Nilpotent Lie algebras obtained by quivers and Ricci solitons

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Abstract

- quiver から nilpotent Lie algebra を構成.
  （他の面白い構成方法を募集中）
- nilsoliton の存在非存在の判定をした.
  （quiver から行列を作る. よい説明募集中）
- nilsoliton が存在 $\Leftrightarrow$ 上の行列が "positive".
  （よい判定法 / 関連する話題を募集中）

- 溝口史華, 北山陽菜との共同研究に基づく.

Background - (1/2)

Def. (ARS = algebraic Ricci soliton)

A metric Lie algebra $(\mathfrak{g}, \langle , \rangle)$ is **ARS**

$$:\Leftrightarrow \quad \exists c \in \mathbb{R},\ \exists D \in \underbrace{\mathrm{Der}(\mathfrak{g})}_{\text{derivation}} : \ \mathrm{Ric} = c \cdot \mathrm{id} + D.$$

derivation, $D[\cdot,\cdot] = [D(\cdot),\cdot] + [\cdot, D(\cdot)]$

Basic Question

Study ARS on **nilpotent** Lie algebras:

- Classification is almost impossible;
- Construct examples;
- Give a condition for the existence.

Note (why ARS? (1))

$(\mathfrak{n}, \langle , \rangle)$ is nilpotent ARS

$\Leftrightarrow$ $(N, \langle , \rangle)$ (1-conn., left-inv.) is **Ricci soliton**

i.e., $\exists c \in \mathbb{R},\ \exists X \in \mathfrak{X}(N) : \ \mathrm{ric} = cg + \mathcal{L}_X g.$

vector field Lie derivative

Note (why ARS? (2))

$(\mathfrak{n}, \langle , \rangle)$ is nilpotent **Einstein**

$\Rightarrow$ $\mathfrak{n}$ is abelian (and it is flat).

Background - (2/2)

Def.

A Lie algebra $\mathfrak{g}$ is

- **2-step nilpotent** if $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]=0$, $[\mathfrak{g},\mathfrak{g}]\neq 0$;
- **3-step nilpotent** if $[\mathfrak{g},[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]]=0$ and not 2-step;
- **solvable** if $[\mathfrak{g},\mathfrak{g}]$ is nilpotent.

Ex.

The following are solvable and (m-step) nilpotent:

$$\mathfrak{sl}(m+1,\mathbb{R}) \supset \underbrace{\left\{\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}\right\}}_{\text{solvable}} \supset \underbrace{\left\{\begin{pmatrix} 0 & \cdots & * \\ & \ddots & \vdots \\ 0 & & 0 \end{pmatrix}\right\}}_{\text{nilpotent}}$$

Thm (Böhm-Lafente 2023)

(M,g) : homogeneous Ricci soliton with $c<0$

$\Rightarrow$ (M,g) is a solvmanifold (simply-connected solvable Lie group with a left-inv Riem metric).

Thm. (Lauret 2011)

Ricci soliton

"Study on solvable RS can reduced to nilpotent RS."

Examples - (1/4)

Ex. 1: rep of Clifford algebra $\to$ nilpotent (H-type)

Note

For two-step nilpotent $(\mathfrak{n}, \langle , \rangle)$,

- $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{z}$ center, $\mathfrak{v} := \mathfrak{z}^{\perp}$;
- $J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$, by $\langle J_Z(X), Y \rangle = \langle Z, [X, Y] \rangle$.

Def. (Kaplan 1980)

$(\mathfrak{n}, \langle , \rangle)$: two-step nilpotent is of **H-type**

$:\Leftrightarrow \quad \forall Z \in \mathfrak{z},\ J_Z^2 = -\langle Z, Z \rangle \cdot \mathrm{id}$

$\Leftrightarrow \quad J$ can be extended to $\widetilde{J} : \mathrm{Cl}\,(\mathfrak{z}, \langle , \rangle) \to \mathrm{End}(\mathfrak{v})$.

$\cup$

$\mathfrak{z} \xrightarrow{J}$

Ex

- $\mathrm{Cl}_1 \cong \mathbb{C} \curvearrowright \mathbb{C}^n \to (2n+1)$-dim Heisenberg;

Thm. (Boggino 1985, Lauret 2003)

- Every H-type Lie algebra is ARS.

Examples - (2/4)

Ex. 2: simple (directed) graph $\to$ two-step nilpotent

Def. (Dani-Mainkar 2005)

For $G = (V, E)$: simple directed graph ($E \subset V \times V$),

- $\mathfrak{n}_G := \mathrm{span}(E \cup V)$ is two-step nilpotent by $[v, w] = e$, when e is an edge from v to w.

Ex.

3-dim Heisenberg $\mathfrak{h}^3$ is obtained by

$G : \underset{v}{\bullet} \xrightarrow{e} \underset{w}{\bullet} \quad \rightsquigarrow \quad \mathfrak{n}_G = \mathrm{span}\{v, w, e\}$ with $[v, w] = e$

Thm. (Lauret-Will 2011)

For $G = (V, E)$, $\mathfrak{n}_G$ admits ARS

$\Leftrightarrow$ $M_G := 3I + \mathrm{Adj}\,(\mathrm{Line}_G)$ is positive (i.e., $\exists v \in (\mathbb{R}_{>0})^n : M_G v = [1]_n$)

$[1]_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$

Examples - (3/4)

Ex. 3: parabolic subalgebra $\to$ nilpotent

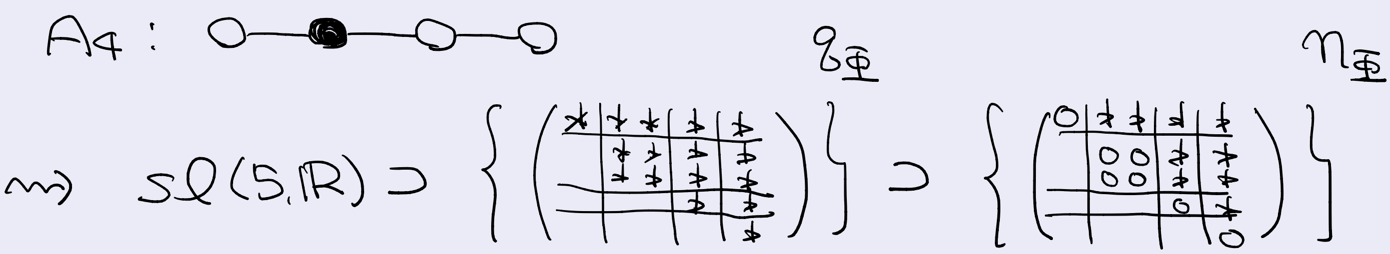
Fact

For a real semisimple Lie algebra $\mathfrak{g}$,

- choosing a subset Φ of simple roots in the restricted root system, one has a parabolic subalgebra $\mathfrak{q}_\Phi$;
- $\mathfrak{q}_\Phi$ has the Langlands decomposition $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ with $\mathfrak{n}_\Phi$ nilpotent.

Ex.

Typical ex (for $\mathfrak{sl}(n, \mathbb{R})$) is given by "block dec.":



Thm. (T. 2011)

- Every $\mathfrak{n}_\Phi$ admits ARS.

Examples - (4/4)

Summary

- H-type: 2-step
- graph $\to$ 2-step
- parabolic $\to$ arbitrary high-step

Note

- nilpotent Lie algebra の分類は不可能;
- 様々な良いクラスの例が見付かると嬉しい;
- “良い対象” から構成できると嬉しい;
- 様々な構成方法を募集中

Quivers - (1/3)

quiver (originally): a container for holding arrows

Def.

$Q = (V, E, s, t)$ is a **quiver** if

- V, E : sets (vertices and edges);
- $s, t : E \to V$: maps (source and target).

Ex.

$$\begin{tikzcd} \underset{s(a)}{\bullet} \arrow[r, "a"] & \underset{t(a)}{\bullet} \arrow[r, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & \underset{t(b_1)}{\bullet} \arrow[r, "c"] & \underset{t(c)}{\bullet} \end{tikzcd}$$

$t(a) = s(b_1) = s(b_2)$

$t(b_1) = t(b_2) = s(c)$

Def.

$\alpha_1 \cdots \alpha_m$ is a **path** of a quiver if

- $\alpha_i \in E$ and $t(\alpha_i) = s(\alpha_{i+1})$ for $\forall i$.

Ex.

$$\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \arrow[r, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & \bullet \arrow[r, "c"] & \bullet \end{tikzcd}$$

path: a, b_1, b_2, c

ab_1, ab_2, b_1c, b_2c

ab_1c, ab_2c

Quivers - (2/3)

Def.

For a quiver Q, the **path algebra** is defined by

- Space: $\mathrm{span}(\mathrm{Path}(Q))$,
  where $\mathrm{Path}(Q) := \{\text{all paths in } Q\}$;
- Product: For $\alpha, \beta \in \mathrm{Path}(Q)$, define

$$\alpha \cdot \beta := \begin{cases} \alpha\beta & (\text{if } t(\alpha) = s(\beta)), \\ 0 & (\text{others}). \end{cases}$$

Def.

For a quiver Q, define the Lie algebra $\mathfrak{n}_Q$ by

- $\mathfrak{n}_Q := \mathrm{span}(\mathrm{Path}(Q))$, $[\alpha, \beta] := \alpha \cdot \beta - \beta \cdot \alpha$.

Ex.

$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \quad \Rightarrow \quad \mathfrak{n}_Q = \mathrm{span}\{a, b, ab\}$

with $[a, b] = a \cdot b - b \cdot a = ab$

$$\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \arrow[r, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & \bullet \arrow[r, "c"] & \bullet \end{tikzcd} \quad \Rightarrow \quad \mathfrak{n}_Q : \text{3-step nilpotent}$$

$$\begin{tikzcd} \bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \end{tikzcd} \quad \Rightarrow \quad \dim \mathfrak{n}_Q = +\infty$$

Quivers - (3/3)

Def

A path α is

- a **cycle** if $t(\alpha) = s(\alpha)$;
- **acyclic** if α does not contain a cycle.

Prop.

For a finite quiver Q without cycles,

- $\exists m =$ max of length of paths;
- $\mathfrak{n}_Q$ is an m-step nilpotent Lie algebra.

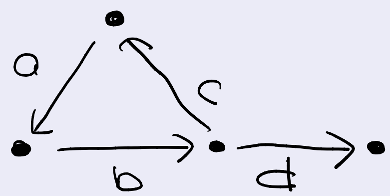
Def.

For a finite quiver Q, denote by $\mathfrak{n}_Q^A$ the span of

- $\mathrm{Apath}(Q) := \{$acyclic path in $Q\}$
- with $[,]$ defined by $\alpha \cdot \beta = \alpha\beta$ if acyclic path

$[\alpha, \beta] = \alpha \cdot \beta - \beta \cdot \alpha$ $\quad$ ($= 0$ otherwise)

Ex.



$\Rightarrow \mathfrak{n}_Q^A = \text{span}\left\{ \begin{matrix} a, & b, & c \\ ab, & bc, & ca \\ abd & & \end{matrix} \right\}$

with $[ab, c] = 0$, ...

Matrices - (1/3)

Thm. (Mizoguchi-T. 2025)

Q : a finite quiver Q without cycles

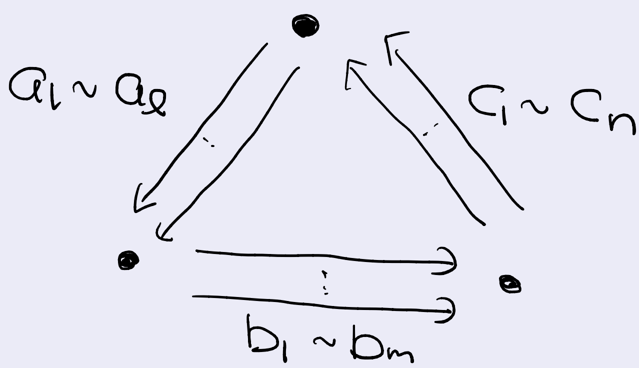
$\Rightarrow$ $\mathfrak{n}_Q$ always admits algebraic Ricci soliton.

Note

This is not true for $\mathfrak{n}_Q^A$.

Prop. (Kitayama)

Consider $\mathfrak{n}_Q^A$ for the following Q:



Then

- $\exists$ ARS $\Leftrightarrow l + m - n, m + n - l, n + l - m \geq 0$;
- when $l = 1$, $\exists$ ARS $\Leftrightarrow |m - n| \leq 1$.

Matrices - (2/3)

Note

We give a condition for $\mathfrak{n}_Q^A$ to admit ARS by **matrices**.

Setting

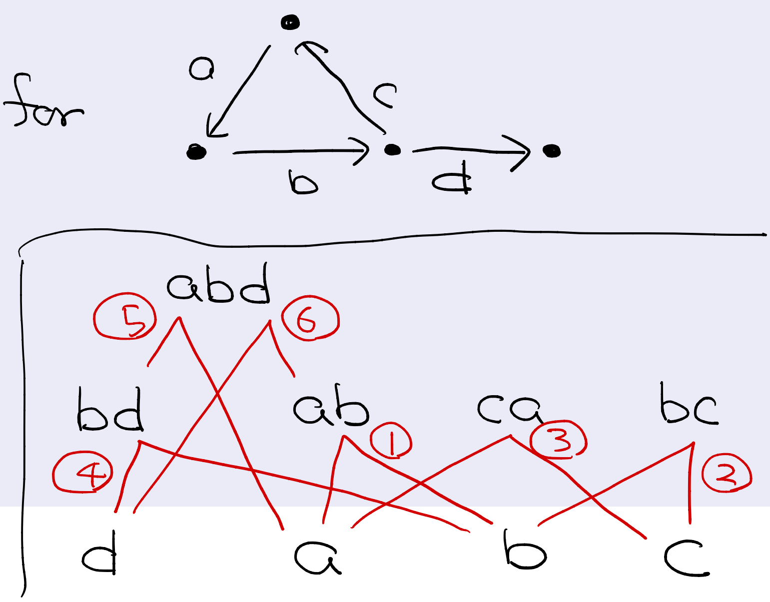
For the Lie algebra $\mathfrak{n}_Q^A$,

- $W_Q := \{\{\alpha, \beta\} \mid [\alpha, \beta] \neq 0\} \subset \mathrm{Apath}(Q)^2$;
- Write $W_Q = \{w_1, \ldots, w_m\}$;
- For $w_j = \{\alpha, \beta\}$,
  define $\overline{w}_j \in \mathrm{Apath}(Q)$ by $\overline{w}_j = \pm[\alpha, \beta]$;

Ex.

$w_1 = \{a, b\}$, $\overline{w}_1 = ab$
$w_2 = \{b, c\}$, $\overline{w}_2 = bc$
$w_3 = \{c, a\}$, $\overline{w}_3 = ca$
$w_4 = \{b, d\}$, $\overline{w}_4 = bd$
$w_5 = \{a, bd\}$, $\overline{w}_5 = abd$
$w_6 = \{ab, d\}$, $\overline{w}_6 = abd$

for

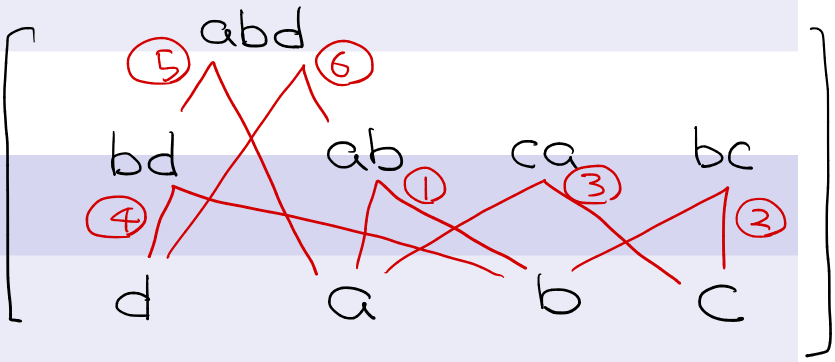


Matrices - (3/3)

Def.

Define $M_Q = (x_{ij}) \in M_m(\mathbb{Z})$ by

$$x_{ij} := \begin{cases} 3 & \text{if } i = j \\ 1 & \text{if } \#(w_i \cap w_j) = 1 \\ 1 & \text{if } \overline{w}_i = \overline{w}_j \\ -1 & \text{if } \overline{w}_i \in w_j \text{ or } \overline{w}_j \in w_i \\ 0 & \text{otherwise} \end{cases}$$



Ex

For the above Q,

$$M_Q = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & -1 \\ 1 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 3 & -1 & 1 \\ 1 & 0 & 1 & -1 & 3 & 1 \\ -1 & 0 & 0 & 1 & 1 & 3 \end{pmatrix}$$

Question

Any better explanation for this construction?

Condition - (1/2)

Thm (Kitayama)

$\mathfrak{n}_Q^A$ admits ARS

$\Leftrightarrow$ M_Q is "positive"
i.e., $\exists v \in (\mathbb{R}_{>0})^m : M_Q v = [1]_m$

Key Notion

Def (Nikolayevsky 2011)

$\mathfrak{n}$ nilpotent, $\{X_1, \dots, X_n\}$ its basis, C_{ij}^k structure const.
This basis is **nice**

$:\Leftrightarrow$ $\forall i, j, \#\{k \mid C_{ij}^k \neq 0\} \leq 1;$
$\forall i, k, \#\{j \mid C_{ij}^k \neq 0\} \leq 1.$

The proof of our thm uses

Thm (Nikolayevsky 2011)

$\mathfrak{n}$ nilpotent with nice basis $\{X_1, \dots, X_n\}$.
Then $\mathfrak{n}$ admits ARS

$\Leftrightarrow$ 構造定数から作った行列が positive;

$\Leftrightarrow$ 構造定数から作った $F \subset \mathbb{R}^n$ (有限) に対し，
0 の $\mathrm{aff}(F)$ への射影が $\mathrm{Conv}(F)$ の内点

Condition - (2/2)

Thm (Nikolayevsky 2011)

Let $\mu \in \mathcal{V} := \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$.
Then the Lie algebra $(\mathbb{R}^n, \mu)$ admits ARS
$\Leftrightarrow$ the orbit $G_\phi.\mu \subset \mathcal{V}$ is closed.

$\Downarrow$ The closedness is cotrolled by $\mathrm{Conv}(F)$

Recall (Nikolayevsky 2011)

$\mathfrak{n}$ (nilpotent with nice basis) admits ARS
$:\Leftrightarrow$ 構造定数から作った行列が positive;
$\Leftrightarrow$ 構造定数から作った $F \subset \mathbb{R}^n$ (有限) の条件.

$\bigcap$

Note (Y. Hashimoto, 2025+)

$\exists$ a criterion for nilpotent $\mathfrak{n}$ without nice basis.

$\Uparrow$ Hilbert-Mumford criterion, GIT, ...

Thm (Y. Hashimoto, 2025+)

$\mathfrak{n}$ nilpotent, $Y := \{\langle , \rangle$ on $\mathfrak{n}$ with some condition$\}$.
Then $\mathfrak{n}$ admits ARS
$\Leftrightarrow$ "Kempf-Ness" $E_\mu : Y \to \mathbb{R}$ has a critical pt.

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Thank you very much!