Two-numbers and Euler characteristics for quandles

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Abstract

We study quandles in view of symmetric spaces.

Outline

- **2-numbers** of symmetric spaces was initiated by Chen-Nagano (vs Euler characteristics).
- We are trying to extend this framework to quandles (2-numbers are easy to define).
- We define **Euler characteristics** for quandles, by extending those for symmetric spaces.
- We calculate them for several examples; prove some properties similar to the usual one.

Ref.

- Kai, Tamaru: "On the Euler characteristics for quandles", Internat. J. Math., 36 (2025)

Preliminaries - (1/3)

Review quandles and Chen-Nagano theory.

Def.

(X, s) : **quandle**
$:\Leftrightarrow X$: a set, $s : X \to \mathrm{Map}(X, X)$ such that

- $\forall x \in X,\ s_x(x) = x$;
- $\forall x \in X,\ s_x$: bijection;
- $\forall x, y \in X,\ s_x \circ s_y = s_{s_x(y)} \circ s_x$.

Fact (Joyce)

- A symmetric space is a quandle.

Def.

For a quandle (X, s) and $A \subset X$,

- A is **antipodal** $:\Leftrightarrow \forall a, b \in A,\ s_a(b) = b$.
- **2-number** $\#_2(X) := \sup\{\#A \mid A \text{ antipodal}\}$.

Preliminaries - (2/3)

Ex (Sphere S^n)

- Any maximal antipodal is $\{\pm p\}$, so $\#_2 S^n = 2$.

Ex (Real projective space $\mathbb{R}\mathrm{P}^n$)

- Any maximal antipodal is congruent to $\{[e_1], \ldots, [e_{n+1}]\}$, so $\#_2 \mathbb{R}\mathrm{P}^n = n+1$;

Note (good points)

- $\#_2$ is an invariant of quandles;
- $\#_2$ generalizes 2-rank initiated by Borel-Serre.
- $\#_2$ gives an obstruction for embeddings ($\exists f : X \to Y$ (embed.) $\Rightarrow \#_2(X) \leq \#_2(Y)$);
- For symmetric spaces, $\#_2$ relates to χ^{top}...

Thm (Chen-Nagano 1988)

M : compact Riemannian symmetric space

$\Rightarrow$ $\#_2 M \geq \chi^{\mathrm{top}}(M)$

($\#_2 M = \chi^{\mathrm{top}}(M)$ if Hermitian)

Preliminaries - (3/3)

Conjecture (Chen-Nagano)

M : compact Riemannian symmetric space

$\Rightarrow \quad \#_2 M \equiv \chi^{\mathrm{top}}(M) \pmod 2$?

Ex (Sphere S^n)

- $\chi^{\mathrm{top}}(S^n) = \begin{cases} 0 & (n \text{ odd}) \\ 2 & (n \text{ even}) \end{cases} \leq 2 = \#_2 S^n$

Ex (Real projective space $\mathbb{R}\mathrm{P}^n$)

- $\chi^{\mathrm{top}}(\mathbb{R}\mathrm{P}^n) = \begin{cases} 0 & (n \text{ odd}) \\ 1 & (n \text{ even}) \end{cases} \leq n+1 = \#_2 \mathbb{R}\mathrm{P}^n$

Note

- We try to generalize the results/conjectures on symmetric spaces to quandles...
- In this talk, we **define** the Euler characteristics for quandles.

Definition - (1/2)

Def (cf. Joyce)

The **displacement group** of a quandle (X, s) is

- $\mathrm{Dis}\,(X, s) := \langle \{ s_x \circ s_y^{-1} \mid x, y \in X \} \rangle_{\mathrm{grp}}$.

Def.

The **Euler characteristic** of a quandle (X, s) is

- $\chi^{\mathrm{qdl}}(X, s) := \inf \{ \#\mathrm{Fix}\,(g) \mid g \in \mathrm{Dis}\,(X, s) \}$.

Thm. (Kai-T. 2025)

(X, s) compact connected Riem symmetric space

$\Rightarrow \quad \chi^{\mathrm{qdl}}(X, s) = \chi^{\mathrm{top}}(X, s)$.

Definition - (2/2)

Idea of Proof (due to Hopf-Samelson)

- (X, s) cpt connected Riem symmetric space
  $\Rightarrow \mathrm{Dis}\,(X, s)$ cpt connected Lie group,
  $\curvearrowright X$ transitive;
- G compact connected Lie group, T maximal torus $\Rightarrow \chi^{\mathrm{top}}(G/K) = \#\mathrm{Fix}\,(T)$;
- Moreover, g $(\in T)$ generator
  $\Rightarrow \chi^{\mathrm{top}}(G/K) = \#\mathrm{Fix}\,(g)$.

Ex. (Sphere S^n)

- $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$;
- n odd: $T = \mathrm{SO}(2)^{(n+1)/2}$, no fixed points;
- n even: $T = \mathrm{SO}(2)^{n/2}$ fixes 2 points.

$\begin{bmatrix} SO(2) & & \\ & SO(2) & \\ & & SO(2) \end{bmatrix} \curvearrowright S^5$, $\begin{bmatrix} 1 & & \\ & SO(2) & \\ & & SO(2) \end{bmatrix} \curvearrowright S^4$

Ex. (Real projective space $\mathbb{RP}^n$)

- $\mathbb{RP}^n = \mathrm{SO}(n+1)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(n))$;
- Then the situation is similar to S^n...

Examples - (1/3)

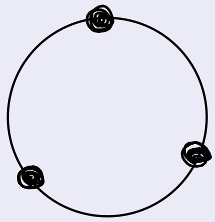
Ex. (trivial quandle)

- (X, s), $s_x := \mathrm{id}$ (**trivial quandle**);
- Then $\mathrm{Dis}\,(X, s) = \{\mathrm{id}\}$;
- Hence $\chi^{\mathrm{qdl}}(X, s) = \# X$.

Ex. (dihedral quandle)

- R_n : n-equal dividing points on S^1;
- R_n is naturally a subquandle of S^1;
- R_1, R_2 are trivial:
- $n \geq 3$: $\chi^{\mathrm{qdl}}(R_n) = 0 = \chi^{\mathrm{top}}(S^1)$.

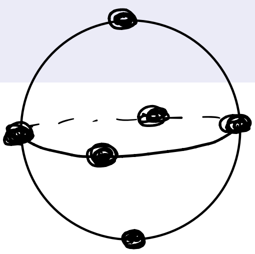
R_3



Ex. (discrete sphere)

- $DS^n := \{\pm e_1, \ldots, \pm e_{n+1}\}$;
- It is naturally a subquandle of S^n;
- This satisfies $\chi^{\mathrm{qdl}}(DS^n) = \chi^{\mathrm{top}}(S^n)$.

DS^2



Examples - (2/3)

Ex. (discrete torus)

- $DT^n := R_{m_1} \times \cdots \times R_{m_n}$;
- It is naturally a subquandle of T^n;
- $R_2 \times \cdots \times R_2$ is trivial;
- $\exists m_i \geq 3$ then $\chi^{\mathrm{qdl}}(DT^n) = 0 = \chi^{\mathrm{top}}(T^n)$.

Question

M compact connected Riemannian symmetric space

$\Rightarrow$ $\exists X$: a finite subquandle of M such that $\chi^{\mathrm{qdl}}(X) = \chi^{\mathrm{top}}(M)$?

Examples - (3/3)

Note

- χ^{top}(cpt connected nontrivial Lie group) $= 0$;
- There are two related results.

Ex. (Core quandle $\mathrm{Core}(G)$)

- A group G is a quandle by $s_g(h) := gh^{-1}g$;
- $G \cong (\mathbb{Z}_2)^k$ then $\mathrm{Core}(G)$ is trivial;
- Otherwise, $\chi^{\mathrm{qdl}}(\mathrm{Core}(G)) = 0$.

Ex. (Generalized Alexander quandle $Q(G, \sigma)$)

- A group G is a quandle by

$$s_g(h) := g\sigma(g^{-1}h) \qquad (\sigma \in \mathrm{Aut}(G)).$$

- $\sigma = \mathrm{id}$ then $Q(G, \sigma)$ is trivial;
- Otherwise, $\chi^{\mathrm{qdl}}(Q(G, \sigma)) = 0$.

Problems - (1/2)

Problems

- Calculate $\#_2$ and χ^{qdl} for more examples;
- Condition for $\chi^{\mathrm{qdl}}(X,s) = 0$?
- $\#_2(X,s) \geq \chi^{\mathrm{qdl}}(X,s)$?
- $\#_2(X,s) \equiv \chi^{\mathrm{qdl}}(X,s) \pmod 2$?

Prop. (cf. Saito-Sugawara 2025)

(V,E): directed simple graph,
$V = \{A_1, \ldots, A_n\}$, A_i abelian group,
Each $(A_i, A_j) \in E$ is labeled by $a_j \in A_j$

$\Rightarrow$ $X := A_1 \sqcup \cdots \sqcup A_n$ becomes a quandle by
$s_{a_i}(a_j) = a_j + \mathrm{label}(A_i, A_j)$

Ex.

(X,s) : quandle constructed from $\mathbb{Z}_m \xrightarrow{1} \mathbb{Z}_n$

$\Rightarrow$ $\#_2(X,s) = \underset{\text{max}}{\cancel{\min}}\{m,n\}$, $\chi^{\mathrm{qdl}}(X,s) = m$

Problems - (2/2)

Note

- $\#_2(X,s) \geq \chi^{\mathrm{qdl}}(X,s)$ for the above example;
- $\#_2(X,s) \not\equiv \chi^{\mathrm{qdl}}(X,s) \pmod 2$ in general...

Problem

Find a condition for quandles such that

- $\#_2(X,s) \geq \chi^{\mathrm{qdl}}(X,s)$;
- $\#_2(X,s) \equiv \chi^{\mathrm{qdl}}(X,s) \pmod 2$.

Problem (fantasize)

Prove a theorem for compact symmetric spaces using finite quandles, e.g.,

- find a nice finite subquandle
  ($\#_2$ and χ^{qdl} are same with the ambient);
- prove (in)equalities for such finite quandles.

Appendix

Thank you very much!

Ref.

- Bang-Yen Chen, Tadashi Nagano; "A Riemannian geometric invariant and its applications to a problem of Borel and Serre", Trans. Amer. Math. Soc. (1988)
- Konomi Furuki, Hiroshi Tamaru; "Homogeneous quandles with abelian inner automorphism groups and vertex-transitive graphs", Internat. Elect. J. Geom. (2024)
- Takuya Saito, Sakumi Sugawara; "Homogeneous quandles with abelian inner automorphism groups", J. Algebra (2025)
- Ryoya Kai, Hiroshi Tamaru; "On the Euler characteristics for quandles", Internat. J. Math. (2025)
- Hiroshi Tamaru; "Some properties of finite two-point homogeneous quandles", Procedings of TJC7, to appear
- Hiroshi Tamaru; "Discrete symmetric spaces and quandles", Sugaku Expositions, to appear