# A lecture at Osaka City University.
# On 3-dimensional R  iemannian manifolds with prescribed Ricci eigenvalues. (Joint work by O. Kowalski and Z. Vlášek).

## Abstract:

We prove first the following Main Theorem:

**Theorem 1**. Let $\rho_1(x,y,z) > \rho_2(x,y,z) > \rho_3(x,y,z)$ be three real analytic functions defined on an open domain $U \subset$ $\mathbf{R}^3$[x , y, z] .Then the local moduli space of (local) Riemannian metrics with the prescribed principal Ricci curvatures $\rho_1$, $\rho_2, \rho_3$ can be parametrized by three arbitrary functions of two variables.

We are using the following results:

**Theorem 2.** Let $\rho_1(x,y,z) > \rho_2(x,y,z) > \rho_3(x,y,z)$ be three real analytic functions defined on an open domain $U \subset$ $\mathbf{R}^3$[x,y,z]. Then all *diagonal* (local) Riemannian metrics with the principal Ricci curvatures $\rho_1, \rho_2, \rho_3$ depend on six arbitrary functions of two variables.

**Theorem 3.** Let (*M, g*) be a real analytic 3-dimensional Riemannian manifold. Then, in a neighborhood of each point p$\in M$, there is a system of local coordinates in which *g* adopts a diagonal form. All coordinate transformations for which the diagonal form is preserved depend on 3 arbitrary functions of two variables.

Theorem 3 is a classics. Then Theorem 1 follows easily from Theorem 2, which is the main result to be proved. The method of the proof is the classical Cauchy-Kowalewski Theorem. We also show a new proof of Theorem 3 at this occasion.

The problem solved in Theorem 1 remains <u>open</u> if some of the prescribed functions coincide, i.e., if we have $\rho_1(x,y,z) > \rho_2(x,y,z) = \rho_3(x,y,z)$. In this case one obtains an over-determined system of PDEs, and the Cauchy-Kowalewski Theorem cannot be applied.

Theorem 1 was proved earlier by the same authors for *constant* functions, i.e., for prescribed constants $\rho_1 > \rho_2 > \rho_3$. Here also the case $\rho_1 > \rho_2 = \rho_3$ was successfully investigated by different method. The result says, surprisingly, that the local moduli space of (local) Riemannian metrics with the prescribed constant principal Ricci curvatures $\rho_1 > \rho_2 = \rho_3$ can be parametrized by two arbitrary functions of one variable. Thus, we have “much less metrics” in the last case.

We also show that, for each triplet $\rho_1 > \rho_2 > \rho_3$ of prescribed *constants*, there is an *explicit* and *not locally homogeneous* metric *g* with the principal Ricci curvatures $\rho_1, \rho_2, \rho_3$. (We call such metrics "generalized Yamato metrics"). Such explicit metrics exist only exceptionally for the case $\rho_1 > \rho_2 = \rho_3$.

Some other problems of the same kind are also solved or presented.

Recall finally that our results are of different nature than a famous result by *DeTurck* saying that, in dimensions $n \geq 3$, to every prescribed real analytic non-degenerate tensor $\rho$ of type (0, 2) in a coordinate neighborhood of a point p $\in M$, there is a Riemannian metric g (defined in possibly smaller coordinate neighborhood of p $\in M$ ,) such that $\rho$ is the corresponding Ricci tensor.