2022/06/24

Prop 2.3.7 $\operatorname{card} A = \operatorname{card} A'$, $\operatorname{card} B = \operatorname{card} B'$

$$\Longrightarrow \operatorname{card} \mathcal{F}(A,B) = \operatorname{card} \mathcal{F}(A',B')$$

∵ [示すこと: $\exists f : \mathcal{F}(A,B) \longrightarrow \mathcal{F}(A',B')$: bij.]

$\operatorname{card} A = \operatorname{card} A'$ より $\exists \alpha : A \to A'$: bij.

$\operatorname{card} B = \operatorname{card} B'$ より $\exists \beta : B \to B'$: bij.

$$\begin{aligned} f : \mathcal{F}(A,B) &\longrightarrow \mathcal{F}(A',B') \\ F &\longmapsto \beta \circ F \circ \alpha^{-1} \end{aligned}$$ と定める.

$$\left(\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A' \arrow[r, dashed] & B' \end{tikzcd} \right)$$

[示すこと: f : bij.]

$$\begin{aligned} g : \mathcal{F}(A',B') &\longrightarrow \mathcal{F}(A,B) \\ F' &\longmapsto \beta^{-1} \circ F' \circ \alpha \end{aligned}$$

とおくと、g は f の逆写像、よって f : bij.

Rem g は f の逆写像

∵ [示すこと: $g\circ f = id$, $f\circ g = id$] → 同様

(i.e., $\forall F \in \mathcal{F}(A,B)$, $g\circ f(F) = F$)

$\forall F \in \mathcal{F}(A,B)$ をとる

[示すこと: $g\circ f(F) = F$]

$$g\circ f(F) = g(\beta\circ F\circ \alpha^{-1})$$

$$= \beta^{-1}\circ(\beta\circ F\circ\alpha^{-1})\circ\alpha = F$$

//

Thm 2.38

∵ (1) 略 //

$m = \operatorname{card} M \quad n = \operatorname{card} N, \quad p = \operatorname{card} P$ とする.

(2) $(mn)^p = \operatorname{card}(\mathcal{F}(P, M \times N))$

$m^p n^p = \operatorname{card}(\mathcal{F}(P, M) \times \mathcal{F}(P, N))$

である。

[示すこと: $\exists f : \mathcal{F}(P, M \times N) \to \mathcal{F}(P, M) \times \mathcal{F}(P, N)$: bij.]

$$\begin{aligned} f : \mathcal{F}(P, M \times N) &\longrightarrow \mathcal{F}(P, M) \times \mathcal{F}(P, N) \\ F &\longmapsto (F_M, F_N) \end{aligned}$$

(ただし $\begin{aligned} F : P &\longrightarrow M \times N \\ p &\longmapsto (F_M(p), F_N(p)) \end{aligned}$)

と定める.

すると f: bij. (略:示せ) //

(3) $$(p^m)^n = \mathrm{card}\ \mathcal{F}(N, \mathcal{F}(M,P))$$

$$p^{mn} = \mathrm{card}\ \mathcal{F}(M \times N, P)$$

となる.

[示すこと: $\exists f : \mathcal{F}(N, \mathcal{F}(M,P)) \to \mathcal{F}(M \times N, P)$]

$\qquad F \in \mathcal{F}(N, \mathcal{F}(M,P))$

$$f(F) : M \times N \longrightarrow P$$

$$(m, n) \longmapsto (F(n))(m)$$

と定める.

$$\begin{tikzcd} & M \ni m \\ M \times N \arrow[ur] \arrow[dr] & \\ & N \ni n \arrow[r, "F"] & \mathcal{F}(M,P) \ni F(n) : M \to P \end{tikzcd}$$

$$M \times N \xrightarrow{F'} P$$

$$g(F') : N \longrightarrow \mathcal{F}(M,P)$$

$$n \longmapsto F'(\cdot, n)$$

$$\mathcal{F}(N, \mathcal{F}(M,P)) \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{F}(M \times N, P)$$

[示すこと: $\forall F \in \mathcal{F}(N, \mathcal{F}(M,P))$, $g \circ f(F) = F$]

$\forall F \in \mathcal{F}(N, \mathcal{F}(M,P))$ をとる

[示すこと: $g \circ f(F) = F \;:\; N \to \mathcal{F}(M,P)$

i.e, $\forall n \in N$, $(g \circ f(F))(n) = F(n)$]

$\forall n \in N$ をとる

[示すこと: $(g \circ f(F))(n) = F(n) \;:\; M \to P$

i.e, $\forall m \in M$, $((g \circ f(F))(n))(m) = (F(n))(m)$]

$\forall m \in M$ をとる

[示すこと: $((g \circ f(F))(n))(m) = (F(n))(m) \in P$]

$$\begin{aligned}
((g \circ f(F))(n))(m) &= ((g(f(F)))(n))(m) \\
&= (f(F)(\cdot, n))(m) \\
&= (f(F))(m, n) \\
&= (F(n))(m)
\end{aligned}$$