}$. The purpose in this section is to show

Proposition 7.1.

$$\{\mathbf{q}\circ\tau_{\mathbb{O}}{}^{-1}\}^*(dv_{P^2\mathbb{O}})(A)\bigwedge\overline{\Omega_{\mathbb{O}}}(A)$$

$$= C_{RC}(A)\cdot\{\tau_{\mathbb{O}}{}^{-1}\}^*\left(\frac{1}{16!}\left(\omega^{P^2\mathbb{O}}\right)^{16}\right)(A) = 2^{26}||A||^3\{\tau_{\mathbb{O}}{}^{-1}\}^*\left(\frac{1}{16!}\left(\omega^{P^2\mathbb{O}}\right)^{16}\right)(A), \tag{7.1}$$

$$A = \tau_{\mathbb{O}}(X,Y)\in\tau_{\mathbb{O}}(T_0^*(P^2\mathbb{O})) = \mathbb{X}_{\mathbb{O}}. \tag{7.2}$$

The homogeneity order is determined by comparing their orders in the both sides (see the relation (6.6)).

7.1. **A local coordinates.** For the determination of the constant $C_{RC}(A/||A||)$ we choose a local coordinates around the point $X_1 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\in P^2\mathbb{O}$.

The condition $X^2 = X = \begin{pmatrix}t_1&c&\theta(b)\\\theta(b)&t_2&a\\b&\theta(a)&t_3\end{pmatrix}$ for $X\in P^2\mathbb{O}\subset\mathcal{J}(3)$ is expressed as

$$\begin{cases}(t_3+t_2)a+\theta(bc)=a,\,(t_1+t_3)b+\theta(ca)=b,\,(t_2+t_1)c+\theta(ab)=c,\\ {t_1}^2+c\theta(c)+\theta(b)b=t_1,\,{t_2}^2+\theta(c)c+a\theta(a)=t_2,\,{t_3}^2+\theta(a)a+b\theta(b)=t_3 \text{ and}\\ \operatorname{tr}X\;=\;t_1+t_2+t_3=1.\end{cases} \tag{7.3}$$

where $a,b,c\in\mathbb{O}, t_i\in\mathbb{R}$. Using the last equation in (7.3), first 6 conditions are rewritten in the forms of

$$\begin{cases}t_1a=\theta(bc),\quad t_2b=\theta(ca),\quad t_3c=\theta(ab),\\ (t_1-1/2)^2+c\theta(c)+\theta(b)b=(t_1-1/2)^2+|c|^2+|b|^2=1/4,\\ (t_2-1/2)^2+\theta(c)c+a\theta(a)=(t_2-1/2)^2+|c|^2+|a|^2=1/4,\\ (t_3-1/2)^2+\theta(x)x+b\theta(b)=(t_3-1/2)^2+|a|^2+|b|^2=1/4.\end{cases} \tag{7.4}$$

Let

$$\mathcal{W}_1=\left\{(b,c)\in\mathbb{O}^2\;\middle|\;|c|^2+|b|^2\;<\;\frac{1}{8}\right\}. \tag{7.5}$$

Then we can solve the equations (7.4) in the following order:

First, we solve the fourth equation in (7.4) with respect to t_1 under the condition $|c|^2+|b|^2\;<\;\frac{1}{8}$ with the solution

$$t_1=\frac{1}{2}+\sqrt{\frac{1}{4}-|b|^2-|c|^2}>\frac{1}{2}.$$

Then the component a is given by (b,c) by the first equation in (7.4) as

$$a=\frac{\theta(bc)}{t_1}.$$

This solution a satisfies the inequality:

$$|a|=\frac{|bc|}{t_1}<2\cdot\frac{|b|^2+|c|^2}{2}<\frac{1}{8}.$$

With these we can solve the variable t_2 in the fifth equation in (7.4) with the solution

$$t_2=\frac{1}{2}-\sqrt{\frac{1}{4}-|c|^2-|a|^2},$$

where $|c|^2+|a|^2<\frac{1}{8}+\frac{1}{64}<\frac{1}{4}$ implies that $t_2<\frac{1}{2}$.

Now, with these solutions expressed in terms of the variables $(b,c)\in\mathcal{W}_1$ we define a map

$$\mathcal{M}:\mathcal{W}_1\ni(b,c)\longmapsto X=\begin{pmatrix}t_1&c&\theta(b)\\\theta(c)&t_2&a\\b&\theta(a)&1-t_1-t_2\end{pmatrix}\in P^2\mathbb{O}. \tag{7.6}$$

Then the matrix $\mathcal{M}(b,c)$ satisfies the condition (7.4), so that we can choose components (b,c) as a local coordinates around the point X_1. We denote by $\widetilde{\mathcal{W}_1}=\mathcal{M}(\mathcal{W}_1)$. The point X_1 corresponds to $(0,0)\in\mathcal{W}_1$.

Lemma 7.2. *In terms of the local coordinates*

$$(b,c)=\left(\sum_{i=0}^{7}\{b\}_i\mathbf{e}_i,\sum_{i=0}^{7}\{c\}_i\mathbf{e}_i\right)$$

introduced above around the point X_1, the Riemann volume form $dv_{P^2\mathbb{O}}$ at the point X_1 is

$$dv_{P^2\mathbb{O}}(0,0)=d\{b\}_0\wedge\cdots\wedge d\{b\}_7\wedge d\{c\}_0\wedge\cdots\wedge d\{c\}_7. \tag{7.7}$$

Proof. We can see this by

$$d\mathcal{M}_{(0,0)}\left(\frac{\partial}{\partial\{b\}_0}\right)=\left(\frac{\partial}{\partial\{b\}_0}\right)_{X_1}+\sum_{i=1}^{3}\frac{\partial t_i(0,0)}{\partial\{b\}_0}\left(\frac{\partial}{\partial t_i}\right)_{X_1}+\sum_{i=0}^{7}\frac{\partial\{a\}_i(0,0)}{\partial\{b\}_0}\left(\frac{\partial}{\partial\{a\}_i}\right)_{X_1}=\left(\frac{\partial}{\partial\{b\}_0}\right)_{X_1},$$

where we know

$$\frac{\partial t_1(0,0)}{\partial\{b\}_0}=\frac{-\{b\}_0}{\sqrt{1/4-|b|^2-|c|^2}}\Big|_{b=0,c=0}=0,\quad \frac{\partial t_2(0,0)}{\partial\{b\}_0}=\frac{-2b_0-2\sum_{i=0}^{7}\{a\}_i\frac{\partial\{a\}_i}{\partial\{b\}_0}}{2\sqrt{1/4-|b|^2-|a|^2}}\Big|_{b=0,c=0}=0,\ etc.,$$

since $a(0,0)=\sum\{a\}_i\mathbf{e}_i=0$. Other derivatives are also

$$\frac{\partial t_i}{\partial\{b\}_j}\Big|_{(0,0)}=0,\ \frac{\partial t_i}{\partial\{c\}_j}\Big|_{(0,0)}=0,\ \frac{\partial\{a\}_j\{a\}_k}{\partial\{b\}_i}\Big|_{(0,0)}=0,\ \frac{\partial\{a\}_j\{a\}_k}{\partial\{c\}_i}\Big|_{(0,0)}=0.$$

Hence

$$d\mathcal{M}_{(0,0)}\left(\frac{\partial}{\partial\{b\}_i}\right)=\left(\frac{\partial}{\partial\{b\}_i}\right)_{X_1},\quad d\mathcal{M}_{(0,0)}\left(\frac{\partial}{\partial\{c\}_i}\right)=\left(\frac{\partial}{\partial\{c\}_i}\right)_{X_1}.$$

Then the metric tensor g_{ij} with respect to the coordinates (b,c) at the point $(b,c)=(0,0)$ is $g_{ij}=\delta_{ij}$. ☐

7.2. **Explicit determination of the pairing with the Riemann volume form.** Let

$$A=\begin{pmatrix}\xi_1 & z & \theta(y)\\ \theta(z) & \xi_2 & x\\ y & \theta(x) & \xi_3\end{pmatrix}\in\mathbb{X}_{\mathbb{O}},\text{where } \xi_i\in\mathbb{C},\ z,y,x\in\mathbb{C}\otimes_{\mathbb{R}}\mathbb{O}.$$

Put ${\tau_{\mathbb{O}}}^{-1}(A)=(X(A),Y(A))$, then

$$X(A)=\frac{A+\overline{A}}{2||A||}+\frac{A\circ\overline{A}}{||A||^2}\ \text{(see (3.10))}$$

$$=\begin{pmatrix}\frac{\xi_1+\overline{\xi_1}}{2||A||}+\frac{|\xi_1|^2+|z|^2+|y|^2}{||A||^2} & \frac{z+\overline{z}}{2||A||}+\frac{-\xi_3\overline{z}-\overline{\xi_3}z+\theta(\overline{x}y+x\overline{y})}{2||A||^2} & \frac{\theta(y+\overline{y})}{2||A||}+\frac{-\theta(\xi_2\overline{y}+\overline{\xi_2}y)+z\overline{x}+\overline{z}x}{2||A||^2}\\ \frac{\theta(z+\overline{z})}{2||A||}+\frac{-\theta(\xi_3\overline{z}+\overline{\xi_3}z)+\overline{x}y+x\overline{y}}{2||A||^2} & \frac{\xi_2+\overline{\xi_2}}{2||A||}+\frac{|\xi_2|^2+|z|^2+|x|^2}{||A||^2} & \frac{x+\overline{x}}{2||A||}+\frac{-\xi_1\overline{x}-\overline{\xi_1}x+\theta(\overline{y}z+y\overline{z})}{2||A||^2}\\ \frac{y+\overline{y}}{2||A||}+\frac{-\xi_2\overline{y}-\overline{\xi_2}y+\theta(z\overline{x}+\overline{z}x)}{2||A||^2} & \frac{\theta(x+\overline{x})}{2||A||}+\frac{-\theta(\xi_1\overline{x}+\overline{\xi_1}x)+\overline{y}z+y\overline{z}}{2||A||^2} & \frac{\xi_3+\overline{\xi_3}}{2||A||}+\frac{|\xi_3|^2+|x|^2+|y|^2}{||A||^2}\end{pmatrix}.$$

From the above expression of ${\tau_{\mathbb{O}}}^{-1}(A)=(X(A),Y(A))$ we consider two components of the matrix $X(A)\in P^2\mathbb{O}$ for $A\in U_{z_1}$:

$$c=\frac{z+\overline{z}}{2||A||}+\frac{-\xi_3\overline{z}-\overline{\xi_3}z+\theta(\overline{x}y+x\overline{y})}{2||A||^2},\quad b=\frac{y+\overline{y}}{2||A||}+\frac{-(\xi_2\overline{y}+\overline{\xi_2}y)+\theta(z\overline{x}+\overline{z}x)}{2||A||^2}.$$

Take a point $A_1=\begin{pmatrix}1 & \sqrt{-1}\mathbf{e}_0 & 0\\ \sqrt{-1}\mathbf{e}_0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\in O_{z_1}$, then $\mathbf{q}\circ{\tau_{\mathbb{O}}}^{-1}(A_1)=X_1$. On the other hand the point $A_1\in\mathbb{X}_{\mathbb{O}}$ corresponds to the matrices

$$A_1\longleftrightarrow(\xi_1,\xi_2,\xi_3,Z,W,Y,V,X,U)=\left(1,-1,0,\begin{pmatrix}\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix}\right).$$

(see the matrix representation (4.3) of the octanions and and vector representation (4.6) of elements in $\mathcal{J}(3)^{\mathbb{C}}$).

So we consider points $A\in U_{z_1}$ around a point

$$P_{z_1}(A)=(\xi_2,z_1,z_2,z_3,w_1,w_2,w_3,w_4,y_2,y_4,v_1,v_3,x_3,x_4,u_2,u_4)$$

$$= (-1, \sqrt{-1}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

By the explicit expression (4.16) of the other dependent variables $(\xi_1, \xi_3, z_4, y_1, y_3, v_2, v_4, x_1, x_2, u_1, u_2)$ in the matrix representation $(\xi_1, \xi_2, \xi_3, Z, W, Y, V, X, U)$ of A_1 is $(0, 0, 0, 0, 0, 0, 0, 0, \sqrt{-1}, 1, 0)$.

For avoiding the confusion of the expression of octanion and its matrix expression by the mp ρ_O, recall the correspondence (2.2) and (2.3)).

Now we determine the differentials *modulo anti-holomorphic differentials*

$$\{\mathbf{q} \circ \tau_{\mathbb{O}}^{-1}\}^*(dc) = \sum_{i=0}^{7} \{\mathbf{q} \circ \tau_{\mathbb{O}}^{-1}\}^*(d\{c\}_i) \otimes \mathbf{e}_i = \sum_{i=0}^{7} d\left(\{\mathbf{q} \circ \tau_{\mathbb{O}}^{-1}\}^*(\{c\}_i)\right) \otimes \mathbf{e}_i, \text{ and}$$
$$\{\mathbf{q} \circ \tau_{\mathbb{O}}^{-1}\}^*(db) = \sum_{i=0}^{7} \{\mathbf{q} \circ \tau_{\mathbb{O}}^{-1}\}^*(d\{b\}_i) \otimes \mathbf{e}_i = \sum_{i=0}^{7} d\left(\{\mathbf{q} \circ \tau_{\mathbb{O}}^{-1}\}^*(\{b\}_i)\right) \otimes \mathbf{e}_i,$$

at the point A_1.

Each component of b and c is given by

$$\{c\}_i = \frac{\{z\}_i + \{\overline{z}\}_i}{2||A||} + \frac{-\xi_3\{\overline{z}\}_i - \overline{\xi_3}\{z\}_i + \{\theta(\overline{x}y + x\overline{y})\}_i}{2||A||^2}, \text{ and}$$
$$\{b\}_i = \frac{\{y\}_i + \{\overline{y}\}_i}{2||A||} + \frac{-\xi_2\{\overline{y}\}_i - \overline{\xi_2}\{y\}_i + \{\theta(\overline{z}x + z\overline{x})\}_i}{2||A||^2}.$$

The pull-back $\{\mathbf{q} \circ \tau_{\mathbb{O}}{}^{-1}\}^*(dv_{P^2\mathbb{O}})$ is expressed as

$$\{\mathbf{q} \circ \tau_{\mathbb{O}}{}^{-1}\}^*(dv_{P^2\mathbb{O}}) = \sum_{i=0}^{16} \Sigma_i, \quad \text{with} \quad \Sigma_i \in \Gamma\left(\bigwedge^{16-i} T^{*'}(\mathbb{X}_{\mathbb{O}})^{\mathbb{C}} \otimes \bigwedge^{i} T^{*''}(\mathbb{X}_{\mathbb{O}})^{\mathbb{C}}\right).$$

In particular,

$$\Sigma_i \wedge \overline{\Omega} = 0 \text{ for } i \geq 1, \text{ and } \overline{\Sigma}_j = \Sigma_{16-j}.$$

Hence for the determination of the constant $C_{RC}(Y/||Y||)$, it is enough to consider the terms consisting of holomorphic differentials

$$d\xi_2, dz_1, dz_2, dz_3, dw_1, dw_2, dw_3, dw_4, dy_2, dy_4, dv_1, dv_3, dx_3, dx_4, du_2, du_4$$

and may ignore the anti-holomorphic differentials $d\overline{x}_3, d\overline{x}_4$, *etc*, so that in the expression of equalities below we denote them as $* \equiv *$, which means both sides coincide *modulo anti-holomorphic differentials.*

Here we gather up relations of the holomorphic differentials of dependent variables by independent variables at the point A_1. See (4.16) for the explicit expression of each variable $\xi_1, \xi_3, \cdots\cdots, x_1, x_2, u_1, u_2$ in terms of independent variables $\xi_2, z_1, \cdots, x_3, x_4, u_2, u_4$.

All the equalities in the Lemmas blow hold at the point A_1.

Lemma 7.3.

$$||A_1|| = 2, \ d||A||^2_{|A_1} \equiv \{\overline{z_1}dz_1 + \overline{z}_4 dz_4 + \overline{\xi_1}d\xi_1 + \overline{\xi_2}d\xi_2\}_{|A_1} = -2d\xi_2,$$
$$z_4(A_1) = \sqrt{-1}, \ dz_{4|A_1} = -dz_1 - 2\sqrt{-1}d\xi_2, \ \xi_3(A_1) = 0, d\xi_{3|A_1} = 0, \ d\xi_{1|A_1} = -d\xi_2,$$
$$dx_{1|A_1} = \sqrt{-1}dy_4, \ dy_{1|A_1} = -\sqrt{-1}dx_4, dy_{3|A_1} = \sqrt{-1}dx_3, \ dx_{2|A_1} = -\sqrt{-1}dy_2,$$
$$dv_{2|A_1} = \sqrt{-1}du_2, \ dv_{4|A_1} = \sqrt{-1}du_4, \ du_{1|A_1} = -\sqrt{-1}dv_1, \ du_{3|A_1} = -\sqrt{-1}dv_3.$$

Lemma 7.4.

$$d\{c\}_{i|A_1} \equiv \frac{d\{z\}_i}{2||A_1||} - \frac{\{z\}_i + \{\overline{z}_i\}}{||A_1||^3} \cdot d\xi_2,$$

and for each $i = 0, \cdots, 7$

$$d\{c\}_{0|A_1} \equiv \frac{-\sqrt{-1}d\xi_2}{2^2}, \ d\{c\}_{1|A_1} \equiv \frac{d\xi_2 - \sqrt{-1}dz_1}{2^2}, \ d\{c\}_{2|A_1} \equiv \frac{dz_2 - dz_3}{2^3}, \ d\{c\}_{3|A_1} \equiv \frac{dz_2 + dz_3}{2^3\sqrt{-1}},$$
$$d\{c\}_{4|A_1} \equiv \frac{dw_1 + dw_4}{2^3}, \ d\{c\}_{5|A_1} \equiv \frac{dw_1 - dw_4}{2^3\sqrt{-1}}, \ d\{c\}_{6|A_1} \equiv \frac{dw_2 - dw_3}{2^3}, \ d\{c\}_{7|A_1} \equiv \frac{dw_2 + dw_3}{2^3\sqrt{-1}},$$

$$d\{b\}_{0|A_1} \equiv \frac{dy_4 - \sqrt{-1}dx_4}{2^2},\ d\{b\}_{i|A_1} \equiv \frac{d\{y\}_i}{2^3} + \sqrt{-1}\frac{d\{x\}_i}{2^3},$$

where we can ignore the term $\{z\overline{x}\}$*, since* $\{x\}_{i|A_1} = 0$ *and* $d\{\overline{z}x\}_{i|A_1} \equiv \sum_{\mathbf{e}_\alpha \mathbf{e}_\beta = \mathbf{e}_i} \{\overline{z}\}_\alpha d\{x\}_\beta = \{\overline{z}\}_0 d\{x\}_{i|A_1} = -\sqrt{-1}d\{x\}_i$ *and for* $i = 1, \cdots, 7$,

$$d\{b\}_{1|A_1} \equiv \frac{dx_4 - \sqrt{-1}dy_4}{2^2},\ d\{b\}_{2|A_1} \equiv \frac{dy_2 - \sqrt{-1}dx_3}{2^2},\ d\{b\}_{3|A_1} \equiv \frac{dx_3 - \sqrt{-1}dy_2}{2^2},$$

$$d\{b\}_{4|A_1} \equiv \frac{dv_1 + \sqrt{-1}du_4}{2^2},\ d\{b\}_{5|A_1} \equiv -\frac{du_4 + \sqrt{-1}dv_1}{2^2},$$

$$d\{b\}_{6|A_1} \equiv \frac{-dv_3 + \sqrt{-1}du_2}{2^2},\ d\{b\}_{7|A_1} \equiv \frac{du_2 - \sqrt{-1}dv_3}{2^2}.$$

Based on these data

Proposition 7.5. *At the point* A_1*, the holomorphic component of the pull-back* $\{\mathbf{q}\circ{\tau_{\mathbb{O}}}^{-1}\}^*(dv_{P^2\mathbb{O}})$ *is equal to*

$$\begin{aligned}
&\{\mathbf{q}\circ{\tau_{\mathbb{O}}}^{-1}\}^*(dv_{P^2\mathbb{O}})_{|A_1} = \{\mathbf{q}\circ{\tau_{\mathbb{O}}}^{-1}\}^*(d\{c\}_0 \wedge \cdots \wedge d\{c\}_7 \wedge d\{b\}_0 \wedge \cdots \wedge d\{b\}_7)_{|A_1} \\
&\equiv \frac{-\sqrt{-1}d\xi_2}{2^2} \wedge \frac{d\xi_2 - \sqrt{-1}dz_1}{2^2} \wedge \frac{dz_2 - dz_3}{2^3} \wedge \frac{dz_2 + dz_3}{2^3\sqrt{-1}} \wedge \frac{dw_1 + dw_4}{2^3} \wedge \frac{dw_1 - dw_4}{2^3\sqrt{-1}} \\
&\quad \wedge \frac{dw_2 - dw_3}{2^3} \wedge \frac{dw_2 + dw_3}{2^3\sqrt{-1}} \wedge \frac{dy_4 - \sqrt{-1}dx_4}{2^2} \wedge \frac{dx_4 - \sqrt{-1}dy_4}{2^2} \\
&\quad \wedge \frac{dy_2 - \sqrt{-1}dx_3}{2^2} \wedge \frac{dx_3 - \sqrt{-1}dy_2}{2^2} \wedge \frac{dv_1 + \sqrt{-1}du_4}{2^2} \wedge -\frac{du_4 + \sqrt{-1}dv_1}{2^2} \\
&\quad \wedge \frac{-dv_3 + \sqrt{-1}du_2}{2^2} \wedge \frac{du_2 - \sqrt{-1}dv_3}{2^2} \\
&= \frac{1}{2^{31}\sqrt{-1}} \cdot dx_3 \wedge dx_4 \wedge du_2 \wedge du_4 \wedge dy_2 \wedge dy_4 \wedge dv_1 \wedge dv_3 \wedge dz_1 \wedge dz_2 \wedge dz_3 \\
&\qquad \wedge dw_1 \wedge dw_2 \wedge dw_3 \wedge dw_4 \wedge d\xi_2.
\end{aligned}$$

Hence

Corollary 7.6.

$$\begin{aligned}
\{\mathbf{q}\circ{\tau_{\mathbb{O}}}^{-1}\}^*(dv_{P^2\mathbb{O}}) \bigwedge \overline{\Omega}\,(A) \;&=\; C_{RC}\frac{1}{16!}\{{\tau_{\mathbb{O}}}^{-1}\}^*\left(\left(\omega^{P^2\mathbb{O}}\right)^{16}\right)(A) \\
&= 2^6 \cdot ||A||^3 \cdot \frac{1}{16!}\{{\tau_{\mathbb{O}}}^{-1}\}^*\left(\left(\omega^{P^2\mathbb{O}}\right)^{16}\right)(A). \qquad (7.8)
\end{aligned}$$

By this formula (7.8) we have an expression of the Bargmann type transformation (5.12).

Corollary 7.7.

$$\begin{aligned}
\mathfrak{B}(h)(X)\cdot dv_{P^2\mathbb{O}}(X) &= \{\mathbf{q}\circ{\tau_{\mathbb{O}}}^{-1}\}_*\left(h\cdot g_0\cdot\{\mathbf{q}\circ{\tau_{\mathbb{O}}}^{-1}\}^*(dv_{P^2\mathbb{O}}) \bigwedge \overline{\Omega_{\mathbb{O}}}\right) \\
&= \{\mathbf{q}\circ{\tau_{\mathbb{O}}}^{-1}\}_*\left(h\cdot g_0\cdot 2^6\cdot||A||^3\cdot\frac{1}{16!}\{{\tau_{\mathbb{O}}}^{-1}\}^*\left(\left(\omega^{P^2\mathbb{O}}\right)^{16}\right)\right) \\
&= 2^6\cdot\mathbf{q}_*\left(h(\tau_{\mathbb{O}}(X,*))\cdot e^{-\sqrt{2}\pi||*||}||*||^6\cdot dV_{T^*(P^2\mathbb{O})}(X,*)\right). \qquad (7.9)
\end{aligned}$$

8. Invariant polynomials and harmonic polynomials on the Jordan algebra $\mathcal{J}(3)$

In this section we describe invariant polynomials on $\mathcal{J}(3)$ and commuting differential operators with constant coefficients under the action by the automorphism group F_4 of the Jordan algebra $\mathcal{J}(3)$ (see [He] and [HL] for the framework here and [Yo] for necessary properties of F_4 in relation with $P^2\mathbb{O}$).

8.1. **Correspondence between polynomials and differential operators with constant coefficients.** Let

$$\mathbb{R}^N \times \mathbb{R}^N \ni (x,\xi) = (x_1,\ldots,x_N,\xi_1\ldots,\xi_N) \longmapsto < x,\, \xi > = \sum x_i\xi_i \in \mathbb{R},$$

be the standard non-degenerate symmetric bi-linear form. We also use the same notation for its extension to the complex bi-linear form defined on $\mathbb{C}^N \times \mathbb{C}^N$.

Differential operators D_x with constant (complex) coefficients are expressed in the form

$$D = D_x = \sum_{|\alpha|\le k} a_\alpha \frac{\partial^{|\alpha|}}{\partial x^{\,\alpha}} = \sum a_\alpha\, D_x^\alpha,$$

where $a_\alpha \in \mathbb{C}$ and $D_x^\alpha := \dfrac{\partial^{\,|\alpha|}}{\partial x^{\,\alpha}} = \dfrac{\partial^{\,\alpha_1+\cdots+\alpha_N}}{\partial {x_1}^{\alpha_1}\cdots\partial {x_N}^{\alpha_N}}$.

Let $D = \sum a_\alpha\, D_x^\alpha$ be a constant coefficient partial differential operator defined on $\mathbb{R}^N$, then by the relation

$$e^{-<x,\xi>} D_x(e^{<\bullet,\xi>})(x) = \sum a_\alpha\, \xi^\alpha := Q^D(\xi), \tag{8.1}$$

where $\xi^\alpha = {\xi_1}^{\alpha_1}\cdots{\xi_N}^{\alpha_N}$, the correspondence $D \longleftrightarrow Q^D(\xi)$ is bijective, that is, the algebra $\mathbb{C}[\mathbb{R}^N] = \mathbb{C}[x_1,\ldots,x_N] = \sum_{k=0}^{\infty} \mathcal{P}_k[x_1,\ldots,x_N]$ of (complex coefficient) polynomials on $\mathbb{R}^N$ and the algebra $\mathcal{D}[x_1,\ldots,x_N] = \sum_{k=0}^{\infty} \mathcal{D}_k[x_1,\ldots,x_N]$ of linear differential operators with constant (complex) coefficients are isomorphic. Here we denote by $\mathcal{P}_k[x_1,\ldots,x_N]$ the subspace of homogeneous polynomials of degree k and by $\mathcal{D}_k = \mathcal{D}_k[x_1,\ldots,x_N]$ the subspace consisting of homogeneous differential operators with constant coefficients of order k.

We will denote the differential operator corresponding to a polynomial $Q \in \mathbb{C}[x_1,\ldots,x_N]$ by D^Q.

Let $g \in \mathrm{GL}(N,\mathbb{R})$ and define $\mathcal{P}_g : \mathbb{C}[x_1,\ldots,x_N] \longrightarrow \mathbb{C}[x_1,\ldots,x_N]$ an algebra isomorphism in the natural way:

$$Q = Q(x) = \sum a_\alpha \cdot x^\alpha \;\longmapsto \mathcal{P}_g(Q)(x) = Q(g^{-1}(x)) = \sum a_\alpha \cdot (g^{-1}(x))^\alpha, \text{ where}$$

$$g^{-1} = \Big(\,\{g^{-1}\}_{i,j}\,\Big), \text{ and } (g^{-1}(x))^\alpha = \left(\sum_i \{g^{-1}\}_{1,i}\, x_i\right)^{\alpha_1} \cdots \left(\sum_i \{g^{-1}\}_{N,i}\, x_i\right)^{\alpha_N}.$$

The following relation will be seen easily.

Lemma 8.1. *Let $g \in \mathrm{GL}(N,\mathbb{R})$ and D a linear differential operator with constant coefficients. Then, $\mathcal{P}_g \circ D = D \circ \mathcal{P}_g$ on the space of the whole polynomial functions, if and only if $Q^D(\xi) = Q^D({}^t g^{-1}(\xi))$.*

Next, we introduce an Hermitian inner product $\ll \cdot\,,\cdot \gg$ on the space of polynomials $\mathbb{C}[x_1,\ldots,x_N]$ by the following way:

we fix the coordinates $(x_1,\cdots,x_N) \in \mathbb{R}^N$ and define the inner product between monomials x^α and x^β by

$$\ll x^\alpha\,,\, x^\beta \gg := \alpha_1!\cdots\alpha_N! \cdot \delta_{\alpha_1,\beta_1}\cdots\delta_{\alpha_N,\beta_N} := \alpha! \cdot \delta_{\alpha,\beta}. \tag{8.2}$$

The inner product on the space $\mathbb{C}[x_1,\ldots,x_N]$ introduced above is used only in this section.

Then the following properties will be seen easily too.

Lemma 8.2. *Let $D = \sum a_\alpha D_x^\alpha$ be a differential operator with constant coefficients and P the polynomial corresponding to D according to the correspondence* (8.1)*, that is*

$$e^{-<x,\,\xi>} D^P(e^{<\bullet,\,\xi>})(x) = P(\xi) = \sum a_\alpha\, \xi^\alpha.$$

Then for any polynomial $Q = \sum C_\beta x^\beta$

$$\ll P,\, Q \gg = \ll \sum a_\alpha\, x^\alpha\,,\, \sum C_\beta\, x^\beta \gg = \sum_\alpha \alpha! \cdot \delta_{\alpha,\beta} \cdot a_\alpha \cdot \overline{C_\beta} \tag{8.3}$$

$$= D^P(\overline{Q})(0) =: (\!(D^P,\, Q)\!) = \overline{(\!(D^Q\,,\, P)\!)}, \tag{8.4}$$

where we replace the variables ξ_i to x_i. In particular, if the order of the differential operator D and the degree of a polynomial Q coincides, then

$$D(\overline{Q})(x) \equiv D(\overline{Q})(0),$$

and the spaces $\mathcal{P}_k$ and $\mathcal{P}_\ell$ are always orthogonal, if $k \neq \ell$.

It holds a kind of associativity:

$$((D_1 \circ D_2,\, Q)) = D_1 \circ D_2(Q)(0) = \ll P_1 \cdot P_2,\, Q \gg = \ll P_1, D_2(Q) \gg = ((D_1,\, D_2(Q))). \tag{8.5}$$

The equation (8.3) can be understood that the Hermitian inner product we introduced is a pairing between the space $\mathcal{D}[x_1, \dots, x_N]$ of differential operators with constant coefficients and the space of polynomials, especially by this pairing the space $\mathcal{D}(\mathbb{R}^N)$ is identified with the (restricted) dual space $\mathcal{D}[x_1, \dots, x_N] \cong \sum_{k=0}^{\infty} \mathcal{P}_k[x_1, \dots, x_N]^*$ of $\mathbb{C}[x_1, \dots, x_N]$. With respect to the action of $g \in \mathrm{GL}(N, \mathbb{R})$ on $\mathbb{C}[x_1, \dots, x_N]$ the dual action of g on $\mathcal{D}(\mathbb{R}^N)$ is

$$\mathcal{D}(\mathbb{R}^N) \ni D \longmapsto {\mathcal{P}_g}^{-1} \circ D \circ \mathcal{P}_g =: \mathcal{P}_g^*(D)$$

and satisfies the relation

$$((\, \mathcal{P}_g^*(D),\, f\,)) = ((\, D,\, \mathcal{P}_g(f))),\ \ D \in \mathcal{D}[x_1, \dots, x_N],\ \ f \in \mathbb{C}[x_1, \dots, x_N]. \tag{8.6}$$

8.2. **Trace function and invariant polynomials.** We recall two important properties Theorems 8.3 and 8.4 on the action of the group F_4 on $\mathcal{J}(3)$. Also the properties (3.5), (3.6) should be reminded in this section (see [SV] and [Yo]).

Theorem 8.3. *For any $A \in \mathcal{J}(3)$, there exists an element $\alpha \in F_4$ such that*

$$\alpha(A) = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \tag{8.7}$$

where the triple of quantities $\{\xi_i\}$ depends only on A and does not depend on such an element $\alpha \in F_4$ which send A to a diagonal matrix in $\mathcal{J}(3)$.

Theorem 8.4. *The representation of F_4 to $\mathcal{J}(3)$ is decomposed into two mutually orthogonal irreducible subspaces, that is*

$$\mathcal{J}(3) = \mathcal{J}_0(3) \oplus \mathbb{R} \cdot Id,$$

where $\mathcal{J}_0(3) = \{\ A \in \mathcal{J}(3) \mid \mathrm{tr}\,(A) = 0\ \}$ and Id is the 3×3 identity matrix which is the fixed point in $\mathcal{J}(3)$ under the action of F_4.

It holds the same decomposition in the complexified Jordan algebra $\mathcal{J}(3)^{\mathbb{C}}$ by the action of the complex group ${F_4}^{\mathbb{C}}$.

In this section we express

$$\mathcal{J}(3) \ni A = \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \longleftrightarrow (z_0, \cdots, z_7, y_0, \cdots, y_7, x_0, \cdots, x_7, \xi_1, \xi_2, \xi_3)$$

with the coefficients of

$$z = \sum \{z\}_i \mathbf{e}_i = \sum z_i \mathbf{e}_i,\ \ y = \sum \{y\}_i \mathbf{e}_i = \sum y_i \mathbf{e}_i,\ \ x = \sum \{x\}_i \mathbf{e}_i = \sum x_i \mathbf{e}_i$$

and do not use the notation $\{z\}_i$, $\{y\}_i$, $\{x\}_i$ (see (4.6)). *We also denote these coordinates as*

$$(z_0, \cdots, z_7, y_0, \cdots, y_7, x_0, \cdots, x_7, \xi_1, \xi_2, \xi_3) = (s_1, \cdots\cdots, s_{24}, s_{25}, s_{26}, s_{27}) \tag{8.8}$$

or

$$(z_0, \cdots, z_7, y_0, \cdots, y_7, x_0, \cdots, x_7, \xi_1, \xi_2, \xi_3) = (s_1, \cdots\cdots, s_{24}, \xi_1, \xi_2, \xi_3). \tag{8.9}$$

We denote the (complex valued) polynomial algebra over $\mathcal{J}(3)$ by $\mathbb{C}[\,\mathcal{J}(3)\,]$ with the independent variables $\{z_i, \dots, x_i, \xi_1, \xi_2, \xi_3\}$ and also regard it as the the algebra of polynomial functions. It is equipped with an Hermitian inner product explained in the preceding subsection § 8.1.

Then, we can identify by the isometric way the space $\mathcal{P}_1[\mathcal{J}(3)] \cong \left(\mathcal{J}(3)^{\mathbb{C}}\right)^*$ with the space $\mathcal{J}(3)^{\mathbb{C}}$ through the correspondence

$$\mathcal{J}(3)^{\mathbb{C}} \ni A \longleftrightarrow h_A \in \mathcal{P}_1[\mathcal{J}(3)],\ \ h_A(X) = \mathrm{tr}\,(X \circ A) = < X\,, A >^{\mathcal{J}(3)^{\mathbb{C}}}. \tag{8.10}$$

The action of the group F_4 is extended to the space $\mathbb{C}[\mathcal{J}(3)]$ as denoted in §8.1 :

$$\mathbb{C}[\mathcal{J}(3)] \ni Q \longmapsto (\mathcal{P}_g(Q)(X) := Q(g^{-1}(X))$$

and the extended action leaves the degree of the polynomials and the inner product.

Definition 8.5. *We denote a subspace in each $\mathcal{P}_k[\mathcal{J}(3)]$ by I_k consisting of invariant polynomials under the extended action of the group F_4 and put $I = I_{F_4} = \sum_{k\geq 0} I_k$, the algebra of invariant polynomials under the action of the Lie group F_4 on $\mathcal{J}(3)$.*

By the property (3.4), the functions $\mathcal{J}(3) \ni A \longmapsto \mathrm{tr}\,(A^k) := T_k(A)$ is well-defined and are invariant polynomials (*off course, these are also well defined on* $\mathcal{J}(3)^{\mathbb{C}}$). Then,

Proposition 8.6. *All the invariant polynomials in $\mathcal{P}_k[\mathcal{J}(3)]$ are given by the linear sums of polynomials of the products*

$$T_1{}^{i_1} \cdot T_2{}^{i_2} \cdot T_3{}^{i_3}$$

under the condition that $i_1 + 2i_2 + 3i_3 = k$ $(0 \leq i_1, i_2, i_3 \leq k)$ and

$$\dim_{\mathbb{C}} I_k = \{\textit{number of the solutions } (i_1, i_2, i_3) \textit{ under the condition } i_1 + 2i_2 + 3i_3 = k\}$$

$$= \sum_{\ell=0}^{[k/3]} \left(\left[\frac{k-3\ell}{2} \right] + 1 \right). \tag{8.11}$$

Proof. Let $f \in I_k$ be an invariant polynomial. Then by the property (8.7) in Theorem 8.3 and the invariance of the trace function (3.5), $f(A) = f(\alpha(A)) = f\left(\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}\right)$ depend only on the triple $\{\xi_i\}_{i=1}^3$ which appears when it is expressed as a diagonal matrix given in the above Theorem 8.3.

Let $\sigma_1 : \mathcal{J}(3) \to \mathcal{J}(3)$ be a permutation defined by

$$\sigma_1 : \mathcal{J}(3) \ni A \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{J}(3). \tag{8.12}$$

Likewise we can define other two permutations σ_2 and σ_3 among the quantities $\{\xi_i\}$ by the matrices $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, respectively. These are elements in F_4 and satisfy $f(\sigma_i(A)) = f(A)$.

Hence the invariant polynomial ring $I = I_{F_4} = \sum\limits_{k\geq 0} I_k$ in $\mathbb{C}[\mathcal{J}(3)]$ is generated by three elementary symmetric polynomials

$$\xi_1 + \xi_2 + \xi_3, \quad \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 \quad \text{and} \quad \xi_1\xi_2\xi_3.$$

This is equivalent to say that the subalgebra of invariant polynomials of positive degree $I_+ := \sum_{k\geq 1} I_k$ (i.e., *without constant terms*) is generated by three invariant polynomials

$$T_1(A) = \mathrm{tr}\,(A) = \sum_{i=1}^{3} \xi_i, \qquad T_2(A) = \mathrm{tr}\,(A^2) = 2\sum_{i=0}^{7} \left({z_i}^2 + {y_i}^2 + {x_i}^2\right) + \sum_{i=1}^{3} {\xi_i}^2 = ||A||^2,$$

$$T_3(A) = \mathrm{tr}\,(A \circ (A \circ A)) = < A, A \circ A >^{\mathcal{J}(3)} = < A \circ A\,, A >^{\mathcal{J}(3)}$$

$$= \sum_{i=1}^{3} {\xi_i}^3 + 3\left(|z|^2(\xi_1 + \xi_2) + |y|^2(\xi_3 + \xi_1) + |x|^2(\xi_2 + \xi_3)\right)$$

$$+ \frac{zx \cdot y + \theta(zx \cdot y) + xy \cdot z + \theta(xy \cdot z) + yz \cdot x + \theta(yz \cdot x)}{2}$$

$$+ \frac{x \cdot yz + \theta(x \cdot yz) + y \cdot zx + \theta(y \cdot zx) + z \cdot xy + \theta(z \cdot xy)}{2}$$

$$= \sum_{i=1}^{3} {\xi_i}^3 + 3\left(|z|^2(\xi_1 + \xi_2) + |y|^2(\xi_3 + \xi_1) + |x|^2(\xi_2 + \xi_3)\right) + 6 \cdot \mathfrak{Re}(x \cdot yz),$$

($\mathfrak{Re}(x \cdot yz) = \{x \cdot yz\}_0$ is the real part of the octanion $x \cdot yz$).

The last formula (8.11) is given by solving the equation $i_1 + 2 \cdot i_2 + 3 \cdot i_3 = k$ (see Appendix). □

In the proof above we used a property of the multiplication low in the octanion $\mathbb{O}$:

$\mathfrak{Re}(x\cdot yz)=\mathfrak{Re}(y\cdot zx)=\mathfrak{Re}(z\cdot xy)$, $\mathfrak{Re}(zx\cdot y)=\mathfrak{Re}(\theta(y)\cdot\theta(zx))=\mathfrak{Re}(y\cdot zx)$ and similar identities.

Next, we mention (see Lemma 8.1 and Theorem 8.3)

Proposition 8.7. *The invariant polynomial ring $I=\sum I_k$ and differential operators with constant coefficients commuting with the F_4 action are isomorphic. Especially, the differential operators corresponding to the generators T_1, T_2 and T_3 of the invariant polynomial ring are*

$$T_1 = e^{-<x,\xi>}L(e^{<\bullet,\xi>})(x) \longleftrightarrow L = L(z,y,x,\xi_1,\xi_2,\xi_3) := \frac{\partial}{\partial\xi_1}+\frac{\partial}{\partial\xi_2}+\frac{\partial}{\partial\xi_3},$$

$$T_2 \longleftrightarrow -\Delta := 2\sum_{i=0}^{7}\left(\frac{\partial^2}{\partial {z_i}^2}++\frac{\partial^2}{\partial {y_i}^2}+\frac{\partial^2}{\partial {x_i}^2}\right)+\sum_{j=1}^{3}\frac{\partial^2}{\partial {\xi_j}^2},$$

$$T_3 \longleftrightarrow \Gamma := \sum \frac{\partial^3}{\partial {\xi_j}^3}+3\left(\frac{\partial}{\partial\xi_1}+\frac{\partial}{\partial\xi_2}\right)\circ\sum_{i=0}^{7}\frac{\partial^2}{\partial {z_i}^2}+3\left(\frac{\partial}{\partial\xi_3}+\frac{\partial}{\partial\xi_1}\right)\circ\sum_{i=0}^{7}\frac{\partial^2}{\partial {y_i}^2}$$
$$+3\left(\frac{\partial}{\partial\xi_2}+\frac{\partial}{\partial\xi_3}\right)\circ\sum_{i=0}^{7}\frac{\partial^2}{\partial {x_i}^2}+6\sum_{i,j,k=0}^{7}\pm\frac{\partial^3}{\partial x_i\partial y_j\partial z_k}.$$

The second operator is the Laplacian on the Euclidean space $\mathbb{R}^{27}\cong\mathcal{J}(3)$.

The last term of the operator Γ consists of 8^3 partial differential operators of the form $\dfrac{\partial^3}{\partial x_i\partial y_j\partial z_k}$ with suitable signs.

We define an F_4-invariant subspace H_k in $\mathcal{P}_k[\mathcal{J}(3)]$ inductively and call polynomials therein "*Cayley harmonic polynomial*".

Definition 8.8.

(0) *H_0 is the space of the constant functions $=\mathcal{P}_0$,*

(1) *$H_1=\{$the linear functions: $\sum_{i=0}^{7}(a_iz_i+b_iy_i+c_ix_i)+\sum_{i=1}^{3}d_i\xi_i$ with $\sum d_i=0\}$,*
this space is isomorphic to $\left\{B\in\mathcal{J}(3)^{\mathbb{C}}\mid \mathrm{tr}\,(B)=0\right\}$,
and we have an orthogonal decomposition $\mathcal{P}_1=H_1\oplus_\perp H_0I_1$,

⋮

(k) *H_k is the orthogonal complement of the space $\left(\sum_{i=0}^{k-1}H_i\cdot I_{k-i}\right)$ taken in $\mathcal{P}_k$,*

$$\mathcal{P}_k[\mathcal{J}(3)]=H_k\oplus_\perp\sum_{i=1}^{k}H_{k-i}\cdot I_i. \tag{8.13}$$

The subspace H_k can be seen as the space corresponding to the space of harmonic polynomials for the case of $SO(n)$ acting on $\mathbb{R}^n$. In fact

Proposition 8.9. *Let $\mathfrak{H}_k$ be*

$$\mathfrak{H}_k:=\{Q\in\mathcal{P}_k[\mathcal{J}(3)]\mid L(Q)=0,\Delta(Q)=0,\Gamma(Q)=0\}.$$

Then $\mathfrak{H}_k=H_k$.

Before proving this Proposition we show a

Lemma 8.10. *For each k the space*

$$I_k+H_1\cdot I_{k-1}+\cdots+H_{k-1}\cdot I_1=I_k+\mathcal{P}_1\cdot I_{k-1}+\cdots+\mathcal{P}_{k-1}\cdot I_1. \tag{8.14}$$

The right hand side need not be a direct sum, while the left hand side is a direct sum which will be proved later after several preparations.

Proof. It is apparent

$$\sum_{i=0}^{k-1} H_i \cdot I_{k-i} \ \subset \ \sum \mathcal{P}_i \cdot I_{k-i}.$$

Since $I_2 \supset I_1 \cdot I_1$,

$$H_1 \cdot I_1 + I_2 \supset (H_1 \cdot I_1 + I_1 \cdot I_1) + I_2 = \mathcal{P}_1 \cdot I_1 + I_2.$$

Hence, $H_1 \cdot I_1 + I_2 = \mathcal{P}_1 \cdot I_1 + I_2$.

Assume

$$\sum_{i=0}^{j-1} H_i \cdot I_{j-i} \ = \ \sum_{i=0}^{j-1} \mathcal{P}_i \cdot I_{j-i}, \text{ for } j \le k.$$

Then using the property $I_j \supset I_a \cdot I_b$ for $a + b = j$, we can show inductively

$$\sum_{i=0}^{j-1} H_i \cdot I_{j-i} \ \supset \ \sum_{i=0}^{j-1} \mathcal{P}_i \cdot I_{j-i}, \text{ for } j \le k.$$

□

Proof of the Proposition 8.9.

At this stage, it will be apparent that the conditions

$$\Gamma(Q) = 0, \quad \Delta(Q) = 0, \quad \text{and } L(Q) = 0$$

are together equivalent to the condition that a polynomial $Q \in \mathcal{P}_k$ is orthogonal to the subspace

$$I_k + H_1 \cdot I_{k-1} + \cdots + H_{k-1} \cdot I_1.$$

□

Lemma 8.11.

$$L(T_1) = 3, \quad L(T_2) = 2T_1, \quad L(T_3) = 3T_2,$$
$$\Delta(T_2) = 198, \quad \Delta(T_3) = 198T_1, \quad \Gamma(T_3) = 562.$$

Remark 9. *Invariant polynomials above need not be orthogonal. For example*

$$\ll T_2\,, {T_1}^2 \gg = L^2(T_2)(0) = L(L(T_2))(0) = 2L(T_1)(0) = 6.$$

Next we show that the sum (8.14) is a direct sum as mentioned in the above Lemma 8.10(summed up in Proposition 8.17).

By definition it is enough to show the sum

$$H_{k-1} \cdot I_1 + \cdots + H_1 \cdot I_{k-1} + I_k$$

is a direct sum. For this purpose we prepare several lemmas.

Lemma 8.12. *The map* $L : I_k \longrightarrow I_{k-1}$ *is surjective for all* $k = 1, 2, \cdots,$

Proof. Let $\mathbf{t} : I_k \longrightarrow I_{k+1}$ be a map defined by

$$\mathbf{t}(T) = T_1 \cdot T,$$

then $\mathbf{t}$ is injective. In fact, if there is an element $T \in I_k$ satisfying

$$L \circ \mathbf{t}(T) = L(T_1 \cdot T) = 3T + T_1 \cdot L(T) = 0,$$

then again we have

$$3L(T) + 3L(T) + T_1 \cdot L^2(T) = 0 \text{ and } 3T - \frac{1}{6}T_1^2 \cdot L^2(T) = 0.$$

By iterating this procedure we have

$$T = 0.$$

Hence the map $L \circ \mathbf{t}$ is injective, which means that the map $L : I_{k+1} \to I_k$ is already surjective (in fact isomorphic) on $\mathbf{t}(I_k)$. □

Based on the equality (8.14) and the Lemma below (8.14), we can construct an orthogonal basis of the space I_k inductively $\{\varphi_k(i)\}_{i=1}^{\dim I_k}$ in the following way:

Definition 8.13.

$$\begin{aligned}
I_1 =& [\{\varphi_1(1) = T_1\}],\\
I_2 =& [\{\varphi_2(1) = T_1^2 = T_1\cdot\varphi_1(1),\ \varphi_2(2) = T_2 - 1/3T_1^2\}],\\
&\quad \textit{where } \varphi_2(2) \textit{ is taken to be orthogonal to } \varphi_2(1) \textit{ and equivalently } L(\varphi_2(2)) = 0,\\
I_3 =& [\{\varphi_3(1) = T_1^3 = T_1\varphi_2(1),\ \varphi_3(2) = T_1\varphi_2(2),\ \varphi_3(3) = T_3 - T_2T_1 + 2/9T_1^3\}],\\
&\quad \textit{where } \varphi_3(3) \textit{ is taken to be orthogonal to } \varphi_3(1) \textit{ and } \varphi_3(2)\textit{, which is also taken}\\
&\quad \textit{to satisfy } L(\varphi_3(3)) = 0 \textit{ and is determined uniquely up to constant multiples.}
\end{aligned}$$

Likewise we can continue the construction in such a way that if $\{\varphi_k(i)\}_{i=1}^{\dim I_k}$ *is constructed as above for* $k = 1, 2, 3, 4$*, then we define for* $k \geq 5$

$$\begin{aligned}
&\varphi_{k+1}(i) = T_1\varphi_k(i) \textit{ for } i = 1\cdots, \dim I_k \textit{ and for } j = 1,\cdots,\dim I_{k+1} - \dim I_k,\\
&\varphi_{k+1}(\dim I_k + j) \textit{ is chosen as being orthogonal to all } \varphi_{k+1}(i),\ i = 1,\cdots, \dim I_k + j - 1.
\end{aligned}$$

The orthogonality condition $\ll \varphi_{k+1}(\dim I_k + j)\,,\, T_1\varphi_k(i) \gg= 0$ *implies that* $L(\varphi_{k+1}(\dim I_k + j)) = 0$.

Lemma 8.14. *The construction is guaranteed by the property that if* f *and* $g \in I_k$ *is orthogonal and* $L(g) = 0$*, then* $T_1\cdot f$ *and* $T_1\cdot g$ *is orthogonal, since*

$$\ll T_1f, T_1g \gg=\ll f, L(T_1g) \gg= 3 \ll f\,,\, g \gg= 0.$$

Lemma 8.15. *Put* $\mathcal{N}_k = \{T \in I_k \mid L(T) = 0\}$*, then*

$$\dim\ \mathcal{N}_{k+1} = \dim I_{k+1} - \dim I_k,$$

and is equal to the number of the non-negative integer solutions (a, b) *of the equation*

$$2a + 3b = k + 1. \tag{8.15}$$

Proof. Put $\varphi_2(2) = T_2 - 1/3T_1^2 := \varphi_2$ and $\varphi_3(3) = T_3 - T_2T_1 + 2/9T_1^3 := \varphi_3$. Then both of these are irreducible polynomials, since they are not decomposed into lower degree polynomials even on the subspace $z = y = x = 0$.

By $L(\varphi_2\cdot\varphi_3) = 0$, products of any powers of these two polynomials are in the kernel of the map L. So corresponding to the non-negative integer solutions (a, b) of (8.15) we have a basis of the kernel $\mathcal{N}_{k+1}$. □

Lemma 8.16. *For any* j *and* ℓ

$$\dim H_j\cdot I_\ell = \dim H_j\cdot\dim I_\ell.$$

Proof. It will be apparent if $\dim I_\ell = 1$.

We prove the property by induction and we show that the natural map $H_j\otimes\mathcal{N}_\ell \longrightarrow H_j\cdot\mathcal{N}_\ell$ is isomorphic. So we assume for $^\forall\ell \leq k$ and any $j \geq 0$ it holds the isomorphism

$$H_j\otimes\mathcal{N}_\ell \cong H_j\cdot\mathcal{N}_\ell. \tag{8.16}$$

Let

$$\sum_{(a,b) \text{ run through the solutions of (8.15)}} h_{a,b}\cdot{\varphi_2}^a\cdot{\varphi_3}^b = 0,\ h_{a,b} \in H_j. \tag{8.17}$$

Let $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ be all the solutions of (8.15):

$$2a_i + 3b_i = k + 1,$$

Assume $a_1 > a_2 > \cdots > a_n$, then $b_1 < b_2 < \cdots < b_n$. Then the we can assume the expression (8.17) has one of the following two forms:

$$[1]:\ \text{if } a_n > 0, {\varphi_2}^{a_n}\cdot p + {\varphi_3}^{b_n}\cdot h_{a_n,b_n} = 0,\ \text{or} \tag{8.18}$$

$$[2]:\ \text{if } a_n = 0, {\varphi_2}^{a_n-1}\cdot p + {\varphi_3}^{b_n}\cdot h_{a_n,b_n} = 0. \tag{8.19}$$

In any case the polynomial φ_2 does not divide the polynomial φ_3, so that we may put $h_{a_n,b_n} = \varphi_2\cdot Q$ with a polynomial $Q \in \mathcal{P}_{j-2}$. Then by the equality (8.14) the polynomial $Q\cdot\varphi_2 \in H_{j-1}\cdot I_1 + H_{j-2}\cdot I_2 + \cdots + H_1\cdot I_{j-1} + I_j$. On the other hand $Q\cdot\varphi_2 = h_{a_n,b_n} \in H_j$. Hence by the definition of the space H_j which is orthogonal complement of the space $H_{j-1}\cdot I_1 + H_{j-2}\cdot I_2 + \cdots + H_1\cdot I_{j-1} + I_j$, hence $h_{a_n,b_n} = 0$ and also $p = 0$. By iterating the arguments we see that in the expression

$$\sum_{(a_i,b_i)\text{ run through the solutions of (8.15)}} h_{a_i,b_i}\cdot {\varphi_2}^{a_i}\cdot {\varphi_3}^{b_i}$$

all the coefficient polynomials h_{a_i,b_i} must be zero.

Finally we see from the sequences

$$\begin{array}{ccccccccc} \{0\} & \longrightarrow & H_j\otimes\mathcal{N}_{k+1} & \longrightarrow & H_j\otimes I_{k+1} & \xrightarrow{Id\otimes L} & H_j\otimes I_k & \longrightarrow & \{0\} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \{0\} & \longrightarrow & H_j\cdot\mathcal{N}_{k+1} & \xrightarrow{inclusion} & H_j\cdot I_{k+1} & \longrightarrow & H_j\cdot I_k & \longrightarrow & \{0\} \end{array}$$

two spaces

$$H_j\otimes I_{k+1}\cong H_j\cdot I_{k+1}$$

are isomorphic. □

Proposition 8.17. *For each k, the sum $H_k + H_{k-1}\cdot I_1 + \cdots H_1\cdot I_{k-1} + I_k$ is a direct sum.*

Proof. First we remark that the sums $\mathcal{P}_1 = H_1 + I_1$ and $\mathcal{P}_2 = H_2 + H_1\cdot I_1 + I_2$ are orthogonal sums. The first one is included in the definition and the second one is shown as

$$\ll h_1T_1\,,\,T_2 \gg = \ll h_1\,,\,L(T_2) \gg = \ll h_1\,,\,2T_1 \gg = \ll L(h_1), 2T_1 \gg = 0,\ \text{ where } h_1\in H_1.$$

Then we assume that the sum

$$H_j + H_{j-1}\cdot I_1 + \cdots H_1\cdot I_{j-1} + I_j$$

are direct sums for $j\le k$.

We express

$$\begin{aligned} &T\in H_k\cdot I_1 + H_{k-1}\cdot I_2 + \cdots + H_1\cdot I_k + I_{k+1}\text{ as} \\ &T = h_k\cdot T_1 + \sum_{i=1}^{2} h_{k-1}(i)\cdot\varphi_2(i) + \cdots + \sum_{i=1}^{\dim I_k} h_1(i)\varphi_k(i) + \sum_{i=1}^{\dim I_{k+1}} h_0(i)\varphi_{k+1}(i) = 0, \end{aligned}$$

where $h_j(i)\in H_j$ and $\varphi_j(i)$ are the basis polynomials of I_j constructed in the Definition 8.13. Then by the induction hypothesis, $L(T) = 0$ implies

$$h_k = 0, h_{k-1}(1)\varphi_2 = 0, h_{k-2}(1)\varphi_3(1) + h_{k-2}(2)\varphi_3(2) = 0, \cdots, \sum_{i=1}^{\dim I_k}\lambda_i\varphi_k(i) = 0,$$

that is, the coefficient polynomials $h_j(i)$ of the basis included in the orthogonal complement of $\mathcal{N}_j$ are zero.

Hence it will be enough to show

$$h_{k-1}(2)\varphi_2 + h_{k-2}(3)\varphi_3 + \cdots + \sum_{i=\dim I_{k-1}+1}^{\dim I_k} h_1(i)\varphi_k(i) + \sum_{i=\dim I_k+1}^{\dim I_{k+1}}\lambda_i\varphi_{k+1}(i) = 0 \tag{8.20}$$

implies all the coefficient polynomials $h_j(i) = 0$ and constants $\lambda_i = 0$. As in the proof of the Lemma 8.16, the equation (8.20) can be rewritten as

$$\varphi_2\cdot P = -\varphi_3\cdot Q \tag{8.21}$$

where the polynomial $P = h_{k-1}(2) + \cdots$ is the sum of all the terms including some power (≥ 0) of φ_2 and $Q = g_1 + g_2\varphi_3 + \cdots$ (especially $g_1 = h_{k-2}(3)\in H_{k-2}$) is a polynomials of the polynomial φ_3 with the coefficient polynomials $g_i\in H_j$ with the degree of $g_i = k+1-3i$. Since φ_2 does not divide φ_3, Q must be divided by φ_2, that is we have

$$\varphi_2\cdot Q_1 = Q = g_1 + g_2\varphi_3 + \cdots,$$

where $Q_1\in\mathcal{P}_{k-4}$. Hence by Lemma 8.10

$$Q = g_1 + g_2\varphi_3 + \cdots \in H_{k-3}I_1 + H_{k-4}I_2 + \cdots + I_{k-2},$$

which implies that $g_1 = 0$. Hence we can rewrite (8.21) as

$$\varphi_2\cdot P = -{\varphi_3}^2\cdot Q_2.$$

By iterating the same arguments as above we see that $Q = 0$. Hence $P = 0$.

Then we can apply the same argument to the polynomial P by expressing P as

$$P = \varphi_2 P_1 + R = 0,$$

where P_1 is the sum of terms in P of the form $h_{a,b} \cdot \varphi_2^a {\varphi_3}^b$, $a > 0$, $b \geq 0$, $h_{a,b} \in H_{k-1-2a-3b}$ and R is a polynomial of φ_3,

$$R = h'_0 + h'_1 \varphi_3 + h_2 \varphi_3^2 + \cdots$$

with coefficients $h'_a \in H_{k-1-3a}$.

Again by the same argument as above we see that $P = 0$, which proves our assertion. □

We put $\mathcal{H} := \sum_{k\geq 0} H_k$, and denote by $I_+(\mathcal{J}(3)^{\mathbb{C}}) = \sum_{k>0} I_k(\mathcal{J}(3)^{\mathbb{C}})$ invariant polynomial functions extended to the complexification $\mathcal{J}(3)^{\mathbb{C}}$ in the natural way.

Since the function taking the trace $A \mapsto \mathrm{tr}\,(A)$ is linear and $A \mapsto A^k$ is an operation inside the Jordan algebra $\mathcal{J}(3)$ (according to the definition of products) and also its complexification $\mathcal{J}(3)^{\mathbb{C}}$, the extensions of the invariant polynomials T_i to $\mathcal{J}(3)^{\mathbb{C}}$ coincide with the trace functions on the complexification $\mathcal{J}(3)^{\mathbb{C}}$:

$$\mathcal{J}(3)^{\mathbb{C}} \ni A \longmapsto \mathrm{tr}\,(A),\ A^2 \longmapsto \mathrm{tr}\,(A^2) \text{ and } A^3 \longmapsto \mathrm{tr}\,(A^3).$$

Let $N_{\mathcal{J}(3)^{\mathbb{C}}}$ be the common null set (other than zero) of the invariant polynomial functions (with respect to F_4 action) considered on the complexified space $\mathcal{J}(3)^{\mathbb{C}}$:

$$N_{\mathcal{J}(3)^{\mathbb{C}}} := \left\{ A = \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \in \mathcal{J}(3)^{\mathbb{C}} \;\middle|\; A \neq 0,\ T_1(A) = T_2(A) = T_3(A) = 0 \right\}.$$

Remark 10. *Let $A \in N_{\mathcal{J}(3)^{\mathbb{C}}}$. Then at least one of the three components z, y, x does not vanish. Since if $A \in N_{\mathcal{J}(3)^{\mathbb{C}}}$ and assume $z = y = x = 0$, then $T_1(A) = \sum \xi_i = 0$, $T_2(A) = \sum {\xi_i}^2 = 0$ and $T_3(A) = \sum {\xi_i}^3 = 0$. Hence these imply that $\xi_i = 0$ too.*

By the Proposition 4.1

Proposition 8.18. *$\mathbb{X}_{\mathbb{O}} = \tau_{\mathbb{O}}(T_0^*(P^2\mathbb{O})) \subset N_{\mathcal{J}(3)^{\mathbb{C}}}$ and the non-singular part of the space $N_{\mathcal{J}(3)^{\mathbb{C}}}$ has* $\dim N_{\mathcal{J}(3)^{\mathbb{C}}} = 24$.

Proof. Let $A \in \mathbb{X}_{\mathbb{O}}$. Then $T_1(A) = \eta_1 + \eta_2 + \eta_3 = 0$ (Proposition 4.1), and $A^2 = 0$ implies $T_2(A) = 0$ and $T_3(A) = 0$ trivially. Hence $\mathbb{X}_{\mathbb{O}} = \tau_{\mathbb{O}}(T_0^*(P^2\mathbb{O})) \subset N_{\mathcal{J}(3)^{\mathbb{C}}}$.

The second assertion is seen by noting that at the points $z = y = x = 0$ the three differentials

$$dT_1,\ dT_2,\ dT_3$$

are linearly independent. □

Let $A \in \mathcal{J}(3)^{\mathbb{C}}$ and consider the functions of the form

$$\mathcal{J}(3) \ni X \longmapsto \mathrm{tr}\,(X \circ A) = h_A(X). \tag{8.22}$$

Since $\mathbb{X}_{\mathbb{O}}$ is F_4 invariant, the nontrivial subspace in H_1 linearly spanned by the functions

$$\mathcal{J}(3) \ni X \longmapsto \mathrm{tr}\,(X \circ A) := h_A(X),\ A \in \mathcal{J}(3)^{\mathbb{C}},\ \mathrm{tr}\,(A) = 0,$$

is an invariant subspace in H_1. Here note that $\mathrm{tr}\,(g(X) \circ A) = \mathrm{tr}\,(X \circ {}^t g(A))$ for $g \in F_4$ and $\mathrm{tr}\,({}^t g(A)) = \mathrm{tr}\,(A) = 0$.

However the representation of the group F_4 to H_1 is irreducible (Theorem 8.4), the space H_1 must be spanned by these functions. Also the same holds that the subspace in H_1 linearly spanned by the functions

$$\left\{ \mathrm{tr}\,(X \circ A) = h_A(X) \mid A \in N_{\mathcal{J}(3)^{\mathbb{C}}} \right\}$$

coincides with H_1 (see Proposition 4.2 and Corollary 4.3). These facts imply

Proposition 8.19. *All the point in $N_{\mathcal{J}(3)^{\mathbb{C}}}$ can be expressed as a linear sum of points in $\mathbb{X}_{\mathbb{O}}$ and this fact implies that the space $N_{\mathcal{J}(3)^{\mathbb{C}}}$ is path-wise-connected.*

Proof. Since any linear function $\mathcal{J}(3) \ni X \longmapsto \mathrm{tr}\,(X \circ A) = h_A(X)$ with $A \in N_{\mathcal{J}(3)^{\mathbb{C}}}$ is a linear sum of functions of the form $\mathrm{tr}\,(X \circ B_i) = h_{B_i}(X)$ with $B_i \in \mathbb{X}_{\mathbb{O}}$,

$$\mathrm{tr}\,(X \circ A) = \sum c_i \mathrm{tr}\,(X \circ B_i) \text{ on } \mathcal{J}(3), \text{ where } B_i \in \mathbb{X}_{\mathbb{O}},$$

$A = \sum c_i B_i$ with these $B_i \in \mathbb{X}_{\mathbb{O}}$.

Let A and $A' \in N_{\mathcal{J}(3)^{\mathbb{C}}}$. Assume $A = \sum c_i B_i$ and $A' = \sum c'_i B'_i$ where $B_i, B'_i \in N_{\mathcal{J}(3)^{\mathbb{C}}}$. Then the second assertion is proved by connecting points B_i and B'_i suitably in $\mathbb{X}_{\mathbb{O}}$. □

In general, the space H_k of "Cayley-harmonic polynomials" is an orthogonal sum of two subspaces $H_k^{(1)}$ and $H_k^{(2)}$, $H_k^{(1)}$ is the subspace linearly spanned by the powers $\mathrm{tr}(X \circ A)^k$ with $A \in N_{\mathcal{J}(3)^{\mathbb{C}}}$ and $H_k^{(2)}$ is the orthogonal complement of $H_k^{(1)}$ in H_k. The orthogonality is equivalent to the property that Cayley-harmonic functions in $H_k^{(2)}$ are vanishing on the subset $N_{\mathcal{J}(3)^{\mathbb{C}}}$ (see [He]).

In our case the second subspace $H_k^{(2)}$ is always $\{0\}$, that is,

Proposition 8.20.

$$H_k^{(2)} = H_k \bigcap \sqrt{I_+(\mathcal{J}(3)^{\mathbb{C}})} = H_k \bigcap I_+(\mathcal{J}(3)^{\mathbb{C}}) = \{0\}.$$

Proof. The first equality is a consequence of Hilbert Nullstellensatz and the irreducibility of $N_{\mathcal{J}(3)^{\mathbb{C}}}$ implies the second equality.

We see the latter one by the following observation that the equation $T_1(A) = 0$ is linear so that if we replace the variable ξ_3 by $\xi_3 = -\xi_1 - \xi_2$, then the space $N_{\mathcal{J}(3)^{\mathbb{C}}}$ can be seen as a subset defined by $T_3(A) = 0$ in the quadrics $Q_2 = \{A \in \mathbb{C}^{26} \backslash \{0\} \mid T_2(A) = 0\}$ and the polynomial T_3 restricted on the space $z = y = x = 0$ is irreducible even modulo T_2, i.e., there are no decomposition such that $T_3(A) = {\xi_1}^2\xi_2 + \xi_1{\xi_2}^2 = (a\xi_1 + b\xi_2)(\alpha{\xi_1}^2 + \beta\xi_1\xi_2 + \gamma{\xi_2}^2)$ on ${\xi_1}^2 + {\xi_2}^2 + \xi_1\xi_2 = 0$. Hence the space $N_{\mathcal{J}(3)^{\mathbb{C}}}$ must be irreducible and we have

$$\sqrt{I_+(\mathcal{J}(3)^{\mathbb{C}})} = I_+(\mathcal{J}(3)^{\mathbb{C}}).$$

□

In fact, our space $N_{\mathcal{J}(3)^{\mathbb{C}}}$ is an irreducible algebraic manifold and a complete intersection. In particular, there are points in $N_{\mathcal{J}(3)^{\mathbb{C}}}$ at which the differentials $dT_1,\ dT_2,\ dT_3$ are linearly independent (see the Lemma 4 on page 345 [Ko] for these aspects).

Especially, as a corollary of Proposition 8.19 we have

Proposition 8.21. *The representation of F_4 to the space $H_k = H_k^{(1)}$ is irreducible for each k.*

Proof. Since $\mathbb{X}_{\mathbb{O}}$ is connected, if the space H_k is decomposed into two invariant subspaces, $H_k = G_1 \oplus G_2$, then they are orthogonal.Consequently, according to this decomposition the space $\mathbb{X}_{\mathbb{O}}$ must be separated into two non intersecting closed subsets and this is a contradiction.

Hence each H_k must be irreducible under the action by the group F_4. □

Now we sum up a conclusion as

Theorem 8.22. *Since the functions in the invariant polynomials I_k are constant on the manifold $P^2\mathbb{O}$, by restricting polynomial functions in $\mathcal{P}_k[\mathcal{J}(3)]$ to $P^2\mathbb{O}$ the decompositions $\mathcal{P}_k[\mathcal{J}(3)] = H_k + I_1 H_{k-1} + \cdots + I_k$ for each k give totally a decomposition of a subspace in $C^\infty(P^2\mathbb{O})$ as*

$$\sum_{k=0}^{\infty} H_{k|P^2\mathbb{O}},$$

which is dense in $C^\infty(P^2\mathbb{O})$.

Proof. Based on the preceding arguments it will be enough to remark the last assertion, which is a standard argument.

Since any smooth function on $P^2\mathbb{O}$ can be extended to a smooth function on an open neighborhood of $P^2\mathbb{O}$ and the Weierstrass approximation theorem guarantees that any smooth function can be approximated in the $C^\infty-$topology by polynomials. Hence the space $\sum H_{k|P^2\mathbb{O}}$ is dense in $C^\infty(P^2\mathbb{O})$. □

Before interpreting the decomposition stated in Theorem 8.22 in the framework of the Peter-Weyl theorem for a symmetric space of our case $P^2\mathbb{O}$ we remark about the Riemannian metric on $P^2\mathbb{O}$.

Proposition 8.23. *The Cayley projective plane $P^2(\mathbb{O}) \cong F_4/Spin(9)$ is an irreducible Riemannian symmetric space, that is, the stationary subgroup $Spin(9)$ acts irreducibly on the tangent space $T_{X_1}P^2(\mathbb{O})$. By Schur's lemma this implies that $P^2(\mathbb{O})$ has an essentially unique F_4-invariant Riemannian metric. Thus, $(\,\cdot\,,\,\cdot\,)^{P^2\mathbb{O}}$ coincides with the metric on $P^2(\mathbb{O})$ induced from the Killing form of the Lie algebra of F_4 up to a constant factor.*

Let $\Phi_k : H_k \otimes {H_k}^* \longrightarrow C^\infty(F_4)$ be a map defined by

$$H_k \otimes {H_k}^* \ni h \otimes \varphi \longmapsto \Phi_k(h \otimes \varphi)(g) = \varphi(\mathcal{P}_{g^{-1}}(h)),\ g \in F_4,$$

then the Peter-Weyl theorem says that the image of the map Φ_k is a subspace consisting of the $\dim H_k$ number of the spaces, all of which are isomorphic to H_k.

Recall we explained the identification (3.2) of the quotient space $F_4/\mathrm{Spin}(9)$ with $P^2\mathbb{O}$ through the correspondence $F_4 \ni g \longmapsto g(X_1) \in P^2\mathbb{O}$.

If we consider a subspace ${H_k}^*{}_{|\mathrm{Spin}(9)}$ consisting of linear forms in ${H_k}^*$ which are invariant under the action by Spin(9), then the functions in

$$\Phi_k(H_k \otimes {H_k}^*{}_{|\mathrm{Spin}(9)})$$

are Spin(9) invariant, so that it can be descended naturally to functions on $F_4/\mathrm{Spin}(9) \cong P^2\mathbb{O} \subset \mathcal{J}(3)$.

For $X \in \mathcal{J}(3)$ we denote the linear form $J_X \in {H_k}^*$

$$H_k \ni h \longmapsto J_X(h) = h(X),$$

that is, this is an evaluation at $X \in \mathcal{J}(3)$. In particular, we take a linear form $J_{X_1} \in {H_k}^*{}_{|\mathrm{Spin}(9)}$, then it can be written as

$$J_{X_1}(\mathcal{P}_{g^{-1}}(h)) = \mathcal{P}_{g^{-1}}(h)(X_1) = h(g(X_1)).$$

Hence through the identification $F_4/\mathrm{Spin}(9) \cong P^2\mathbb{O}$ the function $J_{X_1}(\mathcal{P}_{g^{-1}}(h))$ is the restriction of the original polynomial function $h \in H_k$ to $P^2\mathbb{O}$. Then we have

$$\sum_{k=0}^{\infty} \Phi_k(H_k \otimes \{J_{X_1}\}) = \sum_{k=0}^{\infty} H_{k|P^2\mathbb{O}}.$$

Since $\dim H_{k+1} > \dim H_k$ (see Appendix) and the space $\sum_{k=0}^{\infty} H_{k|P^2\mathbb{O}}$ is already dense in $C^\infty(P^2\mathbb{O})$, a fundamental theorem on compact symmetric spaces gives us

Proposition 8.24. *Each irreducible representation of the group F_4 appears in $C^\infty(P^2\mathbb{O})$ with multiplicity one as in the above way and incidentally $\dim {H_k}^*{}_{|Spin(9)} = 1$. Moreover by the Proposition 8.23 we can see that this decomposition is the eigenspace decomposition of the Laplacian on $P^2\mathbb{O}$.*

The dimension of the space ${H_k}^*{}_{|Spin(9)}$ is always one and the linear form J_{X_1} can be seen as a base vector of the space ${H_k}^*{}_{|Spin(9)}$ for any k.

9. Inverse of Bargmann type transformation

In this section, based on the data obtained until §8 we consider our Bargmann type transformation

$$\mathfrak{B} : \sum \mathcal{P}_k[\mathbb{X}_\mathbb{O}] \longrightarrow C^\infty(P^2\mathbb{O})$$

with respect to the parameter family of the inner products $\{(*\,,*)_\varepsilon\}_{-22<\varepsilon}$ on the space $\sum \mathcal{P}_k[\mathbb{X}_\mathbb{O}]$ on its boundedness and invertibility. It has a dense image from $\sum \mathcal{P}_k[\mathbb{X}_\mathbb{O}]$ always for a possible value of the parameter ε, but unlike the cases of spheres and other projective spaces (see [Ra2],[Fu1], [FY]), it need not be an isomorphism when $\varepsilon = 0$. This means in cases of the values of the parameter $\varepsilon > -47/2$ there are quantum states in $L_2(P^2\mathbb{O})$ which can not be seen by classical observables.

9.1. **Inverse transformation.** Let $\mathcal{A}_k$ be a transformation defined by

$$\mathcal{A}_k : H_k \ni \varphi \longmapsto \int_{P^2\mathbb{O}} \varphi(X)\cdot\left(\mathrm{tr}\,(X\circ A)\right)^k dv_{P^2\mathbb{O}}(X) \in \mathcal{P}_k[\mathbb{X}_\mathbb{O}]. \tag{9.1}$$

and

$$\mathbb{A}_k : H_k \ni \varphi \longrightarrow \mathbb{A}_k(\varphi) = \gamma\circ\mathcal{A}(\varphi) = \mathcal{A}_k(\varphi)\cdot \mathbf{t}_0\otimes\Omega_\mathbb{O} \in \Gamma_\mathcal{G}\left(\mathbb{L}\otimes K^\mathcal{G}, \mathbb{X}_\mathbb{O}\right). \tag{9.2}$$

The correspondence by γ is defined in (5.4).

Proposition 9.1. *For any inner product defined on the space $\mathcal{P}_k[\mathbb{X}_\mathbb{O}]$ according to the value of the parameter ε, the operator $\mathcal{A}_k$ is a constant times a unitary operator.*

Proof. For $\varphi \in H_k$ the inner product

$$(\,\mathcal{A}_k(\varphi), \mathcal{A}_k(\varphi))_\varepsilon \tag{9.3}$$

is expressed as

$$\begin{aligned}
&(\,\mathcal{A}_k(\varphi), \mathcal{A}_k(\varphi))_\varepsilon \\
&= \int_{\mathbb{X}_\mathbb{O}} \left|\int_{P^2\mathbb{O}} \varphi(X)(\,\mathrm{tr}\,(X\circ A)\,)^k dv_{P^2\mathbb{O}}(X)\right|^2 \cdot e^{-2\sqrt{2}\pi||A||^{1/2}}||A||^\varepsilon \Omega_\mathbb{O}\wedge\overline{\Omega_\mathbb{O}} \\
&= \int_{P^2\mathbb{O}}\int_{P^2\mathbb{O}} \left(\int_{\mathbb{X}_\mathbb{O}} \left(\mathrm{tr}\,(\tilde{X}\circ A)\,\right)^k\left(\,\mathrm{tr}\,(X\circ \overline{A})\,\right)^k\cdot e^{-2\sqrt{2}\pi||A||^{1/2}}\cdot||A||^\varepsilon\cdot\Omega_\mathbb{O}\wedge\overline{\Omega_\mathbb{O}}\right)\times \\
&\qquad\qquad\qquad \times \varphi(\tilde{X})\overline{\varphi(X)}dv_{P^2\mathbb{O}}(X)dv_{P^2\mathbb{O}}(\tilde{X}).
\end{aligned}$$

Here we consider the operator $\mathcal{B}_k$

$$\mathcal{P}_k[\mathbb{X}_\mathbb{O}] \ni h \longmapsto \mathcal{B}_k(h) := \int_{\mathbb{X}_\mathbb{O}} h(A)\cdot\left(\mathrm{tr}\,(X\circ\overline{A})\right)^k e^{-2\sqrt{2}\pi||A||^{1/2}}\cdot||A||^\varepsilon\cdot\Omega_\mathbb{O}(A)\wedge\overline{\Omega_\mathbb{O}(A)} \in H_k.$$

Since H_k consists of linear sums of functions of the form $\left(\mathrm{tr}\,(X\circ A)\right)^k$ by arbitrary $A \in \mathbb{X}_\mathbb{O}$ (see Proposition 8.20), we see that $\mathcal{B}_k(h) \in H_k$. Then the inner product (9.3) is understood as

$$(\,\mathcal{A}_k(\varphi), \mathcal{A}_k(\varphi))_\varepsilon = (\mathcal{B}_k\circ\mathcal{A}_k(\varphi), \varphi)^{P^2\mathbb{O}}.$$

Then the operator $\mathcal{B}_k\circ\mathcal{A}_k$ commutes with the F_4 action on H_k. Hence it must be a constant times identity operator (which constant we put b_k) so that the kernel function defined by the integral

$$L_k(X,\tilde{X}) := \left(\int_{\mathbb{X}_\mathbb{O}} \left(\,\mathrm{tr}\,(\tilde{X}\circ A)\,\right)^k\left(\,\mathrm{tr}\,(X\circ\overline{A})\,\right)^k\cdot e^{-2\sqrt{2}\pi||A||^{1/2}}\cdot||A||^\varepsilon\cdot\Omega_\mathbb{O}\wedge\overline{\Omega_\mathbb{O}}\right)$$

must satisfies the invariance:

$$L_k(g(X),\tilde{X}) = L_k(X, g^{-1}(\tilde{X})),\ \text{for } g\in F_4,\ X,\tilde{X}\in P^2\mathbb{O}. \tag{9.4}$$

Then the constant b_k is given by

$$\text{Trace of the operator } \mathcal{B}_k\circ\mathcal{A}_k = \int_{P^2\mathbb{O}} L_k(X,X)dv_{P^2\mathbb{O}} = b_k\dim H_k. \tag{9.5}$$

and the integral $\int_{P^2\mathbb{O}} L_k(X,X)dv_{P^2\mathbb{O}}$ is given by

$$\int_{P^2\mathbb{O}} L_k(X,X)dv_{P^2\mathbb{O}} \equiv L_k(X,X)\cdot \mathit{Vol}(P^2\mathbb{O}),$$

since by the invariance (9.4) the function $L_k(X,X)$ is a constant function and apparently is non-zero.

Now we know $\mathcal{B}_k$ is injective and so

$$\dim H_k \le \dim\mathcal{P}_k[\mathbb{X}_\mathbb{O}].$$

On the other hand, degree k polynomials generated by the invariant polynomials which are naturally extended to the complexification $\mathcal{J}(3)^{\mathbb{C}}$, that is, the polynomials

$$\sum_{i=0}^{k} \mathcal{P}_{k-i}[\mathcal{J}(3)^{\mathbb{C}}] \cdot I_{k-i} = \sum_{i=0}^{k} H_{k-i} \cdot I_i$$

(see Lemma 8.10) are all vanishing on the manifold $\mathbb{X}_{\mathbb{O}}$ so that

$$\dim \mathcal{P}[\mathbb{X}_{\mathbb{O}}] \leq \dim \mathcal{P}[\mathcal{J}(3)^{\mathbb{C}}] - \sum_{i=0}^{k} \dim H_k \cdot \dim I_k,$$

(see Proposition 8.17).

Hence the operator $\mathcal{B}_k$ is also surjective to the space $\mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]$. Consequently, the operator $\mathcal{B}_k$ is a constant times a unitary operator. □

Next, we determine the concrete value of the constant b_k:

Proposition 9.2.

$$L_k(X,X) = b_k \cdot \dim H_k = 2^{26} \cdot Vol(S(P^2\mathbb{O})) \cdot \frac{\Gamma(4k+44+2\varepsilon)}{2^{8k+66+3\varepsilon}\pi^{4k+44+2\varepsilon}},$$

where the constant $Vol(S(P^2\mathbb{O}))$ *is the volume of the unit cotangent sphere bundle* $S(P^2\mathbb{O})$ *of* $P^2\mathbb{O}$ *with respect to the volume form*

$$d\sigma_{S(P^2\mathbb{O})} := \frac{1}{16!} \cdot \theta^{P^2\mathbb{O}} \wedge \left(\omega^{P^2\mathbb{O}}\right)^{15}\Big|_{S(P^2\mathbb{O})}.$$

Proof. Since $L_k(X,X) = \displaystyle\int_{\mathbb{X}_{\mathbb{O}}} \Big|\operatorname{tr}(X \circ A)\Big|^{2k} \cdot e^{-2\sqrt{2}\pi||A||^{1/2}}||A||^{\varepsilon} \cdot \Omega_{\mathbb{O}}(A) \wedge \overline{\Omega_{\mathbb{O}}(A)}$ does not depend on the point $X \in P^2\mathbb{O}$, we have

$$\begin{aligned}
L_k(X,X) &= \int_{\mathbb{X}_{\mathbb{O}}} \Big|\operatorname{tr}(X \circ A)\Big|^{2k} \cdot e^{-2\sqrt{2}\pi||A||^{1/2}} \cdot ||A||^{\varepsilon} \cdot \Omega_{\mathbb{O}}(A) \wedge \overline{\Omega_{\mathbb{O}}(A)} \\
&= \int_{F_4} \left(\int_{\mathbb{X}_{\mathbb{O}}} \Big|\operatorname{tr}(g^{-1}(X) \circ A)\Big|^{2k} \cdot e^{-2\sqrt{2}\pi||A||^{1/2}} \cdot ||A||^{\varepsilon} \cdot \Omega_{\mathbb{O}}(A) \wedge \overline{\Omega_{\mathbb{O}}(A)} \right) dv_{F_4}(g)
\end{aligned}$$

$$= \int_{\mathbb{X}_{\mathbb{O}}} \left(\int_{F_4} \left|\operatorname{tr}\left(X \circ g\left(\frac{A}{||A||}\right)\right)\right|^{2k} dv_{F_4}(g) \right) \cdot ||A||^{2k+\varepsilon} \cdot e^{-2\sqrt{2}\pi||A||^{1/2}} \Omega_{\mathbb{O}}(A) \wedge \overline{\Omega_{\mathbb{O}}(A)}, \tag{9.6}$$

where dv_{F_4} is the normalized Haar measure on F_4.

The function

$$\int_{F_4} \left|\operatorname{tr}\left(X \circ g\left(\frac{A}{||A||}\right)\right)\right|^{2k} dv_{F_4}(g) \tag{9.7}$$

does not depend neither on $X \in P^2\mathbb{O}$ nor on $A \in \mathbb{X}_{\mathbb{O}}$, since the trace function $A \longmapsto \operatorname{tr}(A)$ is F_4-invariant, the group F_4 acts both on the spaces $P^2\mathbb{O}$ and the cotangent sphere bundle $S(P^2\mathbb{O}) \overset{\tau_{\mathbb{O}}}{\cong} S(\mathbb{X}_{\mathbb{O}})$ transitively and the Haar measure dv_{F_4} is bi-invariant.

Let $(X,Y) \in T_0^*(P^2\mathbb{O})$. Put $A_g(X,Y) := g(\tau_{\mathbb{O}}(X,Y))$, then

$$g(\tau_{\mathbb{O}}(X,Y)) = g\left(||Y||^2X - Y^2 + \sqrt{-1} \otimes \frac{||Y||}{\sqrt{2}}Y\right) = g(||Y||)^2 g(X) - g(Y)^2 + \sqrt{-1} \otimes \frac{||g(Y)||}{\sqrt{2}} g(Y).$$

Hence

$$\tau_{\mathbb{O}}{}^{-1}(A_g(X,Y)) = (\, X(A_g(X,Y)), Y(A_g(X,Y)\,) = (\, g(X), g(Y)\,) \in T_0^*(P^2\mathbb{O}).$$

The integral (9.7) is expressed as

$$\begin{aligned}
&\frac{1}{||A_g(X,Y)||^{2k}} \int_{F_4} \left|\operatorname{tr} X(A_g(X,Y)) \circ A_g(X,Y)\,\right|^{2k} dv_{F_4}(g) \\
&= \frac{1}{||Y||^{4k}} \int_{F_4} \left|\operatorname{tr} g(X) \circ \left(||g(Y)||^2 g(X) - g(Y)^2 + \sqrt{-1} \otimes \frac{||g(Y)||}{\sqrt{2}} g(Y)\right)\right|^{2k} dv_{F_4}(g)
\end{aligned}$$

$$= \frac{1}{||Y||^{4k}} \int_{F_4} \left(\frac{1}{2}||g(Y)||^2\right)^{2k} dv_{F_4} = \frac{1}{2^{2k}},$$

since

$$g(X)^2 = g(X),\ \mathrm{tr}\, g(X) = 1\,,\ g(X)\circ g(Y) = \frac{1}{2}g(Y)$$

and we used the property

$$\mathrm{tr}\,(X\circ Y)\circ Z = \mathrm{tr}\, X\circ (Y\circ Z).$$

Now the integral (9.6) is

$$\begin{aligned}
&\int_{\mathbb{X}_{\mathbb{O}}}\int_{F_4}\Big|\mathrm{tr}\, X\circ g(A)\Big|^{2k} dv_{F_4}(g)\cdot e^{-2\sqrt{2}\pi||A||^{1/2}}||A||^{\varepsilon}\cdot \Omega_{\mathbb{O}}(A)\wedge\overline{\Omega_{\mathbb{O}}(A)}\\
&= \int_{\mathbb{X}_{\mathbb{O}}}\frac{1}{2^{2k}}||A||^{2k+\varepsilon}\cdot e^{-2\sqrt{2}\pi||A||^{1/2}}\cdot \Omega_{\mathbb{O}}(A)\wedge\overline{\Omega_{\mathbb{O}}(A)}\\
&= \frac{2^{26}}{2^{2k}}\int_{T_0^*(P^2\mathbb{O})}||Y||^{4k+28+2\varepsilon}\cdot e^{-2\sqrt{2}\pi||Y||}\cdot dV_{T^*(P^2\mathbb{O})}.
\end{aligned}\tag{9.8}$$

where we used the relation (6.8). Then according to the decomposition of the space $T_0^*(P^2\mathbb{O})\cong\mathbb{R}_+\times S(P^2\mathbb{O})$, we can decompose the Liouville volume form $dV_{T^*(P^2\mathbb{O})}$ as

$$dV_{T^*(P^2\mathbb{O})} = t^{15}dt\wedge d\sigma_{S(P^2\mathbb{O})},$$

where $d\sigma_{S(P^2\mathbb{O})}$ is the volume form on the unit cotangent sphere bundle $S(P^2\mathbb{O})$. Finally we have the integral (9.8) as

$$\begin{aligned}
&\frac{2^{26}}{2^{2k}}\cdot\int_{T_0^*(P^2\mathbb{O})}||Y||^{4k+28+2\varepsilon}\cdot e^{-2\sqrt{2}\pi||Y||}dV_{P^2\mathbb{O}} = \frac{2^{26}}{2^{2k}}\cdot\int_{S(P^2\mathbb{O})}d\sigma_{S(P^2\mathbb{O})}\int_0^{\infty}t^{4k+28+2\varepsilon}e^{-2\sqrt{2}\pi t}\cdot t^{15}dt\\
&= \frac{2^{26}}{2^{2k}}\cdot Vol(S(P^2\mathbb{O}))\cdot\frac{\Gamma(4k+44+2\varepsilon)}{(2\sqrt{2}\pi)^{4k+44+2\varepsilon}} = \frac{1}{2^{40}\pi^{44}}\cdot Vol(S(P^2\mathbb{O}))\cdot\frac{\Gamma(4k+44+2\varepsilon)}{2^{8k+3\varepsilon}\pi^{4k+2\varepsilon}},
\end{aligned}$$

and

$$b_k = \frac{1}{2^{40+3\varepsilon}\pi^{44+2\varepsilon}}\cdot Vol(S(P^2\mathbb{O}))\cdot\frac{\Gamma(4k+44+2\varepsilon)}{2^{8k}\pi^{4k}\dim H_k}.\tag{9.9}$$

□

Proposition 9.3. *Since both of the transformations $\mathcal{A}_k$ and the restriction of the transformation $\mathfrak{B}$ to the space $\mathcal{P}[\mathbb{X}_{\mathbb{O}}]$ (for short we denote it by $T_k := \mathfrak{B}_{|\mathcal{P}[\mathbb{X}_{\mathbb{O}}]}$) commute with F_4 action and the representation of F_4 on H_k is irreducible* (see (8.24))*, the composition $T_k\circ\mathcal{A}_k$ on $H_{k|P^2\mathbb{O}}$ is a constant multiple operator $T_k\circ\mathcal{A}_k = a_k Id$ and the constant a_k is given by*

$$\begin{aligned}
a_k &= 2^6\cdot Vol(S^{15})\cdot Vol(P^2\mathbb{O})\cdot\frac{\Gamma(2k+22)}{2^{k+11}\cdot\pi^{2k+22}\dim H_k}\\
&= \frac{1}{2^5\pi^{22}}\cdot Vol(S^{15})\cdot Vol(P^2\mathbb{O})\cdot\frac{\Gamma(2k+22)}{2^{2k}\cdot\pi^{2k}\dim H_k}.
\end{aligned}\tag{9.10}$$

Proof. Let $f\in H_k$ then by Corollary 7.9

$$\begin{aligned}
&T_k\Big(\mathcal{A}_k(f)\Big)(X)\cdot dv_{P^2\mathbb{O}}(X)\\
&= 2^6\mathbf{q}_*\left(\int_{P^2\mathbb{O}}f(\tilde{X})\cdot\{\mathrm{tr}\,(\tilde{X}\circ\tau_{\mathbb{O}}(X,*)\,)\}^k\cdot dv_{P^2}(\tilde{X})\cdot e^{-\sqrt{2}\pi\cdot||*||}\cdot||*||^6\cdot dV_{T^*(P^2\mathbb{O})}(X,*)\right)\\
&= 2^6\int_{P^2\mathbb{O}}f(\tilde{X})\mathbf{q}_*\Big(\{\mathrm{tr}\,(\tilde{X}\circ\tau_{\mathbb{O}}(X,*)\,)\}^k\cdot e^{-\sqrt{2}\pi\cdot||*||}\cdot||*||^6 dV_{T^*(P^2\mathbb{O})}(X,*)\Big)dv_{P^2\mathbb{O}}(\tilde{X})\\
&= 2^6\int_{P^2\mathbb{O}}f(\tilde{X})K_k(\tilde{X},X)dv_{P^2\mathbb{O}}(\tilde{X})\cdot dv_{P^2\mathbb{O}}(X),
\end{aligned}$$

where we put the fiber integral as

$$K_k(\tilde{X},X)\cdot dv_{P^2\mathbb{O}}(X) := \mathbf{q}_*\Big(\{\mathrm{tr}\,\tilde{X}\circ\tau_{\mathbb{O}}(X,*)\}^k\cdot e^{-\sqrt{2}\pi\cdot||*||}\cdot||*||^6 dV_{T^*(P^2\mathbb{O})}(X,*)\Big).$$

The kernel function $K_k(\tilde{X}, X)$ satisfies the property similar to the kernel function $L_k(\tilde{X}, X)$:

$$K_k(g \cdot \tilde{X},\, X) = K_k(\tilde{X},\, g^{-1}(X)). \tag{9.11}$$

Then by this property (9.11) that $K_k(X,X)$ is constant and we have

$$\mathrm{tr}\,(T_k \circ \mathcal{A}_k) = a_k \cdot \dim H_k = 2^6 \cdot \int_{P^2\mathbb{O}} K_k(X,X) dv_{P^2\mathbb{O}}(X) = 2^6 \cdot K_k(X,X) \cdot \mathit{Vol}(P^2\mathbb{O}).$$

Since $\mathrm{tr}\,\big(X \circ \tau_{\mathbb{O}}(X,Y)\big) = 1/2||Y||^2$,

$$\begin{aligned}&\mathbf{q}_*\Big(\{\mathrm{tr}\,(X \circ \tau_{\mathbb{O}}(X,*))\}^k \cdot e^{-\sqrt{2}\pi\cdot||*||} \cdot ||*||^6 \cdot dV_{T^*(P^2\mathbb{O})}(X,Y)\Big)\\ &= (1/2)^k \cdot \mathbf{q}_*\Big(||*||^{2k+6} e^{-\sqrt{2}\pi\cdot||*||} \cdot dV_{T^*(P^2\mathbb{O})}(X,*)\Big).\end{aligned}$$

If we choose a point $X = X_1$, then the above fiber integral is expressed as

$$\begin{aligned}&(1/2)^k \cdot \mathbf{q}_*\Big(||*||^{2k+6} e^{-\sqrt{2}\pi\cdot||*||} \cdot dV_{P^2\mathbb{O}}(X_1,*)\Big)\\ &= (1/2)^k \cdot \int_{\mathbf{q}^{-1}(X_1)} ||Y||^{2k+6} e^{-\sqrt{2}\pi||Y||} d\beta_0\wedge\cdots\wedge d\beta_7\wedge d\gamma_0\wedge\cdots\wedge d\gamma_7\wedge\\ &\qquad\qquad\wedge db_0\wedge\cdots\wedge db_7\wedge dc_0\wedge\cdots\wedge dc_7,\end{aligned}$$

$$= (1/2)^k \cdot \int_{\mathbf{q}^{-1}(X_1)} ||Y||^{2k+6} e^{-\sqrt{2}\pi||Y||} d\beta_0\wedge\cdots\wedge d\beta_7\wedge d\gamma_0\wedge\cdots\wedge d\gamma_7\wedge dv_{P^2\mathbb{O}}(X_1), \tag{9.12}$$

where we express the integral using the local coordinates on $\widetilde{\mathcal{W}}_1$ (see (7.5)) around the point X_1 and the dual coordinates $(X,Y) = (b,c,\beta,\gamma) \longleftrightarrow \sum_i \beta_i db_i + \gamma_i dc_i \in T^*_X(\widetilde{\mathcal{W}}_1)$. Then the integral (9.12) over the point X_1 is

$$\begin{aligned}&(1/2)^k \cdot \int_{\mathbf{q}^{-1}(X_1)} ||Y||^{2k+6} e^{-\sqrt{2}\pi||Y||} d\beta_0 \wedge\cdots\wedge d\beta_7 \wedge d\gamma_0 \wedge\cdots\wedge d\gamma_7\\ &= (1/2)^k \cdot \int_{\mathbb{R}^{16}} \Big(\sum {\beta_i}^2 + {\gamma_i}^2\Big)^{k+3} e^{-\sqrt{2}\pi\sqrt{\sum ({\beta_i}^2+{\gamma_i}^2)}} d\beta_0\cdots d\beta_7 d\gamma_0\cdots d\gamma_7 = \frac{\Gamma(2k+22)}{2^{2k+11}\cdot\pi^{2k+22}} \cdot \mathit{Vol}(S^{15}).\end{aligned}$$

Here $\mathit{Vol}(S^{15})$ is the volume of the standard 15-sphere. □

Now we have

$$a_k \dim H_k = 2^6 \cdot \frac{\Gamma(2k+22)}{2^{2k+11}\cdot\pi^{2k+22}} \cdot \mathit{Vol}(S^{15}) \cdot \mathit{Vol}(P^2\mathbb{O}) = \frac{1}{2^5\pi^{22}} \cdot \mathit{Vol}(S^{15}) \cdot \mathit{Vol}(P^2\mathbb{O}) \cdot \frac{\Gamma(2k+22)}{2^{2k}\cdot\pi^{2k}}.$$

Proposition 9.4.

$$\mathfrak{B}_{|\mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]} \circ \mathcal{A}_k = \frac{1}{2^5\pi^{22}} \frac{\Gamma(2k+22)}{2^{2k}\pi^{2k}\cdot\dim H_k} \cdot \mathit{Vol}(S^{15}) \cdot \mathit{Vol}(P^2\mathbb{O})\, Id.$$

Corollary 9.5. *The operator norm* $||\mathfrak{B}^{-1}{}_{|\mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]}||$ *is given by*

$$||\mathfrak{B}^{-1}{}_{|\mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]}|| = \frac{\sqrt{b_k}}{a_k} = \frac{\sqrt{\frac{Vol(S(P^2\mathbb{O})\Gamma(4k+44+2\varepsilon))}{2^{40+3\varepsilon}\pi^{44+2\varepsilon}\cdot 2^{8k}\pi^{4k}\dim H_k}}}{\frac{Vol(S^{15})\cdot Vol(P^2\mathbb{O})\cdot\Gamma(2k+22)}{2^{2k}\pi^{2k}\cdot\dim H_k}} := C(\varepsilon)\cdot N(k), \tag{9.13}$$

where $C(\varepsilon)$ *includes only* ε *and* $N(k)$ *is a function of* k *and*

$$N(k)^2 = \frac{2^{4k}\cdot\dim H_k\cdot\Gamma(4k+44+2\varepsilon)}{2^{8k}\Gamma(2k+22)^2}. \tag{9.14}$$

It is enough to see (9.14) for the behavior of the norm (9.13) when $k \longrightarrow \infty$ and for this purpose we mention two properties of the Gamma function.

Lemma 9.6.

$$\lim_{k\to\infty} \frac{\Gamma(k+\alpha_1)\cdots\Gamma(k+\alpha_\ell)}{\Gamma(k+\beta_1)\cdots\Gamma(k+\beta_\ell)} = \begin{cases} +\infty, & \text{if } \sum\alpha_i > \sum\beta_i,\\ 1, & \text{if } \sum\alpha_i = \sum\beta_i,\\ 0, & \text{if } \sum\alpha_i < \sum\beta_i.\end{cases}$$

Lemma 9.7.

$$\Gamma(nz) = \frac{n^{nz-1/2}}{(2\pi)^{(n-1)/2}} \cdot \prod_{j=0}^{n-1} \Gamma\left(z + \frac{j}{n}\right).$$

Then by Lemma 9.7

$$N(k)^2 = \frac{2^{44+4\varepsilon}}{\sqrt{2\pi}} \cdot \dim H_k \cdot \frac{\prod_{j=0}^{3} \Gamma(k+11+\varepsilon/2+j/4)}{\Gamma(k+11)^2 \cdot \Gamma(k+11+1/2)^2}. \tag{9.15}$$

By the relation of the Poincaré polynomials $PP(t) = PH(t) \cdot PI(t)$ (see (A.2)), the dimension of H_k is given as

$$\dim H_k = {}_{24+k-1}C_k + 2\cdot{}_{24+k-2}C_{k-1} + 2\cdot{}_{24+k-3}C_{k-2} + {}_{24+k-4}C_{k-3}. \tag{9.16}$$

Hence (9.15) is

$$N(k)^2 = \frac{2^{44+4\varepsilon}}{\sqrt{2\pi}} \cdot \left(\frac{\Gamma(24+k)}{\Gamma(k+1)} + 2\frac{\Gamma(23+k)}{\Gamma(k)} + 2\frac{\Gamma(22+k)}{\Gamma(k-1)} + \frac{\Gamma(21+k)}{\Gamma(k-2)}\right) \times$$
$$\times \frac{\prod_{j=0}^{3} \Gamma(k+11+\varepsilon/2+j/4)}{\Gamma(k+11)^2 \cdot \Gamma(k+11+1/2)^2}.$$

Hence finally by Lemma 9.6 have

Theorem 9.8. (1) *Let $\varepsilon = -\frac{47}{4}$, then the Bargmann type transformation*

$$\mathfrak{B} : \mathfrak{F}_{-47/4} \longrightarrow L_2(P^2\mathbb{O}, dv_{P^2\mathbb{O}})$$

is an isomorphism, although it is not unitary.

(2) *If $-22 < \varepsilon < -\frac{47}{4}$, then the inverse of the Bargmann type transformation*

$$\mathfrak{B}^{-1} : L_2(P^2, dv_{P^2\mathbb{O}}) \longrightarrow \mathfrak{F}_\varepsilon$$

is bounded, but and the Bargmann type transformation can not be extended to the whole Fock-like space $\mathfrak{F}_\varepsilon$.

(3) *If $\varepsilon > -\frac{47}{4}$, then the Bargmann type transformation is bounded with the dense image, but not an isomorphism between the spaces $\mathfrak{F}_\varepsilon$ and $L_2(P^2, dv_{P^2\mathbb{O}})$.*

(4) *Let $\varepsilon \le -22$. Then, for such a k that $4k + 44 + 2\varepsilon \le 0$, the integral* (6.9) *does not converge, although the Bargmann type transformation is defined for such polynomials. Hence by defining an inner product on the finite dimensional space $\sum_{4k+44+2\varepsilon\le 0} \mathcal{P}_k[\mathbb{X}_\mathbb{O}]$ in a suitable way, the Bargmann type transformation behave in the same way as the case of* (2) (*see Remark 8*).

Remark 11. *The result in the above theorem differs from the original Bargmann transformation and other cases of the spheres, complex projective spaces and quaternion projective spaces for which the Bargmann type transformations are always isomorphisms* ([Ba], [Ra2], [Fu1], [FY]) *without a modification factor in the weight for defining an inner product in the Fock-like space.*

10. Some additional results

10.1. Reproducing kernel of the Fock-like space $\mathfrak{F}_\varepsilon$.

As an application of the explicit determination of the constant b_k we show our Fock-like space $\mathfrak{F}_\varepsilon$ has the reproducing kernel.

Since the operator $\mathcal{A}_k$ is an isomorphism from H_k to $\mathcal{P}_k[\mathbb{X}_\mathbb{O}]$ and the operator $\mathcal{B}_k \circ \mathcal{A}_k \equiv b_k$, the composition $\mathcal{A}_k \circ \mathcal{B}_k \equiv b_k$ too. The kernel function (we put it as $R_k(A, B)$, $(A, B) \in \mathbb{X}_\mathbb{O} \times \mathbb{X}_\mathbb{O}$) of the composition

$$\frac{\mathcal{A}_k \circ \mathcal{B}_k}{b_k},$$

which is the identity operator on $\mathcal{P}_k[\mathbb{X}_\mathbb{O}]$, is expressed as

$$R_k(A, B)$$
$$= \frac{\int_{P^2\mathbb{O}} (\mathrm{tr}\, X \circ A)^k (\mathrm{tr}\, X \circ \overline{B})^k dv_{P^2\mathbb{O}} \cdot e^{-2\sqrt{2}\pi(||A||^{1/2}+||B||^{1/2})} (||A|| \cdot ||B||)^{14+\varepsilon}}{b_k}$$
$$= \frac{\int_{P^2\mathbb{O}} (\mathrm{tr}\,(X \circ A/||A||))^k (\mathrm{tr}\,(X \circ \overline{B}/||B||))^k \cdot dv_{P^2\mathbb{O}} \cdot e^{-2\sqrt{2}\pi(||A||^{1/2}+||B||^{1/2})} (||A|| \cdot ||B||)^{k+14+\varepsilon}}{b_k}.$$

Hence the sum

$$R(A,B)) := \sum_{k=0}^{\infty} R_k(A,B)$$

$$\sum_{k=0}^{\infty} \frac{\int_{P^2\mathbb{O}} (\operatorname{tr} X\circ A/||A||)^k (\operatorname{tr} X\circ \overline{B}/||B||)^k dv_{P^2\mathbb{O}} \cdot e^{-2\sqrt{2}\pi(||A||^{1/2}+||B||^{1/2})}(||A||\cdot||B||)^{k+14+\varepsilon}}{b_k}.$$

is estimated as

$$\left|\sum_{k=0}^{\infty} \frac{\int_{P^2\mathbb{O}} (\operatorname{tr} X\circ A/||A||)^k (\operatorname{tr} X\circ \overline{B}/||B||)^k \cdot dv_{P^2\mathbb{O}} \cdot e^{-2\sqrt{2}\pi(||A||^{1/2}+||B||^{1/2})}(||A||\cdot||B||)^{k+14+\varepsilon}}{b_k}\right|$$

$$\leq \frac{Vol(P^2\mathbb{O})2^{40+3\varepsilon}\pi^{44+2\varepsilon}}{Vol(S(P^2\mathbb{O}))} \cdot e^{-2\sqrt{2}\pi(||A||^{1/2}+||B||^{1/2})}(||A||\cdot||B||)^{14+\varepsilon} \cdot \sum \frac{2^{8k}\pi^{4k}\cdot(||A||||B||)^k}{\Gamma(4k+44+2\varepsilon)} \times$$
$$\times \left(\frac{\Gamma(24+k)}{\Gamma(k+1)} + 2\frac{\Gamma(23+k)}{\Gamma(k)} + 2\frac{\Gamma(22+k)}{\Gamma(k-1)} + \frac{\Gamma(21+k)}{\Gamma(k-2)}\right)$$

This inequality implies that the series converges locally uniformly on the space $\mathbb{X}_{\mathbb{O}} \times \mathbb{X}_{\mathbb{O}}$ and the function $R(A,\overline{B})$ is holomorphic there. So $R(A,B)$ is the reproducing kernel of the Hilbert space $\mathfrak{F}_\varepsilon$ $(\varepsilon > -22)$.

10.2. **Geodesic flow and eigenspaces of Laplacian on $P^2\mathbb{O}$.** Let ϕ_t $(t\in\mathbb{R})$ be an action on $\mathbb{X}_{\mathbb{O}}$ defined by

$$\mathbb{X}_{\mathbb{O}} \ni A \longmapsto \phi_t(A) = e^{2\sqrt{-1}t}\cdot A.$$

Then this is an interpretation of the geodesic flow action onto the space $\mathbb{X}_{\mathbb{O}}$ through the map $\tau_{\mathbb{O}}$.

Let $p \in \mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]$. Then

$$\phi_t{}^*(p\cdot \mathbf{t}_0 \otimes \Omega_{\mathbb{O}})(A) = e^{2\sqrt{-1}t(11+k)}\cdot p(A)\cdot \mathbf{t}_0(A)\otimes \Omega_{\mathbb{O}}(A). \tag{10.1}$$

Let $p \in \mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]$ and $q \in \mathcal{P}_\ell[\mathbb{X}_{\mathbb{O}}]$ with $k \neq \ell$, then

Lemma 10.1.

$$(p,,q)_\varepsilon = \int_{\mathbb{X}_{\mathbb{O}}} p\cdot\overline{q}\cdot {g_0}^2\cdot ||A||^\varepsilon \cdot \Omega_{\mathbb{O}}\wedge\overline{\Omega_{\mathbb{O}}} = 0.$$

Proof. The transformation $\phi_t{}^*$ on $\Gamma_{\mathcal{G}}(\mathbb{L}\otimes K^{\mathcal{G}}, \mathbb{X}_{\mathbb{O}})$ is unitary, hence

$$(\phi_t{}^*(p), \phi_t{}^*(q))_\varepsilon \equiv (p,q)_\varepsilon \text{ for any } t\in\mathbb{R}.$$

On the other hand

$$\phi_t{}^*\big(p\cdot\overline{q}\cdot < \mathbf{t}_0, \mathbf{t}_0 >^{\mathbb{L}} \Omega_{\mathbb{O}}\wedge\overline{\Omega_{\mathbb{O}}}\big) = e^{2\sqrt{-1}(k-\ell)t}\cdot\big(p\cdot\overline{q}\cdot < \mathbf{t}_0, \mathbf{t}_0 >^{\mathbb{L}} \Omega_{\mathbb{O}}\wedge\overline{\Omega_{\mathbb{O}}}\big).$$

Hence $(p,q)_\varepsilon = 0$. □

Let $\Delta^{P^2\mathbb{O}}$ be the Laplacian on $P^2\mathbb{O}$. Then

Proposition 10.2. *The geodesic flow action on $\mathbb{X}_{\mathbb{O}}$ and the action given by the one parameter group $\{e^{2\sqrt{-1}\,t\,\sqrt{\Delta^{P^2\mathbb{O}}+11}}\}$ of unitary transformations consisting of the Fourier integral operators commute through the Bargmann type transformation.*

Proof. This is shown based on the data that the eigenvalues of the Laplacian $\Delta^{P^2\mathbb{O}}$ is given by k^2+11k and the Bargmann type transformation on each subspace $\mathcal{P}_k[\mathbb{X}_{\mathbb{O}}]$ maps to H_k which coincides with the k-th eigenspace of the Laplacian (Propositions 8.23, 8.24). □

Remark 12. *Finally we mention that in a forthcoming paper the reproducing kernel above will be made clear to relate with a differential equation satisfied by some hypergeometric functions and also a Töplits operator theory on $P^2\mathbb{O}$ will be discussed.*

Appendix A. Appendix:Generating functions of Poincaré series

In this Appendix we consider the generating functions of the Poincaré series of

(1) *the polynomial algebra* : $PP(t) = \sum \dim P_k \, t^k$,

(2) *the algebra of invariant polynomials* : $PI(t) = \sum \dim I_k \, t^k$ and

(3) *the space of the Cayley harmonic polynomials* : $PH(t) = \sum \dim H_k \, t^k$,

and prove the inequality :

$$\dim \ H_{k+1} > \dim \ H_k. \tag{A.1}$$

In fact, these formal power series converge for $|t| < 1$, which will be seen by explicitly determining their generating functions.

The generating function $PI(t)$ of the *Poincaré series* of the dimensions of invariant polynomials $I = \sum I_k$ is determined as

$$PI(t) = \sum_{k=0}^{\infty} \dim I_k \, t^k = \sum_{k=0}^{\infty} \sum_{\ell=0}^{[k/3]} \left(\left[\frac{k-3\ell}{2} \right] + 1 \right) t^k = \sum_{k=0}^{\infty} \sum_{i_1+2i_2+3i_3=k, \ i_1,i_2,i_3 \in \mathbb{N}_0} t^k$$
$$= \sum_{(i_1,i_2,i_3) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0} t^{i_1+2i_2+3i_3} = \frac{1}{1-t} \cdot \frac{1}{1-t^2} \cdot \frac{1}{1-t^3}.$$

The generating function $PP(t)$ of the polynomial algebra $\mathbb{C}[s_1, \cdots, s_N] = \sum \ P_k$ is given by

$$PP(t) = \sum \dim P_k \, t^k = \sum_{k=0}^{\infty} {}_{N+k-1}C_k \, t^k$$
$$= \sum_{(r_1,r_2,\ldots,r_N) \in {\mathbb{N}_0}^N} t^{r_1+r_2+\cdots+r_N} = \left(\frac{1}{1-t} \right)^N, \text{ in which } N = 27 \text{ for our case.}$$

Let $PH(t)$ be the generating function of the Poincaré series of the dimensions of Cayley harmonic polynomials, then by Lemma 8.16 and Proposition 8.17

$$PP(t) = PH(t) \cdot PI(t) \tag{A.2}$$

and we have

$$PH(t) = \left(\frac{1}{1-t} \right)^{24} \cdot (1+t)(1+t+t^2)$$
$$= \sum_{k=0}^{\infty} {}_{24+k-1}C_k \, t^k \cdot (1+2t+2t^2+t^3) = \sum_{k=0}^{\infty} \dim H_k \, t^k. \tag{A.3}$$

Then

Proposition A.1. $\dim H_k < \dim H_{k+1}$.

This can be proved by the following elementary fact:

Lemma A.2. *Let $f(t) = \sum a_k t^k$ and $g(t) = \sum b_k t^k$ be formal power series with positive coefficients and satisfies the condition that*

$$\text{for all } n, \ b_n \le b_{n+1}.$$

Then the coefficients of the product formal power series $f \cdot g$ is increasing.

Proof. Since the n-th coefficient c_n of the product fg is

$$c_n = \sum_{i=0}^{n} a_{n-i} b_i$$
$$c_{n+1} - c_n = a_{n+1} b_0 + a_n(b_1 - b_0) + \cdots + a_0(b_{n+1} - b_n).$$

In the above expression, each term is non-negative by assumption so that $c_{n+1} - c_n \ge 0$. In addition if $\{b_n\}$ is strictly increasing, then $\{c_n\}$ is also strictly increasing at least one of the coefficient being $a_k > 0$. □

Proof of Proposition A.1.

In our case, all the coefficients of the polynomial $(1+t)(1+t+t^2) = 1+2t+2t^2+t^3$ are positive and the coefficients of the power series expansion of the factor $\left(\dfrac{1}{1-t}\right)^{24}$ are positive and strictly increasing. In fact, the k-th coefficient of the power series expansion of the function $\left(\dfrac{1}{1-t}\right)^{24}$ is ${}_{24+k-1}C_k$ and strictly increasing, since it is a generating function of the Poincaré power series of the polynomial algebra $\mathbb{C}[s_1, \cdots, s_{24}]$ of 24 variables. Hence the assertion for our power series $PH(t) = (1+2t+2t^2+t^3)\cdot\left(\dfrac{1}{1-t}\right)^{24}$ is proved. □

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