2022/06/28

Thm 2.3.10

$\because$ (1) $n \leqq \aleph_0$ とする. $(\aleph_0 = \aleph_0 = \mathrm{card}\,\mathbb{N})$

[示すこと: $n + \aleph_0 = \aleph_0$, i.e., $n + \aleph_0 \leqq \aleph_0$]

$\aleph_0 + \aleph_0 \leqq \aleph_0$ を示せばよい

[示すこと: $^{\exists}A, A'$: $\mathrm{card}\,A = \mathrm{card}\,A' = \aleph_0$
$\mathrm{card}\,(A \sqcup A') \leqq \aleph_0$]

$A = \mathbb{N}$, $A' = -\mathbb{N}$ とおくと,

$\mathrm{card}\,A = \mathrm{card}\,A' = \aleph_0$

[示すこと: $\mathrm{card}\,(A \sqcup A') \leqq \aleph_0$]

$A \sqcup A' = (-\mathbb{N}) \sqcup \mathbb{N} = \mathbb{Z} - \{0\}$ より

$$\mathrm{card}\,(A \sqcup A') = \mathrm{card}\,(\mathbb{Z} - \{0\}) \leqq \mathrm{card}\,\mathbb{Z} = \aleph_0 \quad //$$

(2) 同様に $\mathfrak{c}+\mathfrak{c} \leqq \mathfrak{c}$ を示せばよい

$$\left[\text{示すこと: } \exists A, A' : \begin{array}{l} \mathrm{card}\, A = \mathrm{card}\, A' = \mathfrak{c} \\ \mathrm{card}\,(A \sqcup A') \leqq \mathfrak{c} \end{array}\right]$$

$A := (0,1)$, $A' := [1,2)$

とおけばよい //

(3) $1 \leqq n \leqq \aleph_0$ とする.

$$\left[\begin{array}{ll} \text{示すこと: } & n\aleph_0 = \aleph_0 \\ \text{i.e. } & n\aleph_0 \leqq \aleph_0 \\ \text{i.e. } & \exists N, A : \begin{array}{l} \mathrm{card}\, N = n \\ \mathrm{card}\, A = \aleph_0 \\ \mathrm{card}\,(N \times A) \leqq \aleph_0 \end{array} \end{array}\right]$$

N, A を $\begin{array}{l} \mathrm{card}\, N = n \\ \mathrm{card}\, A = \aleph_0 \end{array}$

となるように(何でもよいので)とる.

N, A は高々可算

$\therefore$ 前に示したことから, $N \times A$ も高々可算

$\therefore$ $\mathrm{card}\,(N \times A) \leqq \aleph_0$

//

(4) $2 \leqq n \leqq \aleph_0$ とする。

<u>claim</u> $2^{\aleph_0} = n^{\aleph_0}$

(i.e., $n^{\aleph_0} \leqq 2^{\aleph_0}$)

∵) $n \leqq 2^n$ より

$$n^{\aleph_0} \leqq (2^n)^{\aleph_0} = 2^{n\aleph_0} \quad (\because \text{指数法則})$$

$$= 2^{\aleph_0} \quad (\because (3)\ n\aleph_0 = \aleph_0 \text{ より})$$

//

<u>claim</u> $2^{\aleph_0} \leqq \mathfrak{c}$

∵) [示すこと: $\exists X, C$: $\operatorname{card} X = 2^{\aleph_0}$, $\operatorname{card} C = \mathfrak{c}$, $\operatorname{card} X \leqq \operatorname{card} C$]

$X := \{1, 2\}^{\mathbb{N}} = \mathcal{F}(\mathbb{N}, \{1, 2\})$, $C := \mathbb{R}$

とおくと, $\operatorname{card} X = 2^{\aleph_0}$, $\operatorname{card} C = \mathfrak{c}$

[示すこと: $\operatorname{card} X \leqq \operatorname{card} C$
i.e. $\exists f : X \to C$: inj.]

各 $x \in X$ ($x : \mathbb{N} \to \{1, 2\}$) に対し, $f(x) = 0.x(1)\,x(2)\cdots\cdots$ とおけばよい //

claim $\mathfrak{c} \leqq 10^{\aleph_0} \; (= 2^{\aleph_0})$

$\because$) [示すこと: $\exists C, X$: $\mathrm{card}\, C = \mathfrak{c}$, $\mathrm{card}\, X = 10^{\aleph_0}$, $\mathrm{card}\, C \leqq \mathrm{card}\, X$]

$C := (0, 1)$, $X = \mathcal{F}(\mathbb{N}, \{0, 1, \cdots, 9\})$

とおくと, $\mathrm{card}\, C = \mathfrak{c}$, $\mathrm{card}\, X = 10^{\aleph_0}$.

[示すこと: $\mathrm{card}\, C \leqq \mathrm{card}\, X$

i.e. $\exists f : X \to C$: surj.]

各 $x \in X$ $(x : \mathbb{N} \to \{0, 1, \cdots, 9\})$ に対し,

$$f(x) = 0.x(1)x(2)x(3)\cdots$$

とおくと, f は surj. //

(5), (6) は 略 //