Classical Poincaré conjecture via 4D topology

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ABSTRACT

The classical Poincaré conjecture that every homotopy 3-sphere is diffeomorphic to the 3-sphere is proved by G. Perelman by solving Thurston's program on geometrizations of 3-manifolds. A new confirmation of this conjecture is given by combining R. H. Bing's result on this conjecture with Smooth Unknotting Conjecutre for an S^2-knot and Smooth 4D Poincaré Conjecture.

Keywords: Homotopy 3-sphere, Smooth unknotting, Smooth homotopy 4-sphere.

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1. Introduction

A *homotopy 3-sphere* is a smooth 3-manifold M homotopy equivalent to the 3-sphere S^3. It is well-known that a simply connected closed connected 3-manifold is a smooth homotopy 3-sphere. The following theorem, called the classical Poincaré Conjecture coming from [14, 15] is positively shown by Perelman [12, 13] solving positively Thurston's program [16] on geometrizations of 3-manifolds (see [11] for detailed histrical notes).

Theorem 1.1. Every homotopy 3-sphere M is diffeomorphic to the 3-sphere S^3.

The purpose of this paper is to give an alternative proof to Theorem 1.1 by combining R. H. Bing's result in [1, 2] on the classical Poincaré conjecture with Smooth Unknotting Conjecutre and Smooth 4D Poincaré Conjecture which are explained from now. Smooth Unknotting Conjecture (for a surface-knot) is the following conjecutre.

Smooth Unknotting Conjecture. Every smooth surface-knot F in the 4-sphere S^4 is a trivial surface-knot if the fundamental group $\pi_1(S^4 \setminus F, x_0)$ is an infinite cyclic group.

The positive proof of this conjecture for any surface-knot is claimed in [7], where a revised proof of the uniquness of an O2-handle pair is in preparation. (See also [8] for a generalization to a surfece-link). A *homotopy 4-sphere* is a smooth 4-manifold X homotopy equivalent to the 4-sphere S^4. Smooth 4D Poincaré Conjecture is the following conjecutre.

Smooth 4D Poincaré Conjecture. Every 4D smooth homotopy 4-sphere X is diffeomorphic to the 4-sphere S^4.

The positive proof of this conjecture is claimed in [9]. The result of R. H. Bing in [1, 2] used for the proof of Theorem 1.1 is in the form of the following lemma.

Lemma 1.2. A homotopy 3-sphere M is diffeomorphic to the 3-sphere if for every knot K in M, there is a meridian-longitude-preserving isomorphism from the fundamental gorup $\pi_1(M \setminus K, x_0)$ onto the fumndamental group $\pi_1(S^3 \setminus K', x'_0)$ of a knot K' in S^3.

The proof of Lemma 1.2 is given in Section 2 together with an explanation of Bing's result. The main result of this paper to prove the follwing lemma.

Lemma 1.3. Every homotopy 3-sphere M meets the assumption of Lemma 1.2. Namely, for every knot K in M, there is a meridian-longitude-preserving isomorphism from the fundamental gorup $\pi_1(M \setminus K, x_0)$ onto the fundamental group $\pi_1(S^3 \setminus K', x'_0)$ of a knot K' in the 3-sphere S^3.

For the proof of Lemma 1.3, a homotopy 3-sphere version of Artin's spinning construction on S^3 is used. This explanation is done in Section 3. In Section 4, the proof of Lemma 1.3 is done. Lemmas 1.2 and 1.3 complete the proof of Theorem 1.1.

Troughout the paper, the notations of the unit disk $D = \{x \in \mathbb{C} | \, |x| \leq 1\}$ and the unit circle $S^1 = \partial D$ are fixed.

2. Proof of Lemma 1.2

The proof of Lemma 1.2 is made as follows.

Proof of Lemma 1.2. R. H. Bing's result in [1, 2] is stated as follows:

Bing's Theorem. A homotopy 3-sphere M is diffeomorphic to S^3 if, for every knot K in M, there is a 3-ball in M containing the knot K.

Let K be a knot in M with a meridian-longitude preserving isomorphism

$$\pi_1(M \setminus K, x_0) \cong \pi_1(S^3 \setminus K', x'_0)$$

for a knot K' in S^3. Let $N(K)$ be a tubular neighborhood of the knot K in M. Let $E(K; M) = \mathrm{cl}(M \setminus N(K))$ be the compact exterior of the knot K in M. By H. Kneser's prime factorization of a compact 3-manifold in [10] (see [4, p.31], the compact 3-manifold $E(K; M)$ is written as a connected sum

$$E(K; M) \cong E' \# M_1 \# M_2 \# \ldots \# M_k$$

for an irreducible compact connected 3-manifold E' with torus boundary and finitely many irreducible homotopy 3-spheres M_i $(i = 1, 2, \ldots, k)$. The 3-manifold E' is understood as the compact exterior $E(K; M')$ of a knot K in a homotopy 3-sphere M'. Note that the meridian-longitude system of $E(K; M)$ is taken over by a meridian-longitude system of the irreducible 3-manifold E'. Thus, there is a meridian-longitude-preserving isomorphism

$$\pi_1(E', x_0) \cong \pi_1(E(K'), x'_0)$$

for the compact exterior $E(K') = \mathrm{cl}(S^3 \setminus N(K'))$ of the knot K' in S^3.

If $\pi_1(E', x_0) \cong \mathbf{Z}$, then by Dehn's lemma (cf. e.g. [6]), the compact 3-manifold E' is diffeomorphic to the product $S^1 \times D$ by a diffeomorphism sending the meridian-longitude pair of E' to the pair $(1 \times S^1, S^1 \times 1)$. Then we see the knot K is a trivial knot in a 3-ball in M.

Assume that $\pi_1(E', x_0) \not\cong \mathbf{Z}$. Then the compact 3-manifold E' is a Haken manifold and hence by F. Waldhausen's result in [17] (see [4]), there is a meridian-longitude-preserving diffeomorphism $E' \cong E(K')$ (cf. [6, p. 74]), which means that the knot K is also in a 3-ball in M.

Thus, it is shown that for any knot in M with a meridian-longitude preserving isomorphism $\pi_1(M \setminus K, x_0) \cong \pi_1(S^3 \setminus K', x'_0)$ for a knot K' in S^3, it is shown that there is a 3-ball B containing the knot K. Since the assumption of Lemma 1.2 is that every knot in M admits a meridian-longitude preserving isomorphism $\pi_1(M \setminus K, x_0) \cong \pi_1(S^3 \setminus K', x'_0)$ for a knot K' in S^3, Bing's theorem means that M is diffeomorphic to S^3. This completes the proof of Lemma 1.2. □

3. Artin's pinning construction

Let M be a smooth homotopy 3-sphere, and M^o the compact once-punctured manifold $\mathrm{cl}(M \setminus B)$ of M for a 3-ball B in M. The closed smooth 4-manifold

$$\Sigma_B^M = (\partial B) \times D \cup M^o \times S^1$$

is a smooth homotopy 4-sphere by van Kampen theorem and a homological argument, which is called the *Artin's spinning construction* of the pair (M, B).

The product $\Lambda' \times D$ for a subspace $\Lambda' \subset \partial B$ in $(\partial B) \times D$, the product $\Lambda'' \times S^1$ for a subspace $\Lambda'' \subset M^0$ in $M^o \times S^1$, or the union $\Lambda' \times D \cup \Lambda'' \times S^1$ in $S(M, B)$ is called a *round object*.

By Smooth 4D Poincaré Conjecture, the closed smooth 4-manifold Σ_B^M is deffeomorphic to the 4-sphere S^4. For a knot K in the interior of M^o, the *spun torus-knot* $K \times S^1$ in the 4-sphere Σ_B^M is a round object uniquely given by the inclusions

$$K \times S^1 \subset M^o \times S^1 \subset \Sigma_B^M.$$

Let $c : K \times [0,1] \to M^o$ be a collar embedding of K in M^o such that $c(K, 0) = K$ and $\ell(K) = c(K, 1)$ is a longitude of K in M^o. For a base point $p \in K$, let $\bar{\ell}(S^1) = c(p, 1) \times S^1$ and $\bar{\ell}(K) = c(K, 1) \times 1 \subset M^0 \times S^1$ be the simple loops around the spun torus-knot $K \times S^1$ in the 4-sphere Σ_B^M. Take the point $x_0 = c(p, 1) \in M^o \setminus K$ as the base point of the longitude $\ell(K)$ of K and the point $\bar{x}_0 = c(p, 1) \times 1 \in \Sigma_B^M \setminus K \times S^1$ as the base point of the loops $\bar{\ell}(S^1)$ and $\bar{\ell}(K)$. Let $m(K)$ be a meridian loop of K in M^o with base point x_0, and $\bar{m}(K \times S^1)$ a meridian loop of $K \times S^1$ in Σ_B^M with base point $\bar{x}_0$, respectively. The following lemma is directly obtained by van Kampen theorem.

Lemma 3.1. For every knot K in the interior of M^o and the spun torus-knot $K \times S^1$ in the 4-sphere Σ_B^M, the fundamental group $\pi_1(\Sigma_B^M \setminus K \times S^1, \bar{x}_0)$ is isomorphic to the fundamental group $\pi_1(M^o \setminus K, x_0)$ by an isomprphism sending the homotopy class $[\bar{\ell}(S^1)]$ to the trivial element $\{1\}$, the homotopy class $[\bar{\ell}(K)]$ to the homotopy class $[\ell(K)]$ and the meridian element $[\bar{m}(K \times S^1)]$ to the meridian element $[m(K)]$, respectively.

Proof of Lemma 3.1. Apply van Kampen theorem for $(M^o \setminus K) \times S^1$ and $(\partial B) \times D$ to obtaine the desired result. $\square$

The following lemma is essentially a corollary to Lemma 3.1.

Lemma 3.2. For a 3-ball B and the standard spin structure on the product $B \times S^1$, let $\varphi : B \times S^1 \to S^4$ be a spin-preserving smooth embedding. For a knot K in the interior of B, the torus-knot $K \times S^1$ in S^4 given by the composite embedding

$$K \times S^1 \subset B \times S^1 \xrightarrow{\varphi} S^4$$

has the fundamental group $\pi_1(S^4 \setminus K \times S^1, x_0')$ which is isomorphic to the fundamental group $\pi_1(B \setminus K, x_0)$ by an isomprphism sending the homotopy class $[\bar{\ell}(S^1)]$ to the trivial element $\{1\}$, the homotopy class $[\bar{\ell}(K)]$ to the homotopy class $[\ell(K)]$ and the meridian element $[\bar{m}(K \times S^1)]$ to the meridian element $[m(K)]$, respectively.

Proof of Lemma 3.2. Let $S^4 = (\partial B) \times D \cup B \times S^1$ be the union of round objects $(\partial B) \times D$ and $B \times S^1$. The embedding φ is smoothly isotopic to an embedding $\varphi' : B \times S^1 \to S^4$ with $\varphi'(B \times S^1) = B \times S^1$. By the isotopy classification of the orientable handle $S^1 \times S^2$ by H. Gluck [3] and the spin assumption on φ, we can assume that the restriction of φ' to the orientable handle $(\partial B) \times S^1$ is the identity map. Hence the embedding $\varphi' : B \times S^1 \to S^4$ extends to a diffeomorphism $(\varphi')^+ : S^4 \to S^4$ whose restriction to the round object $(\partial B) \times D$ is the identity map. By Lemma 3.1, the desired result is obtained. $\square$

For a connected spatial graph Γ in M and a maximal tree T of Γ, let B be a regular neighborhood of T in M which is a 3-ball. Let $a(\Gamma) = \mathrm{cl}(\Gamma \setminus B \cap \Gamma)$ be a proper arc system in $M^o = \mathrm{cl}(M \setminus B)$, and $\dot{a}(\Gamma) = (\partial B) \cap \Gamma$ the boundary point system in the 2-sphere $\partial M^0 = \partial B$. The *spun* S^2*-link* L^Γ_T is a round object uniquely constructed from the pair (Γ, T) in the 4-sphere Σ^M_B by taking the union of the annulus system $a(\Gamma) \times S^1$ and the disk system $\dot{a}(\Gamma) \times D$ as follows:

$$L^\Gamma_T = a(\Gamma) \times S^1 \cup \dot{a}(\Gamma) \times D.$$

The following lemma is a generalization of Lemma 3.1 to the spun S^2-link, which was observed in [6, p.204].

Lemma 3.3. For every connected spatial graph Γ in M and the spun S^2-link L^Γ_T in the 4-sphere Σ^M_B, the fundamental group $\pi_1(\Sigma^M_B \setminus L^\Gamma_T, \bar{x}_0)$ is isomorphic to the fundamental group $\pi_1(M \setminus \Gamma, x_0)$ by an isomprphism sending the meridian system of L^Γ_T to a meridian system of the proper arc system $a(\Gamma)$ in M^o.

Proof of Lemma 3.3. Note that there is a canonical isomorphism

$$\pi_1(M^0 \setminus a(\Gamma), x_0) \cong \pi_1(M \setminus \Gamma, x_0).$$

Then apply van Kampen theorem between $(M^o \setminus a(\Gamma)) \times S^1$ and $((\partial B) \setminus \dot{a}(\Gamma)) \times D$. The result is obtained. $\square$

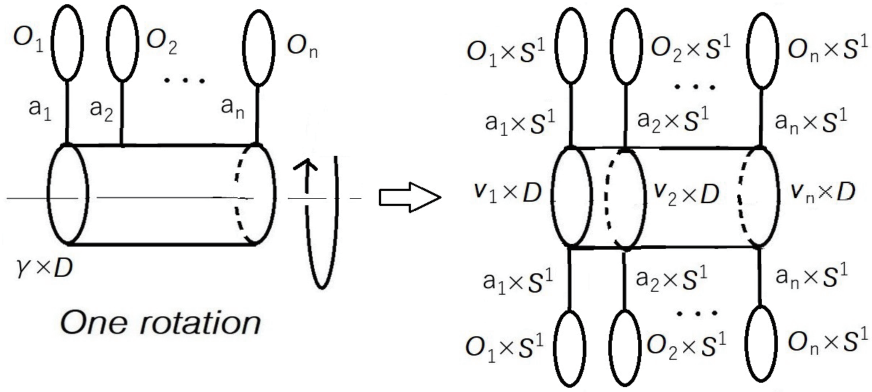

Figure 1: The round object $Y = \Gamma \times S^1 \cup \beta \times D$

Let V be a handlebody of genus n in M, and $F = \partial V$. Notions on graphs in the surface F are introduced as follows:

Let β be a simple arc in F. Let $O_i\,(i = 1, 2, \ldots, n)$ be a system of mutually disjoint simple loops in the surface F with $\beta \cap O_i = \emptyset$ for every i, and $v_i\,(i = 1, 2, \ldots, n)$ a system of distinct points in β enumerated in this order with $\partial\beta = \{v_0, v_n\}$. Let $a_i\,(i = 1, 2, \ldots, n)$ be a system of mutually disjoint simple arcs in F such that a_i spans the point v_i and a point $p_i \in O_i$ with interior $\mathrm{Int}(a_i)$ disjoint from $\beta \cup_{i=1}^{n} O_i$. The graph

$$\Gamma = \beta \cup_{i=1}^{n} (a_i \cup O_i)$$

is called an *n-bouquet graph* with *based arc* β in F. Let $P = N(\Gamma; F)$ be a regular neighborhood of an n-bouquet graph Γ with based arc β in F, which is a planar surface in F. Slide the based arc β of the n-bouquet graph Γ to a boundary component of P. Then the n-bouquet graph Γ is a *spine graph* of the handlebody V with based arc β if the framings of the loops O_i determined by P are 0-framings in M and there is a diffeomorphism

$$c : P \times [0, 1] \to V$$

such that $c(x, 0) = x$ for all $x \in P$. This planar surface P in F is called a *spine surface* of V. We have always a spine graph Γ and a spine surface P of V with based arc β for every handlebody V of genus n. For a maximal tree T of Γ around the arc β and a 3-ball neighborhood B of T in M, the spun S^2-link L_T^Γ in the 4-sphere Σ_B^M is constructed from the pair (Γ, T).

To construct a slightly different construction of a S^2-link in the 4-sphere Σ_B^M

equivalent to the spun S^2-link L_T^Γ, consider a 3-ball B in M so that the intersection

$$B \cap V = (\partial B) \cap F = \Delta$$

is a disk and the intersection

$$B \cap \Gamma = (\partial B) \cap \Gamma = \beta$$

is an arc in the boundary circle $\partial\Delta$ of the disk Δ so that the disk Δ is considered as the product $[0,1] \times \beta$ with $0 \times \beta = \beta$. Let

$$Y = \beta \times D \cup \Gamma \times S^1$$

be a round object in Σ_B^M which is the union of the solid cylinder $\beta \times D$ and the round object $\Gamma \times S^1$ for the spine graph Γ of V with based arc β. The round object Y is illustrated in Fig. 1 where note that the solid cylinder $\beta \times D$ is fixed setwise under the one rotation on Y along the central axis $\beta \times 0$. In Fig. 1, we find mutually disjoint disks

$$D_i = v_i \times D \cup a_i \times S^1 \quad (i = 1, 2, \ldots, n).$$

Let $h_i^2\,(i = 1, 2, \ldots, n)$ be mutually disjoint 2-handles on the tori $O_i \times S^1\,(i = 1, 2, \ldots, n)$ thickening the disks $D_i\,(i = 1, 2, \ldots, n)$, respectively. The 2-sphere components of L_T^Γ are equivalent to the 2-spheres $S_i\,(i = 1, 2, \ldots, n)$ obtained from the tori $O_i \times S^1\,(i = 1, 2, \ldots, n)$ by the surgeries along the 2-handles $h_i^2\,(i = 1, 2, \ldots, n)$, respectively. The S^2-link $L_\beta^\Gamma = \cup_{i=1}^n S_i$ in Σ_B^M constructed in this way is a round object and called the *spun S^2-link constructed from* the n-bouquet graph Γ *with based arc β*. The 2-handle h_i^2 on the torus $O_i \times S^1$ is considered as a 1-handle h_i^{*1} on the 2-sphere component S_i. Note that this construction is done in the 3-manifold

$$A(P) = \beta \times D \cup P \times S^1.$$

In the round object Y, let $\beta_j\,(j = 1, 2, \ldots, n-1)$ be the arcs obtained by dividing the central axis $\beta \times 0$ of the solid cylinder $\beta \times D$ at the points $v_i \times 0\,(i = 1, 2, \ldots, n)$ such that the arc β_j connects $v_j \times 0$ to $v_{j+1} \times 0$ for every $j\,(j = 1, 2, \ldots, n-1)$, so that the arc β_j connects the disk D_j to the disk D_{j+1} (see Fig. 1). The arc β_j is a core arc of the 1-handle $h_j^1 = \beta_j \times D$ on L_β^Γ connecting the component S_j to the component S_{j+1} for all $j\,(j = 1, 2, \ldots, n-1)$. Let $\alpha_i\,(i = 1, 2, \ldots, n)$ be the core arcs of the 1-handles $h_i^{*1}\,(i = 1, 2, \ldots, n)$ in the central axis $\beta \times 0$ of the solid cylinder $\beta \times D$, so that the union

$$\cup_{i=1}^n \alpha_i \cup_{j=1}^{n-1} \beta_j$$

is a division of the central axis $\beta \times 0$ of the solid cylinder $\beta \times D$. Thus the S^2-link L_β^Γ with divided central axis $\beta \times 0 = \cup_{i=1}^n \alpha_i \cup_{j=1}^{n-1} \beta_j$ is obtained (see Fig. 2). Note that this construction is also done in the 3-manifold $A(P)$. We have the following lemma.

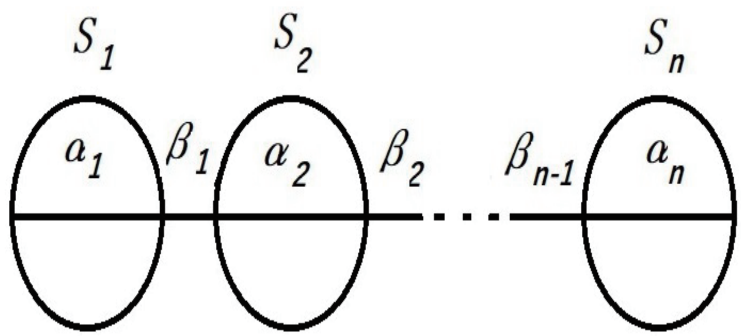

Figure 2: The S^2-link L^Γ_β with divided central axis $\beta \times 0 = \cup_{i=1}^n \alpha_i \cup_{j=1}^{n-1} \beta_j$

Lemma 3.4. The 3-manifold $A(P)$ is obtained from a regular neighborhood of the S^2-link L^Γ_β with divided central axis $\beta \times 0$ in $A(P)$ by adding a boundary collar $(\partial A(P)) \times [0,1]$ of $A(P)$ and the 4-manifold $\Delta \times D \cup V \times S^1$ is a collar $A(P) \times [0,1]$ of the 3-manifold $A(P)$ in Σ^M_B.

Proof of Lemma 3.4. By constructuin, the round object Y is naturally contained in $A(P)$ ant the closed complement $\mathrm{cl}(A(P) \setminus N(Y))$ for a regular neighborhood $N(Y)$ of Y in $A(P)$ is a boundary collar $(\partial A(P)) \times [0,1]$. Since

$$h^2_i \cup O_i \times S^1 = h^{*1}_i \cup S_i$$

for every i, the closed complement of a regular neighborhood of the union

$$L^\Gamma_\beta \cup_{i=1}^n h^{*1}_i \cup_{j=1}^{n-1} h^1_j$$

in $A(P)$ is also a boundary collar $(\partial A(P)) \times [0,1]$. Thus, the closed complement of a regular neighborhood of the union

$$L^\Gamma_\beta \cup_{i=1}^n \alpha_i \cup_{j=1}^{n-1} \beta_j$$

in $A(P)$ is also a boundary collar $(\partial A(P)) \times [0,1]$. Since the handlebodey V is a collar $P \times [0,1]$ in Σ^M_B and the disk Δ is the product $[0,1] \times \beta$ with $0 \times \Gamma = \beta$, we see that the 4-manifold $\Delta \times D \cup V \times S^1$ is a collar $A(P) \times [0,1]$. $\square$

The following lemma is technically important to the proof of Lemma 1.3.

Lemma 3.5. Let L^Γ_β be the S^2-link with divided central axis $\beta\times 0$ in Σ^M_B for a handlebody V in M, and $\bar{S}$ the S^2-knot $\bar{S}$ obtained from the spun S^2-link L^Γ_β by the surgeries along the 1-handles $h^1_j=\beta_j\times D\,(i=1,2,\dots,n-1)$. If the S^2-knot $\bar{S}$ is a trivial S^2-knot in Σ^M_B, then the round object $V\times S^1$ extends to the product $U\times S^1$ for a 3-ball U which is a smooth spin 4-submanifold of Σ^M_B.

Proof of Lemma 3.5. Note that the central axis $\beta\times 0$ spans the trivial 2-sphere $\bar{S}$. Let $\tilde{B}$ be a 3-ball in Σ^M_B bounded by the S^2-knot $\bar{S}$. By [5], the central axis $\beta\times 0$ is moved to an arc $\tilde{\beta}$ in $\tilde{B}$ which is paralell to the 2-sphere $\bar{S}$ by a diffeomorphism of the 4-sphere Σ^M_B isotopic to the identity by an isotopy keeping the 2-sphere $\bar{S}$ fixed and L^Γ_β setwise fixed. Hence there is a diffeomorphism f of the 4-sphere Σ^M_B isotopic to theidentity by an isotopy keeping the 2-sphere $\bar{S}$ fixed and L^Γ_β setwise fixed such that f sends $L^\Gamma_\beta\cup\beta\times 0$ to itself identically and and the central axis $\beta\times 0$ is unknotted in the 3-ball $f(\tilde{B})$. By the spin assumption on the spine P of V, we can assume f sends a tubular neighborhood $\beta\times d$ of $\beta\times 0$ in $\beta\times D$ to a tubular neighborhood $\beta\times d$ of $\beta\times 0$ in $f(\tilde{B})$ identically, so that f sends $L^\Gamma_\beta\cup\beta\times d$ to itself identically. Assume that

$$L^\Gamma_\beta\cup\beta\times d\subset\tilde{B}$$

in Σ^M_B. Let d' be a disk in the interior of d. Let $L^\Gamma_\beta\times[0,1]$ be a collar of L^Γ_β in Σ^M_B so that the collars of the disks $v_i\times d$, $(i=1,2,\dots,n)$ belong to $\beta\times d$. Then the 3-manifold

$$\mathrm{cl}(L^\Gamma_\beta\times[0,1]\cup\beta\times d)\setminus\beta\times d')$$

is canonically diffeomorphic to $P\times S^1$ in Σ^M_B and is included in the solid torus $\mathrm{cl}(\tilde{B}\setminus\beta\times d')$. By the spin assumption on P, this means that there is an inclusion

$$P\times S^1\subset U_1\times S^1$$

for a disk U_1 in Σ^M_B with $U_1\times S^1$ as a spin submanifold. Since $V\times S^1$ is a collar of $P\times S^1$ in $S(M,B)$ by Lemma 3.4, there is an inclusion $V\times S^1\to U\times S^1$ for a 3-ball U in $S(M,B)$. By construction, the 4D solid torus $U\times S^1$ is a smooth spin 4-manifold of $S(M,B)$. $\square$

4. Proof of Lemma 1.3

Now we are in a position to the proof of Lemma 1.3.

Proof of Lemma 1.3. Let M be a homotopy 3-sphere. Let $V\cup V'$ be a Heegaard splitting of M of a Heegaard genus n for some n with Heegaard surface $F=\partial V=\partial V'$. Let Γ be a spine graph of the handlebody V with a based arc β in F. In the notation of Section 3, the n-bouquet graph Γ with based arc β is written as $\Gamma=\beta\cup_{i=1}^n(a_i\cup O_i)$

in F. Similarly, let Γ' be a spine graph of the handlebody V' with the same based arc β as Γ but with different vertices in β. Push the part $\Gamma'\setminus\beta$ into the interior of V' by using a boundary collar of F in V'. Let $\bar{\Gamma}=\Gamma\cup\Gamma'$ be a $2n$-bouquet graph in M with the based arc β. Let B be a 3-ball used for the construction of the 4-sphere Σ_B^M such that

$$B\cap V=(\partial B)\cap F=\Delta$$

is a disk and

$$B\cap\bar{\Gamma}=(\partial B)\cap\bar{\Gamma}=\beta$$

is a boundary arc in the disk Δ so that the disk Δ is considered as the product $[0,1]\times\beta$ with $0\times\beta=\beta$. Let $M^o=\mathrm{cl}(M\setminus B)$. Let $N(\bar{\Gamma})$ be a regular neighborhood of $\bar{\Gamma}$ in M^0. The union $\bar{N}(\bar{\Gamma})=N(\bar{\Gamma})\cup B$ is a regular neighborhood of Γ in M. By construction, the closed complement $\mathrm{cl}(M^o\setminus\bar{N}(\bar{\Gamma}))$ is diffeomorphic to the product $F^o\times[0,1]$ for the once-punctured compact surface of F obtained from F by removing the interior of a disk Δ in F. This means that the fundamental group $\pi_1(\mathrm{cl}(M\setminus\bar{N}(\bar{\Gamma})),x_0)$ is a free group of rank $2n$ given by a free product $\mathbf{F}*\mathbf{F}'$ where the free direct product summand $\mathbf{F}$ has a basis represented by a meridian loop system of $O_i\,(i=1,2,\dots,n)$. Note that some meridian loops of $O'_i\,(i=1,2,\dots,n)$ do not always represent elements of the free product summand $\mathbf{F}'$ (see Example 4.1 later for an example). Let $x'_i\,(i=1,2,\dots,n)$ be a basis of the free group $\mathbf{F}'$. Let

$$L_\beta^{\bar{\Gamma}}=L_\beta^{\Gamma}\cup L_\beta^{\Gamma'}$$

be the $2n$-component spun S^2-link of the $2n$-bouquet graph $\bar{\Gamma}$ with based arc β in the 4-sphere Σ_B^M. The n component spun S^2-link L_β^Γ of the n-bouquet graph Γ with based arc β is an S^2-sublink of the S^2-link of $L_\beta^{\bar{\Gamma}}$. By van Kampen theorem, the fundamental group $\pi_1(\Sigma_B^M\setminus L_\beta^{\bar{\Gamma}},\bar{x}_0)$ is canonically isomorphic to the fundamental group $\pi_1(\mathrm{cl}(M\setminus\bar{N}(\bar{\Gamma}),x_0)$ which is the free product $\mathbf{F}*\mathbf{F}'$ where a meridian system of L_β^Γ represents the basis $x_i\,(i=1,2,\dots,n)$ of $\mathbf{F}$.

Let $\alpha_i\,(i=1,2,\dots,n)$ and $\beta_j\,(j=1,2,\dots,n-1)$ be the arc systems in the central axis $\beta\times 0$ used for the spun S^2-link L_β^Γ with central axis $\beta\times 0$. Let $\bar{S}$ be the 2-sphere obtained from the spun S^2-link L_β^Γ by the surgery along the 1-handles $h_j^1\,(j=1,2,\dots,n-1)$ with the core arc system $\beta_j\,(j=1,2,\dots,n-1)$. The 2-sphere $\bar{S}$ is an S^2-knot in Σ_B^M which is a round object. By construction, note that the free product summand $\mathbf{F}$ changes into the infinit cyclic group $\mathbf{Z}$ by the epimorphism sending the basis element x_i to the generator $1\in\mathbf{Z}$ of $\mathbf{Z}$ for every i, so that the S^2-link $L_\beta^{\bar{\Gamma}}$ changes into the $(1+n)$-component S^2-link $\tilde{L}=\bar{S}\cup L_\beta^{\Gamma'}$ in Σ_B^M with a fundamental group isomorphism

$$\pi_1(\Sigma_B^M\setminus\tilde{L},\bar{x}_0)\cong\mathbf{Z}*\mathbf{F}'$$

sending the meridian element of the component $\bar{S}$ to the generator $1 \in \mathbf{Z}$. This means that the fundamental group $\pi_1(\Sigma_B^M \setminus \bar{S}, \bar{x}_0)$ is a quotient group of the infinite cyclic group $\mathbf{Z}$. Since the first homology group $H_1(\Sigma_B^M \setminus \bar{S}; \mathbf{Z}) \cong \mathbf{Z}$, the fundamental group $\pi_1(\Sigma_B^M \setminus \bar{S}, \bar{x}_0)$ must be isomorphic to the infinite cyclic group $\mathbf{Z}$. By Smooth Unknotting Conjecutre, the S^2-knot $\bar{S}$ is a trivial S^2-knot. By Lemma 3.5, the round object $V \times S^1$ extends to a smooth spin 4-submanifold $U \times S^1$ of Σ_B^M for a 3-ball U.

Every knot K in M is isotopic to a knot K' in V, so that we have the following inclusions:

$$K' \times S^1 \subset V \times S^1 \subset U \times S^1 \subset \Sigma_B^M.$$

By Lemmas 3.1 and 3.2, there are meridian-longitude-preserving isomorphisms

$$\pi_1(M \setminus K, x) \cong \pi_1(U \setminus K', x') \cong \pi_1(DU \setminus K', x')$$

for the fundamental groups $\pi_1(M \setminus K, x)$ and $\pi_1(U \setminus K', x')$, where $DU = \partial(U \times [0,1]) = U \times 0 \cup U \times 1$ denotes the double of the 3-ball U with $U \times 0 = U$ which is diffeomorphic to the 3-sphere S^3. This completes the proof of Lemma 1.3. $\square$

This completes the proof of Theorem 1.1.

In the proof of Lemma 1.3, although the basis elements $x_i \, (i = 1, 2, \ldots, n)$ of the free product summand $\mathbf{F}$ are represented by a meridian system of the S^2-sublink L_β^Γ of the S^2-link $L(\bar{\Gamma}, \beta)$, the S^2-sublink L_β^Γ itself is not necessarily a trivial S^2-link. Here is an example.

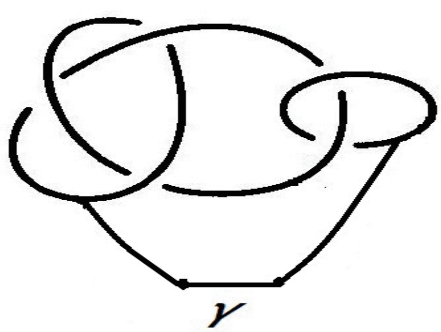

Figure 3: A 2-bouquet graph Γ with based arc β whose funfamental group is a free group of rank 2

Example 4.1. Let Γ be a 2-bouquet graph with based arc β in S^3 illustrated in Fig. 3 whose fundamental group $\pi_1(S^3 \setminus \Gamma, x_0)$ is a free group of rank 2. A regular neighborhood V of Γ in S^3 and the closed complement $V' = \mathrm{cl}(S^3 \setminus V)$ constitute a genus 2 Heegaard splitting $V \cup V'$ of S^3 by noting that the 3-manifold V' is shown to be a handlebody of genus 2 by the loop theorem and the Alexander theorem (cf. e.g. [6]). The graph Γ with based arc β is used to be a spine graph Γ of V by moving the based arc β to the Heegaard surface $F = \partial V = \partial V'$. Let Γ' be any spine graph of V' with base arc β obtained by pushing $\Gamma' \setminus \beta$ into the interior of V'. Let $L_\beta^{\bar{\Gamma}} = L_\beta^{\Gamma} \cup L_\beta^{\Gamma'}$ be the 4-component spun S^2-link in the 4-sphere $\Sigma_B^{S^3}$ constructed from the pair $(\bar{\Gamma}, \beta)$. The fundamental group $\pi_1(\Sigma_B^{S^3} \setminus L_\beta^{\bar{\Gamma}}, \bar{x}_0)$ is canonically isomorphic to the free group $\pi_1(\mathrm{cl}(S^3 \setminus N(\bar{\Gamma}), x_0)$ of rank 4 which is the free product $\mathbf{F} * \mathbf{F}'$ of free groups $\mathbf{F}$ and $\mathbf{F}'$ of rank 2 such that a basis of $\mathbf{F}$ is represented by a meridian system of L_β^{Γ}. The 2-component S^2-sublink L_β^{Γ} is not a trivial S^2-link, because L_β^{Γ} contains a component of the spun trefoil S^2-knot in S^4. Thus, the fundamental group $\pi_1(\Sigma_B^{S^3} \setminus L_\beta^{\Gamma}, \bar{x}_0)$ is not any meridian-based free group. This also implies that there is a meridian element of $L_\beta^{\Gamma'}$ in the fundamental group $\pi_1(\Sigma_B^{S^3} \setminus L_\beta^{\bar{\Gamma}}, \bar{x}_0)$ which is not conjugate to any element of the free product summamd $\mathbf{F}'$, because otherwise the fundamental group $\pi_1(\Sigma_B^{S^3} \setminus L_\beta^{\Gamma}, \bar{x}_0)$ would be a meridian-based free group.

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