ファイバー曲面のスロープ不等式と曲線のモジュライ

<u>Plan</u> … 11/11, 12 Intro. & stack 理論入門.
11/18, 19 slope inequality など.

<u>成績</u> … 講義中に出た問題を解く.
or 講義内容に関係する自分の興味を持っていることをまとめる.
or …

I. Intro. 11/11

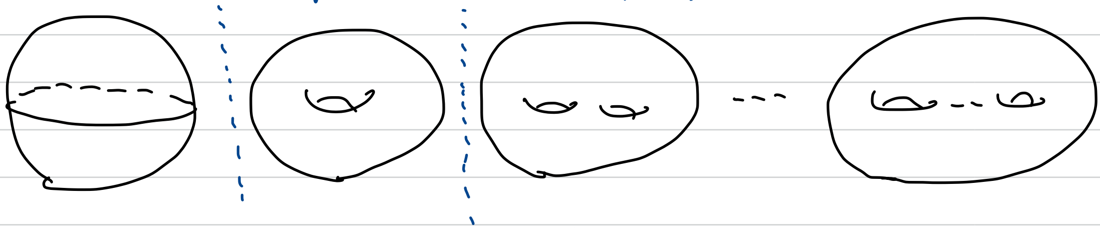
Goal & Direction of AG : Classification of projective varieties

• <u>dim = 1</u> C : smooth proj. curve (= cpt Riemann surface)

$g(C)$: <u>genus</u> of C

$$:= \dim_{\mathbb{C}} H^0(C, K_C) = \frac{1}{2} \dim H^1(C, \mathbb{Q})$$

$\kappa=-\infty$ $\kappa=0$ $\kappa=1$



$g=0$. $C \cong \mathbb{P}^1_{\mathbb{C}}$

$g=1$ elliptic curve $\cong \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$.

$g=2$ $C \xrightarrow{2:1} \mathbb{P}^1_{\mathbb{C}}$ $\cup$ $P_1, \ldots, P_6$ で分岐.

$\mathcal{M}_g$: moduli of smooth proj. curves of genus g.

(isom class 全体に geom. str. いれたもの)

connected. $\dim = 3g-3$. quasi-projective.

$\to$ invariant (genus) で粗く分類して、各々の moduli をしらべる.

X : smooth projective var of dim n.

$\kappa(X)$: Kodaira dimension $\in \{-\infty, 0, 1, \ldots, \underbrace{n}_{\text{general type}}\}$

$-\infty$: Fano etc　0: CY etc

- **dim = 2** S : smooth projective surface.

Enriques–Kodaira classification

$\kappa = -\infty$	0	1	2
ruled surface (genus 0 fibration)	K3, Abelian, Enriques, bielliptic	elliptic surface (genus 1 fibration)	general type.

よく分かっている…

- (Noether inequality)

  S : minimal proj. surface of general type

  Then $K_S^2 \geq 2\chi(\mathcal{O}_S) - 6$ (Noether–Horikawa surfaces)

- Horikawa (1976) classified S with $K_S^2 = 2\chi(\mathcal{O}_S) - 6$ and describe their moduli spaces. (5, 4)

$\rightsquigarrow$ Almost all such surfaces admits genus 2 fibrations.

- (Castelnuovo inequality)
  S : minimal proj. surface of gen. type s.t. canonical map is birational onto image

Then $K_S^2 \geq 3\chi(\mathcal{O}_S) - 10$.

- Ashikaga-Konno (1990) classified such S with $K_S^2 = 3\chi(\mathcal{O}_S) - \underset{(9)}{10}$

$\rightsquigarrow$ Almost all admits non-hyperelliptic genus 3 fibrations.

$\rightsquigarrow$ study fibered surfaces of higher genus.

(cf. 代数曲線束の地誌学)

<u>Def</u> $f : S \to B$: <u>fibered surface of genus g</u>

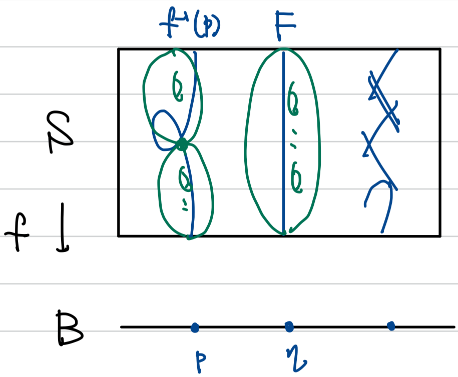
S: sm proj surf, B: sm proj curve

$:\underset{\text{def}}{\Longleftrightarrow}$ surj. morphism s.t. general fiber F is connected & genus g.

Assume f is ~~relatively~~ minimal & $g \geq 2$.

$\omega_f = \mathcal{O}_S(K_f)$: relative canonical sheaf.

$(K_f = K_S - f^* K_B)$



<u>Invariants</u> $\cdot$ $K_f^2 = c_1(\omega_f)^2$

$= K_S^2 - 8(g-1)(g(B)-1)$

alg. geom.

- $\chi_f := \deg f_* \omega_f$
$$= \chi(\mathcal{O}_S) - (g-1)(g(B)-1)$$

topology

- $e_f := \chi_{top}(S) - 4(g-1)(g(B)-1)$
$$= \sum_{p \in B} \left(\chi_{top}(f^{-1}(p)) - \chi_{top}(F)\right) \quad \text{(localization)}$$

- $\sigma(S) :=$ signature of $H^2(S,\mathbb{Q}) \times H^2(S,\mathbb{Q}) \to \mathbb{Q}$.

- (Noether) $\quad 12\chi_f = K_f^2 + e_f$

- (Hirzebruch) $\quad \sigma(S) = K_f^2 - 8\chi_f$

$\therefore \quad K_f^2, \chi_f \quad \leftrightarrow \quad e_f, \sigma(S)$

- $K_f^2, \chi_f, e_f \in \mathbb{Z}_{\geq 0}$.

$K_f^2, \chi_f = 0 \iff$ All fibers are smooth & isom. each other.

$e_f = 0 \iff$ All fibers are smooth.

$\dfrac{K_f^2}{\chi_f}$: slope of f

Thm (Horikawa 1977)

$\exists$ invariant $\mathrm{Ind}(f^{-1}(p)) \in \mathbb{Z}_{\geq 0}$ (Horikawa index)

for genus 2 fiber germs $f^{-1}(p)$

s.t. $K_f^2 - 2\chi_f = \sum_{p \in B} \mathrm{Ind}(f^{-1}(p))$

for $^\forall f : S \to B$: genus 2 fibered surface.

$$K_f^2 \geq 2\chi_f \underset{B=\mathbb{P}^1}{\iff} K_S^2 \geq 2\chi(\mathcal{O}_S) - 6$$

Thm (Xiao, Cornalba–Harris 1987)

$K_f^2 \geq \frac{4(g-1)}{g} \chi_f$ for any fib. surf. f of genus g

Thm (Reid 1990)

$\exists$ invariant $\mathrm{Ind}_H(f^{-1}(p)) \in \mathbb{Z}_{\geq 0}$

s.t. $K_f^2 - 3\chi_f = \sum_{p \in B} \mathrm{Ind}_H(f^{-1}(p))$

for $^\forall f$: non-hyperelliptic genus 3 fib. surf.

Problem establish more sharper slope (In)eq.

for "general" fibered surfaces.

$\exists$ two method :

① use theory of algebraic surfaces.

($g = 2, 3, 4, 5$, hyperelliptic, ---)

good : can treat bad sing. fibers

bad : ad-hoc.

③ use theory of moduli of curves.

$\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g$: Deligne-Mumford cpt'n.

moduli of stable curves.

$f : S \to B$: semistable fibered surf. of genus g

$S \searrow \overline{S} \nearrow \overline{f}$: rel. cano. model

$\rightsquigarrow \varphi_f : B \to \overline{\mathcal{M}}_g$: moduli map.

$p \mapsto [\overline{f}^{-1}(p)]$

• $\exists\, \lambda, \kappa$: line bdl on $\overline{\mathcal{M}}_g$

s.t. $K_f^2 = \deg \varphi_f^* \kappa$, $\chi_f = \deg \varphi_f^* \lambda$.

• If $\exists$ $\mathbb{Q}$-divisor H on $\overline{\mathcal{M}}_g$ s.t.

$H = \kappa - {}^{\exists}a \cdot \lambda$ in $\mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$

& H is "positive" w.r.t. φ_f. (i.e., $H \geq 0$ & $\varphi_f(B) \not\subseteq H$)

then $0 \le \deg \varphi_f^* H = K_f^2 - a\chi_f$

$\therefore K_f^2 \ge a\chi_f.$

今回はこの idea をもっと大きい moduli stack

$\mathcal{M}_g \subseteq \mathcal{M}_g^*$ の上で実行する.

<u>Def</u> $D \subseteq \overline{\mathcal{M}}_g$

$f : S \to B$ is <u>D-general</u>

$:\underset{\text{def}}{\Longleftrightarrow}$ [general fiber F] $\notin D$.

(あとで def slope of D)

<u>Main thm</u> (E, 2023) $D \subseteq \overline{\mathcal{M}}_g$: eff. div. with $s_D \ge 4$, $g \ge 3$.

Then $^\exists$ invariant $\mathrm{Ind}_D(f^{-1}(p)) \in \mathbb{Q}$

s.t. $K_f^2 - (12 - s_D)\chi_f = \sum_{p \in B} \mathrm{Ind}_D(f^{-1}(p))$.

for $^\forall$ D-general genus g fib. surf. f.

(δ 整して、$g \le 5$ の $f^{-1}(p)$ reduced OK)

Moreover, if $f^{-1}(p)$ satisfies Morsification conjecture,

then $\mathrm{Ind}_D(f^{-1}(p)) \geq 0$.

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- $g=2$, $D=\delta_1 \subseteq \overline{\mathcal{M}}_2$

  $s_D = 10$. $\longrightarrow$ Horikawa's slope eq.

- $g=3$, $D=[\text{hyperelliptic}] \subseteq \mathcal{M}_3$

  $s_D = 9$ $\longrightarrow$ Reid's slope eq.

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II. Moduli stacks.

なぜ(今回) moduli stack を考えるのか?

- 今回扱う moduli of curves $\mathcal{M}_g^*$ は coarse moduli space を持たない。
- moduli stack は universal family を持つ:

$$\mathcal{U}_g^* \longrightarrow \mathcal{M}_g^*$$

$$\text{family of curves} \underset{1:1}{\longleftrightarrow} \text{moduli map}$$

$$S \xrightarrow{f} B \quad \begin{matrix} \longmapsto \\ \underset{\text{pullback of univ. fam.}}{\longleftarrow\!\!\!\mid} \end{matrix} \quad \varphi_f : B \to \mathcal{M}_g^*$$

- scheme と同様に、moduli stack 上でも line bdl, divisor, cycle etc が考えられる。

それらを用いて family of geom. 量 をしらべられる。

gluing aff. sch.

in the Zar. top. — schemes

$\cap$

in the Et. top as sheaves — alg. spaces

$\cap$

in the Et. top. as fib. groupoids — DM stacks $\longrightarrow$ Coarse moduli spaces

G finite $\curvearrowright X$

$[X/G] \longmapsto X/G$

$\overline{\mathcal{M}}_g \longmapsto \overline{M}_g$

$\cap$

in the smooth top as fib. groupoids — Artin stacks $\longrightarrow$ good moduli spaces

G = reductive $\curvearrowright X$

$[X^{ss}/G] \longmapsto X^{ss}/G$

$\mathcal{M}_{Fano} \longmapsto M_{Fano}$

参考文献

- Olsson, Alg. Spaces & Stacks.
- Stacks project
- Alper, Stacks & Moduli
- Khan, Lectures on alg. stacks

Def $\mathcal{C}$: category.

<u>Grothendieck topology</u> on $\mathcal{C}$ consists of

a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target in $\mathcal{C}$

($\mathrm{Cov}(\mathcal{C}) \ni \{X_i \to X\}_{i \in I}$: <u>covering</u> of X)

satisfying the following :

(i) $Y \xrightarrow{\sim} X$ isom $\Rightarrow \{Y \xrightarrow{\sim} X\} \in \mathrm{Cov}(\mathcal{C})$

(ii) $\begin{cases} \{X_i \to X\}_{i \in I} \in \mathrm{Cov}(\mathcal{C}) \\ \{X_{ij} \to X_i\}_{j \in J_i} \in \mathrm{Cov}(\mathcal{C}) \quad \forall i \end{cases}$

$\Rightarrow \{X_{ij} \to X_i \to X\}_{\substack{i \in I \\ j \in J_i}} \in \mathrm{Cov}(\mathcal{C})$

(iii) $\begin{cases} \{X_i \to X\}_{i \in I} \in \mathrm{Cov}(\mathcal{C}) \\ Y \to X \text{ : morph in } \mathcal{C} \end{cases} \Rightarrow \begin{array}{l} \exists\, X_i \times_X Y \in \mathcal{C} \\ \text{s.t. } \{X_i \times_X Y \to Y\}_{i \in I} \in \mathrm{Cov}(\mathcal{C}) \end{array}$

$(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$: <u>site</u> ← <u>問題</u> siteのいくつかの定義を比較せよ.

<u>Example</u> (1) X : top. space.

$\mathrm{Op}(X)$: category of open sets of X :

obj : $U \subseteq X$ open

morph : $U \subseteq V$ inclusion.

$$\{U_i \to U\}_i \in \mathrm{Cov}(\mathrm{Op}(X)) \iff \bigcup_{i \in I} U_i = U$$

により、$(\mathrm{Op}(X), \mathrm{Cov}(\mathrm{Op}(X)))$ は site.

(<u>small Zariski site</u> of X)

(2) S : scheme.

Sch/S : category of schemes over S.

$$\{f_i : X_i \to X\}_{i \in I} : \left\{\begin{array}{c} \text{Zariski} \\ \text{étale} \\ \text{fppf} \end{array}\right\} \text{covering}$$

$:\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ f_i : $\begin{bmatrix} \text{open immersion} \\ \text{étale} \\ \text{flat of fin pres.} \end{bmatrix}$ $\forall i$

& $\coprod_{i \in I} X_i \longrightarrow X$ surj.

$(Sch/S)_{\substack{Zar \\ (et) \\ (fppf)}} := Sch/S$ with Zar. covering $\begin{pmatrix} \text{étale} \\ \text{fppf} \end{pmatrix}$

$\left(\underbrace{\text{big} \begin{pmatrix} \text{Zariski} \\ \text{étale} \\ \text{fppf} \end{pmatrix} \text{site of } S} \right)$

<u>問題</u> (1), (2) が site なことを確かめよ。

<u>問題</u> (2) を集合論的問題を回避する形に精密化せよ。

<u>Def</u> $\mathcal{C}$: site.

<u>presheaf</u> on $\mathcal{C}$ $:\underset{\text{def}}{\Longleftrightarrow}$ functor $\mathcal{C}^{op} \to Set$

$$Psh(C) = Fun(C^{op}, Set)$$

$F : C^{op} \to Set$ is a <u>sheaf</u>

$\underset{def}{:\Longleftrightarrow}$ For $\forall$ covering $\{X_i \xrightarrow{f_i} X\}_i$ of C,

natural map

$$F(X) \to Eq\left(\prod_i F(X_i) \underset{pr_2^*}{\overset{pr_1^*}{\rightrightarrows}} \prod_{i,j} F(X_i \times_X X_j)\right)$$

$$\| $$

$$\left\{ (a_i) \in \prod F(X_i) \;\middle|\; \begin{array}{l} pr_1^* a_i = pr_2^* a_j \\ \text{in } F(X_i \times_X X_j) \end{array} \right\}$$

$$a \mapsto (f_i^* a)_i$$

is bijective.

$Sh(C) \subseteq Psh(C)$: full sub cat. of sheaves.

- This inclusion has left adjoint

$$(-)^{a} : Psh(\mathcal{C}) \longrightarrow Sh(\mathcal{C}) \quad : \underline{\text{sheafification}}$$
$$F \longmapsto F^{a}$$

<u>問題</u> これを示せ（以下 証明しない主張の proof は問題とする）

<u>Rem</u> • $Psh(\mathcal{C})$ has all limits & colimits since so does Set :

$$(\lim_i F_i)(X) = \lim_i (F_i(X)) \subseteq \prod_i F_i(X)$$
$$(\mathrm{colim}_i F_i)(X) = \mathrm{colim}_i (F_i(X)) \twoheadleftarrow \coprod_i F_i(X)$$

• $Sh(\mathcal{C})$ has all limits & colimits :

$$\lim_i F_i = \text{limit of } F_i \text{ as presheaves.}$$
$$\mathrm{colim}_i F_i = (\text{colimit of } F_i \text{ as presheaves})^{a}$$

• Yoneda emb. $h : \mathcal{C} \hookrightarrow Psh(\mathcal{C})$ は limit 保つ.
$$X \mapsto h_X = \mathrm{Hom}(-, X)$$

- morph. of presheaves $E \to F$ is <u>representable by C</u>

$:\overset{\Longleftrightarrow}{\text{def.}}$ ${}^\forall X \in C$, ${}^\forall$ morph $h_X \to F$ に対し

${}^\exists Y \in C$ s.t. $E \times_F h_X \cong h_Y$.

<u>Example</u> $X \in Sch/S$.

$h_X : (Sch/S)^{op} \to Set$ is a sheaf on $(Sch/S)_{fppf}$

$Y \longmapsto Hom_{Sch/S}(Y, X)$

(もっと強く sheaf in fpqc topology もいえる.)

(e.g., Zariski covering $Y = \bigcup_i Y_i$ に対しては、

$$Hom(Y, X) \to \prod_i Hom(Y_i, X) \rightrightarrows \prod_{i,j} Hom(Y_i \cap Y_j, X)$$

$$Y \xrightarrow{f} X \mapsto (f|_{Y_i})_i$$

$$(f_i)_i \quad \begin{matrix} \longmapsto \\ \longmapsto \end{matrix} \quad \begin{matrix} (f_i|_{Y_i \cap Y_j})_{ij} \\ (f_j|_{Y_i \cap Y_j})_{ij} \end{matrix}$$
)

$$\therefore\ Sch/S \overset{h_{(-)}}{\hookrightarrow} Sh((Sch/S)_{fppf})$$

Def $X : (Sch/S)^{op} \longrightarrow Set$ is an <u>algebraic space over S</u>

$:\overset{}{\underset{def}{\Longleftrightarrow}}$ (i) X : sheaf in the étale topology,

(ii) diagonal $\Delta : X \to X \times X$ is representable by Sch/S, $\left(\begin{array}{l} X(T) \to X(T) \times X(T) \\ a \mapsto (a, a) \end{array} \right)$

& (iii) $\exists\ \pi : U \longrightarrow X$: étale surj. morphism from a scheme U.

Rem • (ii) $\Longleftrightarrow$ $\forall$ morph. $Y \xrightarrow{f} X$ from sch. is representable.

$$\left(\text{∵) } (\Rightarrow) \begin{tikzcd} Y \underset{X}{\times} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & Y \times Z \arrow[d, "f \times g"] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd} \qquad (\Leftarrow) \begin{tikzcd} & & Y \arrow[d, "\Delta"'] \arrow[dd, bend left, "h"] \\ & Y \underset{X}{\times} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & Y \times Y \arrow[d] \\ & X \arrow[r] & X \times X \end{tikzcd} \right)$$

• (iii) $\Longleftrightarrow$ $\forall$ morph. $Y \to X$ from sch. に対し、
$U \underset{X}{\times} Y \longrightarrow Y$ is étale & surj. of schemes

- $$\begin{tikzcd} R := U \times_X U \arrow[r, "pr_2", "ét"'] \arrow[d, "pr_1"', "ét"] \arrow[dr, phantom, "\lrcorner"] & U \arrow[d, "\pi = ét"] \\ U \arrow[r, "\pi"'] & X \end{tikzcd}$$

scheme

$R \xrightarrow{(pr_1, pr_2)} U \times_S U$ defines étale equivalence relation

$\because$) $T \in Sch/_S$ に対し

$$R(T) = U(T) \times_{X(T)} U(T)$$

$$= \left\{ (a,b) \in U(T) \times U(T) \text{ s.t. } \pi(a) = \pi(b) \text{ in } X(T) \right\}$$

equiv. relation on $U(T)$

& $R \to U \times_S U \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} U$ is étale.

U/R : sheaf associated to a presheaf

$$T \in Sch/_S{}^{op} \longmapsto U(T)/R(T)$$

$\therefore$ $U/R \cong X$ as sheaves.

Conversely, U, R : schemes$/S$

$R \to U \times_S U$: étale equiv. relation.

Then sheaf U/R is an algebraic space.

例 X : scheme$/S$. G : discrete group

$G \curvearrowright X$ free action.

X/G : sheaf on $(Sch/S)_{et}$ ass. to a presheaf

$$T \longmapsto X(T)/G$$

これは $R := G \times X \longrightarrow X \times X$ étale equiv. rel.

$$(g, x) \longmapsto (x, g\cdot x)$$

となり、$X/G \cong X/R$ algebraic space$/S$

問題 $X = \mathbb{A}^1$, $G = \mathbb{Z}$. $x \mapsto x+n$. $\mathbb{A}^1/\mathbb{Z}$ not sch. を示せ.

問題 alg.sp.$/S$ は fiber product 持つことを示せ.

- stack について、

C : site.

sheaf on C = functor $F : C^{op} \to Set$
with glueing condition
$\{X_i \to X\} \in Cov(C)$
$\Rightarrow F(X) \xrightarrow{\cong} F(\{X_i \to X\})$ in Set

$$\| $$

$$Eq\left(\prod_i F(X_i) \rightrightarrows \prod_{i,j} F(X_i \times_X X_j)\right)$$

stack on C = pseudo-functor

$F : C^{op} \longrightarrow Groupoid$ ← (2,1)-Cat

with glueing condition

$F(X) \xrightarrow{\simeq} F(\{X_i \to X\})$
equiv. in Groupoid

$X \xrightarrow{f} Y \xrightarrow{g} Z$

$$\begin{tikzcd} F(Z) \arrow[r, "g^*"] \arrow[dr, bend right, "(g\circ f)^*"'] & F(Y) \arrow[d, "f^*"] \arrow[dl, phantom, "\phi \, \cong \, \Nearrow"] \\ & F(X) \end{tikzcd}$$

ここで. groupoid := category s.t. $\forall$ morph. is isom.

set S は $\forall x, y \in S$ に対し

$$\mathrm{Hom}_S(x,y) := \begin{cases} \phi & (x \neq y) \\ \{id\} & (x = y) \end{cases}$$ として

groupoid とみれる.

- Grothendieck construction. ↓ について述べる.

$$\begin{array}{ccc} \text{pseudo-functor} & & \text{(right fibration)} \\ & \overset{\mathrm{Un}}{\longrightarrow} & \text{category fibred in} \\ F : C^{op} \to \text{Groupoid} & \underset{\mathrm{St}}{\longleftarrow} & \text{groupoids} \\ & & \mathcal{F} \xrightarrow{p} C \end{array}$$

(equiv. of (2,1)-cat)

$$F \longmapsto \mathrm{ob}(\mathcal{F}) := \left\{ (X, x) \;\middle|\; \begin{array}{l} X \in C \\ x \in F(X) \end{array} \right\}$$

$$(X, x) \xrightarrow{(f, \varphi)} (Y, y)$$

$$f : X \to Y, \quad \varphi : x \to f^* y \text{ in } F(X)$$

$$\left[\begin{array}{l} \mathcal{F} : X \longmapsto p^{-1}(X) \\ \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, bend right, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd} \longmapsto \begin{tikzcd} p^{-1}(Y) \arrow[r, "f^*"] & p^{-1}(X) \\ p^{-1}(Z) \arrow[u, "g^*"] \arrow[ur, phantom, "\Leftarrow \phi"] \arrow[ur, bend right, "(g\circ f)^*"'] & \end{tikzcd} \end{array}\right] \hookleftarrow \mathcal{F}$$

pullback!

Def $p : \mathcal{F} \longrightarrow \mathcal{C}$: functor of cat.

- morphism $\varphi : x \longrightarrow y$ in $\mathcal{F}$ is <u>p-cartesian</u>

$:\Longleftrightarrow_{\text{def}}$

$$\begin{tikzcd} \forall z \arrow[r, dashed, "\exists f"] \arrow[rr, bend left, "\forall \psi"] \arrow[d, mapsto] & x \arrow[r, "\varphi"] \arrow[d, mapsto] & y \arrow[d, mapsto] & \mathcal{F} \arrow[d, "p"] \\ p(z) \arrow[r, "\forall g"'] \arrow[rr, bend right, "p(\psi)"'] & p(x) \arrow[r, "p(\varphi)"'] & p(y) & \mathcal{C} \end{tikzcd}$$

$\Longleftrightarrow$ $\forall z \in \mathcal{F}$ に対し、

$$\mathrm{Hom}_{\mathcal{F}}(z, x) \longrightarrow \mathrm{Hom}_{\mathcal{F}}(z, y) \times_{\mathrm{Hom}_{\mathcal{C}}(p(z), p(y))} \mathrm{Hom}_{\mathcal{C}}(p(z), p(x))$$

is bijective.

- p is a <u>fibered category</u> over $\mathcal{C}$

  (or <u>Cartesian fibration</u>)

$:\underset{\text{def.}}{\Longleftrightarrow}$ $\forall$ morph. $f : X \to Y$ in $\mathcal{C}$

$\forall$ $y \in \mathcal{F}$ with $p(y) = Y$

に対し、$\exists$ p-cart. morph. $\varphi : x \to y$

s.t. $p(\varphi) = f$

$$\begin{tikzcd} x \arrow[r, "\exists \varphi"] \arrow[d, mapsto] & y \arrow[d, mapsto] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$\mathcal{F}(X) := p^{-1}(X)$: sub cat

(morph は $\varphi : x \to x'$ s.t. $p(\varphi) = \mathrm{id}_X$)

$\rightsquigarrow$ $\exists$ functor $f^* : \mathcal{F}(Y) \longrightarrow \mathcal{F}(X)$

(<u>pullback</u> along f) $y \longmapsto x = f^* y$ とおく.

$$y \to y' \longmapsto \begin{tikzcd} f^* y \arrow[r, "\exists!"'] \arrow[d] & f^* y' \arrow[d] \\ y \arrow[r] & y' \end{tikzcd}$$

- p is a <u>category fibred in groupoids</u> over $\mathcal{C}$
  (or <u>right fibration</u>)

$:\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ p is a fib. cat & ${}^\forall X \in \mathcal{C}$
に対し、$\mathcal{F}(X)$ は groupoid.

<u>Example</u> $F : \mathcal{C}^{op} \to \mathrm{Set}$: functor.

$\int F$: cat. s.t. $\mathrm{ob}(\int F) = \left\{ (X, x) \;\middle|\; \begin{array}{l} X \in \mathcal{C} \\ x \in F(X) \end{array} \right\}$

$(X, x) \xrightarrow{f} (Y, y)$ $= (f, \mathrm{id}_x)$

$f : X \to Y$ in $\mathcal{C}$ with $f^* y = x$

$\rightsquigarrow \int F \to \mathcal{C}$; $(X, x) \longmapsto X$
$\qquad f \longmapsto f$

is a cat. fib. in groupoids. //

<u>問題</u> これを示せ.

Def $\mathcal{C}$: site. $p : \mathcal{F} \to \mathcal{C}$: fib. cat.

- $\{X_i \to X\}_i \in \mathrm{Cov}(\mathcal{C})$.

<u>descent data</u> w.r.t. p & $\{X_i \to X\}$

is a pair $((x_i)_i, (\sigma_{ij})_{ij})$,

$x_i \in \mathcal{F}(X_i)$, $\sigma_{ij} : pr_1^* x_i \xrightarrow{\sim} pr_2^* x_j$ in $\mathcal{F}(X_i \times_X X_j)$

$\mathcal{F}(X_i) \ni x_i \xrightarrow{pr_1^*} \mathcal{F}(X_i \times_X X_j) \xleftarrow{pr_2^*} x_j \in \mathcal{F}(X_j)$

s.t. in $\mathcal{F}(X_i \times_X X_j \times_X X_k)$,

$$
\begin{tikzcd}
 & pr_{13}^* pr_1^* x_i \simeq pr_{12}^* pr_1^* x_i \arrow[dl, "pr_{13}^* \sigma_{ik}"'] \arrow[dr, "pr_{12}^* \sigma_{ij}"] & \\
pr_{13}^* pr_2^* x_k \arrow[d, phantom, "\wr"] & \circlearrowright & pr_{12}^* pr_2^* x_j \arrow[d, phantom, "\wr"] \\
pr_{23}^* pr_2^* x_k & & pr_{23}^* pr_1^* x_j \arrow[ll, "pr_{23}^* \sigma_{jk}"]
\end{tikzcd}
$$

(cocycle condition)

<u>morph. of descent datum</u> is

$$(\varphi_i) : ((x_i), (\sigma_{ij})) \to ((x_i'), (\sigma_{ij}')),$$

$\varphi_i : x_i \to x_i'$ in $F(X_i)$

with
$$\begin{tikzcd} pr_1^* x_i \arrow[r, "pr_1^* \varphi_i"] \arrow[d, "\sigma_{ij}"'] \arrow[dr, phantom, "\circlearrowright"] & pr_1^* x_i' \arrow[d, "\sigma_{ij}'"] \\ pr_2^* x_j \arrow[r, "pr_2^* \varphi_j"'] & pr_2^* x_j' \end{tikzcd}$$

$F(\{X_i \to X\})$: cat. of descent datum.

natural functor $F(X) \to F(\{X_i \xrightarrow{f_i} X\})$

is det. by $x \mapsto (f_i^* x, \sigma_{ij}^{cano})$

$$\sigma_{ij}^{cano} : pr_1^* f_i^* x \xrightarrow{\sim} pr_2^* f_j^* x$$

cano. isom.

- $p : \mathcal{F} \to \mathcal{C}$ is <u>descent</u>

  $:\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ $\forall\, \{X_i \to X\} \in \mathrm{Cov}(\mathcal{C})$,

  $F(X) \to F(\{X_i \to X\})$ is cat. equiv.

- $p : \mathcal{F} \to \mathcal{C}$ is a <u>stack</u>

  $:\underset{\text{def}}{\Longleftrightarrow}$ cat. fib. in groupoids & descent.

<u>Example</u> (1) $F \in Sh(\mathcal{C})$ can be regarded as a stack.

(2) $\mathcal{C} = (Sch/S)_{\cancel{fppf}}$

$\mathcal{C}oh$: cat. of pairs (X, F)

$X \in Sch/S$, F : q-coh. shf on X.

$(f, \varphi) : (X, F) \longrightarrow (Y, G)$

$f : X \to Y$, $\varphi : F \xrightarrow{\simeq} f^* G$

$QCoh \to (Sch/S)_{fppf}$ is descent.

$(X, F) \longmapsto X$ (より強く fpqc descent)

(quasi-coh に代にても、coh, loc.free, sheaf of sets, alg.space, etc も fppf descent)

<u>Def</u> $\mathcal{F}, \mathcal{G}, \mathcal{H}$: cat. fib. in groupoids / $\mathcal{C}$.

$\mathcal{F} \xrightarrow{f} \mathcal{H}$, $\mathcal{G} \xrightarrow{g} \mathcal{H}$: functor over $\mathcal{C}$ (単に morph とよぶ)

$\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$: <u>fibered product</u> is def. by

obj : (X, x, y, φ), $X \in \mathcal{C}$

$x \in \mathcal{F}(X)$, $y \in \mathcal{G}(X)$

$\varphi : f(x) \xrightarrow{\simeq} g(y)$ in $\mathcal{H}(X)$

morph : $(X, x, y, \varphi) \xrightarrow{(\alpha, \beta)} (X', x', y', \varphi')$

$\alpha : x \to x'$, $\beta : y \to y'$

with

$$\begin{tikzcd} f(x) \arrow[r, "f(\alpha)"] \arrow[d, "\varphi"', "\cong"] \arrow[dr, phantom, "\circlearrowright"] & f(x') \arrow[d, "\varphi'", "\cong"'] \\ g(y) \arrow[r, "g(\beta)"'] & g(y') \end{tikzcd}$$

$$\therefore \quad \begin{tikzcd} \mathcal{F} \times_{\mathcal{H}} \mathcal{G} \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\overset{\varphi}{\Nearrow}\cong"] & \mathcal{G} \arrow[d, "g"] \\ \mathcal{F} \arrow[r, "f"'] & \mathcal{H} \end{tikzcd}$$

& satisfies univ. property (2-cartesian)

Similarly, we can define a product $\mathcal{F} \times \mathcal{G}$.

- $\mathcal{F}, \mathcal{G}, \mathcal{H}$: stack $\Rightarrow$ $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$: stack. ($\mathcal{F} \times \mathcal{G}$)
- $\exists$ stackification of cat. fib. in groupoids.
  $\mathcal{F} \mapsto \mathcal{F}^s$

Def $\mathfrak{X} \to Sch/S$ is an <u>Artin stack</u> over S
(resp. <u>Deligne-Mumford stack</u>)

$:\Leftrightarrow$ (i) $\mathfrak{X}$ is a stack in the étale topology.
def
(ii) $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is representable by algebraic spaces

(i.e., ${}^\forall Y \to \mathfrak{X} \times_S \mathfrak{X}$: morph. from an alg. sp$/S$.
$\mathfrak{X} \times_{\Delta, \mathfrak{X} \times_S \mathfrak{X}} Y \simeq {}^\exists Z$: alg sp$/S$.)

(iii) ${}^\exists \pi : U \to \mathfrak{X}$: smooth surj. morphism from a scheme U.

Rem · (ii) $\Leftrightarrow$ ${}^\forall$ morph $Y \to \mathfrak{X}$ from alg. sp. is rep'ble by alg. spaces.

· (iii) $\Leftrightarrow$ ${}^\forall$ morph. $Y \to \mathfrak{X}$ に対し. ($\rightsquigarrow U \times_{\mathfrak{X}} Y$ alg. sp.)

$\exists$ étale surj. $V \twoheadrightarrow U \times_{\mathfrak{X}} Y$ from a scheme

s.t. $V \twoheadrightarrow U \times_{\mathfrak{X}} Y \longrightarrow Y$ is smooth surj. (étale)

$$\begin{tikzcd}
V & & Y \\
V' \arrow[u] \arrow[rr, two heads, "\exists", "\text{smooth (étale)}"'] & & Y' \arrow[u, "\text{ét}"']
\end{tikzcd}$$

sch. $\rightsquigarrow V'$ $\qquad$ $Y' = $ sch

- $Y \in Sch/S$ に対し

$\mathfrak{X}(Y) \xrightarrow{\sim} \underline{\mathrm{Hom}}(Y, \mathfrak{X})$ "2-Yoneda Lemma"

$x \longmapsto (Y \longrightarrow \mathfrak{X})$

$Y = Sch/Y$

$(T \xrightarrow{f} Y) \mapsto f^* x$

$\varphi(id_Y) \longmapsfrom (Y \xrightarrow{\varphi} \mathfrak{X})$

$Y \longrightarrow \mathfrak{X} \times_S \mathfrak{X} \leftrightarrow (x, y) \in \mathfrak{X}(Y) \times \mathfrak{X}(Y)$

presheaf $\underline{\mathrm{Isom}}(x,y) : (Sch/S)_{et}^{op} \to Set$ を、

$$\underline{\mathrm{Isom}}(x,y)(T) := \left\{ \left(T \xrightarrow{f} Y,\ \varphi : f^*x \underset{\mathrm{Isom}}{\xrightarrow{\sim}} f^*y\right) \right\}$$

と定めると、

$$\mathfrak{X} \times_{\Delta, \mathfrak{X} \underset{S}{\times} \mathfrak{X}, (x,y)} Y \xrightarrow{\sim} \underline{\mathrm{Isom}}(x,y)$$

∵) obj of $\mathfrak{X} \times_{\Delta, \mathfrak{X} \underset{S}{\times} \mathfrak{X}, (x,y)} Y$ は

$$\left(T,\ \underset{\mathfrak{X}(T)}{z \in},\ T \xrightarrow{f} Y,\ (z,z) \xrightarrow[\sim]{(\alpha,\beta)} (f^*x, f^*y)\right)$$

$$\mapsto \left(T \xrightarrow{f} Y,\ \beta \circ \alpha^{-1} : f^*x \xrightarrow{\sim} f^*y\right)$$

$$\in \underline{\mathrm{Isom}}(x,y)(T)$$

$(T, f^*x, f, (id, \varphi)) \longleftarrow\!\shortmid (f, \varphi)$ は quasi-inverse

//

これより、(ii) $\Leftrightarrow$ $^\forall Y \in Sch/S$

$^\forall x, y \in \mathfrak{X}(Y)$ に対し、

$\underline{Isom}(x,y)$ is an alg. space $/S$.

- $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$: Artin stacks $/S$

$\mathfrak{X} \xrightarrow{f} \mathfrak{Z}$, $\mathfrak{Y} \xrightarrow{g} \mathfrak{Z}$: morph.

$\Rightarrow$ $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is also an Artin stack $/S$.

$\because$) (sketch) (i) OK.

(ii) $(T, x, y, \varphi : f(x) \xrightarrow{\sim} g(y)) =: W$

$(T, x', y', \varphi' : f(x') \xrightarrow{\sim} g(y')) =: W'$

in $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}(T)$ に対し、

$$\underline{\mathrm{Isom}}(w,w') \cong \underline{\mathrm{Isom}}(x,x') \times_{\underline{\mathrm{Isom}}(f(w),f(w'))} \underline{\mathrm{Isom}}(y,y')$$

となり、RHS は alg. sp $/S$ の fib. prod. より alg. sp $/S$

(iii) $U \to \mathfrak{X}$, $V \to \mathfrak{Y}$: smooth surj. from alg. sp $/S$

$\Rightarrow$ $U \times_{\mathfrak{Z}} V$: alg. sp $/S$ になり、

$U \times_{\mathfrak{Z}} V \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$: smooth surj. //

$$\begin{tikzcd} U \times_{\mathfrak{Z}} V \arrow[rr] \arrow[dr] & & \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \\ & \mathfrak{X} \times_{\mathfrak{Z}} V \arrow[ur] & \end{tikzcd}$$

<u>Example</u> algebraic space X over S は

Deligne-Mumford stack $/S$:

$\mathfrak{X} = Sch/X \longrightarrow Sch/S$; $(Y \to X) \mapsto (Y \to X \to S)$

$\mathfrak{X}(Y) \cong {}^{\exists}$ a set. ${}^{\forall} Y \in Sch/S$

Conversely, any Artin stack $\mathfrak{X}/S$

with $\mathfrak{X}(Y)\simeq{}^{\exists}$ a set $\forall Y\in Sch/S$

is an algebraic space$/S$.

∵) functor $X : Sch/S^{op}\to Set$ と定めると

$$Y\mapsto \mathfrak{X}(Y)/\simeq$$

仮定 より natural functor $\mathfrak{X}\to X$ is equiv.

~~def. of Artin stacks (ii), (iii) より~~

(もう忘れんぞ.
(ii)は $\Delta_{\mathfrak{X}}$ mono. より ver. hyp. ch.
(iii)は $\Delta_{\mathfrak{X}}$ form. unram. より
DM stack から)

X は alg. space$/S$ //

問題 morph $\mathfrak{X}\xrightarrow{f}\mathfrak{Y}$ of Artin stacks$/S$ is representable by alg. sp

$\Leftrightarrow$ f は faithful as functors を示せ. //

Def C : site. G : sheaf of groups on C.

これは. $G : C^{op}\to Grp$ sheaf

$$\left(\begin{array}{lll} \text{or} & G : \text{sheaf of sets on } \mathcal{C} & \\ \text{with} & m : G \times G \longrightarrow G & : \text{multiplication} \\ & i : G \longrightarrow G & : \text{inverse} \\ & e : * \longrightarrow G & : \text{unit} \\ & \text{satisfying group axioms.} & \end{array}\right)$$

- $\overset{P}{\overset{\shortparallel}{(P, \rho)}}$ is a <u>G-torsor</u> on $\mathcal{C}$

$:\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$
- P : sheaf of sets on $\mathcal{C}$.
- $\rho : G \times P \longrightarrow P$: action

s.t.
- $\forall X \in \mathcal{C}.\ \exists$ covering $\{X_i \to X\}$

  s.t. $P(X_i) \neq \phi \ \forall i$.
- $(\rho, pr_2) : G \times P \xrightarrow{\sim} P \times P$ isom.

  $(g, x) \longmapsto (g \cdot x, x)$

- G-torsor P is <u>trivial</u>

$:\overset{\text{def}}{\Longleftrightarrow}$ $P \overset{\exists}{\simeq} G$: G-equiv. isom.

$\Longleftrightarrow$ $\mathrm{Hom}_{\mathrm{Sh}(C)}(*, P) \neq \phi$.

∵) $\Leftarrow$ は、$* \xrightarrow{\varphi} P$ より、$G \to P$ を

$G(X) \xrightarrow{\Phi} P(X)$: $g \mapsto g^{-1} \cdot \varphi_X(*)$ ととれば

~~inverse right action~~ $\cancel{\circlearrowright G}$

フツーに left action でよかった

$$\Phi(h \cdot g) = \Phi(g \cdot h^{-1}) = h \cdot g^{-1} \cdot \varphi_X(*) = h \cdot \Phi(g)$$

//

- <u>morphism of G-torsors</u> $f : P \to P'$

is a G-equiv. morph. of sheaves on C.

$\rightsquigarrow$ isom.

Example G : smooth group scheme $/ S$.

$\pi : P \to S$: <u>principal G-bundle</u> (一旦, P は scheme)

$:\overset{\text{def}}{\Longleftrightarrow}$ $G \curvearrowright P$, π : G-invariant fppf covering

& $G \times_S P \xrightarrow{\sim} P \times_S P$ isom.

$(g, x) \longmapsto (g \cdot x, x)$

Then P is a G-torsor as $\left(\begin{matrix}\text{étale} \\ \text{fppf}\end{matrix}\right)$ sheaves.

$$\left(\because\right) \quad \begin{tikzcd} P \arrow[d, "\substack{\text{smooth} \\ \text{cov.}}"] & \\ S & S' \arrow[l, hook', "\substack{\exists \\ \text{ét.} \\ \text{cov.}}"'] \arrow[ul, bend right, "\exists"'] \end{tikzcd} \quad \rightsquigarrow P \text{ is trivial étale locally}$$

Princ. G-bdl $/ S \longrightarrow G$-torsor $/ (Sch/S)_{\substack{\text{ét} \\ \text{fppf}}}$

(P として alg. sp. も許すと, equiv.
特に, G : affine のとき, equiv.)

//

<u>Def</u> G : smooth group scheme$/S$ $\curvearrowright$ X : scheme$/S$
(or alg. sp)

<u>quotient stack</u> $[X/G]$ is defined as

obj : $(T, P, \pi : P \to X \times_S T)$

$T \in Sch/S$

P : G_T - torsor on $(Sch/T)_{et}$

$ \shortparallel$

$(G \times_S T$ as group sch$/T)$

π : G_T - equiv. morph.

morph : $(T, P, \pi) \xrightarrow{(f, \varphi)} (T', P', \pi')$

$f : T \to T'$: morph$/S$

$$\begin{array}{ccc} \varphi : P & \xrightarrow{\cong} & P' \times_{T'} T \\ \pi \downarrow & \circlearrowright & \downarrow \pi' \times \mathrm{id}_{T'} \\ X \times_S T & \cong & X \times_S T' \times_{T'} T \end{array}$$

: isom. of G_T-torsors

$[X/G] \to Sch/S \ ; \ (T, P, \pi) \mapsto T$

$\qquad\qquad\qquad\qquad\qquad (f, \varphi) \mapsto f.$

<u>Prop</u> $[X/G]$ is an Artin stack $/S$.

∵) (cat. fib in groupoids なこと)

$$\begin{tikzcd} (U, P\times_T U, \pi\times id_U) \arrow[r, "(f, id)"] \arrow[d] & (T, P, \pi) \arrow[d, mapsto] \\ U \arrow[r, "f"'] & T \end{tikzcd}$$

は cartesian morph.

〃

(stack なこと) $\{T_i \to T\}$: étale covering

$[X/G](T) \to [X/G](\{T_i \to T\})$ equiv. を示す.

$((T_i, P_i, \pi_i), \sigma_{ij}) \in [X/G](\{T_i \to T\})$

$\sigma_{ij} : P_i \times_{T_i} (T_i \times_T T_j) \xrightarrow{\sim} P_j \times_{T_j} (T_i \times_T T_j)$ G-equiv.

sheaf の étale descent より、

$^\exists P$: sheaf on $(Sch/T)_{et}$

s.t. $P \times_T T_i \cong P_i$.

同様に $G_T \times P \to P$ action も、

descent により定まり、P は G_T-torsor となる。

G_T-equiv. morph. $\pi : P \to X \times_S T$ も

同様に定まり、対応

$((T_i, P_i, \pi_i), \varphi_{ij}) \mapsto (T, P, \pi)$ は quasi-inverse

を与える。 //

quotient morphism $q : X \to [X/G]$ を、

$(X, G_X, G_X \to X \times_S X) \in [X/G](X)$ に対応する
$(g, x) \mapsto (g \cdot x, x)$

ものとする。

Claim $\forall (T, P, \pi) \in [X/G]$ に対し、

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T \arrow[d, "(T,P,\pi)"] \\ X \arrow[r, "q"'] & {[X/G]} \end{tikzcd}$$

i.e., $P \cong X \times_{[X/G]} T$

$\because)$ $X \times_{[X/G]} T(U) \ni (U, U \to X, U \to T, \varphi)$

$$\begin{tikzcd} \varphi : G_U \arrow[rr, "\cong"] \arrow[dr] & & P \times_T U \arrow[dl, "\pi \times \mathrm{id}_U"] \\ & X \times_T U & \end{tikzcd} \quad : G\text{-equiv.}$$

$e \in G$

$\longmapsto \varphi_U(e) \in P(U)$.

Conversely, $x \in P(U)$ に対し、

$$\begin{tikzcd} P \arrow[r] \arrow[rr, bend left, "b"] \arrow[dr, "a"'] & X \times_S T \arrow[r] \arrow[d] & T \\ & X & \end{tikzcd}$$

$(U, a(x), b(x), \varphi : g \mapsto g \cdot x)$

$\in X \times_{[X/G]} T(U)$

//

∴ $q : X \to [X/G]$ は smooth surj. → (iii)

(ii): $T \to [X/G]$, $T' \to [X/G]$: morph. from sch.

に対し、$Y := T \times_{[X/G]} T'$ alg. sp. を示す。

$$
\begin{tikzcd}
 & Z \arrow[rr] \arrow[dl, "sm"'] \arrow[dd] & & P' \arrow[dl, "sm"'] \arrow[dd] \\
Y \arrow[rr] \arrow[dd] & & T' \arrow[dd] & \\
 & P \arrow[rr] \arrow[dl, "sm."'] & & X \arrow[dl, "q"] \\
T \arrow[rr] & & {[X/G]} &
\end{tikzcd}
$$

→ Z: alg. sp. / S.

同様に、$R := Z \times_Y Z$: alg. sp. / S であり、

$R \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} Z$ étale surj.

∴ $Y \cong Z/R$ alg. space / S (alg. sp. の étale descent) //

Rem ・先の claim より、

$$\begin{tikzcd} G \times_S X \arrow[r, "\text{action}"] \arrow[d, "pr_2"'] \arrow[dr, phantom, "\perp"] & X \arrow[d, "q"] \\ X \arrow[r, "q"'] & {[X/G]} \end{tikzcd}.$$

$$\therefore\ G \times_S X \xrightarrow{\simeq} X \times_{[X/G]} X$$

$G \curvearrowright X$ free のとき、$[X/G] \cong X/G$: alg. sp.

・ $\mathfrak{X}$: stack in $(Sch/S)_{et}$

$\Phi : X \longrightarrow \mathfrak{X}$: G-invariant morph. $/S$.

i.e.,

$$\begin{tikzcd} G \times_S X \arrow[r, "\text{action}"] \arrow[d, "pr_2"'] \arrow[dr, phantom, "\overset{\simeq}{\Leftarrow}"] & X \arrow[d, "\Phi"] \\ X \arrow[r, "\Phi"'] & \mathfrak{X} \end{tikzcd}$$

Then $\exists$ morph. $\Phi : [X/G] \to \mathfrak{X}$

s.t.
$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[dr, "q"'] \arrow[r, phantom, shift right=4ex, "\cong\Downarrow"] & \mathfrak{X} \\ & {[X/G]} \arrow[u, "\Phi"'] \end{tikzcd}$$

Moreover, if $\forall$ morph. $T \to \mathfrak{X}$ from sch,

$X \times_{\mathfrak{X}} T$ is a G_T-torsor,

$\Phi : [X/G] \xrightarrow{\sim} \mathfrak{X}$: equiv.

- moduli problem

$\mathcal{M} : (Sch/S)^{op} \longrightarrow Set$: moduli functor

$\mathcal{M}(T) = \{$ "family of geom. objects" over $T \}$

$(T \xrightarrow{f} U) \longmapsto (\mathcal{M}(U) \xrightarrow{f^*} \mathcal{M}(T))$

<u>Q.</u> $\exists M \in Sch/S$ s.t. $\mathcal{M} \cong h_M$?

もしあれば、このMを <u>fine moduli space</u> of $\mathcal{M}$ という。すると、

$$\mathcal{M}(M) \cong h_M(M) = \mathrm{Hom}(M, M)$$
$$\text{univ.} \longleftrightarrow \mathrm{id}_M$$

(<u>universal family</u>)

$$\mathcal{M}(T) \cong \mathrm{Hom}(T, M)$$
$$f \longleftrightarrow \varphi_f : T \to M \quad \underline{\text{moduli map}}$$

$$\begin{tikzcd}
\mathrm{univ} \in \hat{\mathcal{M}}(M) \arrow[d, "\rho_f^*"'] \arrow[r, phantom, "\cong"] \arrow[dd, bend right, start anchor=west, end anchor=west] & \mathrm{Hom}(M, M) \ni \mathrm{id}_M \arrow[d, "\rho_f^*"] \arrow[dd, bend left] \\
\mathcal{M}(T) \arrow[r, phantom, "\cong"] & \mathrm{Hom}(T, M) \\
f \in \mathcal{M}(T) & \rho_f \in \mathrm{Hom}(T,M)
\end{tikzcd}$$

$\mathrm{univ} \longleftrightarrow \mathrm{id}_M$, $f \longleftrightarrow \rho_f$

$\therefore\ f = \rho_f^* \mathrm{univ}$. つまり、

全ての fam. of geom. obj. は univ. fam. の moduli map による pullback である。

<u>Example</u> $S = \operatorname{Spec}\mathbb{Z}$. $P \in \mathbb{Q}[t]$ poly.

$$\mathrm{Hilb}_N^P : \mathrm{Sch}^{op} \longrightarrow \mathrm{Set}$$

$$T \longmapsto \left\{ \begin{array}{l} Z \subseteq \mathbb{P}^N \times T \ \text{: closed sub.} \\ \quad {}_{\pi}\searrow \quad \downarrow \qquad \pi\text{: flat} \\ \qquad\quad\ T \qquad\ \text{Hilb. poly of fibers of } \pi = P \end{array} \right\}$$

is representable by a projective scheme Hilb^P_N
(<u>Hilbert scheme</u>)

しかし多くの場合、fine moduli space を持たない。

<u>Example</u> $\exists$ fine moduli space M_g of
smooth proj. curves of genus g $/\mathbb{C}$ と仮定。

$f_i : C_i \to T$ $(i=1,2)$
locally trivial fibered bdl with fiber $C \in M_g$

$f_1 : C \times T \to T$ trivial
f_2 : non-trivial とすると、

moduli maps $\varphi_{f_i} : T \to M_g$ $(i=1,2)$ は
どちらも const. map with value C. $\lightning$

よって、moduli stack $\mathcal{M} \to (Sch/S)_{et}$

lift して考えて、これが Artin or DM stack

を示すことで fine moduli とみなす、

univ. family は stack として存在：

$U \xrightarrow{\pi} \mathcal{M}$: smooth covering from sch.

$\longleftrightarrow$ "fam. of geom. obj. over U" $\in \mathcal{M}(U)$

$\shortparallel$

univ$_U$

$R := U \times_{\mathcal{M}} U$: alg. space $\overset{pr_1}{\underset{pr_2}{\rightrightarrows}} U \to \mathcal{M}$

$pr_1^*\, \mathrm{univ}_U \cong pr_2^*\, \mathrm{univ}_U$ in $\mathcal{M}(R)$

$\overset{\text{descent}}{\rightsquigarrow}$ $\exists$ universal fam. over $\mathcal{M}$ "$\in \mathcal{M}(\mathcal{M})$"

今回　$S = \mathrm{Spec}\,\mathbb{Z}$　(あとで $\mathrm{Spec}\,\mathbb{C}$)

condition of curves ~~$k=\bar{k}$~~　$*$　を.

$*$: ~~smoothable~~ Gorenstein genus g curves
with very ample n-canonical linear system
($n \geq 4$ fix)　とする.　(+ α : open condition)

$\mathcal{M}_g^{*} \to Sch$: moduli stack of $*$-curves

を、obj : $(T, f : C \to T)$
$f : C \to T$: $\underbrace{\text{flat fam of } *\text{-curves}}_{\forall \text{ geom. fib. is } *\text{-curve}}$

morph : $(T, f : C \to T) \xrightarrow{(\alpha, \beta)} (T', f' : C' \to T')$

$\alpha : T \to T'$: morph.
$\beta : C \xrightarrow{\simeq} C' \times_{T'} T$　isom $/T$.

と定める.

Prop $\mathcal{M}_g^*$ is an Artin stack of fin. ~~type~~.

$\because$ $H^* : Sch^{op} \to Set$ を.

$$H^*(T) := \left\{ \left(f : \underset{\mathcal{M}_g^*(T)}{\overset{\cap}{C}} \to T,\ \sigma : \mathbb{P}_T^N \underset{\text{isom}/T}{\overset{\simeq}{\to}} \mathbb{P}_T(f_*\omega_f^{\otimes n}) \right) \right\} \Big/ \simeq$$

ここで. $N := (2n-1)(g-1) - 1$

$(f : C \to T, \sigma) \simeq (f' : C' \to T, \sigma')$

$\Longleftrightarrow$
$$\begin{tikzcd} C \arrow[rr, "\simeq"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowright"] & C' \arrow[dl, "f'"] \\ & T & \end{tikzcd}$$
compatible with σ & σ'.

$P_{g,n}(t) := (2nt-1)(g-1)$

$= \chi(\omega_{*\text{-curve}}^{\otimes nt})$: Hilb. poly.

(↙ $*$-condition の openness より)

$\mathrm{Hilb}^* \subseteq \mathrm{Hilb}^{P_{g,n}}_N$: the open subsch. parametrizing $*$-curves embedded in $\mathbb{P}^N$.

$H^* \longrightarrow \mathrm{Hilb}^*$ を functor として、

$$(f : C \to T, \sigma) \longmapsto \left(\begin{tikzcd} C \arrow[r, hook] \arrow[drr, bend right] & \mathbb{P}_T(f_* \omega_f^{\otimes n}) \arrow[r, "\sigma^{-1}", "\simeq"'] & \mathbb{P}^N_T \arrow[d] \\ & & T \end{tikzcd} \right)$$

と定めると、これは rep'ble by closed emb.

つまり、$^\forall T \longrightarrow \mathrm{Hilb}^*$: morph. from sch.

に対し、$H^* \times_{\mathrm{Hilb}^*} T \longrightarrow T$: closed emb.

$$\begin{tikzcd} H^* \times_{\mathrm{Hilb}^*} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T \arrow[d] \\ H^* \arrow[r] & \mathrm{Hilb}^* \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} C \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}^N_T \arrow[d] \\ & T \end{tikzcd}$$

「一般に、$\omega_{C/T}^{\otimes n} \not\cong \mathcal{O}_C(1)$ である 」

$H^* \times_{Hilb^*} T \subseteq T$ は、local on T で

$f_* \omega_{C/T}^{\otimes n} \cong \mathcal{O}_T^{\oplus N+1} = f_* \mathcal{O}_C(1)$ をとったとき

$$
\begin{tikzcd}
f^* f_* \omega_{C/T}^{\otimes n} \arrow[r, "\cong"] \arrow[d, "\phi"] & f^* f_* \mathcal{O}_C(1) \arrow[d] \\
\omega_{C/T}^{\otimes n} \arrow[r, dashed, "\exists"', "\cong"] & \mathcal{O}_C(1)
\end{tikzcd}
$$

(つまり $\ker \phi \to \mathcal{O}_C(1)$ 0-map)

なる closed locus

よって H^* は quasi-proj. scheme.

natural morph. $H^* \longrightarrow \mathcal{M}_g^*$ は

$$(C \xrightarrow{f} T, \sigma) \longmapsto f : C \longrightarrow T$$

PGL_N-invariant であり、

$$[H^* / PGL_N] \xrightarrow{\cong} \mathcal{M}_g^*$$ がいえる。

//

III. Intersection theory on Artin stacks.

$\mathfrak{X}$: Artin stack, reduced, equidim'l & of finite type / a field k.

i.e., $\exists\ U \longrightarrow \mathfrak{X}$: smooth surj. from a reduced, equidim'l & of finite type / k.

($\mathcal{M}_g^* / \mathbb{C}$ の $\overline{\mathcal{M}}_g$ を含む reduced & irred comp. を想定)

<u>Def</u>

<u>prime divisor</u> on $\mathfrak{X}$:= irred. & reduced closed substack of codim. 1.

$(\mathbb{Q}\text{-})$ <u>Weil divisor</u> on $\mathfrak{X}$:= finite $(\mathbb{Q}\text{-})$ $\mathbb{Z}$-linear combination of prime divisors on $\mathfrak{X}$.

$D = \sum_i a_i D_i$ Weil div. on $\mathfrak{X}$.

$U \to \mathfrak{X}$: smooth morph. from a sch. に対し、

$U' \to U'$

$D_{i,U} \to D_i$ $\qquad D_U = \sum_i a_i D_{i,U}$ Weil divisor on U.

であり、 $U \xrightarrow{\varphi} V$: flat に対し、

$$\begin{tikzcd} U \arrow[rr, "\varphi"] \arrow[dr, "sm."'] & \Leftarrow\!\!\!\cong & V \arrow[dl, "sm."] \\ & \mathfrak{X} & \end{tikzcd}$$

$D_U = \varphi^* D_V$ をみたす、逆にこのような

family $\{D_U\}_{U \xrightarrow{sm.} \mathfrak{X}}$ は Weil divisor D on $\mathfrak{X}$

を定める。

<u>rational function</u> f on $\mathfrak{X}$:⟺ $^\exists$ dense open substack $\mathfrak{X}' \subseteq \mathfrak{X}$

(<u>regular function</u>) s.t. $f : \mathfrak{X}' \to \mathbb{A}^1$: morph.

(open dense 上一致するものは同一視) $\quad (f : \mathfrak{X} \to \mathbb{A}^1)$

rational function f on $\mathfrak{X}$ に対し、

<u>principal divisor</u> $\mathrm{div}_{\mathfrak{X}}(f)$ を、Weil divisor として

$$\mathrm{div}_{\mathfrak{X}}(f) := \sum_{\mathcal{Y} : \mathrm{prime}} \mathrm{ord}_{\mathcal{Y}}(f) \cdot \mathcal{Y}$$

$$= \{ \mathrm{div}_U(f|_U) \}_{U \to \mathfrak{X} \text{ smooth}}$$

• sheaf on $\mathfrak{X}$ について

<u>Def</u> $\mathrm{Lis\text{-}Et}(\mathfrak{X})$: <u>lisse-étale site</u> of $\mathfrak{X}$ とは、

full sub cat. of $\mathfrak{X}$ s.t. $^\forall a \in \mathfrak{X}(U)$;

$$a \in \mathrm{Lis\text{-}Et}(\mathfrak{X}) \iff a : U \to \mathfrak{X} \text{ smooth.}$$

$$\{ a_i \xrightarrow{f_i} a \}_i \in \mathrm{Cov}(\mathrm{Lis\text{-}Et}(\mathfrak{X}))$$

$\iff$ $a_i : U_i \to \mathfrak{X}$ とおいて、$\{ U_i \xrightarrow{f_i} U \}_i$ は
$a : U \to \mathfrak{X}$ étale covering of U.

(同様に、flat fppf site も考えられる。
cf. stacks project.)

<u>Example</u> ・ $\mathcal{O}_{\mathfrak{X}} : (U \to \mathfrak{X}) \mapsto \{\text{regular functions on } U\}$
$= \Gamma(U, \mathcal{O}_U)$

$K_{\mathfrak{X}} : (U \to \mathfrak{X}) \mapsto \{\text{rational functions on } U\}$

は lisse-étale sheaf of rings.

(注) $K_{\mathfrak{X}} \cong Q(\mathcal{O}_{\mathfrak{X}}) : U \mapsto \mathcal{S}_U^{-1}\mathcal{O}(U)$ a sheaf 化.
$\mathcal{S}_U = \{a \in \mathcal{O}(U) \,;\, \times a : \mathcal{O}_U \to \mathcal{O}_U \text{ 単}\}$)

・ $\mathrm{CDiv}(\mathfrak{X}) := \Gamma(\mathfrak{X}, K_{\mathfrak{X}}^* / \mathcal{O}_{\mathfrak{X}}^*)$

$$:= \lim_{\substack{U \to \mathfrak{X} \\ \text{smooth}}} (K_{\mathfrak{X}}^* / \mathcal{O}_{\mathfrak{X}}^*)(U)$$

の元 D を <u>Cartier divisor</u> on $\mathfrak{X}$ という.

これは, $D = \{(U_i, f_i)\}_i$, where

$$\left\{\begin{array}{l} U_i \to \mathfrak{X} : \text{smooth s.t. } \coprod_i U_i \twoheadrightarrow \mathfrak{X}. \\ f_i : \text{rational function on } U_i \\ \quad \text{with } \left.\dfrac{f_i}{f_j}\right|_{U_i \times_{\mathfrak{X}} U_j} : \text{invertible regular} \end{array}\right.$$

とかける.

$D^W := \{\mathrm{div}_{U_i}(f_i)\}_i$: <u>associated Weil divisor</u>

- $U \mapsto \mathrm{WDiv}(U) = \{\text{Weil divisor on } U\}$

は lisse-étale sheaf of abelian groups.

$\mathrm{WDiv}_{\mathfrak{X}}$ を定める

$\mathrm{WDiv}(\mathfrak{X}) = \Gamma(\mathfrak{X}, \mathrm{WDiv}_{\mathfrak{X}})$ is

the group of Weil divisors on $\mathfrak{X}$.

- $\mathrm{CDiv}(\mathfrak{X}) \to \mathrm{WDiv}(\mathfrak{X})$; $D \mapsto D^W$. hom.

$\mathfrak{X}$ が normal のとき、injective.

Def L : <u>invertible $\mathcal{O}_{\mathfrak{X}}$-module</u>

or <u>line bundle</u> on $\mathfrak{X}$

$:\overset{\text{def}}{\Longleftrightarrow}$ L is étale sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules

s.t. • $f : U \to V$ / $\mathfrak{X}$ に対し、$f^* L|_V \xrightarrow[\text{isom.}]{\simeq} L|_U$

• $\exists$ smooth covering $\{U_i \to \mathfrak{X}\}_i$

s.t. $L|_{U_i} \simeq \mathcal{O}_{U_i}$.

(L, Z, s) : <u>pseudo-divisor</u> on $\mathfrak{X}$ (cf. Fulton, Intersection theory)

$:\overset{\text{def}}{\Longleftrightarrow}$ L : invertible $\mathcal{O}_{\mathfrak{X}}$-module

$Z \subseteq \mathfrak{X}$: closed sub. ($Z = \mathfrak{X}$ でも可)

$s : \mathcal{O}_{\mathfrak{X} \setminus Z} \xrightarrow{\simeq} L|_{\mathfrak{X} \setminus Z}$: isom.

Cartier divisor $D = \{(U_i, f_i)\}_i$ に対し

pseudo-divisor $(\mathcal{O}_X(D), \operatorname{supp} D^W, s_D)$ が定まる：

where $\mathcal{O}_X(D)$: line bdl on X

def. by $\mathcal{O}_X(D)|_{U_i} = \mathcal{O}_{U_i} \cdot f_i^{-1}$ $\forall i$.

s_D : U_i 上 $\mathrm{id} : \mathcal{O}_{U_i \setminus \operatorname{supp} D} \xrightarrow{\simeq} \mathcal{O}_{U_i \setminus \operatorname{supp} D} \cdot \underbrace{f_i^{-1}}_{\text{invertible}}$

により定まる isom.

<u>Rem</u> $\operatorname{Pic}(X) \simeq H^1(X_{\text{Zar}}, \mathcal{O}_X^*)$

= the group of isom classes of line bdls on X.

(<u>Picard group</u> of X)

$\operatorname{CDiv}(X) \to \operatorname{Pic}(X)$; $D \mapsto \mathcal{O}_X(D)$

is induced by $0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^* / \mathcal{O}_X^* \to 0$

(exact)

<u>Lem</u> $(\mathcal{L}, Z, s)$: pseudo-divisor.

Assume $Z \subseteq \mathfrak{X}$ nowhere dense. or

$\mathfrak{X}$ contains a dense open subscheme $\mathfrak{X}' \subseteq \mathfrak{X}$.

Then $\exists$ Cartier divisor D s.t.

$$\mathrm{Supp}\, D \subseteq Z \quad \& \quad \mathcal{O}_{\mathfrak{X}}(D) \cong \mathcal{L}$$

$$s_D \leftrightarrow s.$$

//

$\because$) <u>WMA</u> $\mathfrak{X}$: irreducible.

smooth covering $\{U_i \to \mathfrak{X}\}_{i \in I}$ s.t.

$\varphi_i : \mathcal{O}_{U_i} \xrightarrow{\cong} \mathcal{L}|U_i$ とる.

$U_0 := \mathfrak{X} \setminus Z$, $\varphi_0 := s : \mathcal{O}_{U_0} \xrightarrow{\cong} \mathcal{L}|U_0$.

とおく. (後者のとき. $\mathcal{L}|\mathfrak{X}'$ usual l.b. $\rightsquigarrow$ Zar. loc. triv.
より、$U_0 \subseteq \mathfrak{X}'$ dense open s.t. $\mathcal{O}_{U_0} \xrightarrow{\cong} \mathcal{L}|U_0$
とれる)

$g_{ij} := \varphi_i^{-1} \circ \varphi_j|_{U_{ij}} : \mathcal{O}_{U_{ij}} \xrightarrow{\sim} \mathcal{O}_{U_{ij}}$: trans. funct.

ここで $U_{ij} := U_i \times_{\mathcal{X}} U_j$, $i,j \in I \cup \{0\}$.

$U_{i0} \subseteq U_i$ open とみれ、

$f_i := g_{i0} \in \Gamma(\mathcal{O}_{U_{i0}}^*)$ を rational function on U_i

とみれる。

$\frac{f_i}{f_j} = g_{ij}$, ${}^\forall i,j \in I$ より、

$D := \{(U_i, f_i)\}_{i \in I}$ は Cartier divisor on $\mathcal{X}$

を定め、$\mathcal{O}_{\mathcal{X}}(D) \xrightarrow{\sim} \mathcal{L}$

$s_D \longleftrightarrow s$ //

$f : \mathcal{Y} \to \mathcal{X}$: morph. of Artin stacks / k.

$D = (\mathcal{L}, Z, s)$: pseudo-divisor on $\mathcal{X}$.

に対し、$f^*D := (f^*\mathcal{L}, f^{-1}(Z), f^*s)$ は

pseudo-divisor on Y. <u>pullback</u> of D

D は Cartier div. のとき、

$f^{-1}(\mathrm{supp}\, D^{W}) \subseteq Y$ nowhere dense

or Y has dense open subsch. のとき、

Lem から f^*D は Cartier divisor とみれる。

(後者は、up to linear equivalence で定まる)

- functorial divisors

以下、alg. space X は reduced, equidim'l,

of finite type$/k$ & Zariski locally quasi-separated

と仮定。 $\left(\overset{\text{def}}{\Longleftrightarrow} X = {}^{\exists}\bigcup_i X_i : \text{Zar. open covering} \right.$

s.t. $\left. \Delta X_i \text{ quasi-compact} \right)$

$\Rightarrow {}^{\exists} X' \subseteq X$: dense open subsch.

$D = (Z, \{D_B\}_{B \to \mathfrak{X}})$: functorial divisor on $\mathfrak{X}$ (D~)

:$\Longleftrightarrow_{\text{def}}$ • $Z \subseteq \mathfrak{X}$: nowhere dense closed sub. (codim 1)

• $p : B \to \mathfrak{X}$: morph. from an alg. space B

に対し、$D_B \in A_{\dim B - 1}(p^{-1}(Z)) \otimes \mathbb{Q}$

(つまり、$p^{-1}(Z) \subseteq B$ nowhere dense のとき

D_B は divisor on B with support $\subseteq p^{-1}(Z)$

そうでないとき、dense な comp. 毎に div. class

up to linear equivalence.)

s.t. compatible with alteration pushforward,

flat pullback & Gysin pullback.

つまり、$\varphi : B' \to B$: alteration of same deg. d

(resp. flat morph, regular closed immersion)

に対し、 $\varphi_* D_{B'} = d D_B$

$$\left(\text{resp. } \begin{array}{l} \varphi^* D_B = D_{B'} \\ \varphi^! D_B = D_{B'} \end{array} \right)$$

//

D は <u>effective</u> $:\overset{}{\underset{\text{def}}{\iff}}$ ${}^\forall \rho : B \to \mathfrak{X}$ with

$\rho^{-1}(Z)$ nowhere dense

に対し、 $D_B \geq 0$.

$D_{\mathfrak{X}} := \{ D_U \}_{\underset{\text{smooth}}{U \to \mathfrak{X}}}$: <u>associated Weil divisor</u>

<u>Example</u> $\pi : \mathcal{Y} \to \mathfrak{X}$: representable proper flat morph. of rel. dim. n betw. Artin stacks.

$E_1, \cdots, E_{n+1}$: Cartier divisors on $\mathcal{Y}$

s.t. $Z := \pi\left(\bigcap_{i=1}^{n+1} \operatorname{supp} E_i \right)$ codim. 1.

Then $D := \pi_*(E_1 \cdot \cdots \cdot E_{n+1})$ は、

$p : B \to \mathfrak{X}$ に対し、
$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi_B"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y} \arrow[d, "\pi"] \\ B \arrow[r, "p"'] & \mathfrak{X} \end{tikzcd}$$

(注) C は Zar-loc. $\mathbb{Q}$-sep.
(∵ π_B : sep. $\Rightarrow$ $\mathbb{Q}$-sep.)

$D_B := \pi_{B*}(E_1|_C \cdot \cdots \cdot E_{n+1}|_C)$ として

functorial divisor $(\mathfrak{X}, \{D_B\}_{B \to \mathfrak{X}})$ と

みなせる.

$n = 0$, $\pi = \mathrm{id}$ として、特に Cartier divisor は

functorial divisor である.

(Effectivity Thm とよぶ)

Thm $\mathfrak{X}$ は regular in codim. 1 と仮定.

このとき functorial divisor D on $\mathfrak{X}$ は

effective $\Longleftrightarrow$ ass. Weil divisor $D_{\mathfrak{X}}$ は effective.

∵) 簡単のため, char $k = 0$ のときに示す.
(char $k > 0$ のときも alteration を用いていえる)

(⇒) は明らかなので, (⇐) を示せば良い.

$D_{\mathfrak{X}} \geq 0$ と仮定.

$\forall p : B \to \mathfrak{X}$: morph. s.t. $p^{-1}(Z) \subseteq B$ nowhere dense

に対し $D_B \geq 0$ を示そう.

$\pi : U \to \mathfrak{X}$: smooth surj. morph. from sch.

$$\begin{array}{ccc} B_U & \longrightarrow & U \\ {\scriptstyle \pi_B}\downarrow & \Box & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow[p]{} & \mathfrak{X} \end{array}$$

(注) B_U は Zar. loc. q-sep.
∵) π は (Zar. loc.) q-sep. なことから, $\mathfrak{X}$: q-cpt & U : (Zar. loc.) q-sep. よりいえる.

$D_{B_U} = \pi_B^* D_B$ であり, $D_{B_U} \geq 0$ ならば

$D_B \geq 0$ なので.

B を B_U (の étale cov.) にとりかえる。

WMA
$$\begin{tikzcd} B \arrow[r, "\exists"] \arrow[dr, "p"'] & U \arrow[d, "\pi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathfrak{X} \end{tikzcd}$$

$\tau : U' \to U$、$\varphi : B' \to B$: resolution of sing

を
$$\begin{tikzcd} B' \arrow[r, "\exists \psi"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & U' \arrow[d, "\tau"] \\ B \arrow[r] & U \end{tikzcd}$$
ととる。

$D_B = \varphi_* D_{B'}$ より、$D_{B'} \geq 0$ をいえば良い。

B'、U' は non-singular より、ψ は loc. comp. int.

(実際、$B' \xrightarrow[\text{reg. closed imm.}]{(id, \psi)} B' \times U' \xrightarrow[\text{smooth}]{pr_2} U'$)

よって $D_{B'} = \psi^! D_{U'}$。

よって $D_{U'} \geq 0$ をいえば良い.

今 仮定 より, $D_U \geq 0$ である.

$C \subseteq U'$: τ-vertical curve

$C' \to C$: normn.

$$\begin{tikzcd} C' \arrow[r, "\nu"] \arrow[d, "\gamma"'] & C \subseteq U' \arrow[d, "\tau"] \\ \mathrm{Spec}\, k' \arrow[r] & U \end{tikzcd}$$

この時 $D_{U'} \cdot C = \deg \nu^! D_{U'}$

$$= \deg D_{C'} = \deg \gamma^* D_{\mathrm{Spec}\, k'} = 0.$$

$\therefore$ $D_{U'}$ は τ-numerically trivial.

特に $-D_{U'}$ は τ-nef.

よって negativity lemma より, $D_{U'} \geq 0$. //

IV. Horikawa index Ind_D （以下 $/\mathbb{C}$ とする。）

条件 $*$ として、

$*$ = local complete intersection curves of genus g with very ample n-cano. lin. sys. $\left(\begin{array}{l} g \geq 2 \\ n \geq 4 \\ \text{fix} \end{array}\right)$

(resp. reduced & $\;\cancel{\;\;}$) とし、

$\mathcal{M}_g^{lci}$ (resp. $\mathcal{M}_g^{rlci}$) := reduced & irred. comp. of $\mathcal{M}_g^{*}$ containing $\overline{\mathcal{M}}_g$.

とおく。

$$\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g \subseteq \mathcal{M}_g^{rlci} \subseteq \mathcal{M}_g^{lci}$$

$\mathcal{M}_g^{lci}$ ← non-separated.

$\mathcal{M}_g$, $\overline{\mathcal{M}}_g$: smooth DM stacks

$\mathcal{M}_g^{rlci}$: smooth Artin stack. ← (by deformation theory)

(by semist reduction)

Lem $f : S \to B$: fibered surface of genus g.

Then rel. cano. model $\overline{f} : \overline{S} \to B$ belongs to $\mathcal{M}_g^{lci}$.

tautological class

$\pi : \mathcal{U}_g^{lci} \to \mathcal{M}_g^{lci}$: universal family.

$\omega_\pi = \mathcal{O}(K_\pi)$: relative dualizing sheaf on $\mathcal{U}_g^{lci}$

$\lambda := \det \underbrace{\pi_* \omega_\pi}_{\text{loc. free of rank } g}$: Hodge line bdl.

$\kappa := \pi_*(K_\pi \cdot K_\pi)$: 1st Morita-Mumford class

これらは functorial divisor on $\mathcal{M}_g^{lci}$ とみれる.

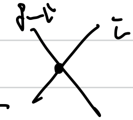
- rel. min. fib. surf. $f : S \to B$ of genus g

に対し、 $\bar{f} : \bar{S} \to B \leftrightarrow \varphi_{\bar{f}} : B \to \mathcal{M}_g^{lci}$

とみて、
$$K_f^2 = \deg \kappa_B$$
$$\chi_f = \deg \lambda_B$$
$$e_f = \deg (12\lambda_B - \kappa_B)$$

$\delta := \overline{M}_g \setminus M_g$: <u>boundary divisor</u> on $\overline{M}_g$

$= \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} \delta_i$: irred. decomp. は simple normal crossing.

$\delta_0 \ni$ [], $\delta_i \ni$ [

] $(i \geq 1)$

- (Harer, Moriwaki) $\mathrm{Pic}(\overline{M}_g)_{\mathbb{Q}}$ is generated by

$$\lambda, \delta_0, \cdots, \delta_{\lfloor \frac{g}{2} \rfloor} \begin{cases} \text{freely} & (g \geq 3) \\ \text{with one relation} & (g = 2) \\ 10\lambda = \delta_0 + 2\delta_1 & \end{cases}$$

- (Noether) $12\lambda = \kappa + \delta$

以下、$g \geq 3$ と仮定。

$D \subseteq M_g$: effective divisor を fix.

$$\overline{D} = a\lambda - \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} b_i \delta_i \quad \text{in } \mathrm{Pic}(\overline{M}_g)_{\mathbb{Q}}$$

for $\exists ! a, b_i > 0$.

$s_D := \dfrac{a}{b}$, $b := \min\{b_i\}$.

$$\kappa - (12 - s_D)\lambda \underset{\mathbb{Q}}{\sim} \frac{1}{b}\left(\overline{D} + \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} (b_i - b)\delta_i\right)$$

on $\overline{\mathcal{M}}_g$

$\therefore$ $\exists$ rational function φ on $\mathcal{M}_g^{lci}$

$\exists$ $q \in \mathbb{Q}$.

$\exists$ δ' : $\mathbb{Q}$-divisor with support $\subseteq \mathcal{M}_g^{lci} \setminus \overline{\mathcal{M}}_g$

s.t. $\kappa - (12 - s_D)\lambda + q \cdot \mathrm{div}(\varphi)$

$$= \underbrace{\frac{1}{b}\left(\overline{D} + \sum_i (b_i - b)\delta_i\right) + \delta'}_{\text{as a Weil div.}} \quad \text{on } \mathcal{M}_g^{lci}$$

LHS は functorial divisor とみれる.

これを H_D とかく.

(rel. min. genus g は常に固定)

<u>Def</u> D-general fiber germ $f : S \to (p \in B)$

に対し、<u>Horikawa index</u> $\mathrm{Ind}_D(f^{-1}(p))$ を、

$\bar{f} : \bar{S} \to (p \in B)$: rel. cano. model

$\longleftrightarrow$ $\rho_f : (p \in B) \to \mathcal{M}_g^{lci}$: moduli map

$\mathrm{Ind}_D(f^{-1}(p)) := \mathrm{Coeff}_p H_{D,B}$ と定義.

・$f : S \to B$: D-general fib. surf.

に対し、

$$K_f^2 - (12 - s_D)\chi_f = \deg(\kappa_B - (12 - s_D)\lambda_B)$$

$$= \deg H_{D,B} = \sum_{p \in B} \mathrm{Ind}_D(f^{-1}(p))$$

が成立。 $\to$ Main thm 前半 が いえた.

Q. H_D は effective ?
(つまり $\delta' \geq 0$?)

もしこれが正しいなら、~~Effectivity~~ Thm より
$\mathrm{Ind}_D(f^{-1}(p)) \geq 0$ がいえる。

これは $\mathcal{M}_g^{\mathrm{red}}$ 上なら正しい:

Thm $\mathrm{codim}(\mathcal{M}_g^{\mathrm{red}} \setminus \overline{\mathcal{M}}_g) \geq 2$.

これより、D-general fiber germ $f^{-1}(p)$ は
$\overline{f}^{-1}(p)$ reduced のとき、$\mathrm{Ind}_D(f^{-1}(p)) \geq 0$.
が分かる。

- Morsification conj.

$f: S \to (p \in B)$: fiber germ.
(← étale local に考える。)

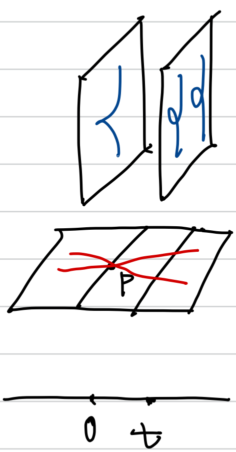
<u>splitting deformation</u> of $f^{-1}(p)$ とは。

$$\mathcal{S} \xrightarrow{\mathbb{f}} (p \in \mathbb{B}) \xrightarrow[\text{sm.}]{h} (0 \in T) \quad \text{s.t.}$$

flat fam. of proj. curves   smooth surf.   smooth curve

$$\mathbb{f}_0 : \mathcal{S}_0 \to (p \in \mathbb{B}_0) \cong f : S \to (p \in B).$$

For $t \in T$, $t \neq 0$, $\mathbb{f}_t : \mathcal{S}_t \to \mathbb{B}_t$ may have many singular fibers $F_1, \cdots, F_\ell$.



$$\mathcal{S}$$
$$\downarrow$$
$$\mathbb{B}$$
$$\downarrow$$
$$T$$

"$f^{-1}(p) \longrightarrow F_1 + \cdots + F_\ell$"

• (Morsification conj. Xiao-Reid)

Any fiber germ is splittable into stable fiber germs with one node & smooth multiple fiber germs by finitely many splitting deformations.

$(x,y) \mapsto xy$

α, X

$(x,y) \mapsto x^m$

$$\left(f^{-1}(p) \xrightarrow{\exists} \cdots \xrightarrow{\exists} F_1 + \cdots + F_l, \quad F_i \text{ as above} \right)$$

<u>Rem</u> • $g \leq 5$ OK (Takamura)

$g \geq 6$ open.

• $f^{-1}(p)$ reduced のとき OK.

(codim $M_g^{rlci} \setminus \overline{M}_g \geq 2$ と M_g^{rlci} smooth なことを用いる)

Lem (deformation invariance of Ind_D)

$\mathbb{S} \xrightarrow{f} (p \in \mathbb{B}) \xrightarrow{h} (0 \in \mathbb{T})$: splitting deformation of $f^{-1}(p)$.

Then $\mathrm{Ind}_D(f^{-1}(p)) = \coprod_{q \in \mathbb{B}_t} \mathrm{Ind}_D(f_t^{-1}(q))$

for $t \in \mathbb{T}$ suff. near 0. //

∵) $\bar{f} : \bar{\mathbb{S}} \to (p \in \mathbb{B})$: rel. cano. model of f

⤳ $(p \in \mathbb{B}) \longrightarrow \mathcal{M}_g^{lci}$: moduli map.

$\mathbb{B}_t \overset{\iota_t}{\hookrightarrow} \mathbb{B}$ regular imm. より、

$H_{D,\mathbb{B}_t} = \iota_t^! H_{D,\mathbb{B}}$ であり、この degree は

$t \in \mathbb{T}$ suff. near 0 で一定。よって

$\coprod_{q \in \mathbb{B}_t} \mathrm{Ind}_D(\bar{f}_t^{-1}(q)) = \deg H_{D,\mathbb{B}_t} = \deg H_{D,\mathbb{B}_0} = \mathrm{Ind}_D(\bar{f}^{-1}(p))$

//

- semistable reduction との関係.

$f : S \to B$: fibered surface. (B not nece. proper)

$f' : S' \to B'$: the semistable reduction

$$\begin{tikzcd}
 & & \tilde{S} \arrow[dl, "\text{resol}"'] \arrow[dr, "\text{rel. min. model}"] & \\
S \arrow[d, "f"'] & S \times_B B' \arrow[l] \arrow[d] \arrow[dl, phantom, "\lrcorner", very near start] \arrow[rr, dashed, "\sim"] & & S' \arrow[dll] \\
B & B' \arrow[l, hook', "\varphi"', "\text{finite}"] & &
\end{tikzcd}$$

f' : semistable.

$N := \deg \varphi$.

Then moduli maps are

$$\begin{tikzcd}
B \arrow[d, "\rho_f"'] & B' \arrow[l, hook', "\varphi"'] \arrow[d, "\rho_{f'}"] \\
\mathcal{M}_g^{\mathrm{lct}} & \overline{\mathcal{M}_g} \arrow[l, phantom, "\supseteq"]
\end{tikzcd}$$

$$\rightsquigarrow K_{(f,B)} - \frac{1}{N}\varphi_* K_{(f',B')} = \sum_{p\in B}^{\exists} c_1^2(f^{-1}(p))\cdot p$$

$$\lambda_{(f,B)} - \frac{1}{N}\varphi_* \lambda_{(f',B')} = \sum_{p\in B}^{\exists} \chi(f^{-1}(p))\cdot p$$

$c_1^2(f^{-1}(p)), \chi(f^{-1}(p))$: Tan's Chern invariant

Properties (1) $c_1^2(f^{-1}(p)), \chi(f^{-1}(p)) \geq 0$.

「=」 $\Leftrightarrow$ $f^{-1}(p)$ semistable.

(2) $c_1^2(f^{-1}(p)), \chi(f^{-1}(p))$ are topological invariants.

(can be written by the data of topological monodromy)
(Ashikaga)

(3) $c_1^2(f^{-1}(p)) \leq 8\chi(f^{-1}(p))$

「=」 $\Leftrightarrow$ $f^{-1}(p) = m\cdot C$, C: nodal curve.

Lem (comparison formula of Ind_D for semist. red.)

$$\mathrm{Ind}_D(f^{-1}(p)) = \frac{1}{N}\sum_{q\in\varphi^{-1}(p)} \mathrm{Ind}_D(f'^{-1}(q))$$

$$+ \ c_1^2(f^{-1}(p)) - (12 - s_D)\cdot\chi(f^{-1}(p))$$

Proof of Main thm

Morsification conj. 満たす D-gen. fib germ $f^{-1}(p)$ に対し

$\mathrm{Ind}_D(f^{-1}(p)) \geq 0$ 示せば良い.

deformation invariance of Ind_D より、

WMA $f^{-1}(p)$ は stable with one node

or smooth multiple fib germ.

前者の場合は H_D が $\overline{M}_g$ 上 effective より OK.

後者のとき、$f^{-1}(p) = m\cdot C$ は

mth cyclic covering $B' \to B$ branched at p

$q \mapsto p$

により semist. red. $f' : S' \to B'$ with f' smooth

がとれる。よって comparison formula より

$$\mathrm{Ind}_D(f^{-1}(p)) = \frac{1}{m}\mathrm{Ind}_D(f'^{-1}(q)) + c_1^2(f^{-1}(p)) - (12 - s_D)\chi(f^{-1}(p))$$

$$c_1^2(f^{-1}(p)) = 8\chi(f^{-1}(p)) \quad \text{by (3)}$$

よって $s_D \geq 4$ のとき $\mathrm{Ind}_D(f^{-1}(p)) \geq 0$. //

$$s_g := \inf_{D \subset M_g} s_D$$ とおく.

<u>Known</u>

- $g \leq 11$ のとき s_g の値と $s_g = s_D$ となる D

e.g.) $s_2 = 10$, $s_3 = 9$, $s_5 = 8$, $s_{10} = s_{11} = 7$.
$(D = \delta_1)$

- $\exists \{D_g\}$ s.t. $s_{D_g} \to 6$ $(g \to \infty)$.

- (Farkas–Morisson conj) $s_D > 6 \quad \forall D \subseteq \overline{M}_g$.

<u>Application</u>

<u>Cor</u> (partial answer to Lu-Tan's conj.)

$f^{-1}(p)$: Mosification conj. を満たす fiber germ に対し.

$c_1^2(f^{-1}(p)) \geq \chi(f^{-1}(p))$.

∵) semist. red.

$$\begin{tikzcd}
S \arrow[d, "f"'] & S \times_B B' \arrow[l] \arrow[d] \arrow[r, dashed, "\sim"] & S' \arrow[dl, "f'"] \\
B & B' \arrow[l, "\varphi"'] & \\
p \in B & q \in B' \arrow[l, mapsto] &
\end{tikzcd}$$

をとる.

- (Cornalba–Harris) $D = a\lambda - b\delta$ on $\overline{M}_g$ は

ample $\iff$ $b > 0$ & $s_D = \frac{a}{b} > 11$.

これより $^\forall \varepsilon > 0$ に対し、general ample eff. $\mathbb{Q}$-div. D_ε を、

$[\bar{f}^{-1}(q)] \notin D_\varepsilon$ & $s_{D_\varepsilon} = 11+\varepsilon$

ととれる。

Comparison formula より、

$$\underset{\substack{\vee\!| \\ 0}}{\mathrm{Ind}_{D_\varepsilon}(f^{-1}(p))} = \frac{1}{N}\underbrace{\mathrm{Ind}_{D_\varepsilon}(\bar{f}^{-1}(q))}_{=0} + c_1^2(f^{-1}(p)) - (1-\varepsilon)\chi(f^{-1}(p)).$$

by Main thm.

(∵ $[\bar{f}^{-1}(q)] \notin D_\varepsilon$ & $b_i = b$)

$$\therefore \quad c_1^2(f^{-1}(p)) \geq (1-\varepsilon)\chi(f^{-1}(p))$$

$$\xrightarrow[\varepsilon \to 0]{} \quad c_1^2(f^{-1}(p)) \geq \chi(f^{-1}(p)).$$

//

同様のセロニで、C: smooth proj. curve

$\sigma \in \mathrm{Aut}(C)$ に対し、

σ の "type" ($C \to C/\sigma$ の分岐のデータ)

が C の "speciality" ($[C] \in \mathcal{M}_g$ がどんな locus D に含まれるか)

により制限されるという主張がいえる。

e.g. C: genus 3 curve. with $\sigma \in \mathrm{Aut}(C)$.

s.t. order of $\sigma = 7$

$$\begin{tikzcd}
C \arrow[r] \arrow[dr, "\text{normaliz.}"'] & C/\sigma \cong \mathbb{P}^1 & \ni p_1 + p_2 + 5p_3 \\
& C' \arrow[u, "7\text{-cyclic}"'] & |\mathcal{O}(7)| \arrow[u]
\end{tikzcd}$$

$\Rightarrow$ C: hyperelliptic.