数学演習　B.2.5

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$$F_{i,j} = \begin{cases} E_{i,j}, & \text{if } i \neq j, 1 \leq i,j \leq n \\ E_{i+1,j+1} - E_{i,j}, & \text{if } i = j, 1 \leq i,j < n \end{cases}$$

と定義するこの時これが $M`_n(\mathbf{R})$ の基底となることを示す

$$\sum_{i,j} c_{i,j} F_{i,j} = 0 (i = j = n \text{ のときは除く})$$

を考える

展開し成分を比べることにより

$$c_{i,j} = 0 (i \neq j)$$

$$-c_{1,1} = 0, c_{n,n} = 0, c_{i-1,i-1} - c_{i,i} = 0 (2 \leq i < n-1)$$

従って、$c_{i,j} = 0$ で $F_{i,j}$ は線形独立
n=1 の時、明らか $M`_1(\mathbf{R}) = \{\emptyset\}$ で次元は 0
n=2 の時、$\forall X \in M`_2(\mathbf{R})$ をとる $\exists a,b,c \in \mathbf{R}, X = \begin{bmatrix} -b & c \\ a & b \end{bmatrix}$ この時 $X = bF_{1,1} + aF_{2,1} + cF_{1,2}$
よって基底は $F_{1,1}, F_{1,2}, F_{2,1}$
$n \geq 3$ の時、$\forall X \in M`_n(\mathbf{R})$ をとる $\exists c_{i,j} (0 < i,j \leq n) \in \mathbf{R}, \sum_{k=1}^{n} c_{k,k} = 0, X = \sum_{i,j} c_{i,j} E_{i,j}$
ここで $X = \sum_{i \neq j} c_{i,j} E_{i,j} + \sum_{k=1}^{n} c_{k,k} E_{k,k}$ で
また
$\sum_{k=1}^{n} c_{k,k} E_{k,k} = c_{1,1} E_{1,1} - \sum_{k=1}^{n-1} c_{k,k} E_{n,n} + \sum_{k=1}^{n} c_{k,k} E_{n,n} + \sum_{k=2}^{n-1} \{-(\sum_{i=1}^{k-1} c_{i,i}) E_{k,k} + (\sum_{i=1}^{k} c_{i,i}) E_{k,k}\}$
$= \sum_{k=1}^{n} c_{k,k} E_{n,n} - \sum_{k=2}^{n} (\sum_{i=1}^{k-1} c_{i,i}) E_{k,k} + \sum_{k=1}^{n-1} (\sum_{i=1}^{k} c_{i,i}) E_{k,k}$
$= \sum_{k=1}^{n} c_{k,k} E_{n,n} - \sum_{k=1}^{n-1} (\sum_{i=1}^{k} c_{i,i}) E_{k+1,k+1} + \sum_{k=1}^{n-1} (\sum_{i=1}^{k} c_{i,i}) E_{k,k}$
$= -\sum_{k=1}^{n-1} (\sum_{i=1}^{k} c_{i,i}) E_{k+1,k+1} + \sum_{k=1}^{n-1} (\sum_{i=1}^{k} c_{i,i}) E_{k,k} (\because \sum_{k=1}^{n} c_{k,k} = 0)$
$= -\sum_{k=1}^{n-1} (\sum_{i=1}^{k} c_{i,i}) (E_{k+1,k+1} - E_{k,k})$
$= -\sum_{k=1}^{n-1} (\sum_{i=1}^{k} c_{i,i}) F_{k,k}$
よって $X = \sum_{i \neq j} c_{i,j} E_{i,j} + \sum_{k=1}^{n} c_{k,k} E_{k,k} = \sum_{i \neq j} c_{i,j} F_{i,j} - \sum_{k=1}^{n-1} (\sum_{i=1}^{k} c_{i,i}) F_{k,k}$
よって基底は $F_{i,j} (1 \leq i,j \leq n$ ただし $i = j = n$ を除く) で次元は $n^2 - 1$ これは $n = 1, 2$ の時も成り立つ